Clocking the End of Cosmic Inflation

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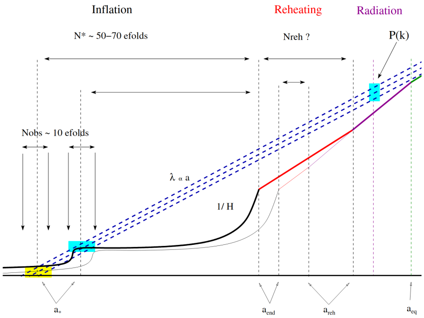
- Encodes deviations from a purely de Sitter background
- Computation of scalar and tensor power-spectra using ϵ_i-expansion, for instance

$$\mathcal{P}_\zeta(k) = \frac{H_*^2}{8\pi^2 M_{\mathrm{Pl}}^2 \epsilon_1^*}\left[1 - 2(C+1)\epsilon_{1*} - C\epsilon_{2*} - (2\epsilon_{1*} + \epsilon_{2*})\ln\left(\frac{k}{k_*}\right) + \ldots\right]$$

Pivot scale

- $k_* \sim \mathrm{Mpc}^{-1}$ is the wave number around which the expansion is made
- Hubble-flow functions are evaluated at the time N_* at which the pivot scale exited Hubble radius during inflation



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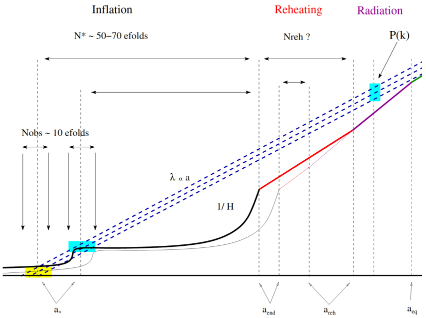

How is determined N_*?[1]

$$\frac{k_*}{a_0} = \boxed{(1+z_{\mathrm{end}})^{-1}} \frac{a(N_*)}{a(N_{\mathrm{end}})} H(N_*)$$

Depends on the reheating era!

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Pivot scale and reheating era

The dependence is such that

$$\Delta N_* \equiv N_* - N_{\mathrm{end}} = -\ln R_{\mathrm{rad}} + \frac{1}{4}\ln\left[\frac{9}{\epsilon_{1*}(3-\epsilon_{1\mathrm{end}})}\frac{V_{\mathrm{end}}}{V_*}\right] + \mathrm{cste}$$

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For a given potential $V(\phi)$ and a reheating history ($R_{\rm rad}$), to determine N_* we need

- the field trajectory $\phi(N) \Longleftrightarrow N(\phi)$
- the e-fold $N_{\rm end}$ at which inflation ended

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Two possible ways:

- Exact numerical integration
- Slow-roll (SR) approximation

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Two possible ways:

- Exact numerical integration
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If we stick with analytical methods

- As SR is violated towards the end of inflation, both the approximated trajectory and N_{end} are plagued with (small) error
- Any errors damaging N_{end} (as the ones building up close to the end of inflation) are folded into all the values of ΔN

Assessing slow-roll accuracy: field trajectory

Let us be more precise on slow-roll accuracy

In full generality

- Friedamnn-Lemaître equations written in number of e-folds N + Klein-Gordon give the equation of motion for $\phi(N)$

$$\underbrace{\frac{2}{6-\Gamma^2}\frac{\mathrm{d}\Gamma}{\mathrm{d}N}}_{\text{Acceleration of the field}} + \underbrace{\Gamma}_{\text{Friction}} = \underbrace{-\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}}_{\text{External force}}, \qquad \Gamma \equiv \frac{\mathrm{d}\phi}{\mathrm{d}N}, \qquad \epsilon_1 = \frac{\Gamma^2}{2}$$

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For a slowly rolling field

- The acceleration term is neglected

$$\Gamma \simeq \Gamma_{\mathrm{sr}} = -\frac{\mathrm{d}\ln V}{\mathrm{d}\phi}$$

- Hence the field trajectory

$$N_{\mathrm{sr}}(\phi) = -\int^{\phi} \frac{V(\psi)}{V'(\psi)}\,\mathrm{d}\psi$$

Assessing slow-roll accuracy: characterization of the end of inflation

- For vanilla single-field models, the accelerated expansion ends through a graceful exit

$$\epsilon_{1\mathrm{end}} \equiv \epsilon_1(N_{\mathrm{end}}) = 1 \qquad \Longleftrightarrow \qquad \Gamma_{\mathrm{end}} \equiv \Gamma(N_{\mathrm{end}}) = \pm\sqrt{2}$$

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- Without knowledge of the exact field trajectory, we resort to solving

$$\Gamma_{\mathrm{sr}}(N_{\mathrm{end}}^{\mathrm{sr}}) = \pm\sqrt{2} \quad \text{with} \quad N_{\mathrm{end}}^{\mathrm{sr}} = N_{\mathrm{sr}}(\phi_{\mathrm{end}}^{\mathrm{sr}})$$

i.e. we extrapolate the slow-roll trajectory up to the end of inflation

$$\Delta N_{\mathrm{sr}}(\phi) \equiv N_{\mathrm{sr}}(\phi) - N_{\mathrm{end}}^{\mathrm{sr}} = \int_{\phi}^{\phi_{\mathrm{end}}^{\mathrm{sr}}} \frac{V(\psi)}{V'(\psi)}\,\mathrm{d}\psi$$

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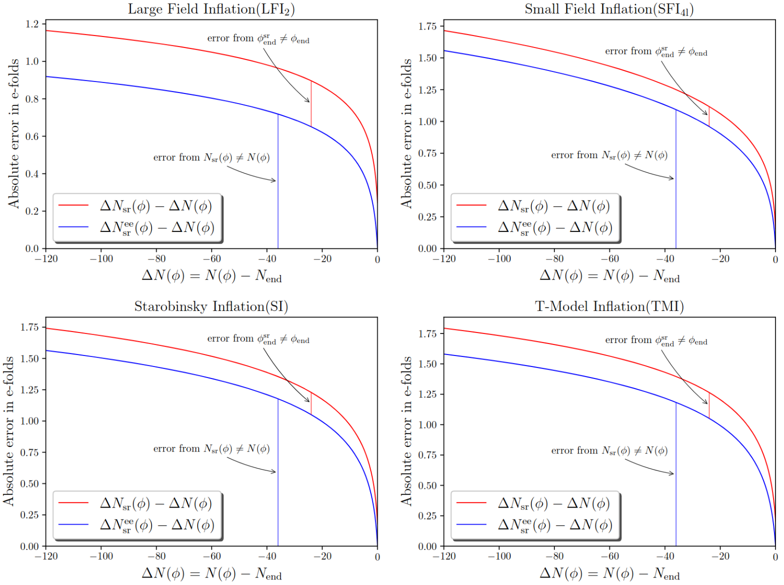
Two source of errors

Trading $\Delta N(\phi) \longrightarrow \Delta N_{\mathrm{sr}}(\phi)$ induces two errors coming from

$$\Gamma \simeq \Gamma_{\mathrm{sr}} \quad \text{and} \quad \phi_{\mathrm{end}} \simeq \phi_{\mathrm{end}}^{\mathrm{sr}}$$

To separate the two effects, define $\Delta N_{\mathrm{sr}}^{\mathrm{ee}}(\phi) \equiv N_{\mathrm{sr}}(\phi) - N_{\mathrm{sr}}(\phi_{\mathrm{end}})$

Assessing slow-roll accuracy



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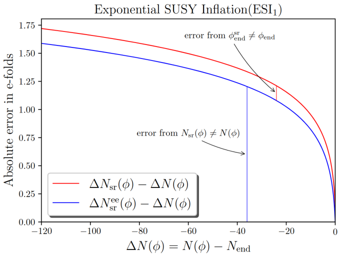

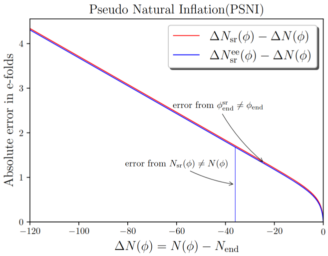

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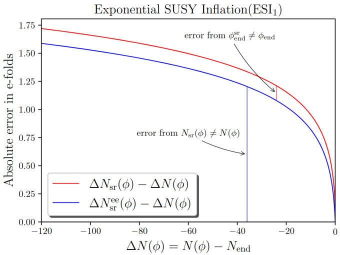

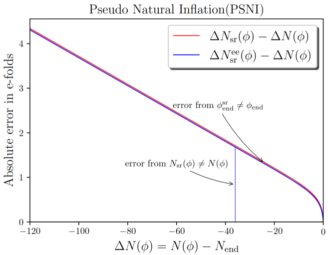

What to conclude?

For the fiducial value $\Delta N \simeq N_0 \sim -61.5$, the typical error on the trajectory are $\mathcal{O}(1)$ e-folds $\Longrightarrow$ Slow-roll approximation performs well!

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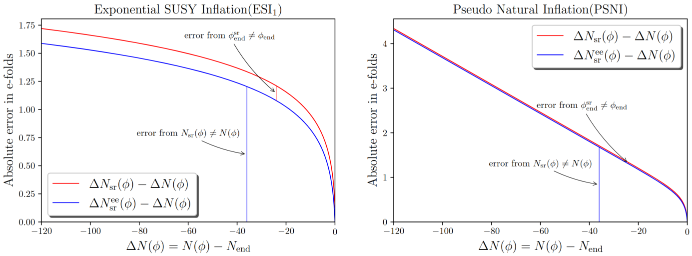

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Why bother then?

- With forthcoming measurements (Euclid, galaxy surveys, LiteBIRD...) this $\mathcal{O}(1)$ error will not be innocuous anymore
- Models differing only through their reheating history can be disambiguated (cf C. Ringeval's talk)
- Conversely, if reheating is unknown, ΔN_* constrains R_{rad}

Correcting slow-roll

Integral constraints

We see Γ as a function of ϕ and define the absolute error

$$\mathcal{E} \equiv \Gamma(\phi) - \Gamma_{\rm sr}(\phi) \quad \text{with E.O.M.} \quad \frac{2\Gamma}{6-\Gamma^2}\frac{\mathrm{d}\Gamma}{\mathrm{d}\phi} + \Gamma = \Gamma_{\rm sr}$$

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$$\int_{\phi}^{\phi_{\rm end}} \mathcal{E}(\psi)\,\mathrm{d}\psi = \ln\left[\frac{4}{6-\Gamma^2(\phi)}\right]$$

- In SR regime, $\Gamma^2 \ll 1$, the integrated error made between ϕ and $\phi_{\rm end}$ is $\ln(2/3) \simeq -0.4$

$$\int_{\phi}^{\phi_{\rm end}} \frac{\mathcal{E}(\psi)}{\Gamma(\psi)}\,\mathrm{d}\psi = \frac{1}{\sqrt{6}}\ln\left[(2\mp\sqrt{3})\frac{\sqrt{6}+\Gamma(\phi)}{\sqrt{6}-\Gamma(\phi)}\right]$$

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Integral constraints on $\mathcal{E}$

- Both the absolute error $\mathcal{E}$ and the relative error $\mathcal{E}/\Gamma$ are bounded, deep in the SR regime as well as at the end of inflation
- One can expand the exact solution with $\mathcal{E}$ or $\mathcal{E}/\Gamma$ as small parameters

New expansion of the field trajectory

We express

$$\Delta N(\phi) = N(\phi) - N_{\rm end} = -\int_\phi^{\phi_{\rm end}} \mathrm{d}\psi \, \frac{1}{\Gamma_{\rm sr}(\psi)} \boxed{\frac{\Gamma_{\rm sr}(\psi)}{\Gamma(\psi)}} \curvearrowright = 1 - \frac{\mathcal{E}}{\Gamma}$$

so that

$$\Delta N(\phi) = \underbrace{N_{\rm sr}(\phi) - N_{\rm sr}(\phi_{\rm end})}_{\text{SR approximation}} + \int_\phi^{\phi_{\rm end}} \mathrm{d}\psi \, \frac{\mathcal{E}(\psi)}{\Gamma^2(\psi)} \boxed{\frac{\Gamma(\psi)}{\Gamma_{\rm sr}(\psi)}} \curvearrowright = \frac{1}{1-\frac{\mathcal{E}}{\Gamma}}$$

and after a Taylor expansion of $1/(1-\mathcal{E}/\Gamma)$

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Expansion in $\mathcal{E}/\Gamma$

$$\Delta N(\phi) = \Delta N_{\rm sr}^{\rm ee}(\phi) + \int_{\phi}^{\phi_{\rm end}} \frac{\mathcal{E}(\psi)}{\Gamma^2(\psi)} \mathrm{d}\psi + \sum_{k=2}^{\infty} \int_{\phi}^{\phi_{\rm end}} \frac{1}{\Gamma(\psi)} \left[\frac{\mathcal{E}(\psi)}{\Gamma(\psi)}\right]^k \mathrm{d}\psi$$

- This expansion is not based on the usual SR expansion, as the condition $|\Gamma| \ll 1$ is not needed
- Hence it safely incorporates effects coming from the end of inflation
- The "small parameter" $\mathcal{E}/\Gamma$ is under control thanks to integral constraints

Velocity correction

The first order correction of this new expansion is exactly calculable!

$$\int_{\phi}^{\phi_{\mathrm{end}}} \frac{\mathcal{E}(\psi)}{\Gamma^2(\psi)}\,\mathrm{d}\psi = \frac{1}{6}\ln\left[\frac{2\Gamma^2(\phi)}{6-\Gamma^2(\phi)}\right]$$

- It acts as a velocity correction
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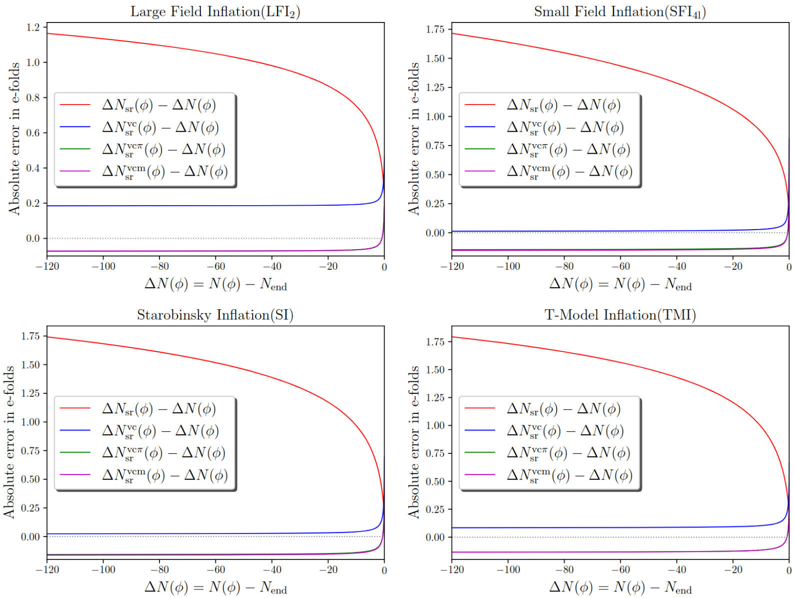
First order correction

As we are interested in $\Delta N(\phi)$ far from the end of inflation, $\Gamma \simeq \Gamma_{\rm sr}$ so that

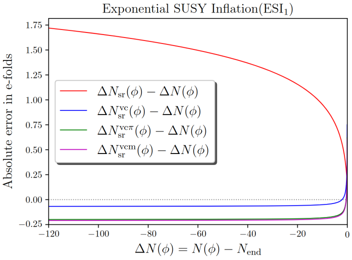
$$\Delta N_{\rm sr}^{\rm vc}(\phi) \equiv \Delta N_{\rm sr}(\phi) + \frac{1}{6}\ln\left[\frac{2\Gamma_{\rm sr}^2(\phi)}{6-\Gamma_{\rm sr}^2(\phi)}\right]$$

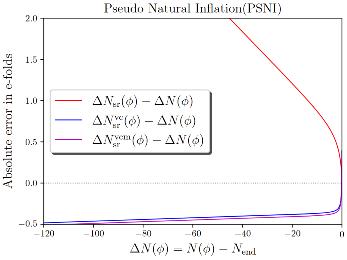
$\implies$ the velocity correction erases the logarithmic error growth w.r.t. ΔN to less than a tenth of an e-fold

Velocity correction



Endpoint correction





New algebraic methods for implementing endpoint corrections on the value of ϕ_{end} compatible with all types of single-field models

- Using our integral constraints and trapezoidal approximation
- Matching with Mukhanov inflation

	LFI_2		SFI_{41}		SI		TMI		ESI_1		PSNI	
ϕ_{end}	1.009	0%	9.657	0%	0.615	0%	0.839	0%	0.271	0%	1.564	0%
ϕ_{end}^{sr}	1.414	40%	9.361	3.1%	0.940	53%	1.208	44%	0.535	97%	1.478	5%
ϕ_{end}^{π}	0.984	2.5%	9.661	0.04%	0.607	1.3%	0.826	1.5%	0.273	0.7%	–	–
ϕ_{end}^{m}	0.986	2.3%	9.678	0.2%	0.594	3.4%	0.825	1.7%	0.238	12%	1.604	2%

Conclusion

- Observable predictions for cosmic inflation are measured in e-folds $\Delta N = N - N_{\rm end}$
- Determining with precision $\Delta N(\phi)$ is crucial to correctly map wavenumbers today to wavenumbers during inflation
    - The approximated trajectory $N(\phi)$ is determined via SR with $\mathcal{O}(1)$ precision in e-folds
    - Same for $N_{\rm end}$

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  - Simple and practical velocity correction to the usual SR trajectory
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- New methods to improve analytical observable predictions for the trajectory
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  - Kills the absolute error on $\Delta N(\phi)$ by an order of magnitude, for all tested models
- Algebraic methods to improve accuracy on the endpoint of inflation $\phi_{\rm end}$
  - Allows a more accurate determination of $\rho_{\rm end}$
  - Computationally not expensive
  - Does not improve the VC correction alone but never degrades it

For the last word

Incidentally, we derived a new exact solution of the field trajectory when the inflationary potential V is expressed in e-folds N

$$\Lambda(N) = e^{-6\Delta N}\frac{V_{\rm end}}{V(N)}\Lambda_{\rm end} - \int_N^{N_{\rm end}} e^{6(n-N)}\frac{V(n)}{V(N)}\,{\rm d}n \quad \text{with} \quad \Lambda(N) \equiv \frac{1}{6-\Gamma^2(N)}$$

Maybe useful, maybe not!

Thank you for your attention !