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The formation of the first nonlinear dark matter structures in the universe

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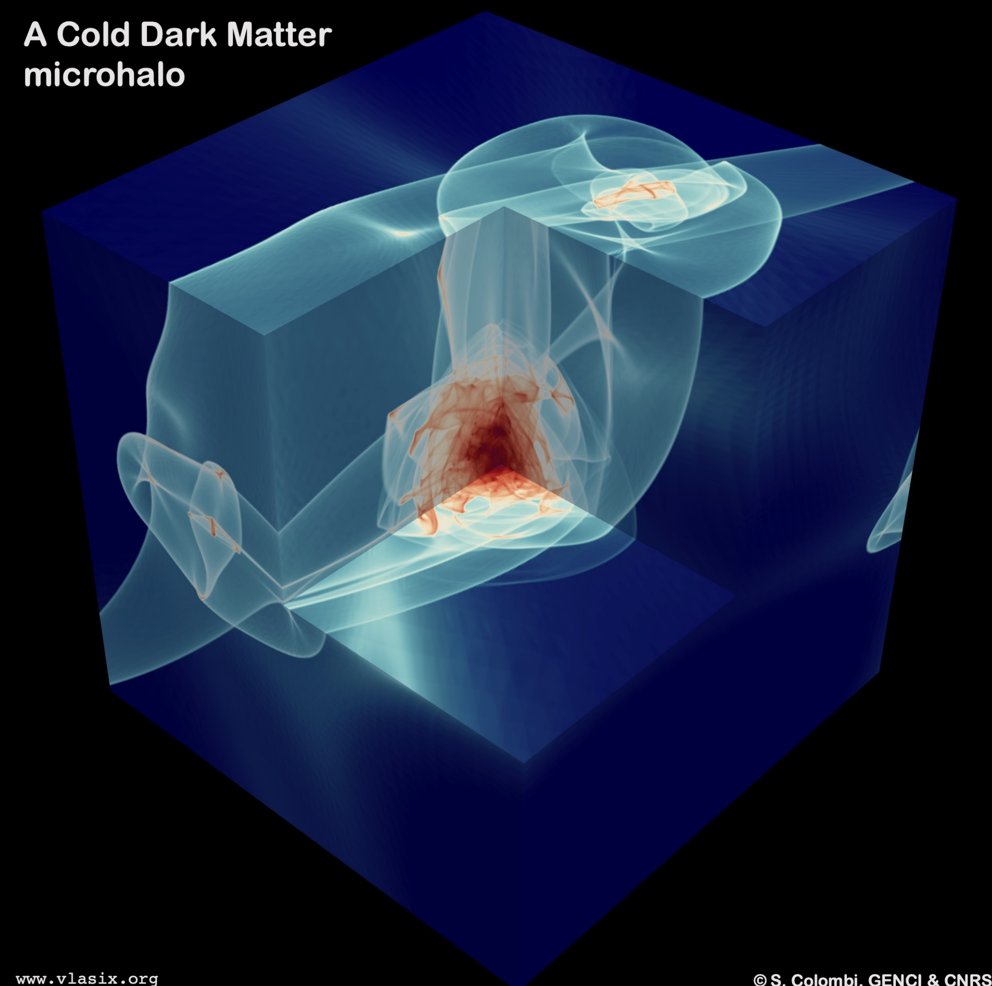
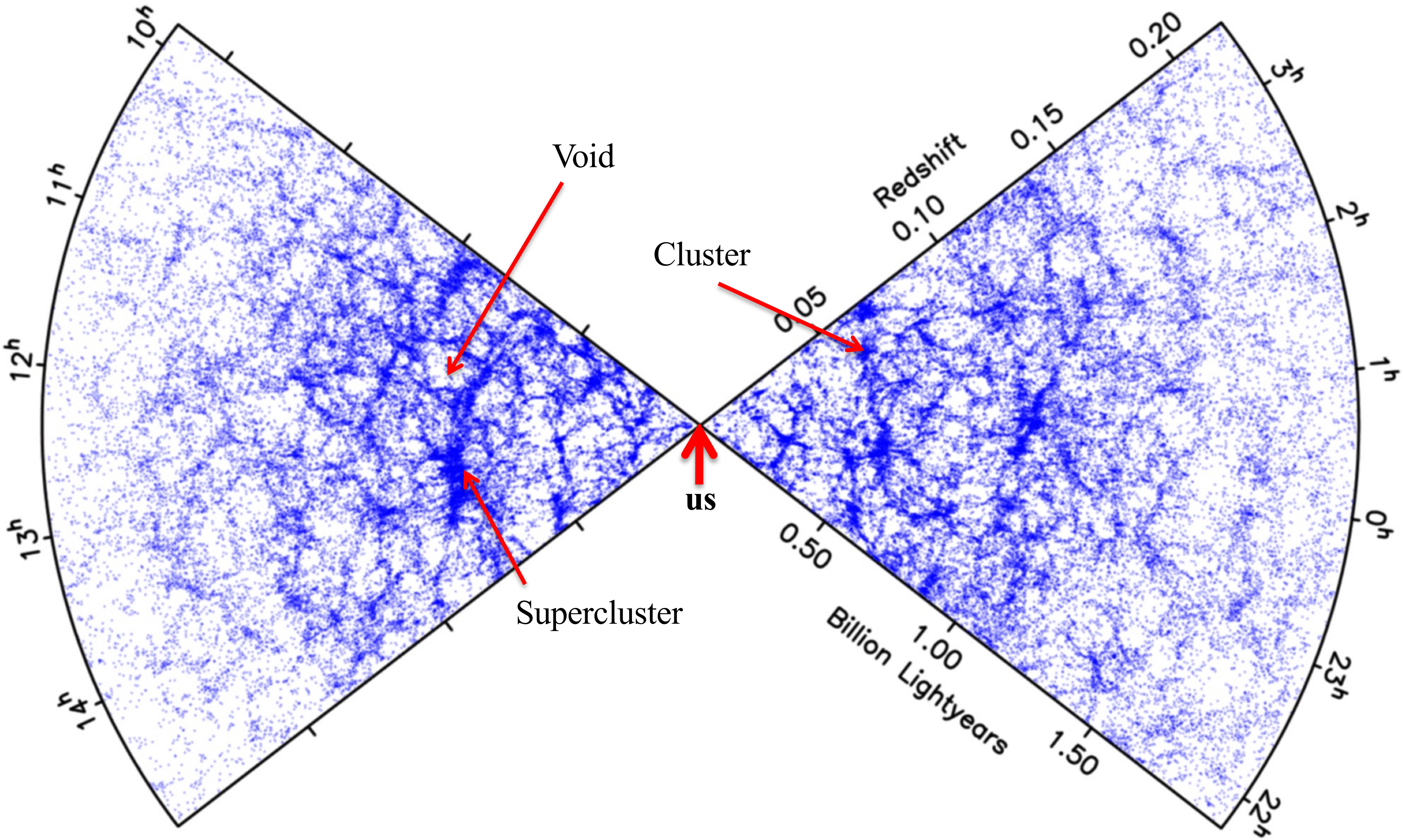


2dF redshift galaxy survey



https://www.roe.ac.uk/~jap/2df/

Hypotheses

- Matter distribution in the Universe: mainly composed of collisionless massive particles interacting mainly through gravitational force (interactions of other nature are at best very weak and neglected)
- Dark energy is embodied by the cosmological constant
- Dark matter distribution follows Vlasov-Poisson equations (Note: the equations below ignore the expansion of the Universe)

$$\frac{\partial f}{\partial t} + \mathbf{u}.\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \phi}{\partial \mathbf{x}}.\frac{\partial f}{\partial \mathbf{u}} = 0$$

$$\Delta_{\mathbf{r}}\phi = 4\pi G\rho = 4\pi G \int f(\mathbf{r}, \mathbf{u}, t)\, \mathrm{d}^3\mathbf{u}$$

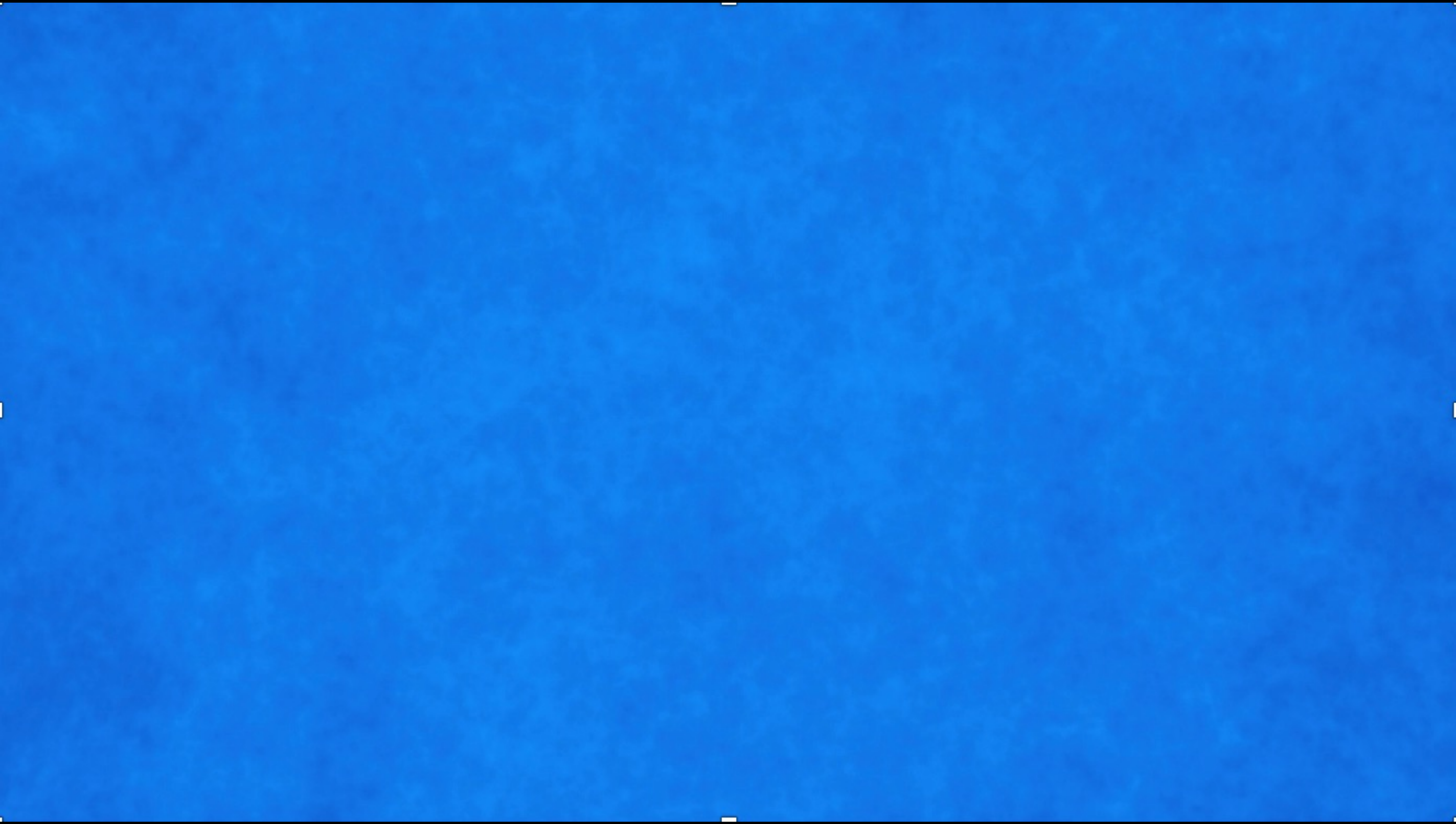
Model considered here: Cold Dark Matter

e.g., standard Cold Dark Matter model with neutralino of mass 100 Gev

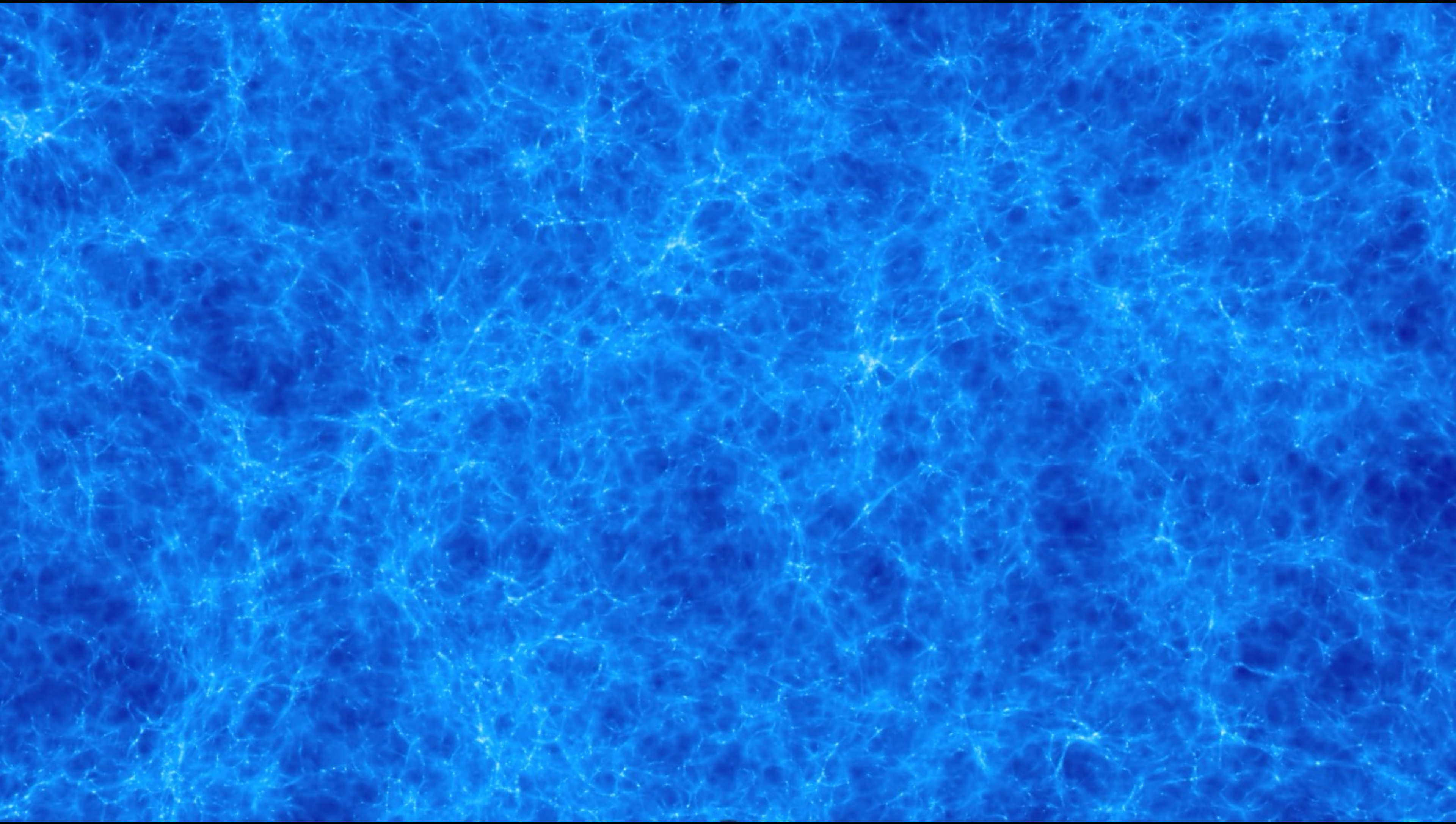
- Smoothness of initial conditions at scales smaller or of the order of a pc
- Initial local velocity dispersion so small that it can be neglected in practice
- ➢ The dark matter distribution can be considered as a 3D sheet folding in 6D phase-space.
  Initial conditions: $f(\mathbf{r}, \mathbf{u}, t = t_{\mathrm{i}}) = \rho_{\mathrm{i}}(\mathbf{r})\, \delta_{\mathrm{D}}[\mathbf{u} - \mathbf{u}_{\mathrm{i}}(\mathbf{r})]$

- ➢ The first dark matter (micro)halos to form are typically of earth mass and solar system size (e.g. Diemand, J., Moore, B., & Stadel, J. 2005, Nature, 433, 389)

Simulation of the dark matter distribution

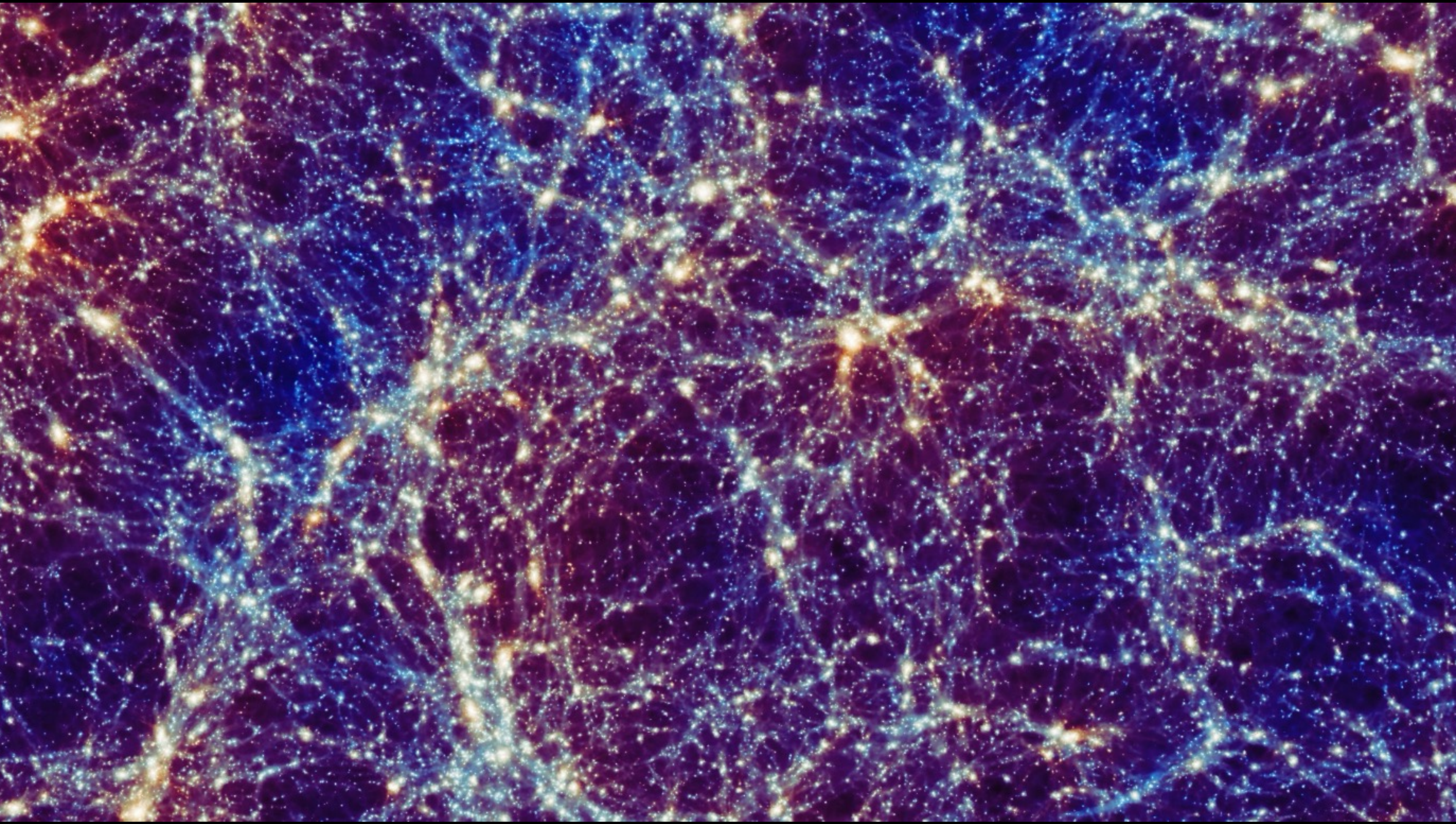


Simulation of the dark matter distribution



Colombi & Dubois, IAP

Simulation of the dark matter distribution

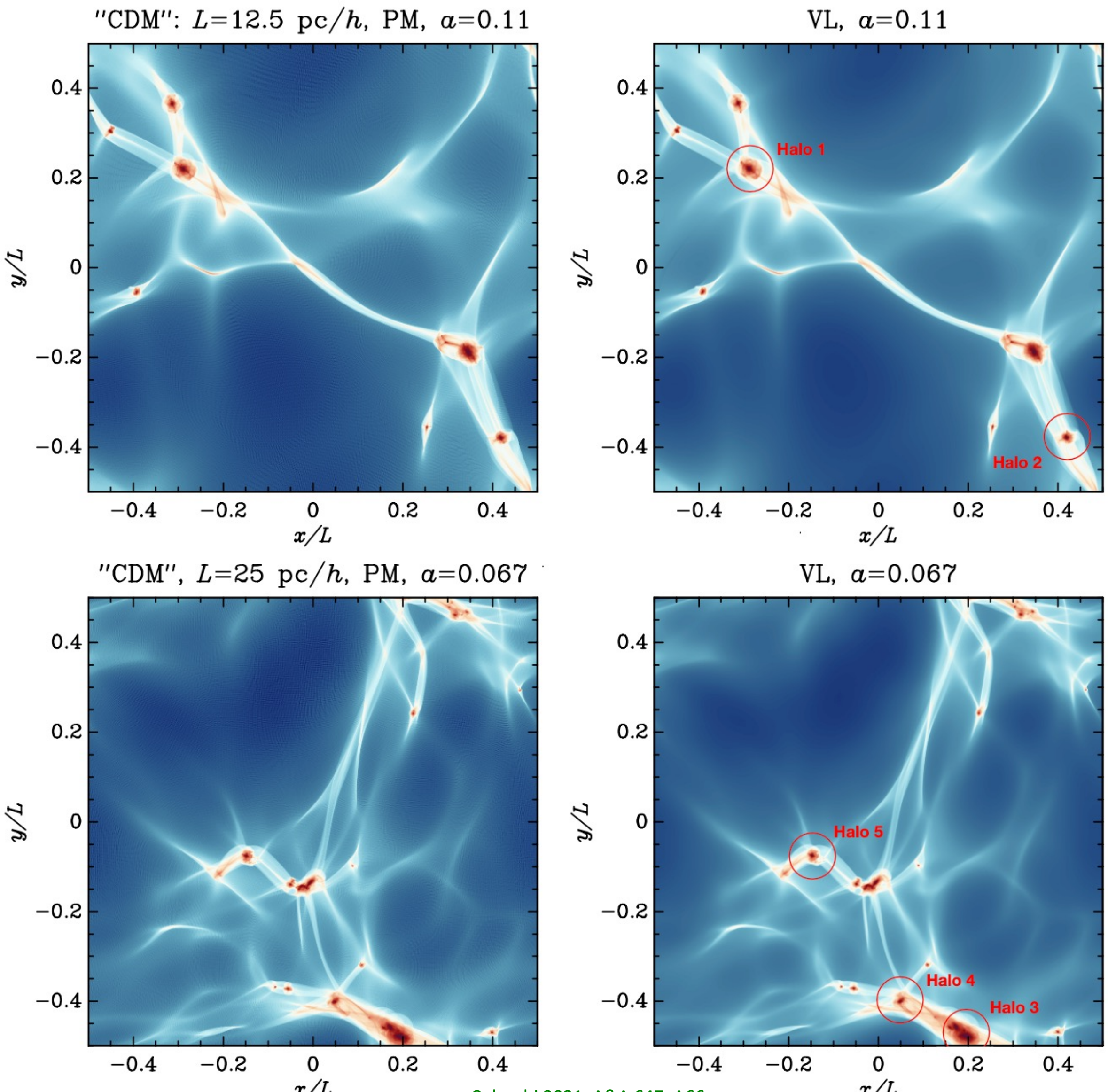


Colombi & Dubois, IAP

« Cold Dark Matter » in a very small box

PM

Vlasov



Colombi 2021, A&A 647, A66

Standard neutralino model

(see, e.g., Diemand, Moore & Stadel 2005, Nature 433, 389)

The 4 steps of dark matter halos history

A. **The pre-collapse phase:** resolvable analytically with perturbation theory, of high order when approaching shell-crossing; Lagrangian perturbation theory can be applied, with the proper settings, slightly beyond shell-crossing (see, e.g., Bernardeau, Colombi, Gaztanaga, Scoccimarro 2002, PhR 367, 1)

B. **The shell-crossing phase:** *Post-collapse* Lagrangian perturbation theory can be applied, with the proper settings, slightly beyond shell-crossing (Refs. Below)

C. **The violent relaxation phase:** establishment of a prompt power-law cusp during a monolithic phase with clear signatures of *self-similarity* (see, e.g., Ishiyama, Makino & Ebisuzaki, 2010, ApJ 723, L195; Colombi 2021, A&A 647, A66; Delos & White 2023, MNRAS 518, 3509). Simulations needed. ***See talk of A. Parichha in the 2D case***

D. **The evolution towards a dynamical attractor**, the so-called NFW profile (Navarro, Frenk & White, 1996, ApJ 462, 563; 1997, ApJ 490, 493), during which mergers can (or not) take place. Signatures of Self-similarity are less evident but still present through e.g. the pseudo-phase-space density. Maximum entropy methods have been proposed to explain convergence to NFW. Simulations needed.

Note: the same process as above is expected to apply for Warm Dark Matter (WDM) models where the particle mass is about a few KeV. The smoothing scale is larger, and the initial velocity dispersion of dark matter particles might not be neglected for accurate calculations.

A. The pre-collapse phase

Lagrangian equations of motion

Lagrangian equations of motion in the expanding Universe:

$$\ddot{\boldsymbol{x}} + 2H\dot{\boldsymbol{x}} = -\frac{1}{a^2}\boldsymbol{\nabla}_x\phi(\boldsymbol{x}) \ ,$$
$$\boldsymbol{\nabla}_x^2\phi(\boldsymbol{x}) = 4\pi G\bar{\rho}_{\rm m}a^2\delta(\boldsymbol{x})$$

a : scale factor of the Universe
ρ_m: average matter density
δ : density contrast

Lagrangian displacement field: $\boldsymbol{\Psi}(t,\boldsymbol{q})$

$$\boldsymbol{x}(t,\boldsymbol{q}) \ = \ \boldsymbol{q} + \boldsymbol{\Psi}(t,\boldsymbol{q})$$
$$\boldsymbol{v}(t,\boldsymbol{q}) \ = \ a(t)\dot{\boldsymbol{\Psi}}(t,\boldsymbol{q})$$

Divergence: longitudinal part: $\boldsymbol{\nabla}_x\cdot\left[\ddot{\boldsymbol{\Psi}}(t,\boldsymbol{q}) + 2H\dot{\boldsymbol{\Psi}}(t,\boldsymbol{q})\right] = -4\pi G\bar{\rho}_{\rm m}\delta(\boldsymbol{x})$

Curl: transverse part: $\boldsymbol{\nabla}_x\times\left[\ddot{\boldsymbol{\Psi}}(t,\boldsymbol{q}) + 2H\dot{\boldsymbol{\Psi}}(t,\boldsymbol{q})\right] = 0 \ ,$

Mass conservation: $1+\delta(\boldsymbol{x}) = 1/\det\left(\delta_{ij}+\Psi_{i,j}\right)$

Lagrangian perturbation theory

See, e.g., Bernardeau, F., Colombi, S., Gaztañaga, E., & Scoccimarro, R. 2002, Phys. Rep., 367, 1
Rampf, C. 2012, JCAP, 2012, 004
Matsubara, T. 2015, Phys. Rev. D, 92, 023534
Below: notations of Saga, Taruya & Colombi 2018, PRL 121, 241302

Perturbative expansion: $\boldsymbol{\Psi}(\tau,\boldsymbol{q}) = \sum_{n=1}^{\infty} \boldsymbol{\Psi}^{(n)}(\tau,\boldsymbol{q})$ $\qquad \boldsymbol{\Psi}^{(n)}(\boldsymbol{q},t) = D_+^n(t)\,\boldsymbol{\Psi}^{(n)}(\boldsymbol{q})$

Einstein de Sitter : $\Omega_{\mathrm{m},0} = 1 \qquad \tau \equiv \ln D_+ = \ln a$

Longitudinal:

$$\left(\hat{\mathcal{T}} - \frac{3}{2}\right)\Psi^{(n)}_{k,k} = -\epsilon_{ijk}\epsilon_{ipq}\sum_{n=m_1+m_2}\Psi^{(m_1)}_{j,p}\left(\hat{\mathcal{T}} - \frac{3}{4}\right)\Psi^{(m_2)}_{k,q}$$
$$-\frac{1}{2}\epsilon_{ijk}\epsilon_{pqr}\sum_{n=m_1+m_2+m_3}\Psi^{(m_1)}_{i,p}\Psi^{(m_2)}_{j,q}\left(\hat{\mathcal{T}} - \frac{1}{2}\right)\Psi^{(m_3)}_{k,r}\,,$$

Transverse:

$$\epsilon_{ijk}\hat{\mathcal{T}}\Psi^{(n)}_{j,k} = -\epsilon_{ijk}\sum_{n=m_1+m_2}\Psi^{(m_1)}_{p,j}\hat{\mathcal{T}}\Psi^{(m_2)}_{p,k}$$

Time operator: $\hat{\mathcal{T}} \equiv \frac{\partial^2}{\partial\tau^2} + \frac{1}{2}\frac{\partial}{\partial\tau}$

Solution reconstruction: $\boldsymbol{\Psi}^{(n)} = \triangle^{-1}\left[\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\boldsymbol{\Psi}^{(n)}\right) - \boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\boldsymbol{\Psi}\right)\right]$

Linear Lagrangian perturbation theory

Exact in 1D up to shell-crossing

Astron. & Astrophys. 5, 84—89 (1970)

Gravitational Instability: An Approximate Theory for Large Density Perturbations

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Institute of Applied Mathematics, Moscow

Received September 19, 1969

An approximate solution is given for the problem of the growth of perturbations during the expansion of matter without pressure. The solution is qualitatively correct even when the perturbations are not small. Infinite density is first obtained on disc-like surfaces by unilateral compression.

The following layers are compressed first adiabatically and then by a shock wave. Physical conditions in the compressed matter are analysed.

Key words: Galaxies formation — Cosmology — Gravitational instability

1. The Approximate Solution

The linear theory of perturbations, applied to the uniform isotropic cosmological solution, is now well understood. It is generally admitted that its predictions are limited by $\delta\varrho/\varrho < 1$, and that further events must be followed by numerical calculations. Such calculations, in three dimensions and with random initial conditions, promise to be tedious. Therefore an approximate method, which gives the right answer at least qualitatively, is of interest.

In this article the linear theory is taken to formulate the answer in terms of lagrangian coordinates: the actual position $\boldsymbol{r}$ of a particle is given as a function of its lagrangian coordinate $\boldsymbol{q}$ (i.e. its initial position) and the time t, $\boldsymbol{r} = \boldsymbol{r}(t, \boldsymbol{q})$. The linear theory is applied to the simplest case of pressure $\mathscr{P} = 0$ ("dust") in the Newtonian approximation. Only the growing perturbations are considered. The answer is of the form

$$\boldsymbol{r} = a(t)\,\boldsymbol{q} + b(t)\,\boldsymbol{p}(\boldsymbol{q})\,. \qquad (1)$$

The first term $a(t)\,\boldsymbol{q}$ describes the cosmological expansion; the second term describes the perturbations. The functions $a(t)$ and $b(t)$ are known; $b(t)$ is growing more rapidly than $a(t)$, as a result of gravitational instability. The vector function $\boldsymbol{p}(\boldsymbol{q})$ depends on the initial perturbation. With given $\boldsymbol{r}(t, \boldsymbol{q})$, it is possible to calculate the distribution of velocity and density in space; $\boldsymbol{r}(t, \boldsymbol{q})$ contains the whole picture of the motion.

The approximation proposed in this article consists in the extrapolation of formula (1) into the region where the perturbations of density $\delta\varrho/\varrho$ are not small.

Let us first investigate the consequences of the approximation; this will help us to analyse its plausibility. In order to follow the behaviour of a small group of particles centered on some definite $\boldsymbol{q}$, we calculate the tensor of deformation

$$\mathscr{D}_{ik} = \frac{\partial r_i}{\partial q_k} = a(t)\,\delta_{ik} + b(t)\,\frac{\partial p_i}{\partial q_k}\,.$$

The derivatives $\frac{\partial p_i}{\partial q_k}$ define a set of fundamental axes. After choosing the coordinate system along the axes, one obtains[1]) for a given $\boldsymbol{q}$

$$D = \begin{Vmatrix} a(t) - \alpha\, b(t) & 0 & 0 \\ 0 & a(t) - \beta\, b(t) & 0 \\ 0 & 0 & a(t) - \gamma\, b(t) \end{Vmatrix}.$$

A volume which was initially a cube (at $t \to 0$) and which would be a cube in the unperturbed motion, is transformed into a parallelepiped. One can always choose the axis of the cube so that it is transformed into a rectangular parallelepiped; the axes are not rotating in solution (1). The density near a particle with given $\boldsymbol{q}$ is given by the conservation of mass

$$\varrho\,(a - \alpha b)\,(a - \beta b)\,(a - \gamma b) = \bar{\varrho}\, a^3. \qquad (2)$$

[1]) It can be shown that $\frac{\partial p_i}{\partial q_k} = \frac{\partial p_k}{\partial q_i}$ in the growing mode of perturbations. Here $\alpha = \xi_1$, $\beta = \xi_2$, and $\gamma = \xi_3$ are the three roots of $\left|\frac{\partial p_k}{\partial q_i} + \xi\delta_{ik}\right| = 0$. The sign of α, β, γ is not defined in the usual manner, for the sake of subsequent convenience.

Tests of Lagrangian Perturbation Theory

Saga, Taruya & Colombi, 2022, A&A 664, A3
Saga, Taruya & Colombi 2018, PRL 121, 241302

- **3 sine waves** in a periodic box (following footpath of e.g. Moutarde et al. 1991, ApJ 382, 377)

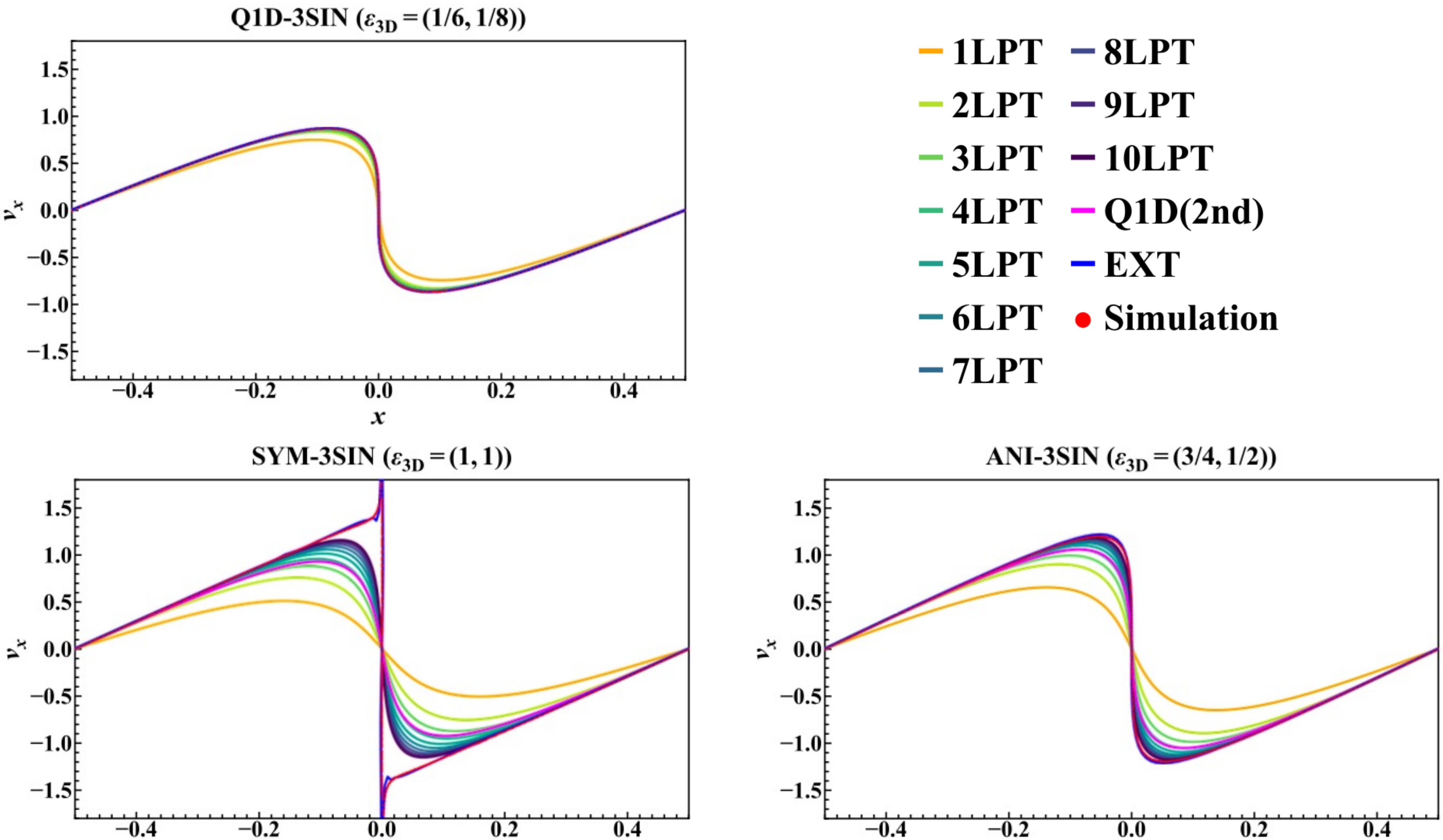
$$\boldsymbol{\Psi}^{\rm ini}(\boldsymbol{q}) = \frac{L}{2\pi}\begin{pmatrix} \epsilon_x \sin\left(\frac{2\pi}{L} q_x\right) \\ \epsilon_y \sin\left(\frac{2\pi}{L} q_y\right) \\ \epsilon_z \sin\left(\frac{2\pi}{L} q_z\right) \end{pmatrix}$$

$(\varepsilon_x, \varepsilon_y, \varepsilon_z)$=(-24, -4, -3) : ''quasi'' 1D
(-24, -18, -12) : "normal"
(-18, -18, -18) : ''Isotropic''

a_{ini}=0.0005

Phase-space diagram at collapse time

The intersection of the phase-space sheet (a 3D hypersurface) with the hyperplane y=z=0 (a 4D hypersurface) is a (set) of curve(s)



Saga, Taruya & Colombi, 2022, A&A 664, A3

B. The post-collapse phase: catastrophe theory

Catastrophe theory: the simple case

See, e.g., V. I. Arnold, S. F. Shandarin, and I. B. Zeldovich, Geophysical and Astrophysical Fluid Dynamics 20, 111 (1982)
J. Hidding, S. F. Shandarin, and R. van de Weygaert, MNRAS 437, 3442 (2014)
Feldbrugge, J., van de Weygaert, R., Hidding, J., & Feldbrugge, J. 2018, JCAP, 2018, 027

The simple case below: notations of Saga, Colombi & Taruya, 2023, A&A 678, A168

$$\boldsymbol{x}(\boldsymbol{q},t) = \boldsymbol{q} + \boldsymbol{\Psi}(\boldsymbol{q},t) \qquad 1+\delta(\boldsymbol{x}) = \frac{1}{J} \qquad J_{ij}(\boldsymbol{q},t) = \frac{\partial x_i(\boldsymbol{q},t)}{\partial q_j} = \delta_{ij} + \Psi_{i,j}(\boldsymbol{q},t)$$

Singular surface (or point/curve): $J = 0$

Symmetric configuration:

$$\begin{aligned}
\Psi_x(q_x,q_y,q_z) &= \Psi_x(q_x,-q_y,q_z) = \Psi_x(q_x,q_y,-q_z) \\
&= -\Psi_x(-q_x,q_y,q_z)\ , \\
\Psi_y(q_x,q_y,q_z) &= \Psi_y(q_x,q_y,-q_z) = \Psi_y(-q_x,q_y,q_z) \\
&= -\Psi_y(q_x,-q_y,q_z)\ , \\
\Psi_z(q_x,q_y,q_z) &= \Psi_z(-q_x,q_y,q_z) = \Psi_z(q_x,-q_y,q_z) \\
&= -\Psi_z(q_x,q_y,-q_z)\ .
\end{aligned}$$

Taylor expansion of the displacement field of around the initial singularity point:

$$x(\boldsymbol{q}) \simeq (1+\psi_{100})\,q_x + \frac{1}{2}\left(\psi_{120}\,q_y^2 + \psi_{102}\,q_z^2\right) q_x + \frac{1}{6}\psi_{300}\,q_x^3\ ,$$

$$y(\boldsymbol{q}) \simeq (1+\psi_{010})\,q_y + \frac{1}{2}\left(\psi_{012}\,q_z^2 + \psi_{210}\,q_x^2\right) q_y + \frac{1}{6}\psi_{030}\,q_y^3\ ,$$

$$z(\boldsymbol{q}) \simeq (1+\psi_{001})\,q_z + \frac{1}{2}\left(\psi_{201}\,q_x^2 + \psi_{021}\,q_y^2\right) q_z + \frac{1}{6}\psi_{003}\,q_z^3\ .$$

Shortly after shell-crossing: h=0 corresponds to the collapse time when a singular point appears. Collapse along x axis:

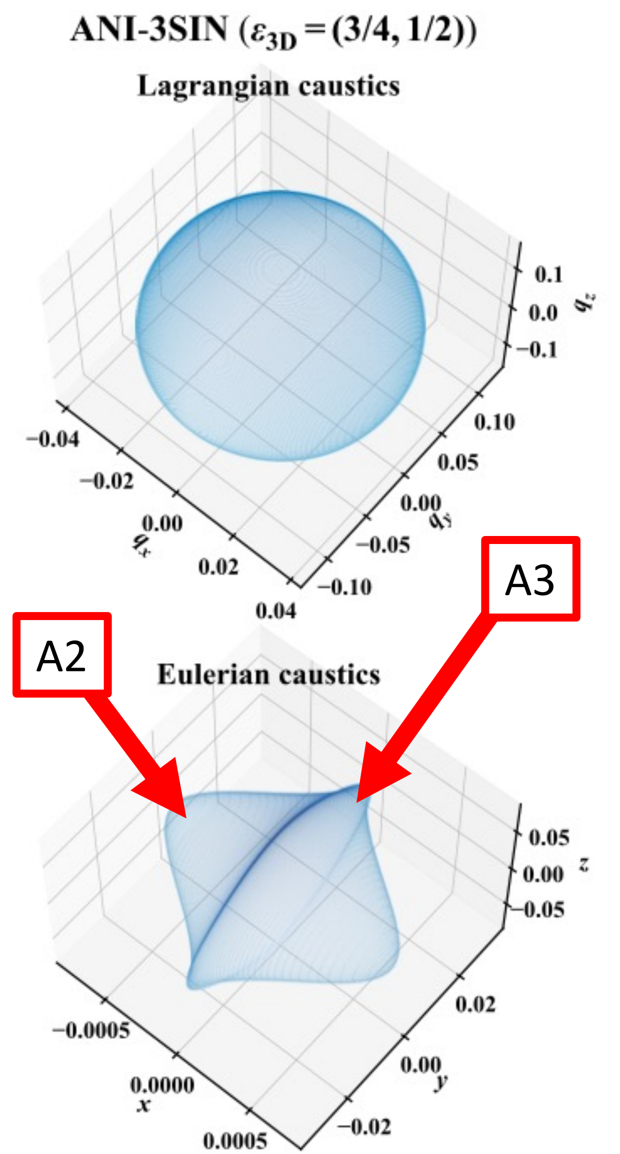
$$\frac{\partial x(\mathbf{0})}{\partial q_x} \equiv -h = 1 + \psi_{100} < 0\,,$$
$$\frac{\partial y(\mathbf{0})}{\partial q_y} = 1 + \psi_{010} > 0\,,$$
$$\frac{\partial z(\mathbf{0})}{\partial q_z} = 1 + \psi_{001} > 0\,.$$

Then:
$$x(\boldsymbol{q}) \simeq (1+\psi_{100})q_x + \frac{1}{2}\left(\psi_{120}\, q_y^2 + \psi_{102}\, q_z^2\right) q_x + \frac{1}{6}\psi_{300}\, q_x^3\,,$$
$$y(\boldsymbol{q}) \simeq (1+\psi_{010})q_y\,,$$
$$z(\boldsymbol{q}) \simeq (1+\psi_{001})q_z\,.$$

$$J \simeq \frac{1}{2}(1+\psi_{010})(1+\psi_{001})\left(-2h + \psi_{120}q_y^2 + \psi_{102}q_z^2 + \psi_{300}q_x^2\right)\,.$$

Extension of the pancake in configuration space:

$$x_{\max} = \sqrt{\frac{8h^3}{9\psi_{300}}}\,,$$
$$y_{\max} = (1+\psi_{010})\sqrt{\frac{2h}{\psi_{120}}}\,,$$
$$z_{\max} = (1+\psi_{001})\sqrt{\frac{2h}{\psi_{102}}}\,.$$



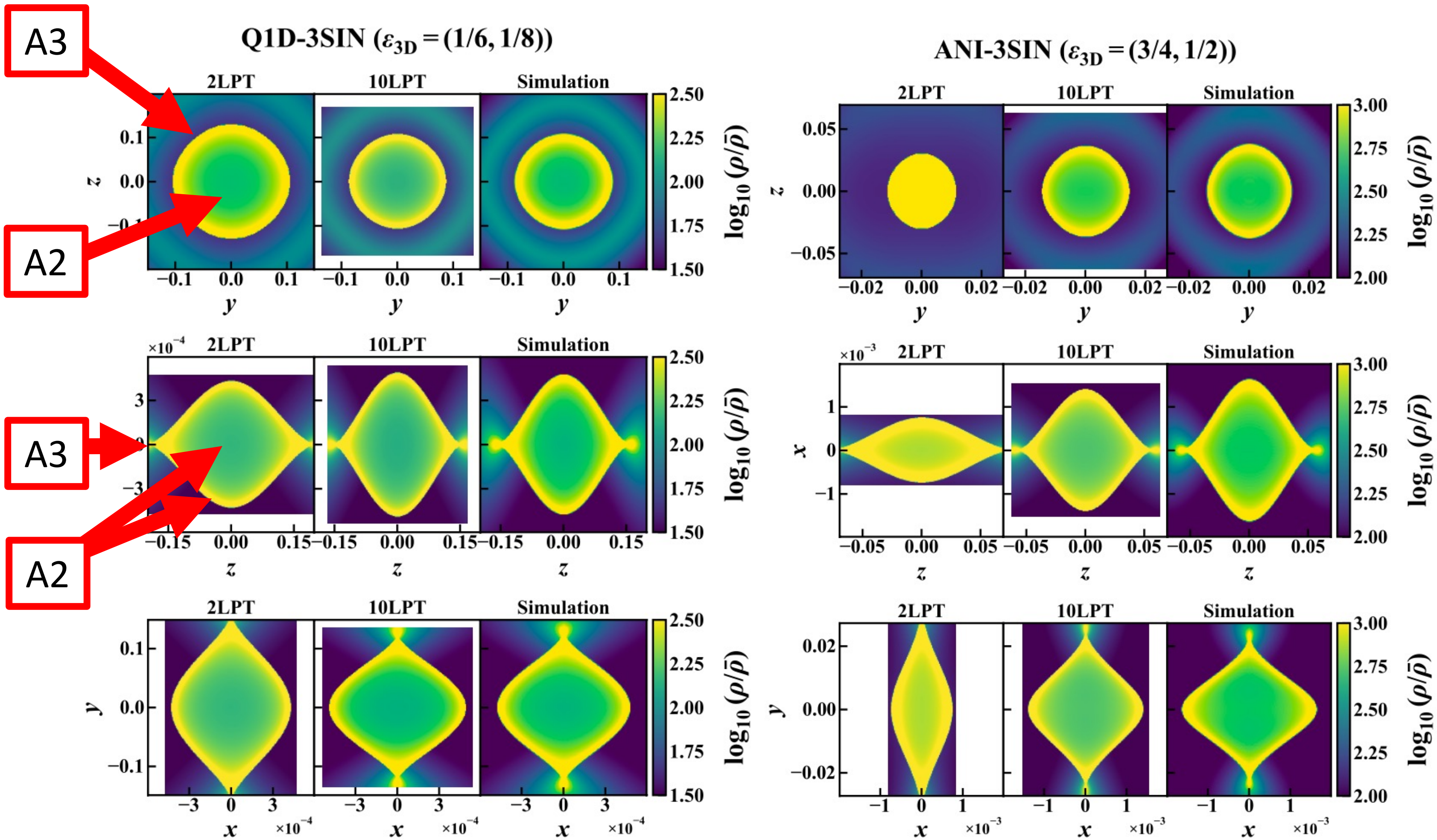
Saga, Taruya & Colombi, 2022, A&A 664, A3

The shape of a pancake just after shell-crossing

using the ballistic approximation: validation of Perturbation theory again

In Lagrangian space: the pancake is an ellipsoid

In Eulerian space: it has a pancake shape with sharp A3 edge

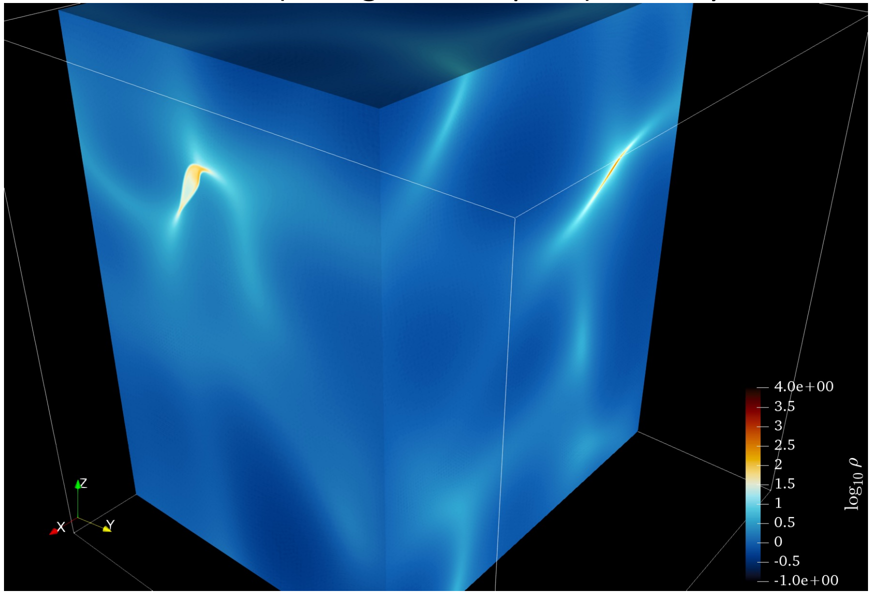


Saga, Taruya & Colombi, 2022, A&A 664, A3

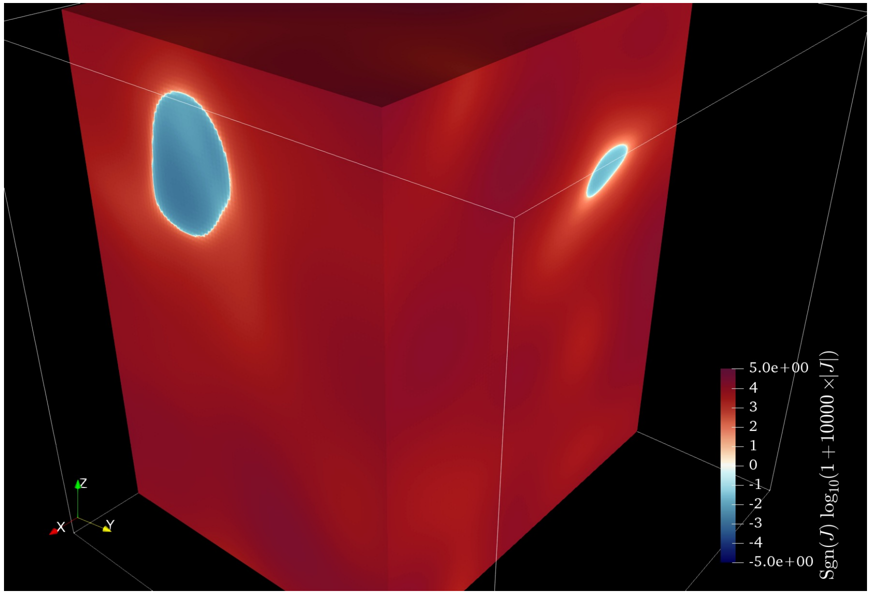
« Cold Dark Matter » in a very small box

(simulation box size 12.5 h^{-1} pc)

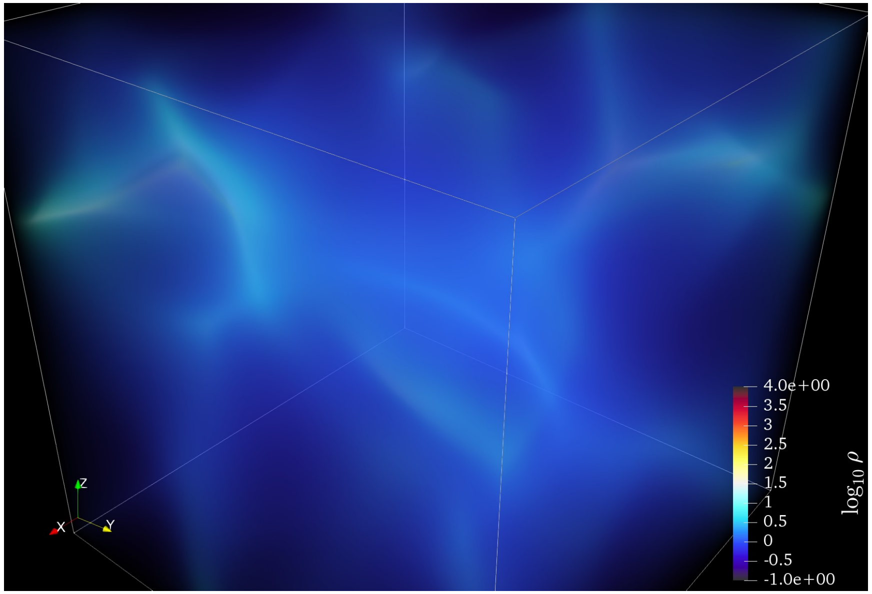
Eulerian (configuration space): density



Lagrangian space: Jacobian



Eulerian (configuration space): ray-traced (cumulated) density



Simulation described in Colombi 2021, A&A 647, A66

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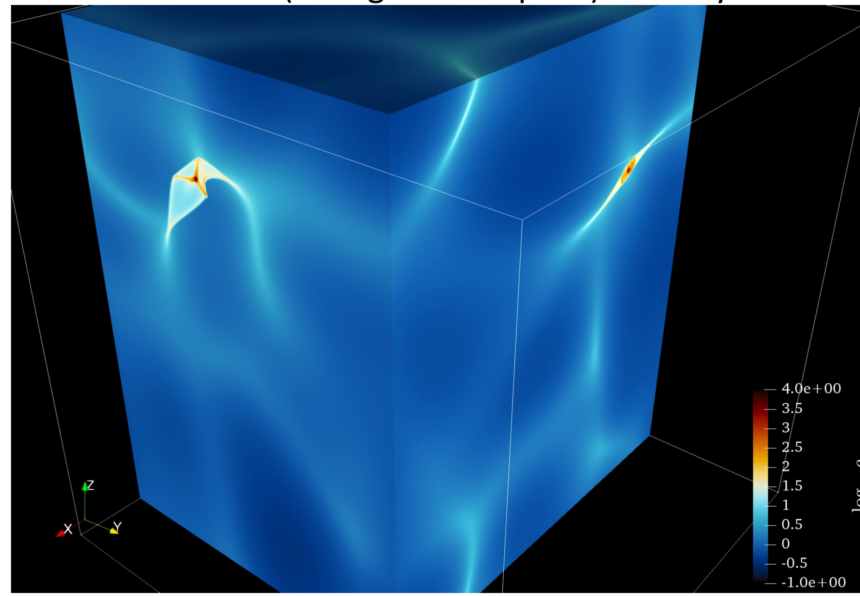
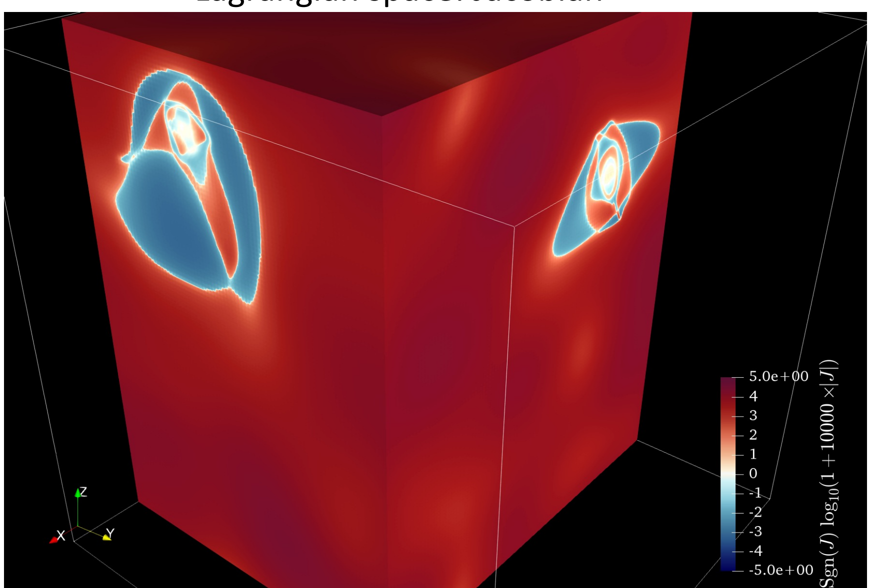
Eulerian (configuration space): density



Lagrangian space: Jacobian



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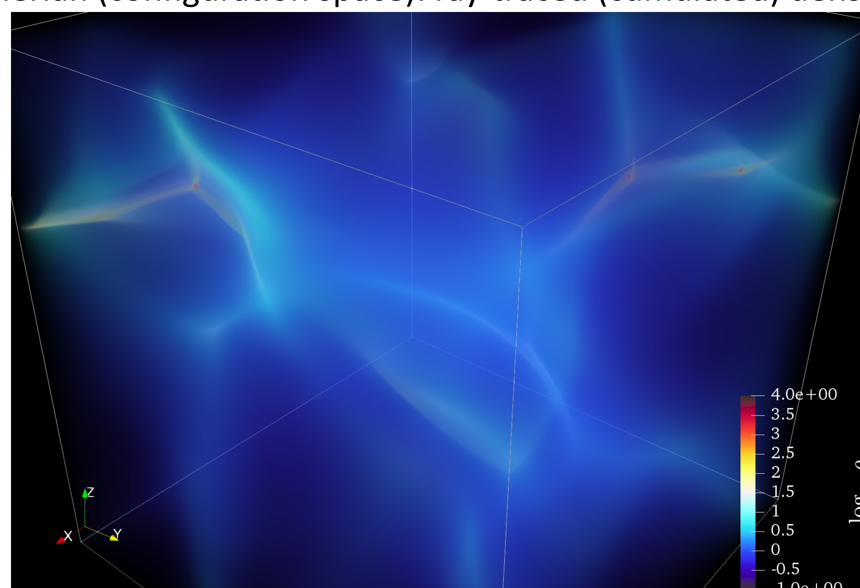

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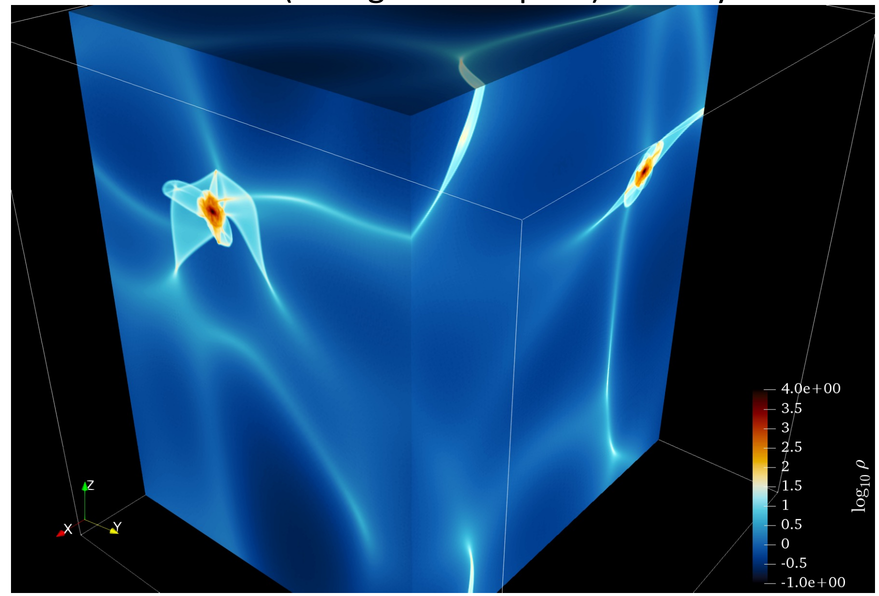

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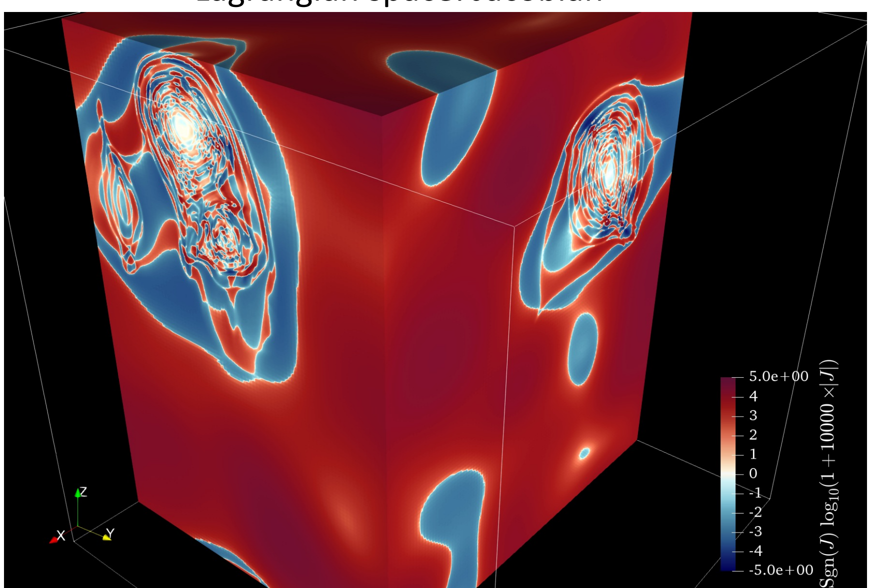
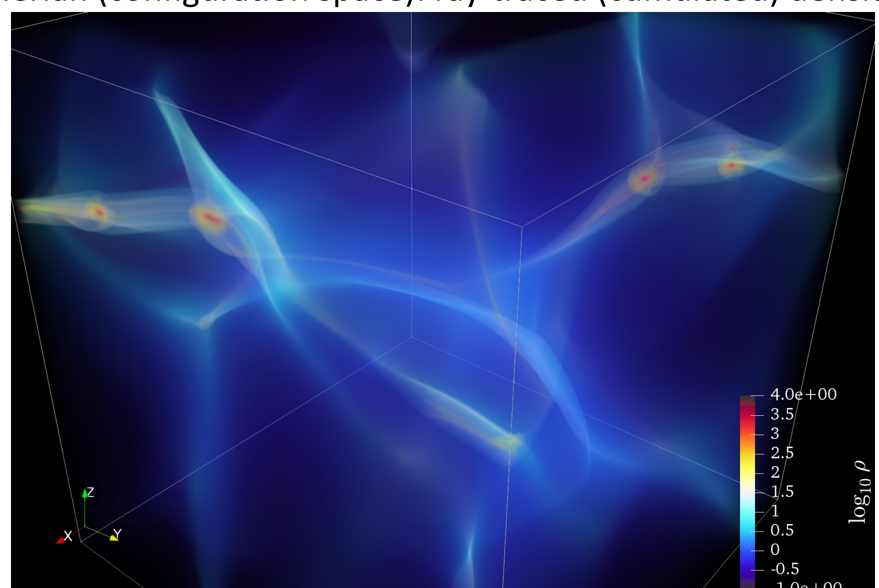


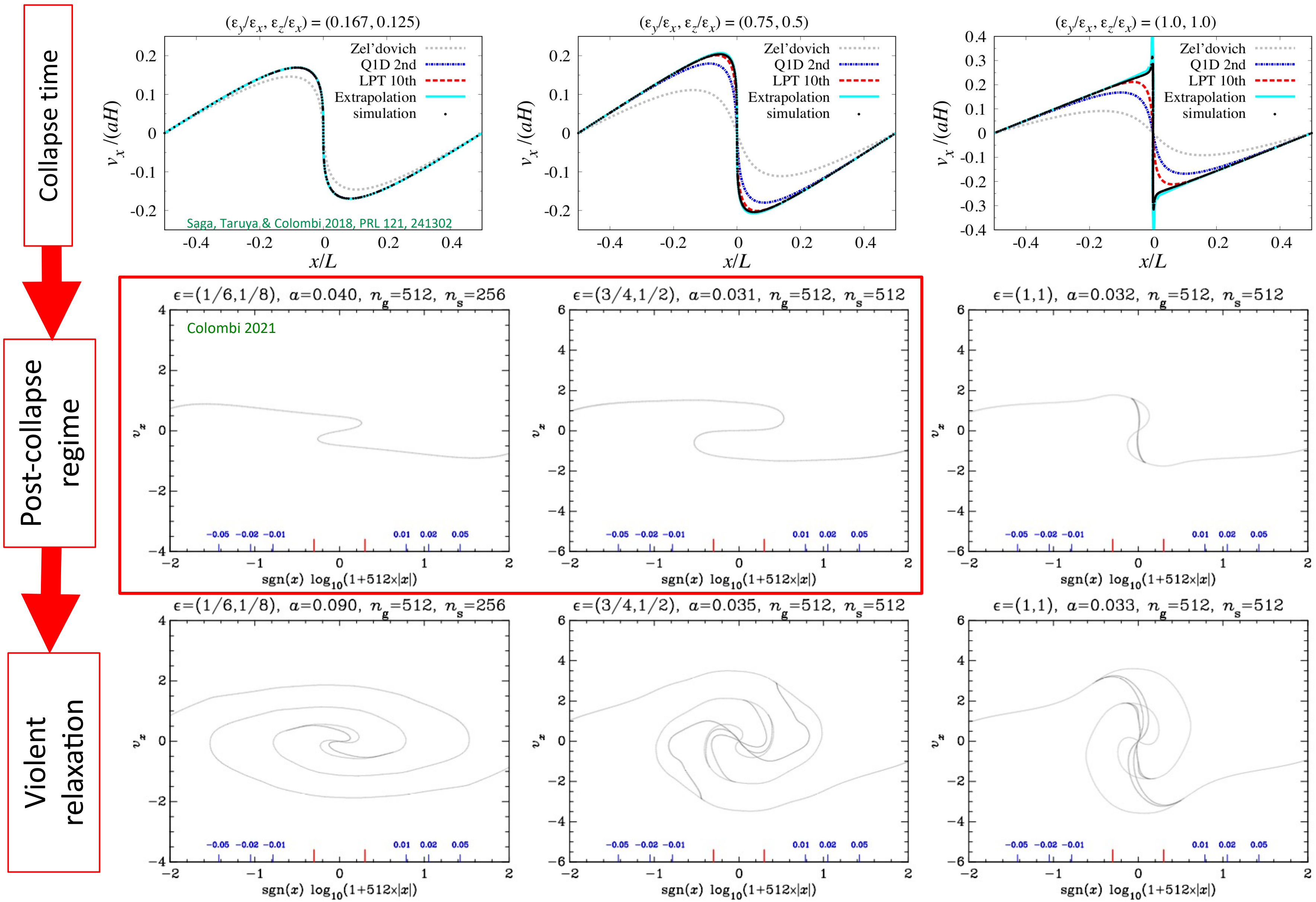
Eulerian (configuration space): ray-traced (cumulated) density



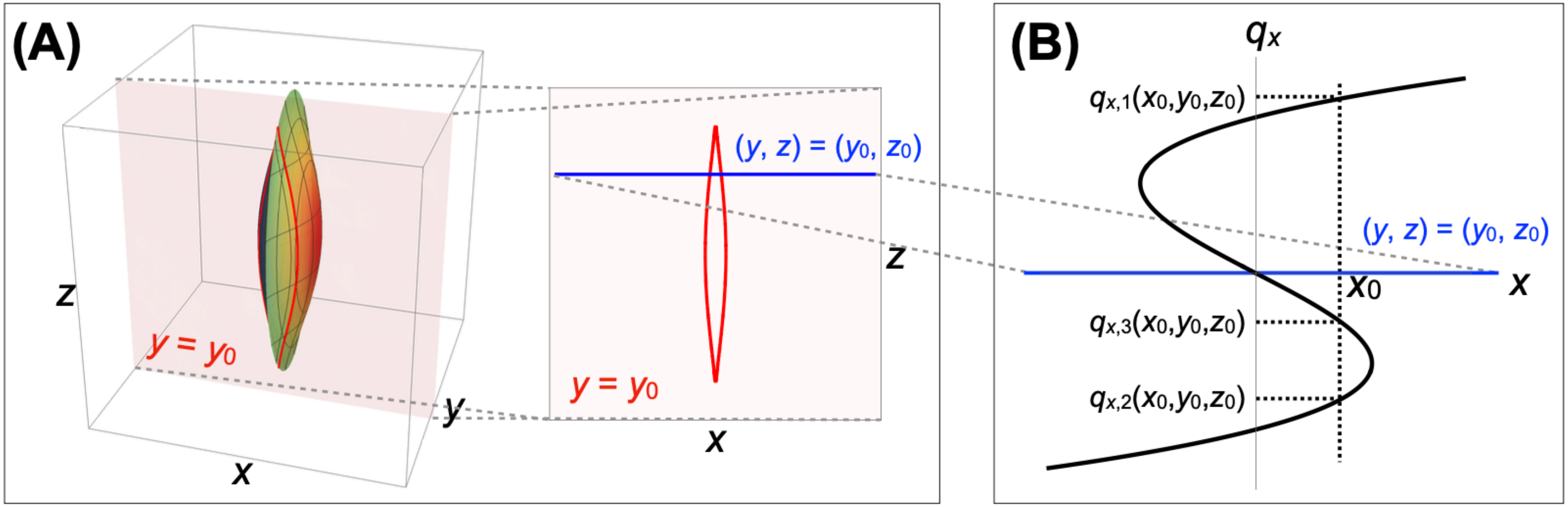
Simulation described in Colombi 2021, A&A 647, A66

B. Post-collapse perturbation theory: dynamics of a pancake

Phase-space diagram: beyond collapse time



The dynamics slightly beyond shell-crossing: Post-collapse perturbation theory



Saga, Taruya & Colombi, 2022, A&A 664, A3

Dynamics of pancakes: post-collapse perturbation theory in 1D

Colombi 2015, MNRAS 446, 2902
Taruya & Colombi 2017, MNRAS 470, 4858
Rampf, C., Frisch, U., & Hahn, O. 2021, MNRAS, 505, L90

- The main idea is that at collapse time the system presents a quasi-1D structure.
- In 1D dynamics, linear Lagrangian perturbation theory provides exact solution prior to collapse time. It can be used as a first approximation of the dynamics just after collapse time.

$$x(Q;\tau) = x_{\rm Zel}(Q;\tau) \equiv q + \psi(q) D_+(\tau),$$
$$u(Q;\tau) = u_{\rm Zel}(Q;\tau) \equiv \psi(q) \frac{{\rm d}D_+(\tau)}{{\rm d}\tau}.$$

- Correction to the force, hence to the motion, is computed just after collapse time assuming that *x(Q)* and *v(Q)* are third order polynomials of Lagrangian position *Q*, which is correct asymptotically in the Zel'dovich motion

Outer part of the "S" shape

$$x(Q;\tau) = x_{\rm Zel}(Q;\, \widehat{\tau}_{\rm c}(Q)) + \Delta x_{\rm out}(Q;\tau,\, \widehat{\tau}_{\rm c}(Q)),$$
$$u(Q;\tau) = u_{\rm Zel}(Q;\, \widehat{\tau}_{\rm c}(Q)) + \Delta u_{\rm out}(Q;\tau,\, \widehat{\tau}_{\rm c}(Q)).$$

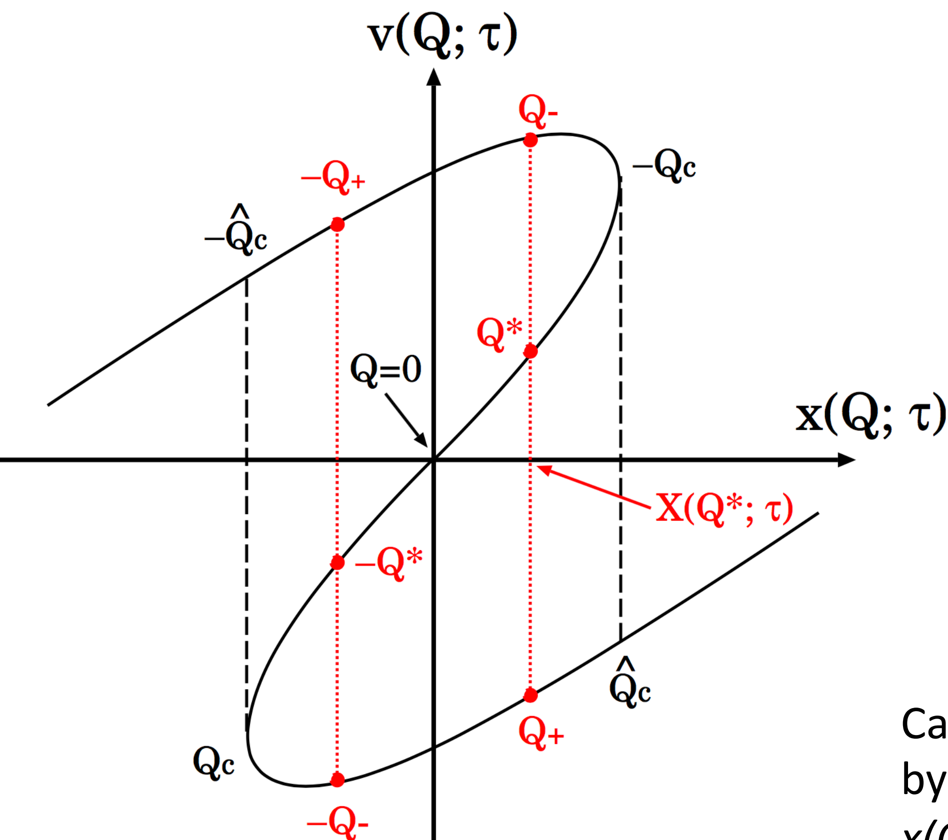
Inner part of the "S" shape

$$x(Q;\tau) = x_{\rm Zel}(Q;\, \widehat{\tau}_{\rm c}(Q)) + \Delta x_{\rm in}(Q;\tau,\, \widehat{\tau}_{\rm c}(Q)),$$
$$u(Q;\tau) = u_{\rm Zel}(Q;\, \widehat{\tau}_{\rm c}(Q)) + \Delta u_{\rm in}(Q;\tau,\, \widehat{\tau}_{\rm c}(Q)).$$

Collapse occurs at peaks of the initial density: $\delta_{\rm L}(q_0) = \dfrac{1}{D_+(\tau_0)}, \quad \left.\dfrac{{\rm d}\delta_{\rm L}(q)}{{\rm d}q}\right|_{q_0} = 0, \quad \left.\dfrac{{\rm d}^2\delta_{\rm L}(q)}{{\rm d}q^2}\right|_{q_0} < 0.$

At collapse the motion can be expanded at third order in the Lagrangian coordinate

$$x(q;\tau) \simeq A(q_0;\tau) - B(q_0;\tau)\,(q-q_0) + C(q_0;\tau)\,(q-q_0)^3$$



$$A(q_0;\tau) \equiv x(q_0;\tau) = q_0 + D_+(\tau)\,\psi(q_0),$$

$$B(q_0;\tau) \equiv -\left.\frac{\partial x}{\partial q}\right|_{q_0} = -1 - D_+(\tau)\psi'(q_0)$$

$$= \{D_+(\tau) - D_+(\tau_0)\}\,\delta_{\rm L}(q_0),$$

$$C(q_0;\tau) \equiv \frac{1}{6}\left.\frac{\partial^3 x}{\partial q^3}\right|_{q_0} = \frac{1}{6}\,D_+(\tau)\,\psi'''(q_0)$$

$$= -\frac{1}{6}\,D_+(\tau)\,\delta''_{\rm L}(q_0),$$

Calculation of the acceleration is then facilitated by the fact that the multivalued equation *x(Q)=x(Q')* has simple solutions

Taruya & Colombi 2017, MNRAS 470, 4858

Final expressions for the corrections of the motion are cumbersome: partial result

$$\Delta u_{\rm out}(Q;\tau,\,\widehat{\tau}_{\rm c}) = -\frac{3}{2}H_0^2\,\Omega_{\rm m,0}\,a(\tau_0)\Big[\widetilde{\alpha}_1(\tau)\,Q+\widetilde{\beta}_1(\tau)\,Q^3 + \widetilde{\gamma}_1(\tau_0)\left\{\widehat{Q}_{\rm c}^2(\tau)-Q^2\right\}^{3/2}+\widetilde{\delta}_1(\tau_0)\,Q^5\Big]+\widetilde{\epsilon}_1(\tau,\widehat{\tau}_{\rm c}),$$

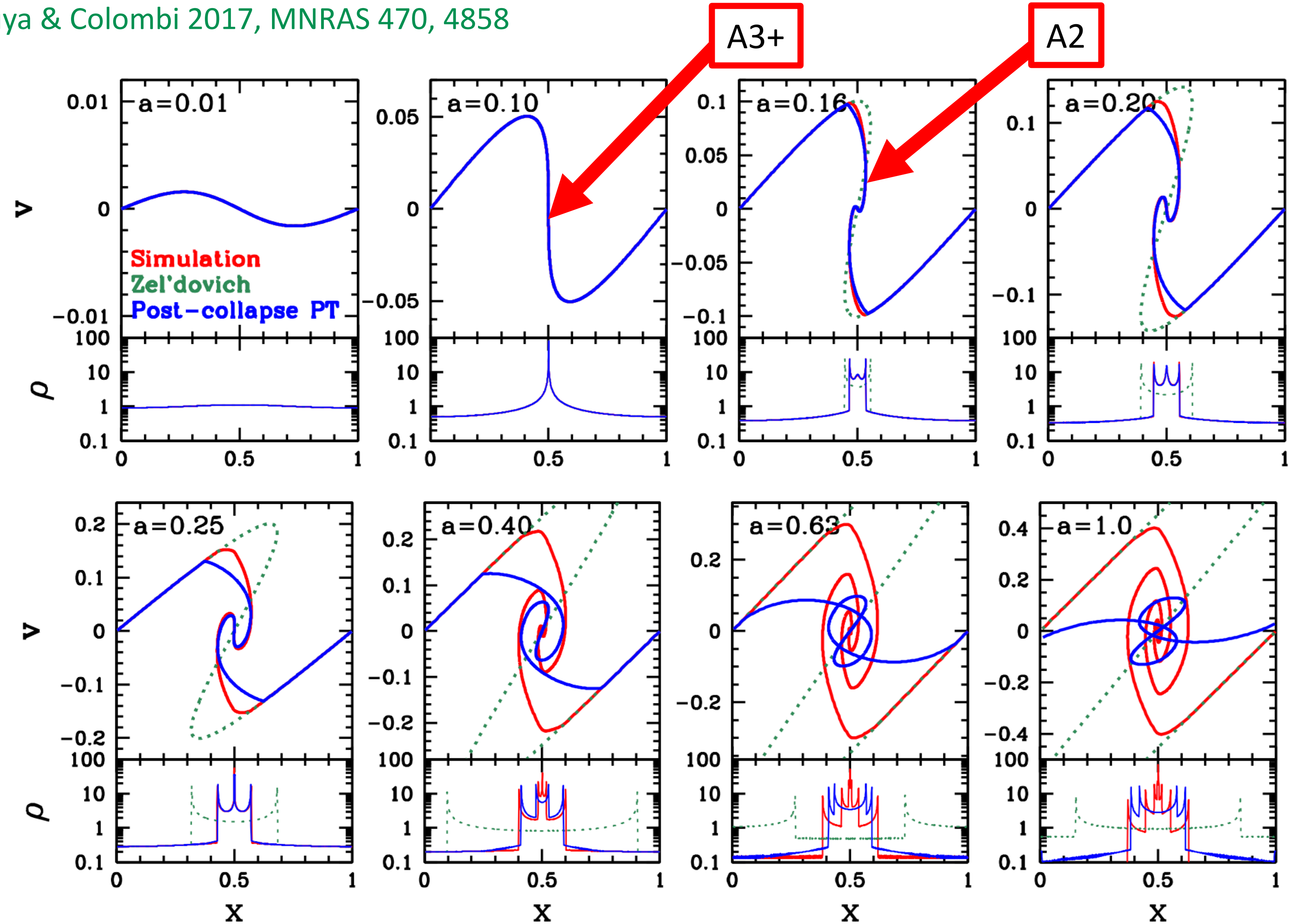
$$\Delta x_{\rm out}(Q;\tau,\widehat{\tau}_{\rm c}) = -\frac{3}{2}H_0^2\,\Omega_{\rm m,0}\,a(\tau_0)\Big[\widetilde{\alpha}_2(\tau)\,Q+\widetilde{\beta}_2(\tau)\,Q^3 + \widetilde{\gamma}_2(\tau_0)\left\{\widehat{Q}_{\rm c}^2(\tau)-Q^2\right\}^{5/2}+\widetilde{\delta}_2(\tau)\,Q^5+\widetilde{\zeta}_2(\tau)\,Q^7\Big] + \widetilde{\epsilon}_2(\tau,\widehat{\tau}_{\rm c}),$$

Coefficients	velocity $\Delta u_{\rm out}$ [Eq. (52)]	position $\Delta x_{\rm out}$ [Eq. (54)]
$\widetilde{\alpha}$	$T\,(\equiv\tau-\tau_0)$	$\frac{T^2}{2}$
$\widetilde{\beta}$	$T\,\frac{\delta_{\rm L}''(q_0)}{6}\,D_+(\tau_0)-\frac{\kappa}{8}$	$-\frac{\kappa}{8}\,T+\frac{\delta_{\rm L}''(q_0)}{12}\,D_+(\tau_0)\,T^2$
$\widetilde{\gamma}$	$-{\rm sgn}(Q)\,\frac{\kappa}{4\sqrt{3}}$	$-{\rm sgn}(Q)\,\frac{\kappa^2}{80\sqrt{3}}$
$\widetilde{\delta}$	$-\frac{\delta_{\rm L}''(q_0)}{48}\,\kappa\,D_+(\tau_0)$	$\frac{1}{2}\left(\frac{\kappa}{8}\right)^2-\left(\frac{\kappa}{8}\right)\,\frac{\delta_{\rm L}''(q_0)}{6}\,D_+(\tau_0)\,T$
$\widetilde{\zeta}$	———	$\frac{1}{2}\left(\frac{\kappa}{8}\right)^2\,\frac{\delta_{\rm L}''(q_0)}{6}\,D_+(\tau_0)$
$\widetilde{\epsilon}$	$\psi(q_0)\left[\frac{{\rm d}D_+(\tau')}{{\rm d}\tau'}\right]_{\widehat{\tau}_{\rm c}(Q)}^{\tau}$	$\psi(q_0)\left\{D_+(\tau)-D_+(\widehat{\tau}_{\rm c}(Q))-\left.\frac{{\rm d}D_+}{{\rm d}\tau}\right\vert_{\widehat{\tau}_{\rm c}(Q)}(\tau-\widehat{\tau}_{\rm c}(Q))\right\}$

Taruya & Colombi 2017, MNRAS 470, 4858

Example in 1D: collapse and evolution of a single sine wave

Taruya & Colombi 2017, MNRAS 470, 4858



Post-collapse perturbation theory in 3D

Saga, Colombi, &Taruya, 2023, A&A 678, A168
Saga et al., in preparation

- Generalization of the previous calculations to the 2D and 3D cases is straightforward, but even more cumbersome
- It uses the fact that shortly after collapse, the structure of the system remains quasi-1D, with the shape of a pancake
- The calculation of the force field along shell-crossing axis is analogous to the 1D case
- In the other axes of the motion, the system does not experience (yet) shell-crossing, which allows one to exploit directly higher order perturbation theory results, as long as convergence is achieved

Example: preliminary results

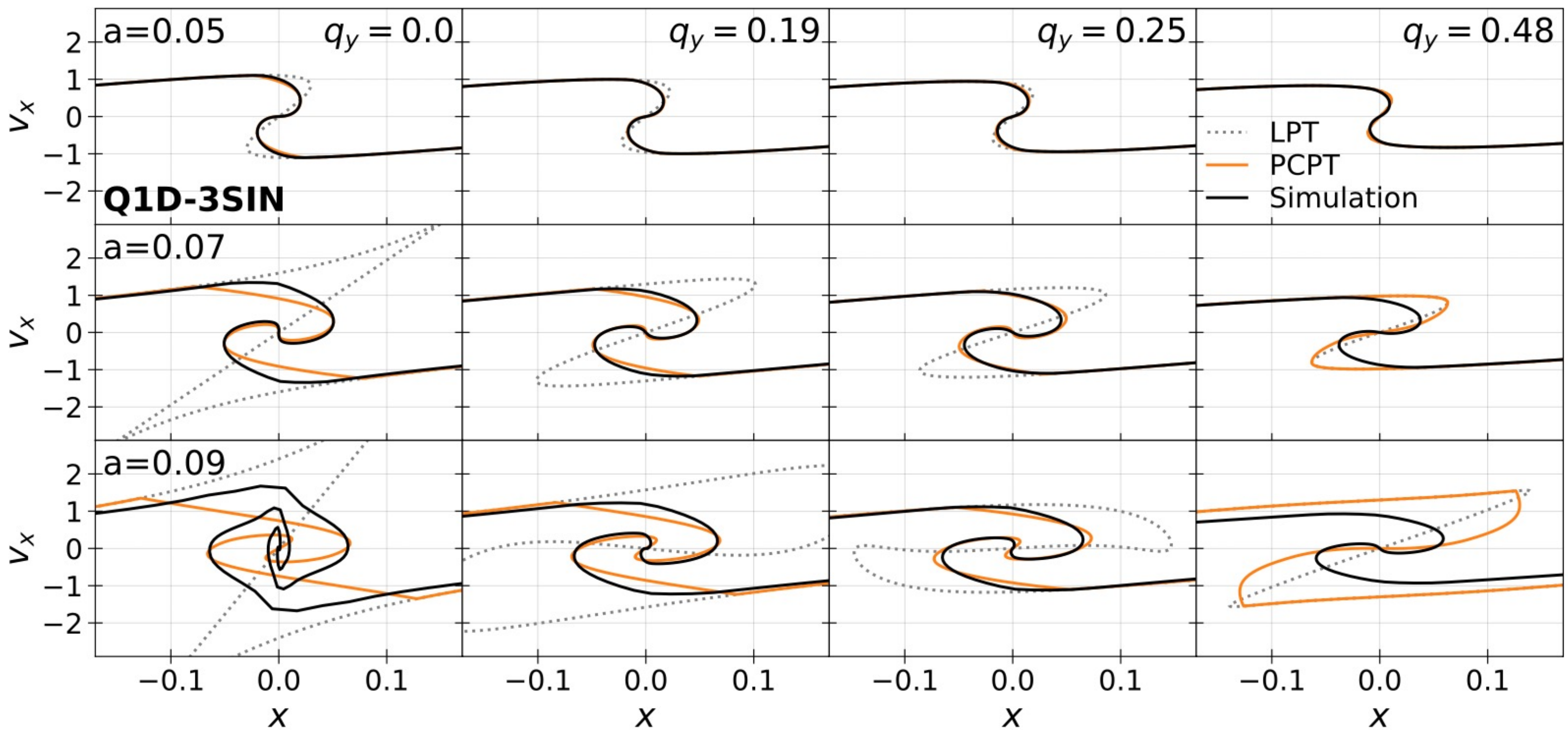
Acceleration along x-axis:
Saga, Colombi & Taruya, 2023, A&A 678, A168

$$F_x(\boldsymbol{x}_0) \simeq \int \mathrm{d}x \, \frac{4\pi G\bar{\rho}a^2}{(1+\psi_{010})(1+\psi_{001})} \sum_{n=0,1,2} \left|\frac{\partial x}{\partial q_{x,n}}\right|^{-1} \frac{1}{2}\left[\Theta_{\mathrm{H}}(x-x_0) - \Theta_{\mathrm{H}}(x_0-x)\right],$$

$$= \int \mathrm{d}q_x \, \frac{4\pi G\bar{\rho}a^2}{(1+\psi_{010})(1+\psi_{001})} \frac{1}{2}\left[\Theta_{\mathrm{H}}\left(x\left(q_x, \frac{y_0}{1+\psi_{010}}, \frac{z_0}{1+\psi_{001}}\right) - x_0\right) - \Theta_{\mathrm{H}}\left(x_0 - x\left(q_x, \frac{y_0}{1+\psi_{010}}, \frac{z_0}{1+\psi_{001}}\right)\right)\right],$$

$$= -\frac{4\pi G\bar{\rho}a^2}{(1+\psi_{010})(1+\psi_{001})}\left[q_{x,1}(\boldsymbol{x}_0) + q_{x,2}(\boldsymbol{x}_0) - q_{x,3}(\boldsymbol{x}_0)\right] \qquad \text{(in the multi-stream region)},$$

Phase-space diagrams:



Saga et al., in preparation

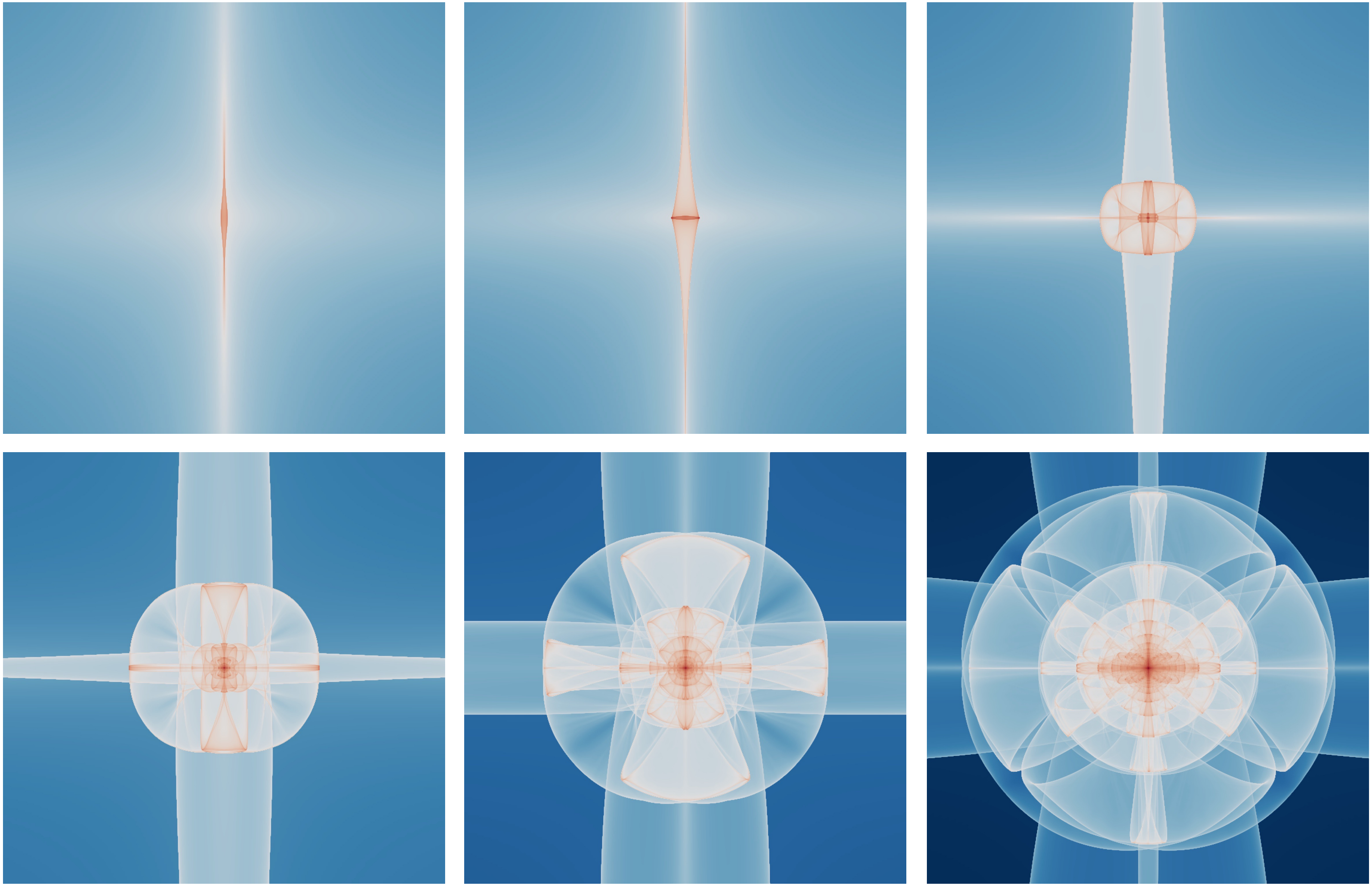
Perspectives

- **The calculations can potentially be iterated** until next crossing time and so on to obain a toy model of first instants of violent relaxation (example in 1D: Colombi 2015, MNRAS 446, 2902)

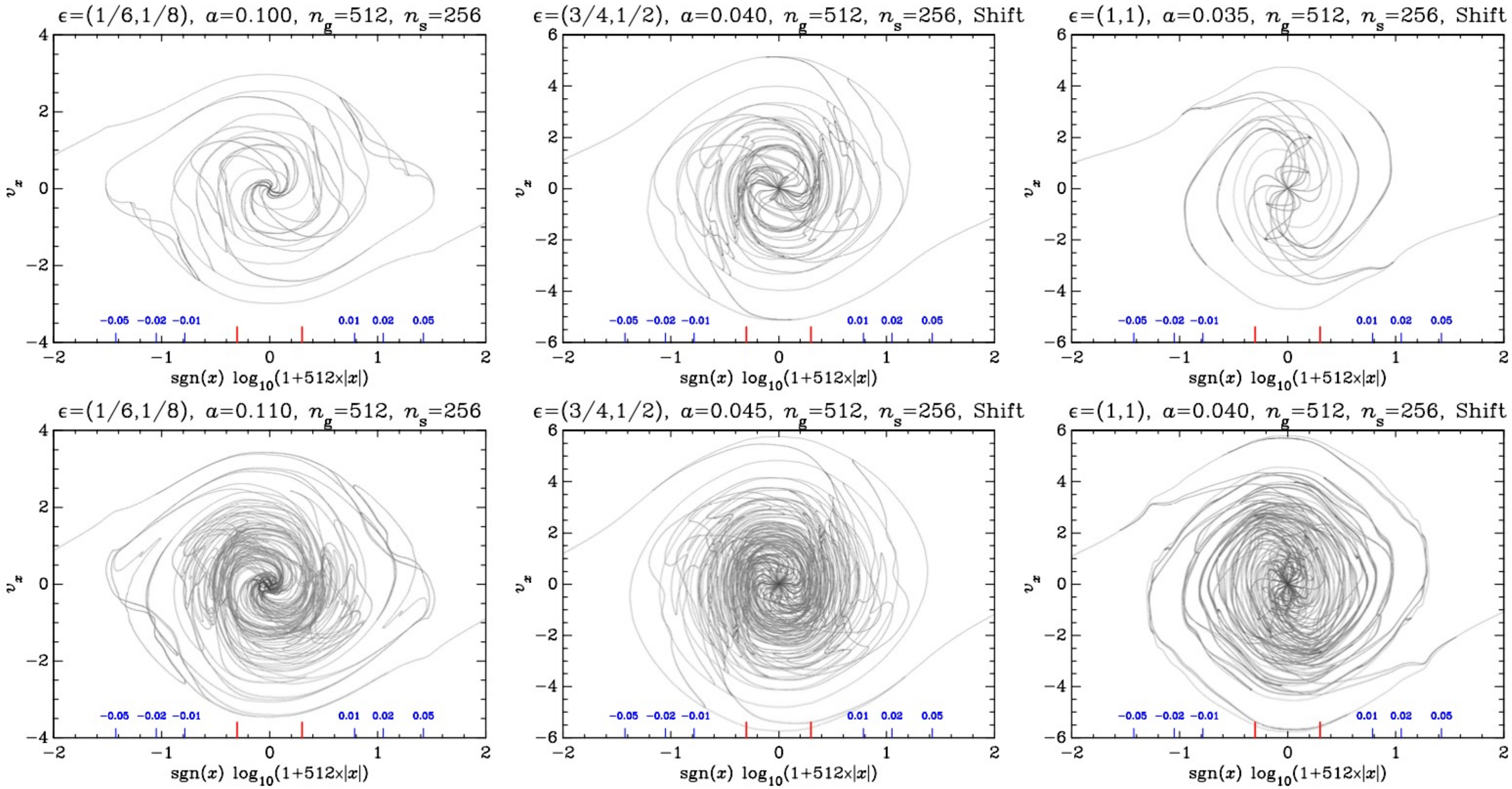
- **However:**
  - Shell crossings can take place along other axes
  - The general case is asymmetrical: the 3 sines wave symmetry is too restrictive
  - A link remains to be established with self-similarity
  - The real life of a dark matter halo is complex, with successive mergers

Beyond post-collapse: 2 sine waves with different amplitudes

Sousbie & Colombi 2016, JCP 321, 644; **Pariccha** et al 2024



The violent relaxation phase: self-similarity?



See talk of A. Parichha for the 2D case

Colombi 2021, A&A 647, A66

Phase-space slices : « CDM » halos