The operational meaning of total energy of isolated systems in general relativity

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based on the eponymous paper with Abhay Ashtekar

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Outline:

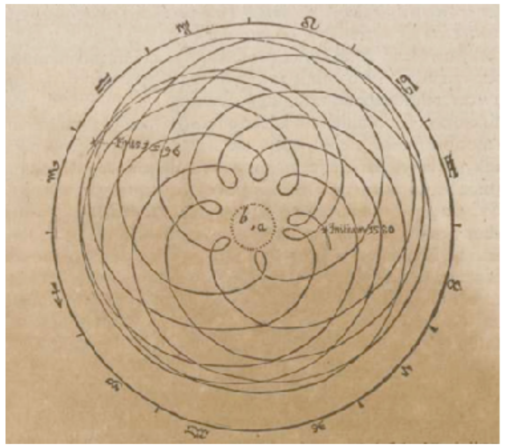
I Introduction and Motivations

II Operational definition of ADM energy

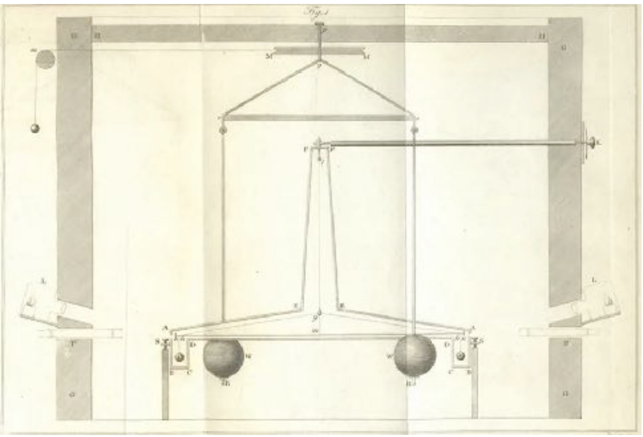
III Operational definition of Bondi-Sachs energy

Total mass in Newtonian gravity

In Newtonian theory, mass is operationally defined through the force it exerts



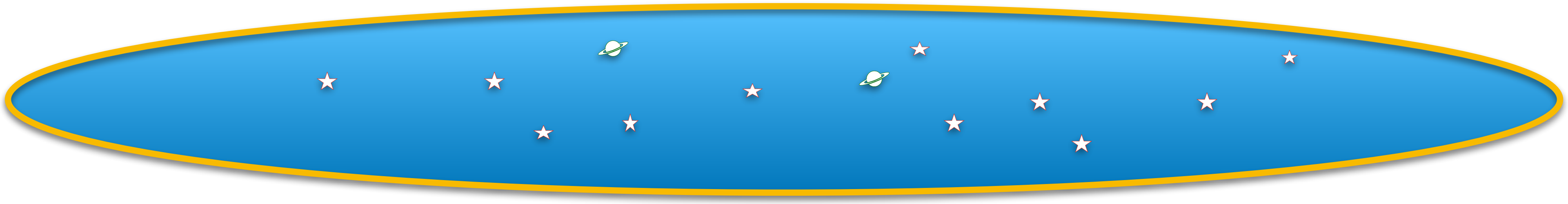
Orbital motion



Cavendish balance

This operation can be performed *locally*, or quasi-locally:

the total mass is given by (the gradient of) the Newton potential across *any* surface surrounding the sources



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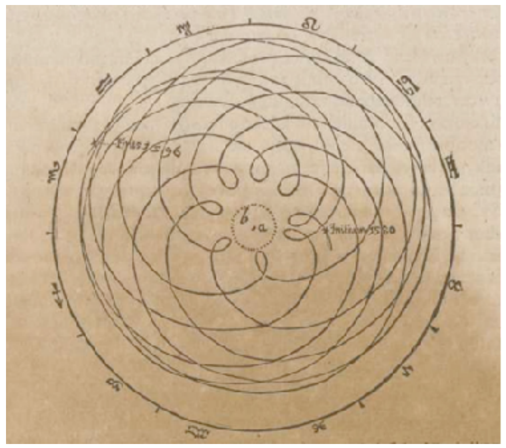

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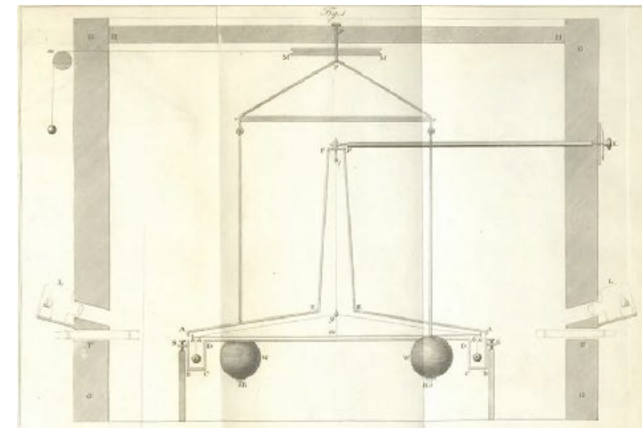

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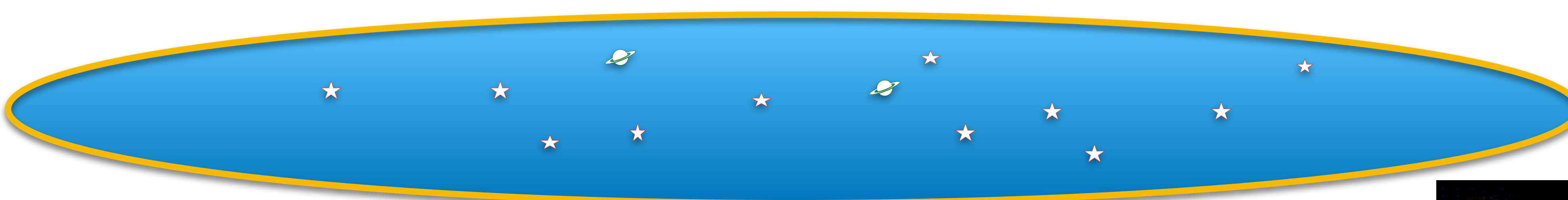
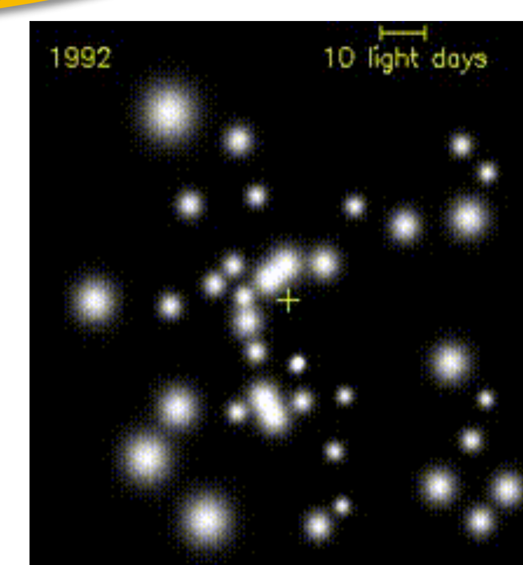


This method applies also to general relativity for **stationary** configurations
(no gravitational waves!)

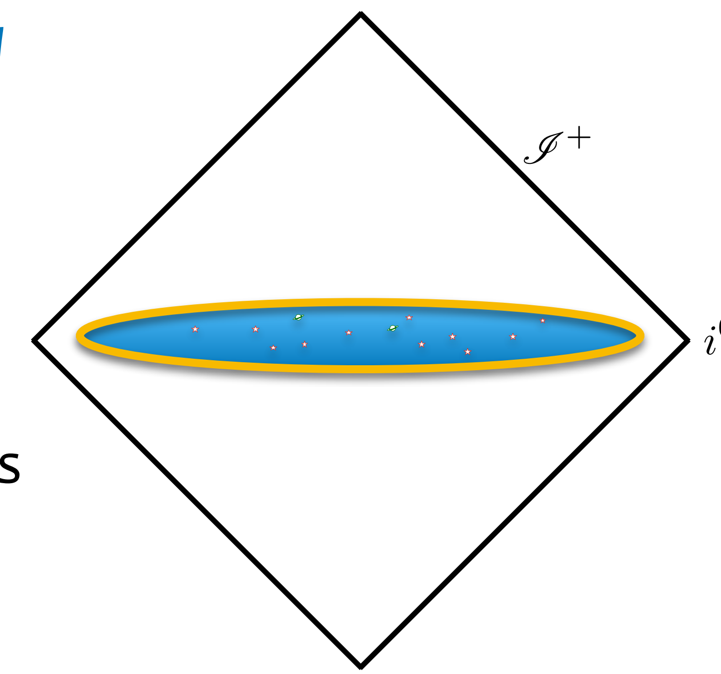
For instance, this is how we estimate the mass of Sagittarius A*



No local observables in general relativity

If we want to include the energy of gravitational waves, the situation becomes more complicated

The difficulties can be seen in many different ways, but at a more formal level, they boil down to the fact that **it is not possible to define a local and covariant energy-momentum tensor for the gravitational field**: the gravitational contribution is necessarily ***non-local***



Famous examples of non-local observables:
the ADM energy, and the Bondi-Sachs energy,
are defined as surface integrals *asymptotically far away* from the sources

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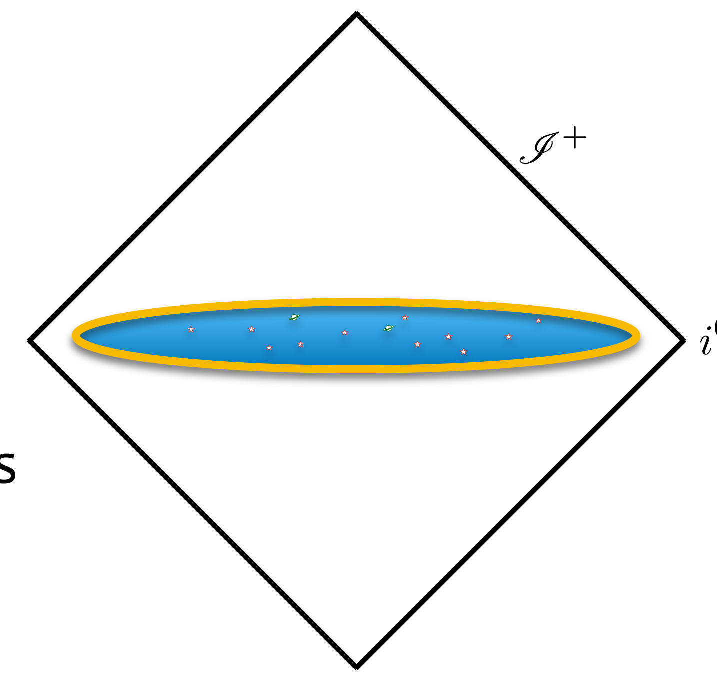

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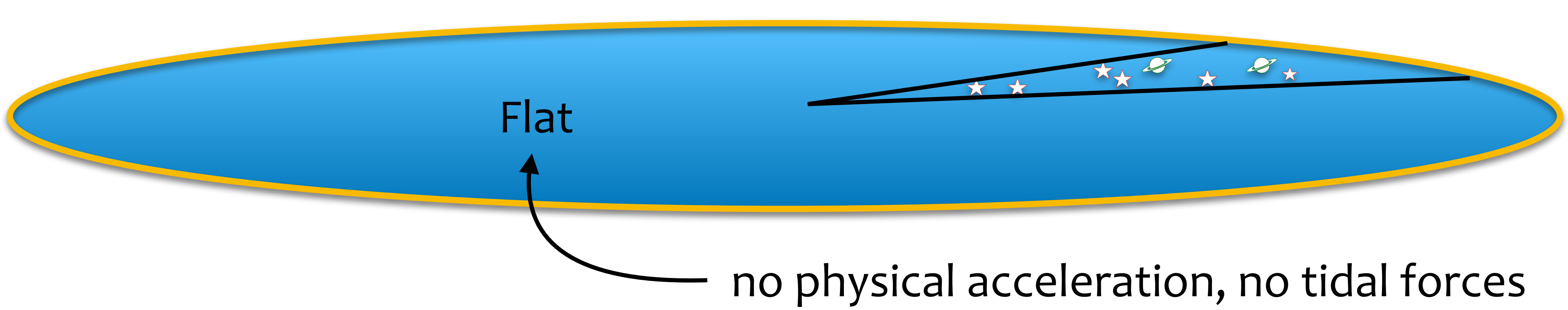
are defined as surface integrals *asymptotically far away* from the sources

Conceptual discussions of this point can be found in textbooks, but there is a very concrete recent argument: the Carlotto-Schoen solutions (2014)

Spacetimes with positive ADM energy which are **everywhere flat** expect inside a sharp cone



Total energy is a truly global quantity in general relativity

The properties of the Carlotto-Schoen spacetimes highlight the ***necessity*** of working with a full 2-sphere's worth of test particles in the asymptotic region in order to take into account correctly the total gravitational energy beyond the Newtonian or stationary cases

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To make the discussion more concrete, consider first the stationary case of the Kerr metric
A coordinate-independent definition of its mass can be given using Komar's 2-form:

$$M = -\frac{1}{8\pi G}\oint_S \epsilon_{\mu\nu\rho\sigma}\nabla^\rho \xi^\sigma dx^\mu \wedge dx^\nu$$

time-translational Killing vector

S here can be any surface : **the integral is independent of this choice**
(the proof is an application of Noether's theorem)

In this context, there is a simple **operational definition** of the mass:
the force required (at infinity) to hold a unit rest mass in place at the horizon

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In the presence of ***GWs****, the situations changes :* **the integral depends on S** (its shape, its location)

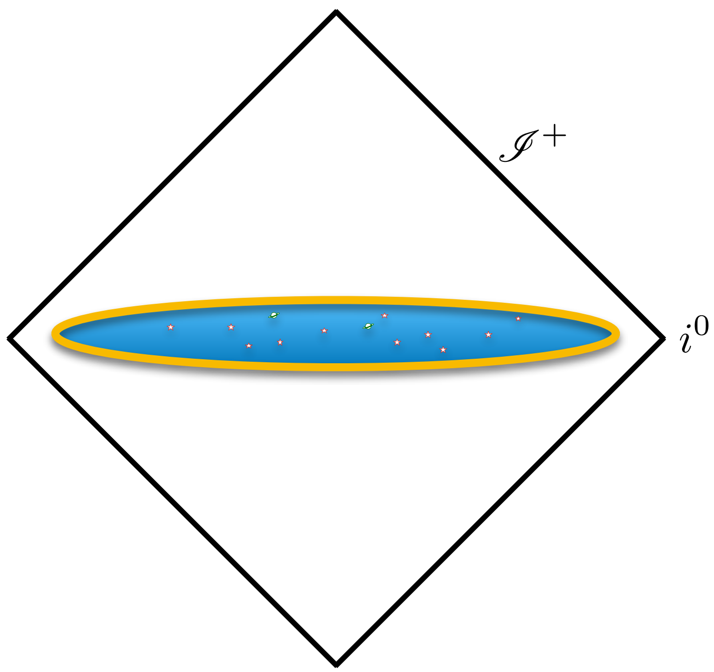
One way out of the ambiguity is to take the limit to the asymptotic region, and then the Komar formula can be related to the ADM and BS energies

This is the sense in which ADM and BS energies provide good gravitational observables, and they are manifestly global

The question we want to address is whether these observables admit an operational definition

Operational definition of the ADM energy

$$E_{\mathrm{ADM}} = \frac{1}{16\pi G} \lim_{\mathring{r}\to\infty} \oint_{r=r_\circ} (\mathring{D}_c\, q_{bc} - \mathring{D}_b\, q_{ac})\, \mathring{q}^{ac}\, \mathring{r}^b \mathrm{d}^2 V$$



Integral over a whole 2-sphere infinitely far away from the source
→ thought experiment

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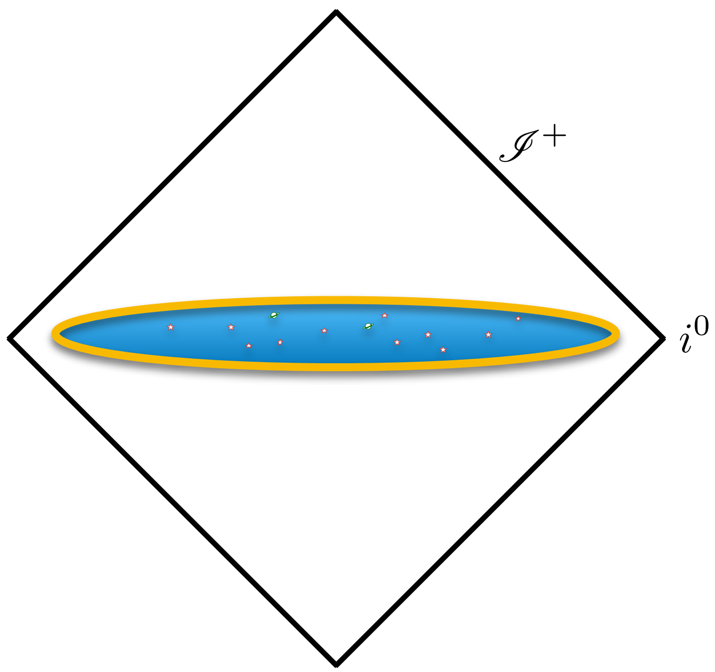

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→ thought experiment

How can this be measured?

Using Stokes' theorem and the Gauss-Codazzi equation :

$$E_{\mathrm{ADM}} = -\frac{1}{8\pi G} \lim_{\mathring{r}\to\infty} \oint_{r=r_\circ} r\, (\mathring{r}^a \tau^b \mathring{r}^c\, \tau^d R_{abcd})\, \mathrm{d}^2 V$$

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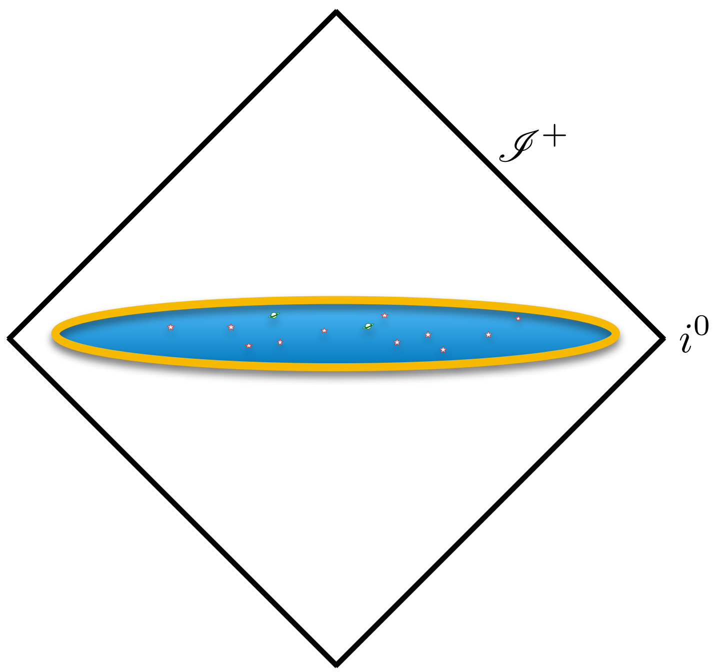

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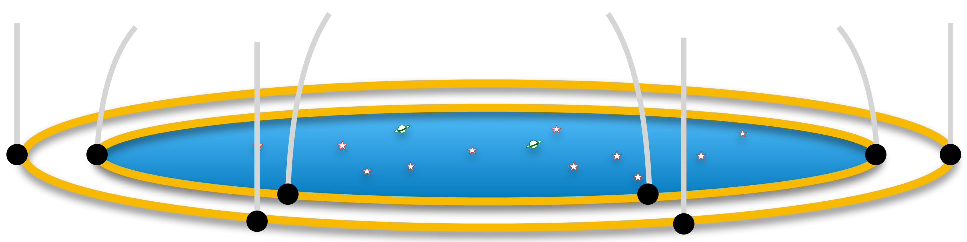
tidal force

Thought experiment to measure the ADM energy: measure the tidal acceleration of a spherical distribution of test masses

- essential to be in the asymptotic region
- essential to cover all angular directions

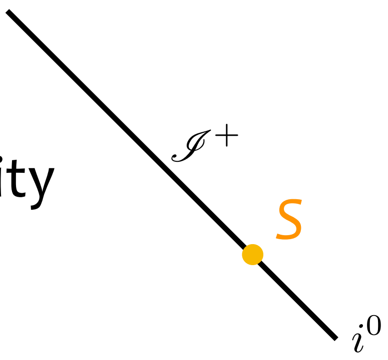


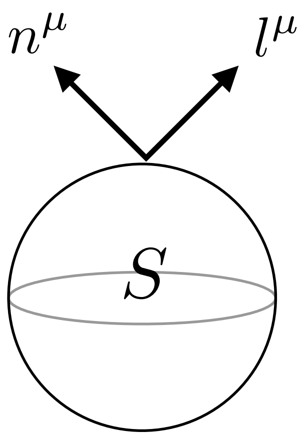
Bondi-Sachs energy in terms of expansions

Can be expressed in a coordinate-independent way using Newman-Penrose formalism

$$E_{\rm BS} = -\frac{1}{4\pi}\oint_S \mathrm{Re}(\psi_2 + \sigma\dot{\bar{\sigma}})\epsilon_S$$

where S is a cross-section of null-infinity





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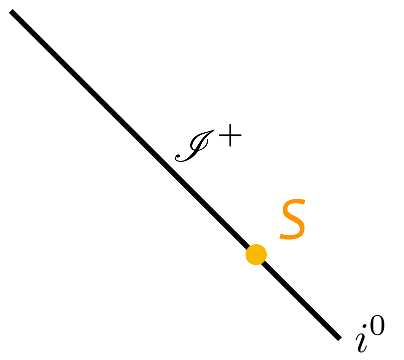

$$E_{\mathrm{BS}} = -\frac{1}{4\pi}\oint_S \mathrm{Re}(\psi_2 + \sigma\dot{\bar{\sigma}})\epsilon_S$$

somewhat mysterious :

radiative component,
captures metric components that fall-off like $\dfrac{1}{r}$

Coulombic component,
captures metric components that fall-off like $\dfrac{1}{r^3}$,
like in Newtonian case

Bondi-Sachs energy in terms of expansions

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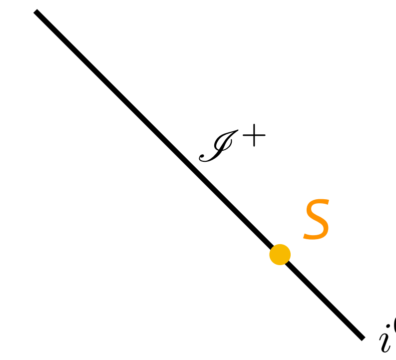

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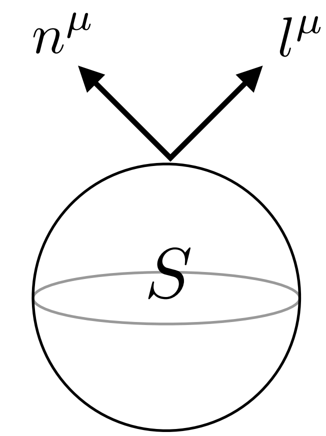

Using a codimension-2 version of the Gauss-Codazzi equation :

$$E_{\text{BS}} = \frac{1}{16\pi}\oint_S r(\mathcal{R}^{(2)} + \theta_l\theta_n)\epsilon_S$$

This relation appears for instance in Hawking '68

Bondi-Sachs energy in terms of expansions

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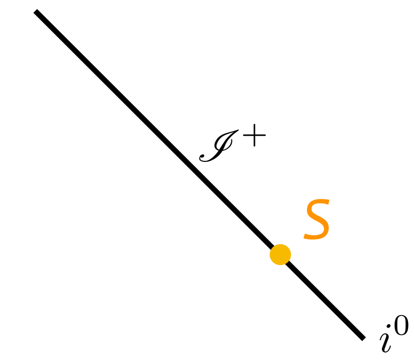

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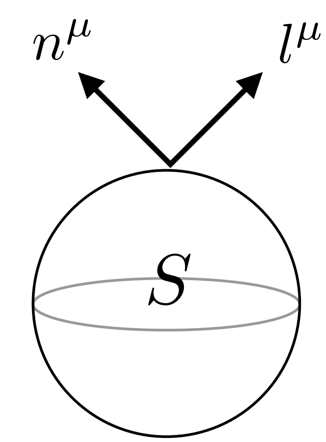

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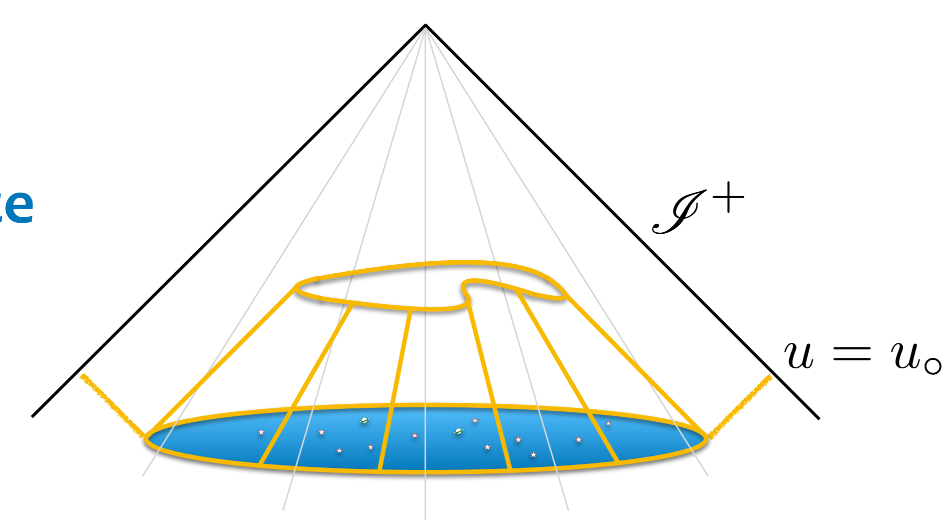
Using the explicit falloff behaviour of the metric :

$$E_{\rm BS} = \frac{1}{8\pi}\lim_{r\to\infty}\oint_{S_r}\left(\theta_n + \frac{1}{r}\right)\epsilon_{S_r}$$

convergence of ingoing null rays

Thought experiment to measure the BS energy:
measure the angular average of the ingoing congruence
in the asymptotic region



See the paper for details on how to do this

Conclusions

The ADM and BS energies are well-defined observables that have played an important role in the understanding and development of general relativity

They are global quantities: to be measured one has to consider a full 2-sphere in the asymptotic region

In spite of being typically written as formal metric expressions, they can be recasted so that their physical meaning becomes manifest:

- **tidal acceleration for the ADM energy**
- **convergence of null geodesic congruences for the BS energy**

Operational definition is then possible

On a more **speculative** side, the relation of the Bondi-Sachs energy to the convergence suggests the possibility that the **energy be quantized** with a discrete spectrum, e.g. loop quantum gravity