Mock modularity of Calabi-Yau threefolds

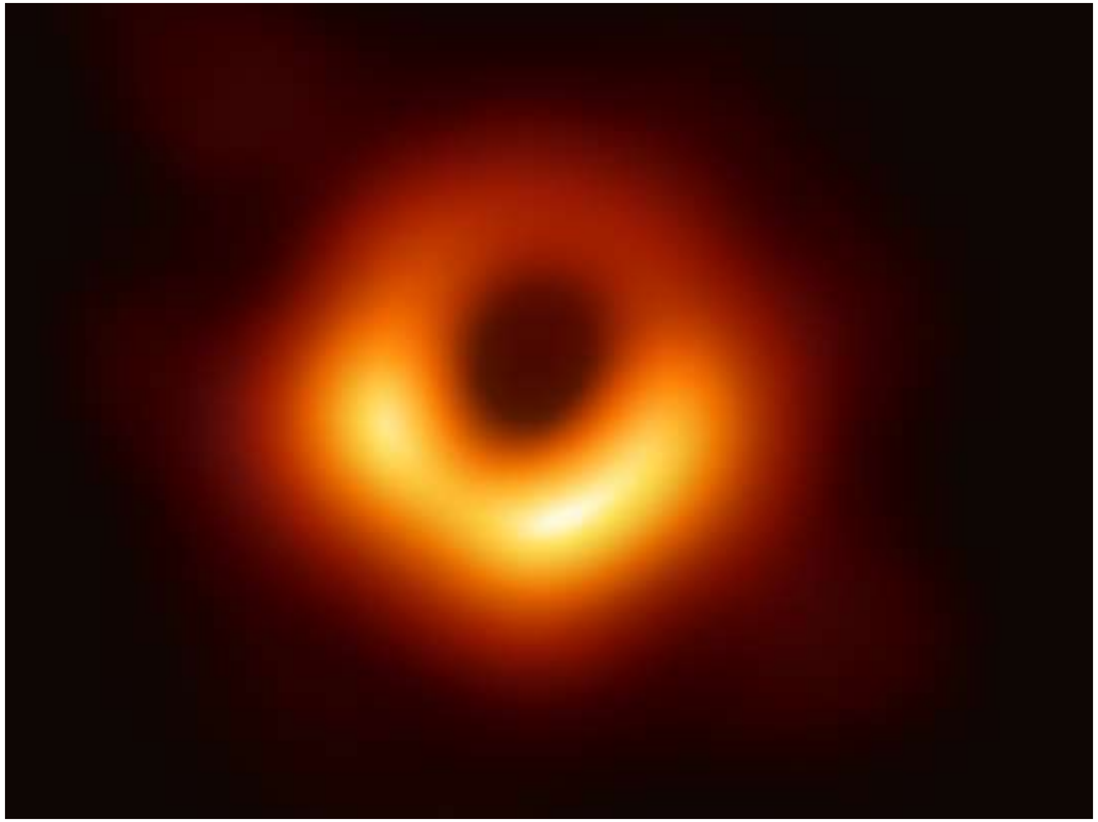
Khalil Bendriss

Laboratoire Charles Coulomb, Montpellier

October 2024

Based on joint work with Sergey Alexandrov to appear soon…

Black holes



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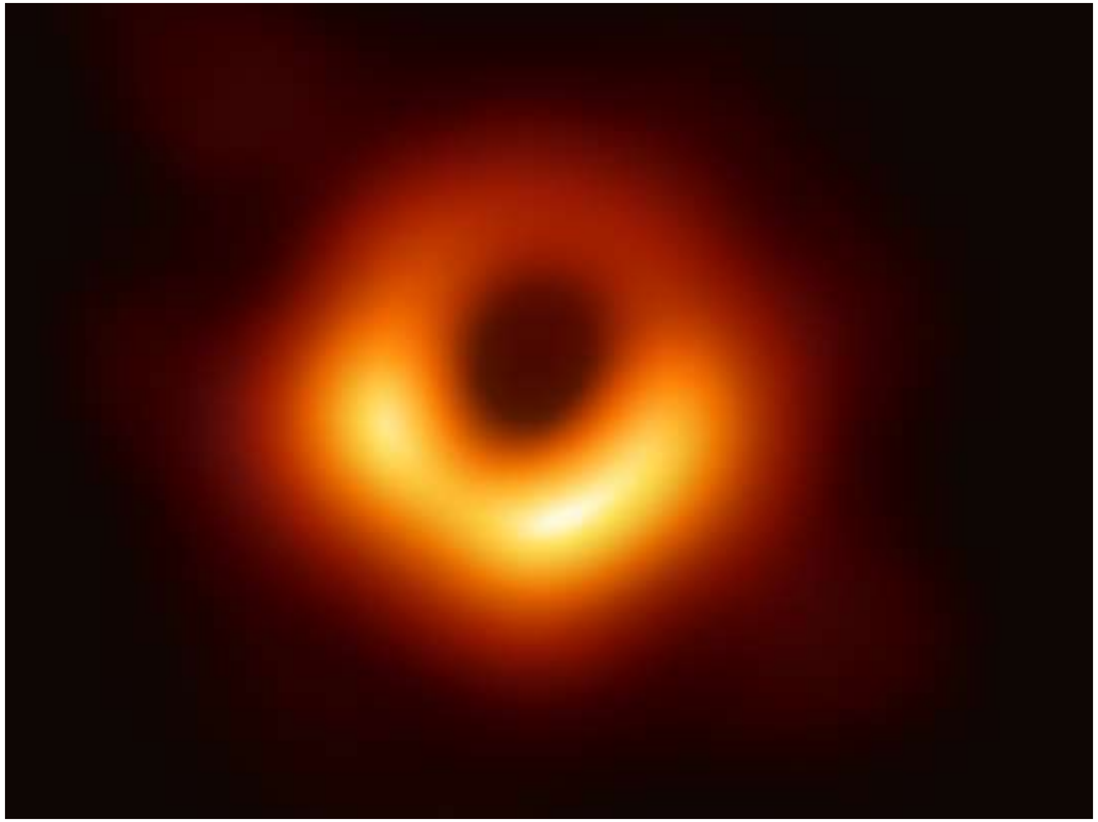

- Thermodynamical objects!
  [Bekenstein ‘72, Hawking ‘74]

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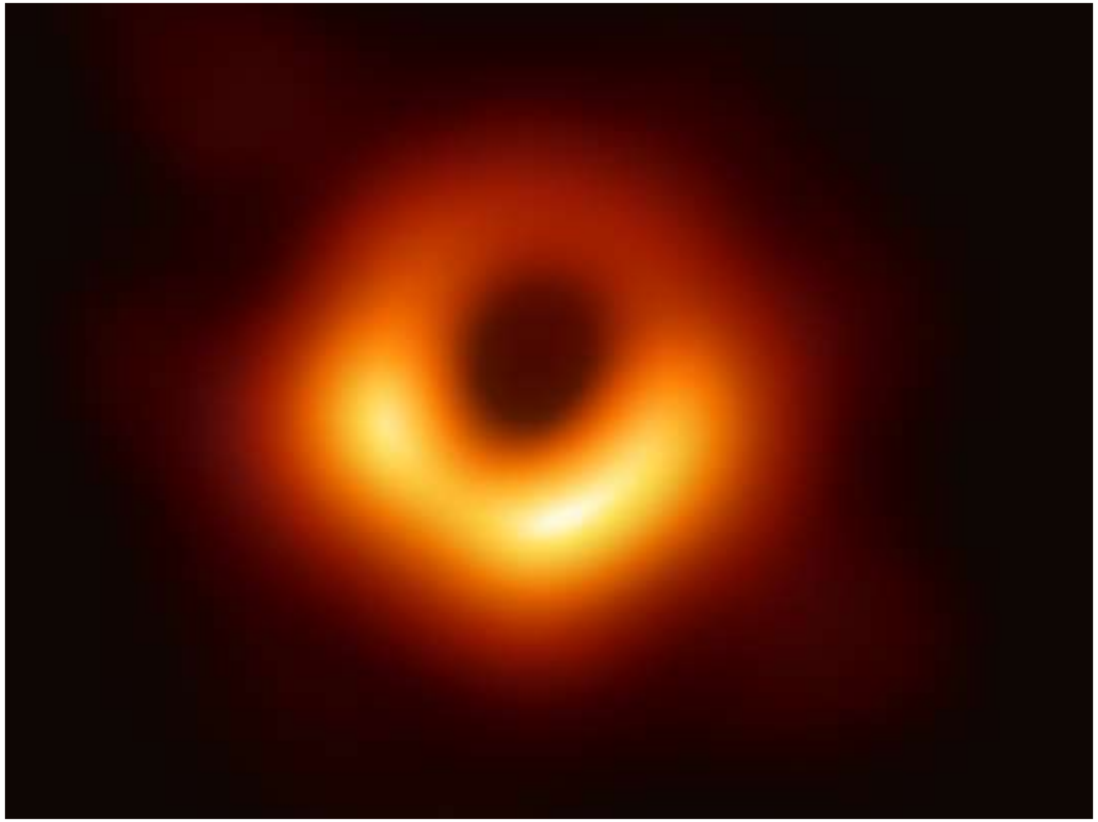

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- Entropy S

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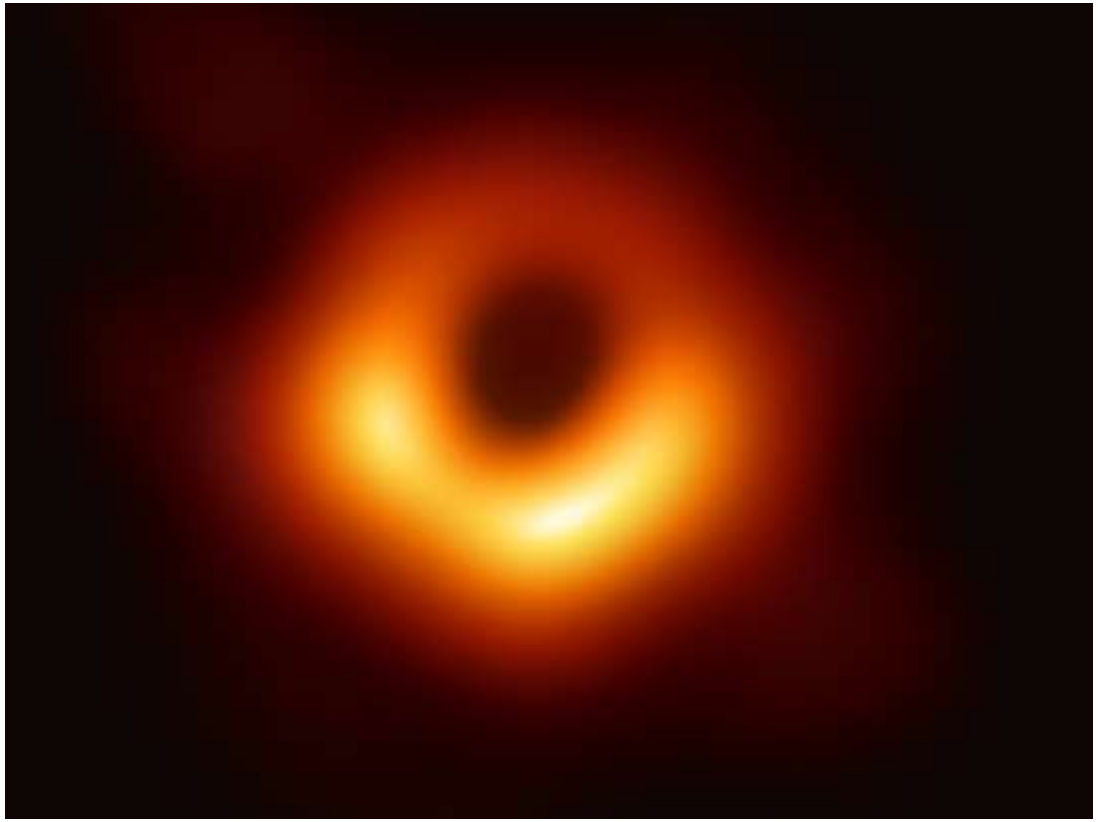
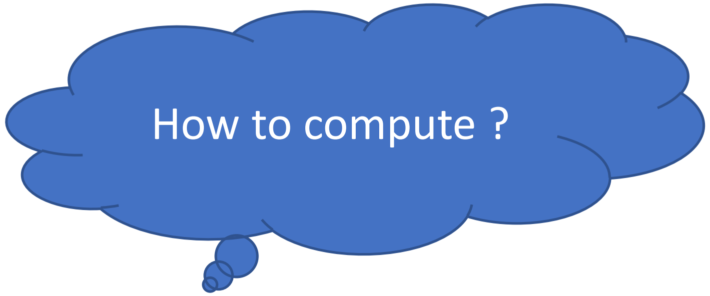


- Thermodynamical objects! [Bekenstein ‘72, Hawking ‘74]
- Entropy S
- $\log(S)$: Black hole microstates.



String theory

- String theory lives in 10 dimensions
- Compactify on a 6d manifold X.
- Often X is a Calabi-Yau threefold.

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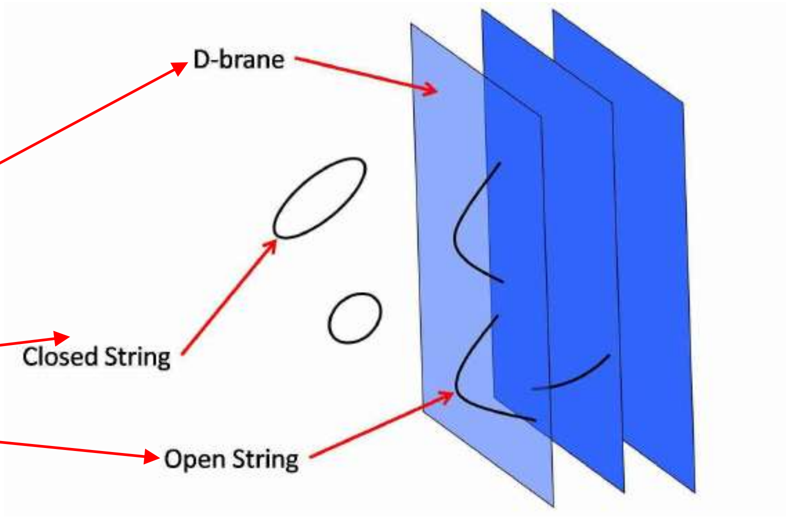

String theory

- String theory lives in 10 dimensions
- Compactify on a 6d manifold X.
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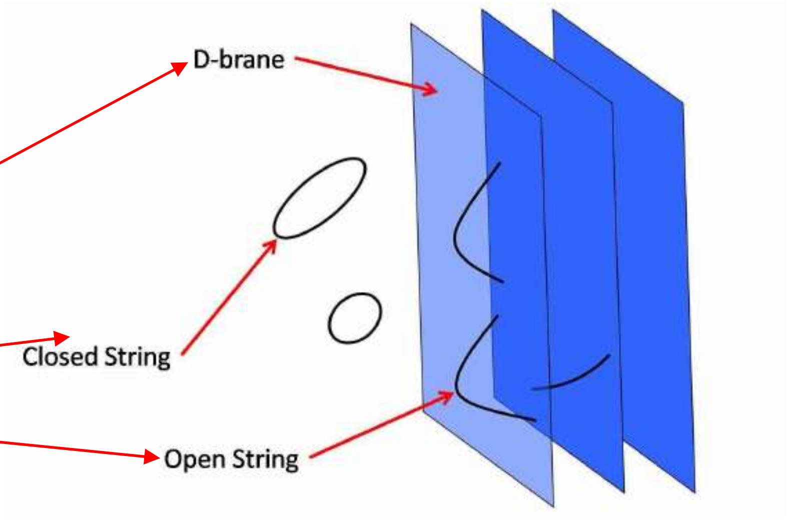
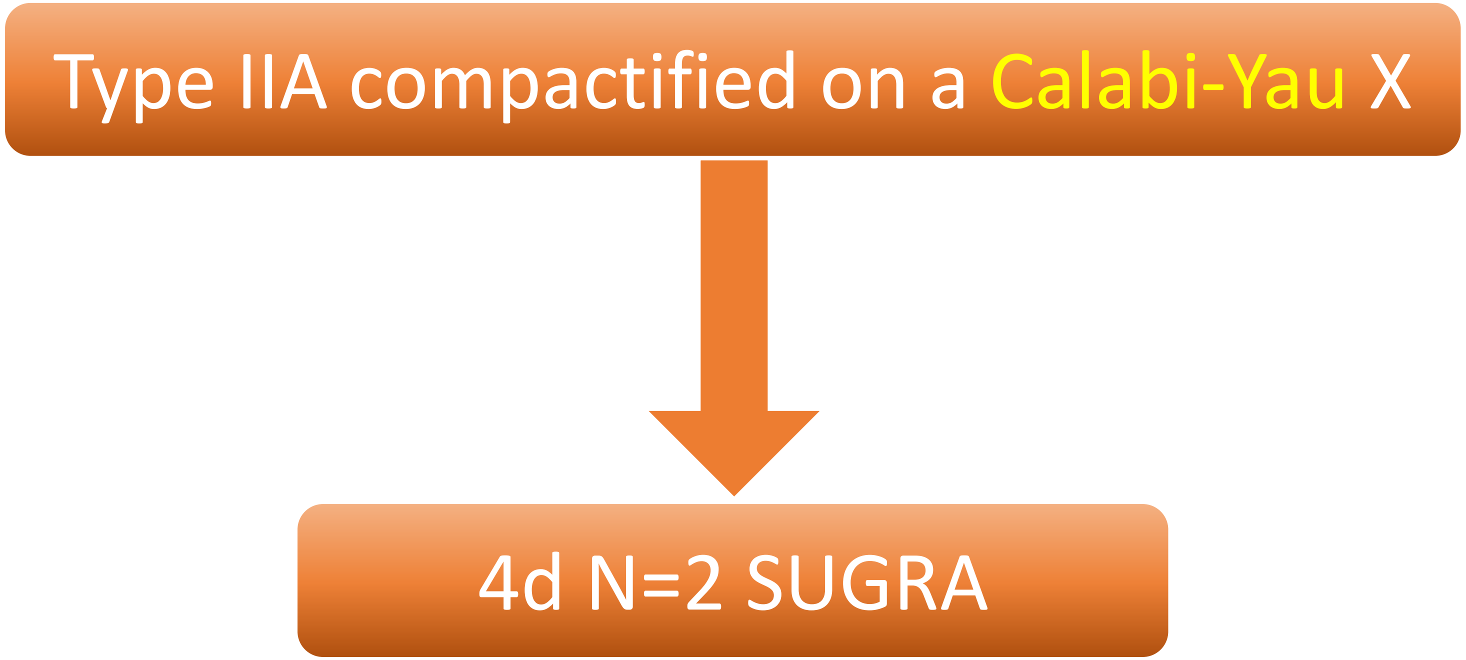
- A theory of extended objects
- Dp-brane: $p+1$ dimensional object



Type IIA compactified on a Calabi-Yau X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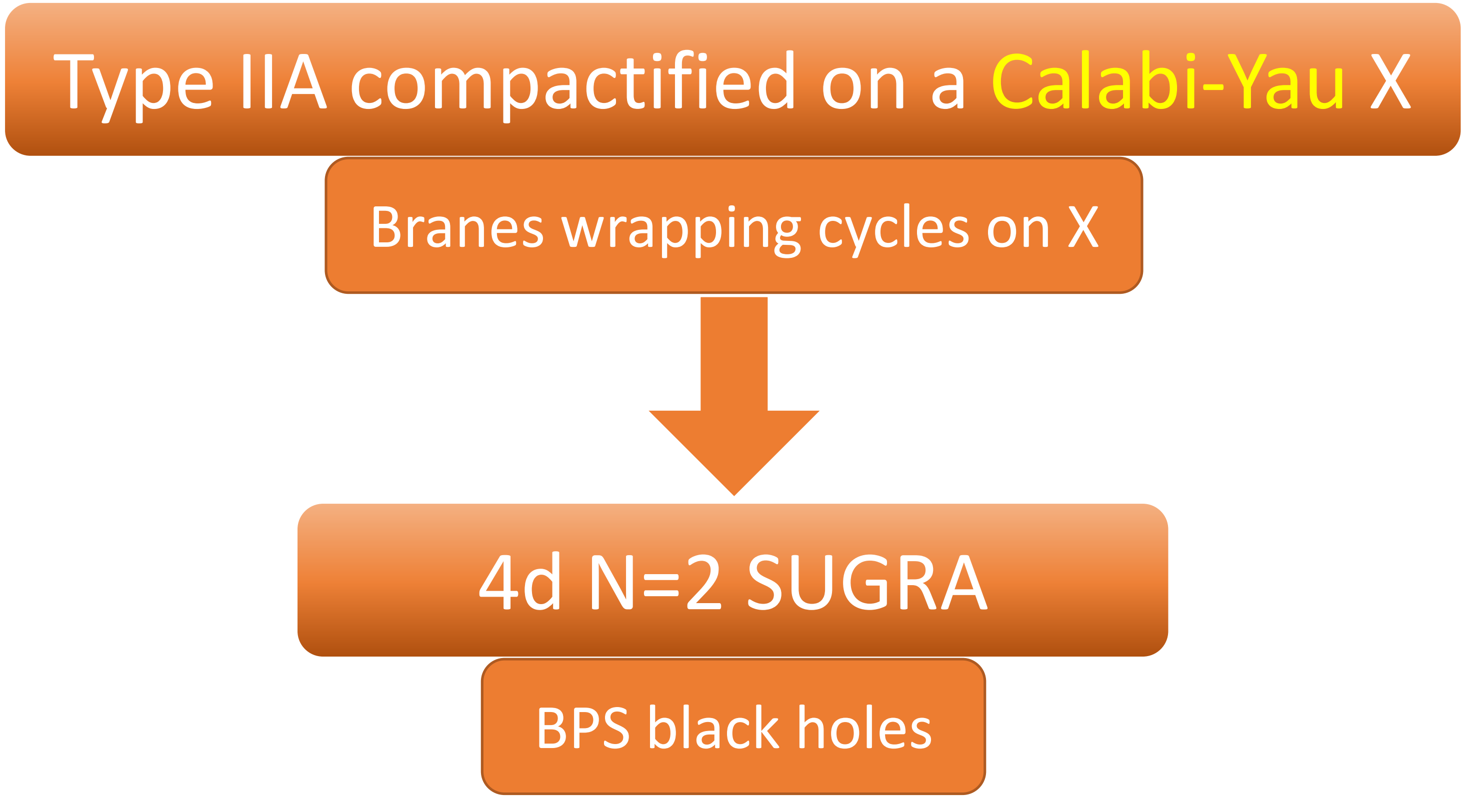
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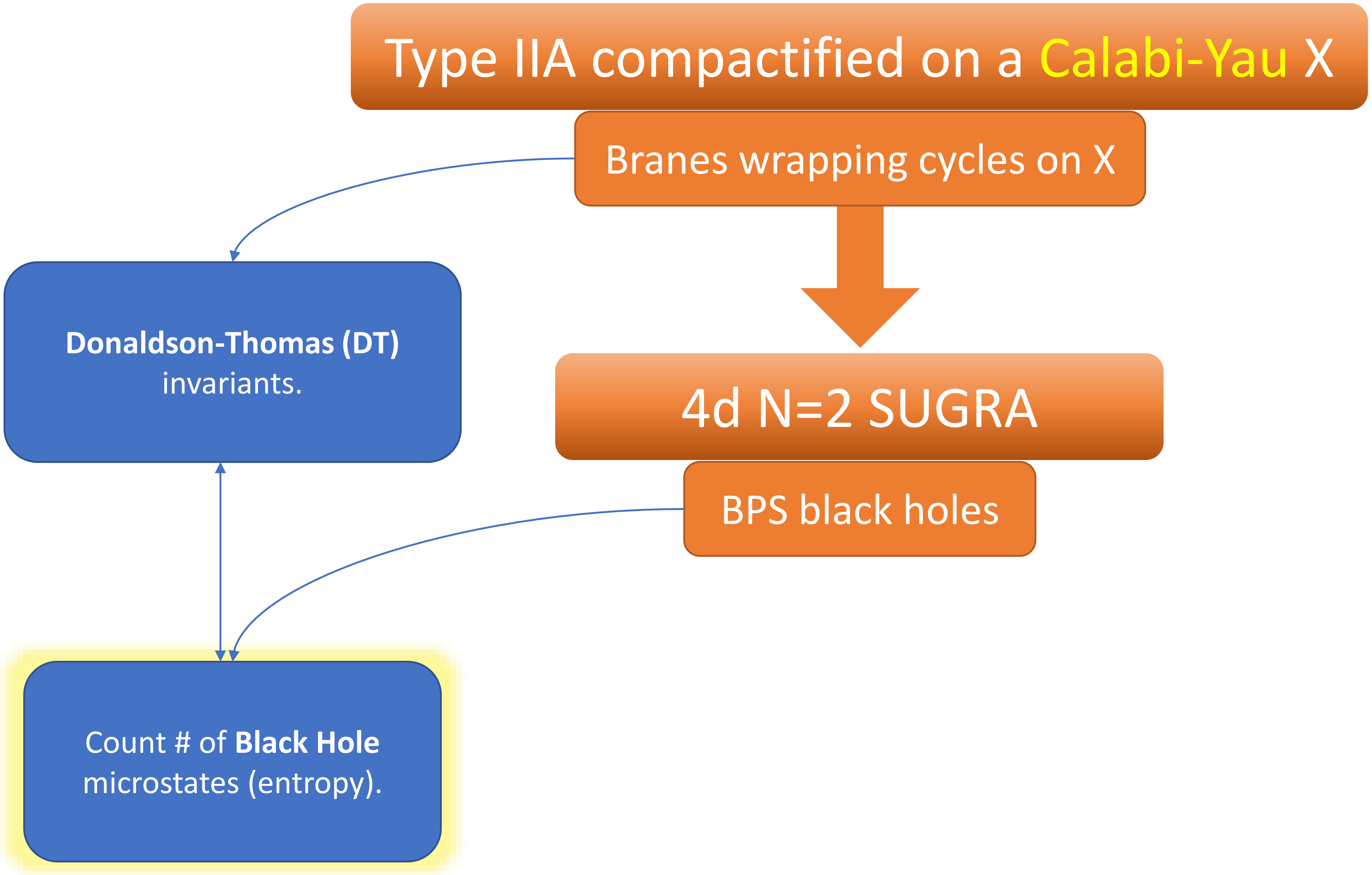
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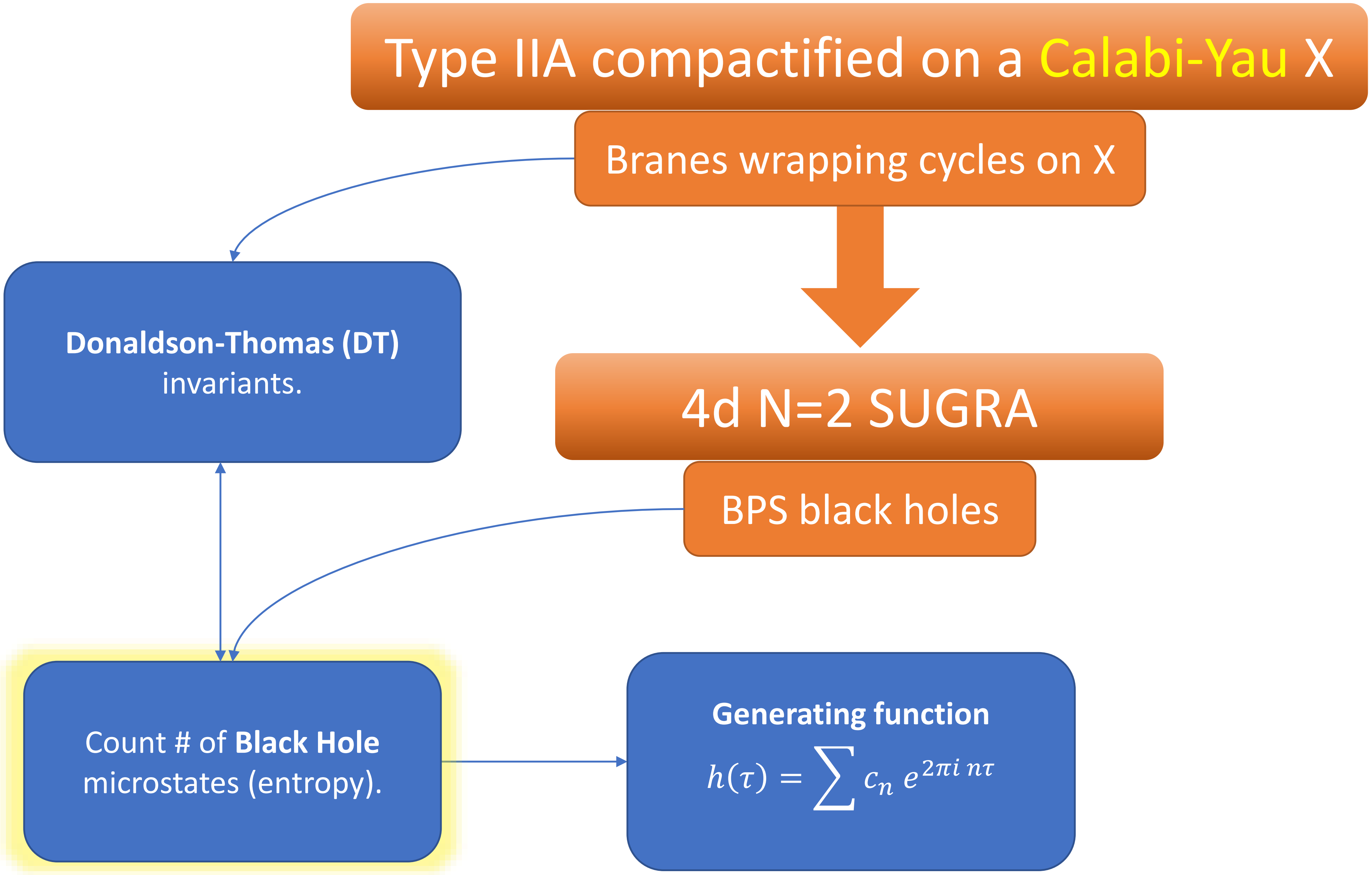
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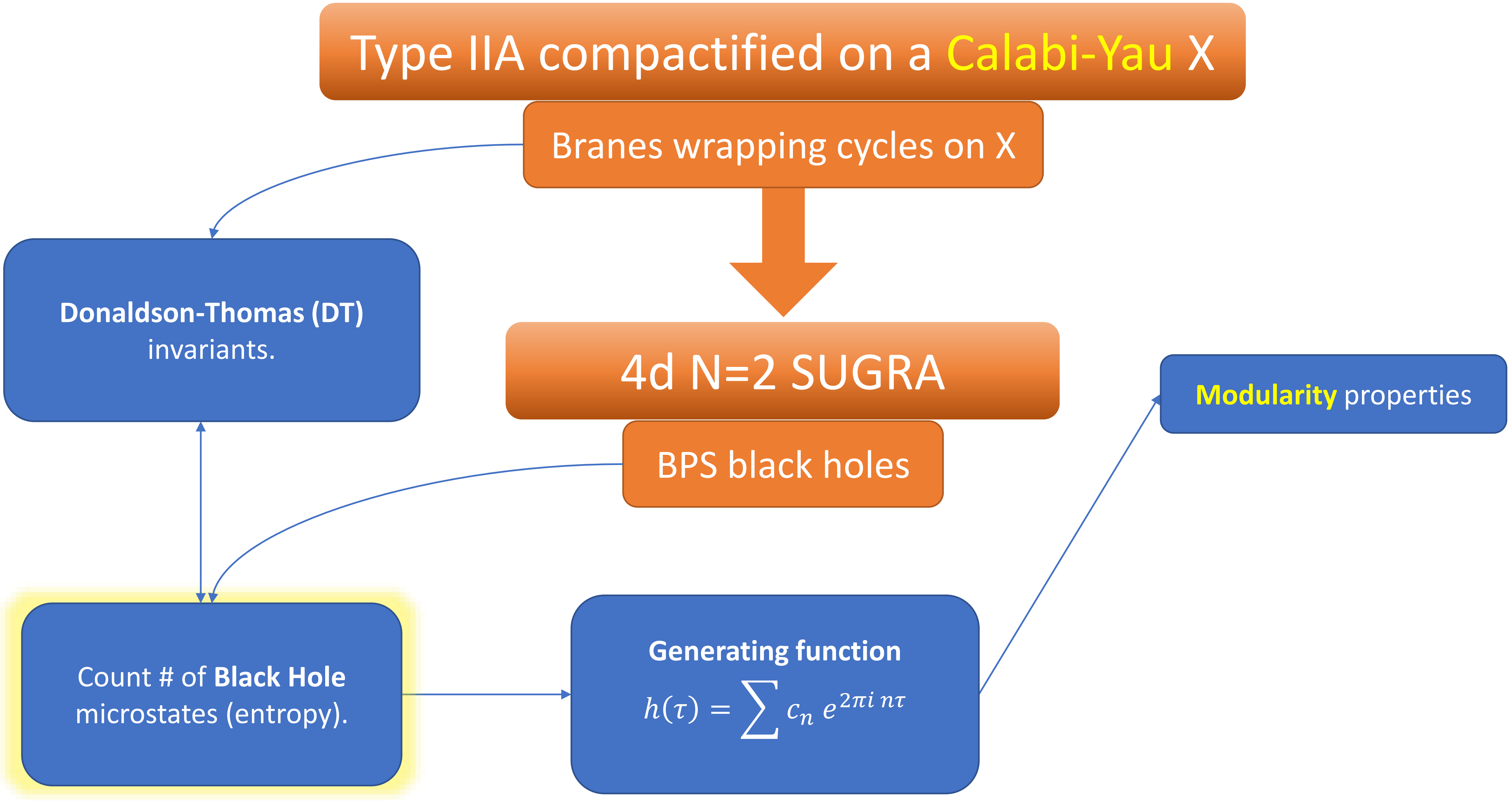
$h(\tau) = \sum c_n \, e^{2\pi i \, n\tau}$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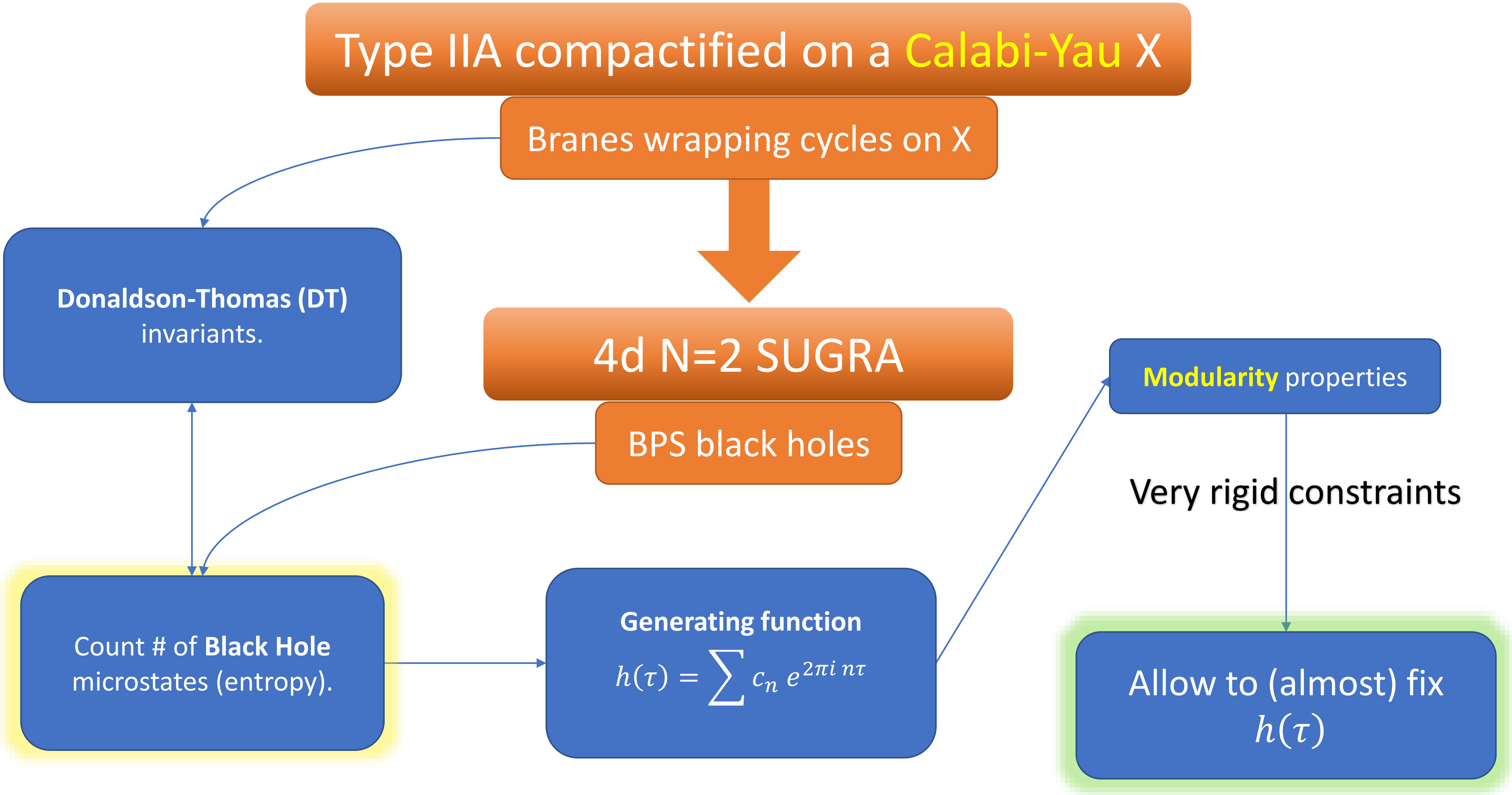
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Modularity properties
Very rigid constraints
Allow to (almost) fix $h(\tau)$

Outline

I. Modularity

II. DT invariants

III. Constraining the generating function

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I. **Modularity**

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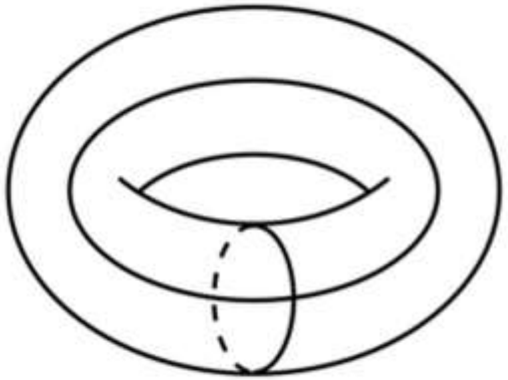
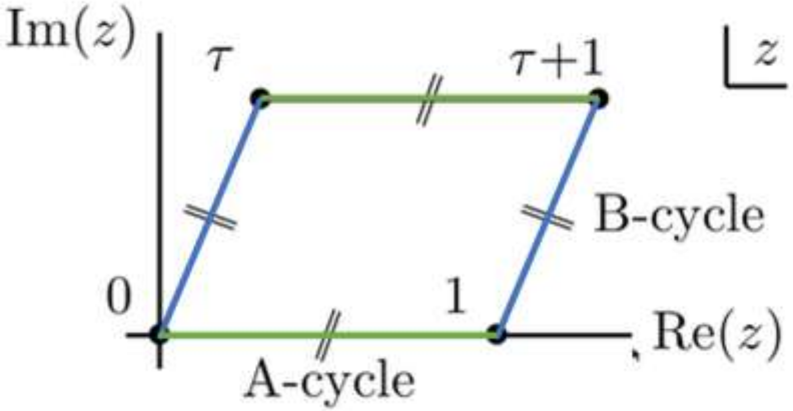
Tori, lattices and the modular group





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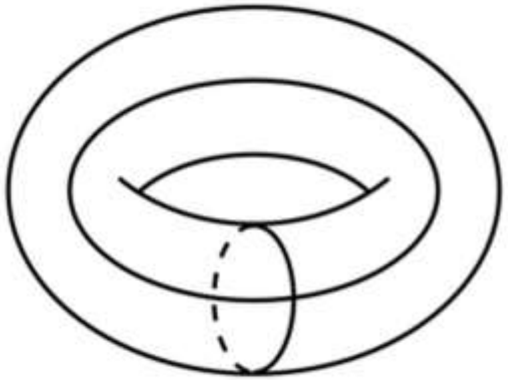

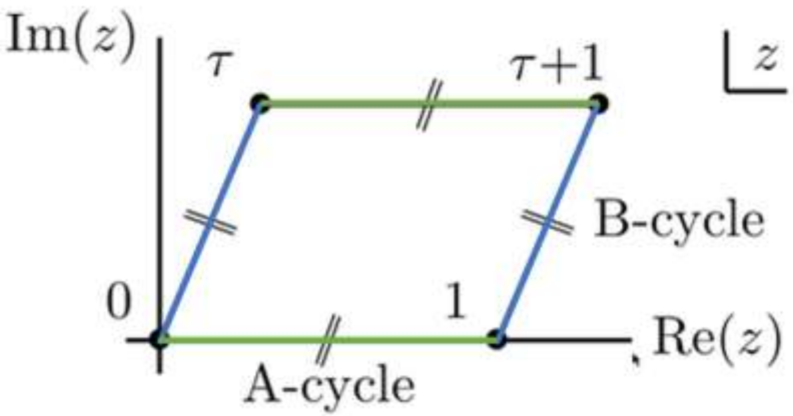

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$SL(2,\mathbb{Z})$ The modular group

Keeps the torus invariant

Preserves the orientation

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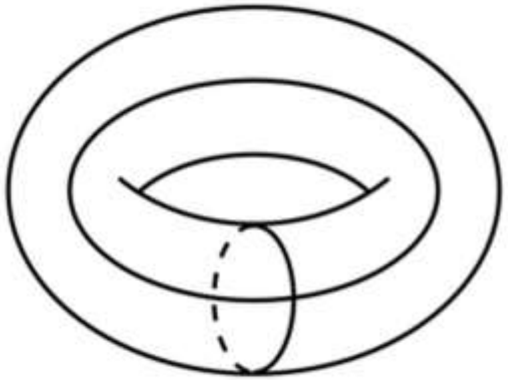

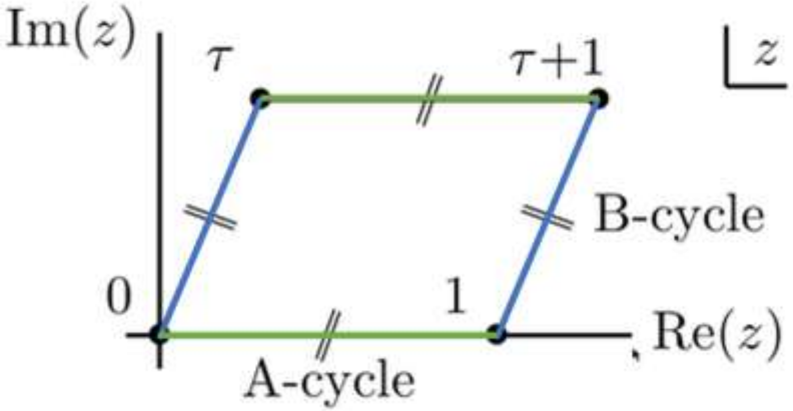

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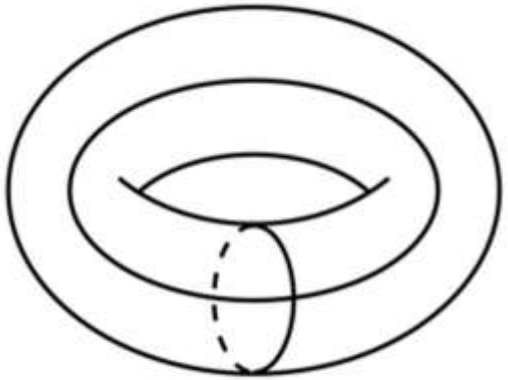

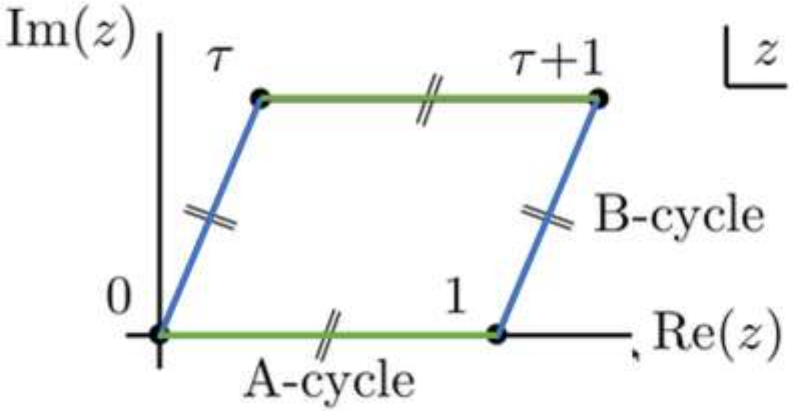

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Modularity ?

Modular forms

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Modular forms have a Fourier expansion: $$f(\tau) = \sum_{n=n_0}^{\infty} c_n e^{2\pi i \, n \, \tau}$$ $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to f(\tau + 1) = f(\tau)$$	

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Mock modular Modular forms

Properties	Characterestics
$f : \begin{matrix} \mathbb{H} \to \mathbb{C} \\ \tau \to f(\tau) \end{matrix}$ holomorphic	$g(\tau)$: Modular form of weight 2-k
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \left(f(\tau) - \int_{-d/c}^{-i\infty} \frac{\overline{g(\bar{z})}}{(\tau-z)^k}\, dz\right)$$	k is the **weight**. $g(\tau)$ is the **shadow**.

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Two mock modular forms of fixed weight k and shadow g are related by a **modular form**.

Depth 1 Mock modular Modular forms

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Depth n Mock modular Modular forms

Properties	Characterestics
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	$g(\tau)$: Depth $(n-1)$ mock modular form of weight 2-k
$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \left(f(\tau) - \int_{-d/c}^{-i\infty} \frac{\overline{g(\bar{z})}}{(\tau-z)^k}\, dz \right)$$	k is the **weight**. $g(\tau)$ is the **shadow**.
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Two (higher depth) mock modular forms of fixed weight k and shadow g are related by a **modular form**.

Outline

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II. DT invariants

III. Constraining the generating function

The Donaldson-Thomas (DT) invariants

Type IIA compactified on a Calabi-Yau X

We restrict to $b_2 = 1$.

Count D6-D4-D2-D0 brane bound states

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Defining the generating functions

Rank 0 DT invariants

D6-brane charge $p^0 = 0$

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Modular properties!!

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A torus in type IIA ?

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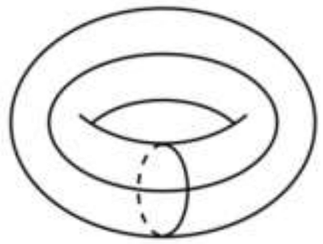
Type IIA $/X \times S^1$ ⟷ M-theory $/X \times \underbrace{S^1 \times S^1}_{T^2}$



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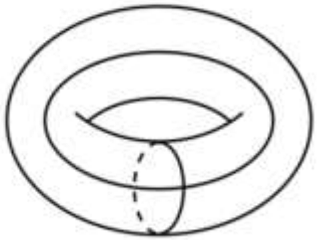

D4-branes $/\Gamma_4 \times S^1$

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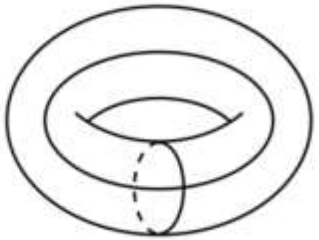

D4-branes $/\Gamma_4 \times S^1$ → Pointlike (instanton)

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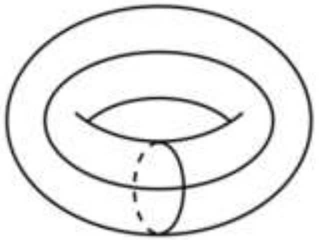

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↓

D4-branes $/\Gamma_4 \times S^1$ → Pointlike (instanton) → (Instanton corrections) $\mathcal{M}$ Moduli space (vector multipliet)

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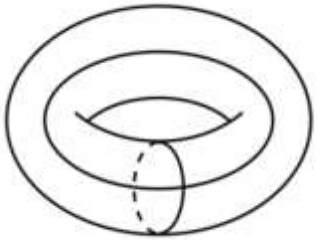

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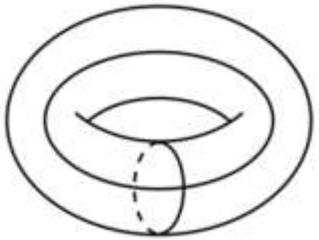

Symmetry under action of the **modular group**

D4-branes $/\Gamma_4 \times S^1$ → Pointlike (instanton) → Instanton corrections → $\mathcal{M}$ Moduli space (vector multipliet)

Donaldson-Thomas invariants.

Imposes modular constraints on h_p

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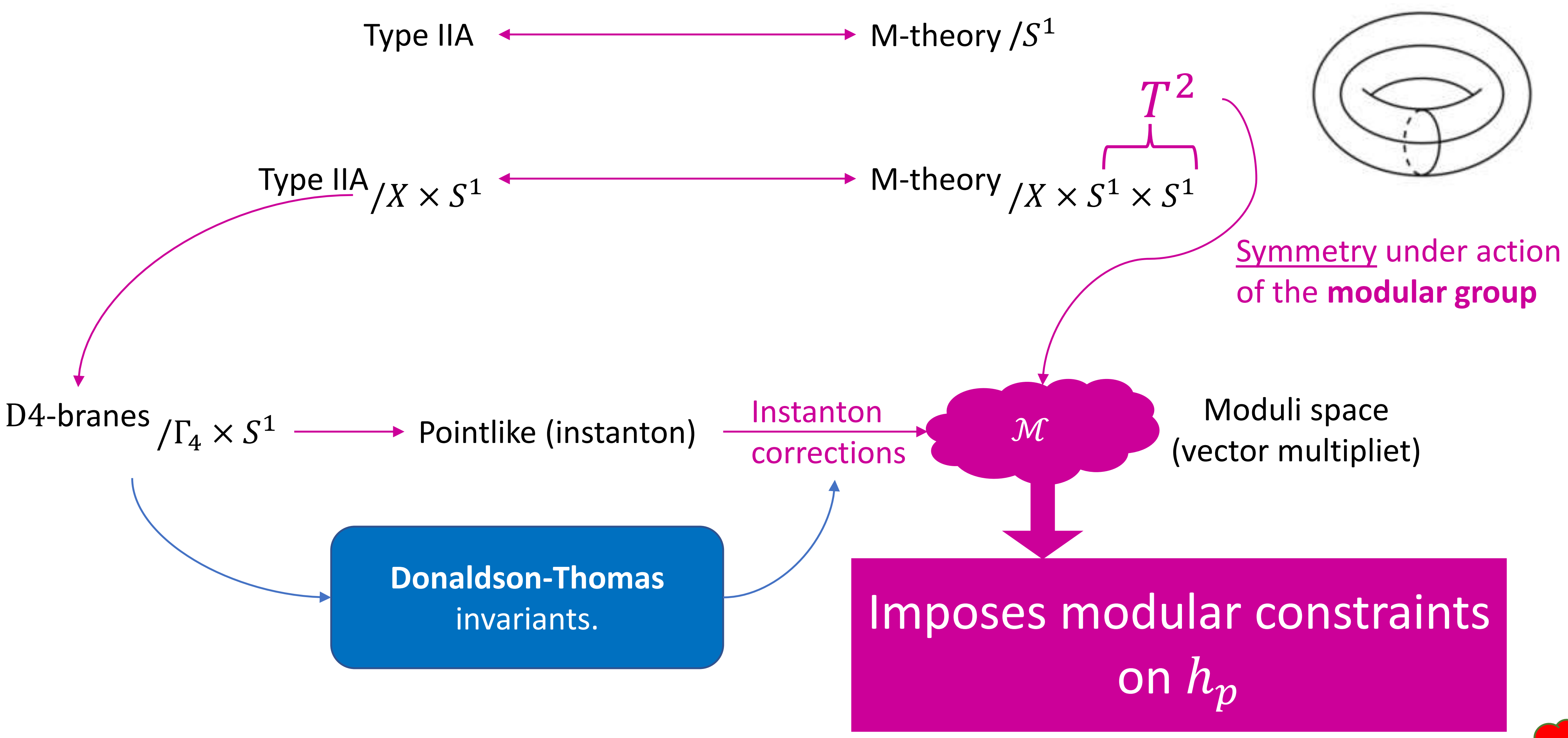

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[S.Alexandrov, B.Pioline '18]

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[S.Alexandrov, B.Pioline '18]

Completion equation

$$\hat{h}_{p,\mu}(\tau,\bar{\tau}) = \sum_{n=1}^{p} \sum_{p_1+\cdots+p_n=p} R_{\mu,\mu_1,\ldots,\mu_n}^{(p_1,\ldots,p_n)}(\tau_2) \prod_{i=1}^{p} h_{p_i,\mu_i}(\tau)$$

$$\tau_2 = Im(\tau)$$

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Let's look at an example

The modular ambiguity

Example: $p=2$

$$\hat{h}_{2,\mu}(\tau,\bar{\tau}) = h_{2,\mu}(\tau) + R^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}(\tau_2) h_{1,\mu_1} h_{1,\mu_2}$$

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Problem: the equation doesn't fix h_2 completely.

Solution: compute a few DT invariants (specifically the polar terms) and fix the modular ambiguity

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Two-step approach

$$h_{p,\mu} = h_{p,\mu}^{(an)} + h_{p,\mu}^{(0)}$$

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Strategy:

1. Find a solution $h_p^{(an)}$
2. Compute a few DT invariants and fix $h_p^{(0)}$

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Challenge: the completion equation for h_p depends on $h_{p_i}^{(0)}$ for lower charges.

$$\hat{h}_{2,\mu}(\tau,\bar{\tau}) = h_{2,\mu}(\tau) + R_{\mu,\mu_1,\mu_2}^{(p_1,p_2)}(\tau_2)(h_{1,\mu_1}^{(an)} + h_{1,\mu_2}^{(0)})(h_{1,\mu_2}^{(an)} + h_{1,\mu_1}^{(0)})$$

Two-step approach

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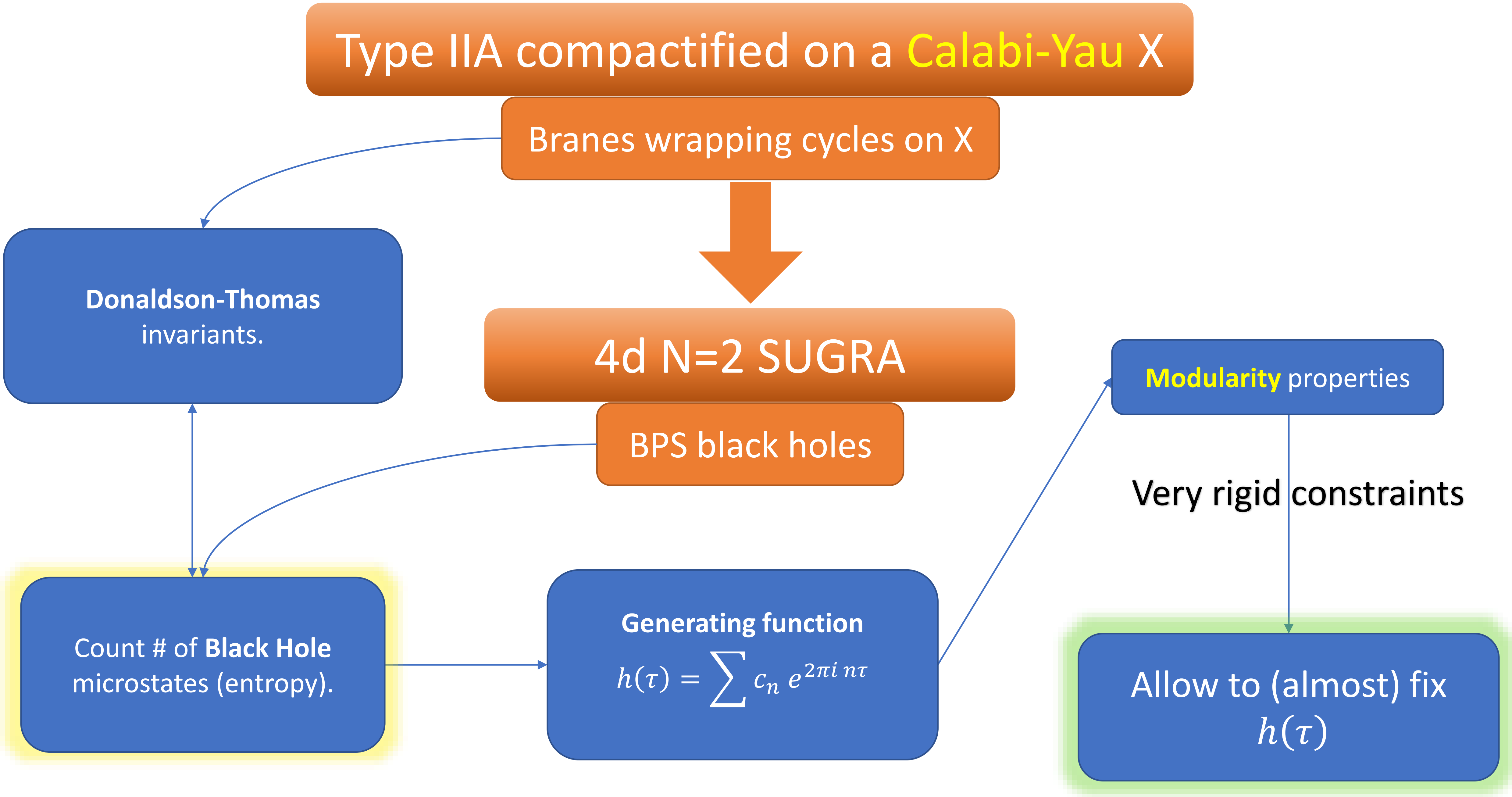
Result: Recipe to compute h_p up to $h_{p_i}^{(0)}$ for all $p_i \leq p$.

(Using indefinite theta series)

Conclusions

- DT invariants of the Calabi-Yau count the number of BPS black hole microstates.
- Generating functions of these invariants at rank 0, posess remarkable modular properties. ⟶ Mock modular
- We fix these functions, by solving their modular anomaly, up to computing a finite number of DT invariants.
- Further directions:
  - Compute polar terms to fix $h_p^{(0)}$(done for $p = 1$ for eleven CYs [S. Alexandrov, S.Feyzbakhsh, A.Klemm, B.Pioline, T.Schimannek '23])
  - Generalize the construction for $b_2 > 1$.


Type IIA compactified on a Calabi-Yau X
Branes wrapping cycles on X
Donaldson-Thomas invariants.
4d N=2 SUGRA
BPS black holes
Modularity properties
Very rigid constraints
Count # of Black Hole microstates (entropy).
Generating function
$h(\tau) = \sum c_n \, e^{2\pi i \, n\tau}$
Allow to (almost) fix $h(\tau)$

Appendix

Disentangling the ambiguity

Ansatz to extract the dependance of the generating functions on lower rank ambiguities

$$h_{p,\mu}(\tau,\bar{\tau}) = \sum_{n=1}^{p} \sum_{p_1+\cdots+p_n=p} g^{(p_1,\ldots,p_n)}_{\mu,\mu_1,\ldots,\mu_n}(\tau) \prod_{i=1}^{n} h^{(0)}_{p_i,\mu_i}(\tau)$$

Anomalous coefficients

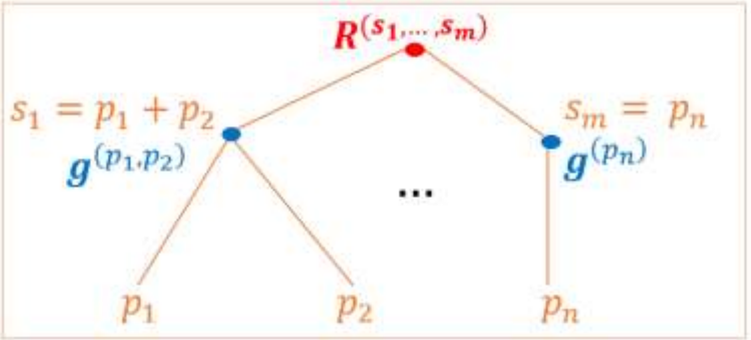
$$g^{(p)}_{\mu\nu} = \delta_{\mu\nu}$$

We trade the conditions on h_p for conditions on $g^{(p_i)}_{\mu,\mu_i}$

The new completion equation

For n charges $p_1, \dots, p_n$ → $g^{(p_1,\dots,p_n)}$ is a VV depth $(n-1)$ mock modular form

Completion equation



$$\hat{g}^{(p_1,\dots,p_n)} = Sym\left\{\sum_{\sum n_i = n} R^{(s_1,\dots,s_m)} \prod_{i=1}^{m} g^{\left(p_{j_i+1},\dots,p_{j_{i+1}}\right)}\right\}$$

Goal: find the anomalous coefficients $g^{(p_1,\dots,p_n)}_{\mu,\mu_1,\dots,\mu_n}(\tau)$

Studying $n = 2$

Example: $n = 2$

$$\hat{g}_{\mu,\mu_1,\mu_2}^{(p_1,p_2)} = g_{\mu,\mu_1,\mu_2}^{(p_1,p_2)} + R_{\mu,\mu_1,\mu_2}^{(p_1,p_2)}(\tau_2)$$

Indefinite theta series

Definite theta series

$$\vartheta_\mu(\tau) = \sum_{k\in\Lambda+\mu} e^{-\pi i\, Q(k)\, \tau}$$

Λ is a d dimensional lattice.
It has quadratic form $Q(x) \in 2\mathbb{Z}$
Q is negative definite

$\vartheta_\mu(\tau)$ is a Vector valued modular form of weight $d/2$

Indefinite theta series

$$\vartheta_\mu(\tau) = \sum_{k\in\Lambda+\mu} \Phi\left(\sqrt{2\tau_2}\, k\right) e^{-\pi i\, Q(k)\, \tau}$$

Λ is a d dimensional lattice.
It has quadratic form $Q(x) \in 2\mathbb{Z}$
Q is indefinite

Kernel: ensures convergence.

	Holomorphic	Modular
$\Phi(x)$	(Product of) difference of sign functions. $(sign(v_1 \cdot x) - sign(v_2 \cdot x))$	(Product of) difference of error functions. $\left(Erf\left(\frac{v_1 \cdot x}{\|\|v_1\|\|}\right) - Erf\left(\frac{v_2 \cdot x}{\|\|v_2\|\|}\right)\right)$

!!!!

When a vector v_i is null then we can have both!

$$Erf\left(\frac{v_1 \cdot x}{||v_1||}\right) \to sign(v_1 \cdot x)$$

Studying $n = 2$

Example:

$n = 2$

$$\hat{g}^{(p_1,p_2)}_{\mu,\mu_1,\mu_2} = g^{(p_1,p_2)}_{\mu,\mu_1,\mu_2} + R^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}(\tau_2)$$

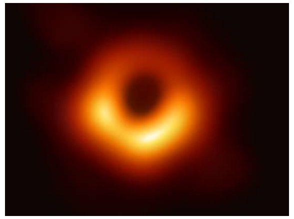
$R^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}$: Positive definite theta series on a 1 dimensional lattice with kernel $Erf(v_1 \cdot k) - sign(v_1 \cdot k)$

Choose $g^{(p_1,p_2)}_{\mu,\mu_1,\mu_2}$ to be an indefinite theta series with kernel $sign(v_1 \cdot k) - sign(w_1 \cdot k)$ where $Q(w_1) = 0$.

Our lattice doesn't have null vectors. Extend the lattice.

Recipe for solution:
Extend to solve.

Black hole microstates



$\boldsymbol{h}_p(\tau)$

Ansatz ↓ ↑ Compute few DT invariants

$\boldsymbol{g}^{(p_1,\ldots,p_n)}(\tau)$

Refinement ↓ ↑ Limit $z \to 0$

$\boldsymbol{g}^{(p_1,\ldots,p_n)}(\tau, z)$

Lattice extension ↓ ↑ Modular derivatives

$\widetilde{\boldsymbol{g}}^{(p_1,\ldots,p_n)}(\tau, z, \{z_i\})$

Solve the equation with
indefinite theta series

Vector-Valued(VV) Modular forms

Properties	Characterestics
$f \quad : \begin{array}{l} \mathbb{H} \to \mathbb{C} \\ \tau \to f(\tau) \end{array}$ holomorphic	
$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$	
$$f_\mu\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k \sum_\nu M_{\mu\nu}(\rho)\, f_\nu(\tau)$$	k is the **weight**. $M_{\mu\nu}$ is the **multiplier system**.
Modular forms have a Fourier expansion: $f_\mu(\tau) = \sum_{n=n_0}^{\infty} c_{n,\mu} q^n, \qquad q = e^{2\pi i\,\tau}$	

VV Modular forms of fixed weight k and multiplier system $M_{\mu\nu}$ form a **finite dimensional vector space**.

Jacobi-like Modular forms

Properties	Characterestics
$f : \begin{matrix} \mathbb{H} \to \mathbb{C} \\ \tau \to f(\tau) \end{matrix}$ holomorphic	
$f\left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \, \boldsymbol{e}^{\frac{2\pi i\, m\, c\, z^2}{c\tau+d}} f(\tau, z)$	Automorphy factor k is the **weight**. m is the **index**.
Jacobi-like forms have a series expansion in z : $f(\tau, z) = \sum_{n>n_0}^{\infty} f_n(\tau)\, z^n$,	

The function $f_{n_0}(\tau)$ is a weight $(k + n_0)$ modular form.

Modularity recap

Term	Math. Object	Charact.
Modular form	$f(\tau)$	Weight k
VV modular form	$f_\mu(\tau)$	Multiplier system $M_{\mu\nu}$
Jacobi-like form	$f(\tau, z); f(\tau, z_1, z_2)$	Index m; indices m_1, m_2
Mock modular form	$f(\tau) \leftrightarrow \hat{f}(\tau, \bar{\tau})$	Shadow $\mathrm{g}(\tau)$

Modularity recap

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Mock modular form	$f(\tau) \leftrightarrow \hat{f}(\tau,\bar{\tau})$	Shadow $g(\tau)$

Modular forms offer control on the **growth** of their Fourier coefficients

$$n_0 > 0 \Longrightarrow \quad c_n \sim n^{\frac{k}{2}}$$
$$n_0 = 0 \Longrightarrow \quad c_n \sim n^{k-1}$$
$$n_0 < 0 \Longrightarrow \quad c_n \sim e^{C\sqrt{n}}$$