Quadratic black hole perturbations: the algebraically special sector

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Introduction

- Study of BH perturbations is crucial to model the GW signal from a binary BH merger in the ringdown phase
- Non-linear perturbations play an important role at the beginning of this phase
- *Algebraically special* perturbations play a key role in the identification of the symmetries of BH perturbations
- They are also the only known analytic solution to the perturbation equations
- Goal of this work: investigate quadratic algebraically special perturbations of the Schwarzschild BH

Linear BH perturbations

Usual perturbation setup

$$g_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\text{Schwarzschild metric}} + \underbrace{\varepsilon}_{\text{Perturbation parameter}} \underbrace{h_{\mu\nu}}_{\text{Perturbation metric}} \,, \qquad \varepsilon \ll 1$$

Main idea: Decomposition of $h_{\mu\nu}$ using the symmetries of $\bar{g}_{\mu\nu}$

- 2+2 scalar-vector-tensor decomposition of $h_{\mu\nu}$
- Gauge fixing [Regge, Wheeler '57; Zerilli '70]

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \begin{array}{l} -\dfrac{\partial^2 Y_{\mathrm{RW}}}{\partial t^2} + \dfrac{\partial^2 Y_{\mathrm{RW}}}{\partial r_*^2} = V_{\mathrm{RW}}\, Y_{\mathrm{RW}} \\[2ex] -\dfrac{\partial^2 Y_{\mathrm{Z}}}{\partial t^2} + \dfrac{\partial^2 Y_{\mathrm{Z}}}{\partial r_*^2} = V_{\mathrm{Z}}\, Y_{\mathrm{Z}} \end{array}$$

Analytical solutions

- Usual resolution procedure: Fourier transformation
- Recover contributions from quasi-normal modes and late-time tail
- Only numerical resolution

Analytical result [Chandrasekhar '84]

$$Y_{\mathrm{RW}} = e^{\kappa_{\mathrm{a}}(t-r_*)} f_{\mathrm{a}}(r)\,, \quad Y_{\mathrm{Z}} = e^{-\kappa_{\mathrm{p}}(t-r_*)} f_{\mathrm{p}}(r)\,, \quad \kappa = \frac{(\ell-1)\ell(\ell+1)(\ell+2)}{12M}$$

- Purely outgoing waves (ingoing solutions also exist by time reversal)
- Divergent at the horizon $r_* \to -\infty$
- Crucial for the proof of isospectrality [Glampedakis, Johnson+ '17]

Petrov classification

Main idea: classify spacetimes by multiplicity of the principal null directions of the Weyl tensor. Higher multiplicity ⟶ higher symmetry

$$\begin{tikzcd}
\text{General BH perturbations} & \mathrm{I} \arrow[l, dashed] \arrow[d] \arrow[dr] & & \text{Black hole spacetime} \\
\text{Analytical perturbations} & \mathrm{II} \arrow[l, dashed] \arrow[r] \arrow[d] \arrow[dr] & \mathrm{D} \arrow[ur, dashed] \arrow[d] \arrow[dr] & \\
 & \mathrm{III} \arrow[r] & \mathrm{N} \arrow[r] & \mathrm{O}
\end{tikzcd}$$

} algebraically special spacetimes

- Black hole perturbations are type I in general [Araneda '18]
- These analytical perturbations are of type II: **algebraically special**

→ Can one recover these analytical perturbations starting from the algebraically special requirement?

Setup for the resolution

Goldberg-Sachs theorem

A vacuum solution of the Einstein field equations will admit a **shear-free null geodesic congruence** if and only if the Weyl tensor is algebraically special.

Construction of the metric [Stephani, Kramer+ ’03]

$$\mathrm{d}s^2 = -2\omega^l\omega^n + 2\omega^m\omega^{\bar{m}}$$

$$\omega^l = \mathrm{d}u + 2\,\mathrm{Re}(L\,\mathrm{d}z)\,, \quad \omega^n = \mathrm{d}r + 2\,\mathrm{Re}(W\,\mathrm{d}z) + H\omega^l\,, \quad \omega^m = \frac{\mathrm{d}\bar{z}}{P\rho}$$

- Metric depends on reals $m(u, z, \bar{z})$, $P(u, z, \bar{z})$ and complex $L(u, z, \bar{z})$
- H, W and ρ can be deduced from these 3 quantities

Full equations of motion

Definitions:

$$\rho = -\frac{1}{r + i\Sigma}\,, \qquad 2i\Sigma = P^2(\bar{\partial} L - \partial \bar{L})\,,$$

$$W = \frac{\partial_u L}{\rho} + \partial(i\Sigma)\,, \qquad \partial = \partial_z - L\partial_u\,,$$

$$H = \frac{K}{2} - r\partial_u(\log(P)) - \frac{mr + N\Sigma}{r^2 + \Sigma^2}\,, \qquad K = 2P^2\,\mathrm{Re}(\partial(\bar{\partial}\log P - \partial_u \bar{L}))\,,$$

$$N = \Sigma K + P^2\,\mathrm{Re}(\partial\bar{\partial}\Sigma - 2\partial_u \bar{L}\partial\Sigma - \Sigma\partial_u\partial\bar{L})\,,$$

Evolution:

$$i\partial N - 3(m + iN)\partial_u L = 0\,,$$

$$P\big[\partial + 2(\partial\log P - \partial_u\bar{L})\big]\partial\big[\bar{\partial}(\bar{\partial}\log P - \partial_u\bar{L}) + (\bar{\partial}\log P - \partial_u\bar{L})^2\big] - \partial_u\big[P^{-3}(m + iN)\big] = 0\,.$$

Linear perturbation setup

Schwarzschild solution

The Schwarzschild BH is recovered by setting spherical symmetry:

$$m(u,z,\bar{z}) = M\,, \quad P(u,z,\bar{z}) = P_0(z,\bar{z}) = \frac{1}{\sqrt{2}}(1+z\bar{z})\,, \quad L(u,z,\bar{z}) = 0\,.$$

- Perturbative study: expand around this background
- Linear case investigated in part in [Couch, Newman '73; Qi, Schutz '93]
- Introduce $\varepsilon \ll 1$ and two real perturbation functions f_1 and F_1:

$$P(u,z,\bar{z}) = P_0(z,\bar{z})e^{\varepsilon F1(u,z,\bar{z})}\,, \quad L(u,z,\bar{z}) = i\varepsilon\partial_z\big[f_1(u,z,\bar{z})\big]$$

Linear perturbation resolution

Evolution equations

Einstein equations imply two evolution equations:

$$\Delta_0\Delta_0 f_1 + 2\Delta_0 f_1 - 12M\partial_u f_1 = 0 \quad \text{and} \quad \Delta_0\Delta_0 F_1 + 2\Delta_0 F_1 + 12M\partial_u F_1 = 0$$

- To solve, decompose on spherical harmonics since $\Delta_0\, Y_\ell^m = -\ell(\ell+1)\, Y_\ell^m$
- Solutions are of the form

$$\boxed{f_1 = E_{\mathrm{a}}\, e^{\kappa_{\mathrm{a}} u}\, Y_{\ell_{\mathrm{a}}}^{m_{\mathrm{a}}}} \quad \text{and} \quad \boxed{F_1 = E_{\mathrm{p}}\, e^{-\kappa_{\mathrm{p}} u}\, Y_{\ell_{\mathrm{p}}}^{m_{\mathrm{p}}}}$$

→ recover Chandrasekhar’s result of outgoing waves obtained as analytical solutions of the perturbation equations!

Motivation

Investigation of non-linear GR

- Backreaction of perturbations on the BH properties (mass, spin)
- Non-linear deformability (Love numbers)

Consistency checks of numerical simulations

- Analytical solution convenient for verification of the simulated amplitudes [Bucciotti, Juliano+ '24]

Experimental detection of quadratic perturbations

- Next-gen detectors will be sensitive to quadratic perturbations [Yi, Kuntz+ '24]

Principle of the computation

Perturbation setup

Introduce two new functions F_2 and f_2:

$$P = P_0 e^{\varepsilon F_1 + \varepsilon^2 F_2}, \quad L = i\varepsilon \partial_z f_1 + \varepsilon^2 \left[i\partial_z f_2 + \kappa_{\mathrm{a}} \partial_z (f_1^2) \right]$$

Evolution equations become

$$\Delta_0 \Delta_0 f_2 + 2\Delta_0 f_2 - 12M\partial_u f_2 = \mathcal{A}(f_1, F_1)$$
$$\Delta_0 \Delta_0 F_2 + 2\Delta_0 F_2 + 12M\partial_u F_2 = \mathcal{P}(F_1, F_1) + \mathcal{Q}(f_1, f_1) + \mathcal{R}(f_1, F_1)$$

- Non-homogeneous linear equations imply amplitudes are fixed
- Can still be solved by decomposition onto spherical harmonics and use of Clebsch-Gordan coefficients

Result for the axial sector

- Analytic form of the source term (non-trivial to obtain)
  $\mathcal{A} = -2F_1\Delta_0\Delta_0 f_1 - 2\Delta_0 F_1 \Delta_0 f_1 - 2\Delta_0(F_1\Delta_0 f_1) - 8F_1\Delta_0 f_1$
- Decomposed onto spherical harmonics: $\mathcal{A} = E_\mathrm{p} E_\mathrm{a}\, e^{(\kappa_\mathrm{a}-\kappa_\mathrm{p})u} \sum_{\ell,m} \mathcal{A}^\ell_m\, Y^m_\ell$
- Decomposition coefficients:
  $\mathcal{A}^\ell_m = -2\ell_\mathrm{a}(\ell_\mathrm{a}+1)\left[\ell_\mathrm{a}(\ell_\mathrm{a}+1) + \ell_\mathrm{p}(\ell_\mathrm{p}+1) + \ell(\ell+1) - 4\right] k^{m_\mathrm{a} m_\mathrm{p} \ell}_{\ell_\mathrm{a} \ell_\mathrm{p} m}$

Clebsch-Gordan

Analytical solution

$$f_2 = E_\mathrm{p} E_\mathrm{a}\, e^{(\kappa_\mathrm{a}-\kappa_\mathrm{p})u} \sum_{\ell,m} a^\ell_m\, Y^m_\ell\,, \qquad a^\ell_m = \frac{\mathcal{A}^\ell_m}{\kappa - \kappa_\mathrm{a} + \kappa_\mathrm{p}}$$

→ from this, one can recover the metric content and check that it explicitly solves the quadratic perturbation equations!

Time-independent perturbations

- Perturbations are time-independent if and only if $\kappa_{\rm a} = \kappa_{\rm p} = 0$, so $\ell_{\rm a}, \ell_{\rm p} \in \{0, 1\}$
- They correspond to **non-linear zero modes** of the Schwarzschild BH
- Linear level: axial monopole is gauge and dipole adds spin, polar monopole changes mass and polar dipole is gauge [Martel, Poisson '05]
- Quadratic level, dipole-dipole interaction: modification of spin due to the axial dipole + of mass due to the polar dipole $\rightarrow$ *change of BH horizon*

Conclusion

- First investigation of algebraically special quadratic perturbations of the Schwarzschild BH
- Obtain an **analytical solution** of the quadratic perturbation equations using a general Petrov type II solution of Einstein's equations
- Can be used to investigate non-linear zero modes and check numerical simulations
- Open the road to look for hidden symmetries of BH perturbations as in the linear case
- Next step: generalisation to Kerr (work in progress)

Thank you for your attention!