Small-scale clustering of Primordial Black Holes [arxiv:2402.00600] (PRD)

Pierre Auclair, Baptiste Blachier

Théorie, Univers, Gravitation (TUG) LAPTh, November 6th 2024

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Journal of Cosmology and Astroparticle Physics
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Primordial black holes from metric preheating: mass fraction in the excursion-set approach

Pierre Auclair and Vincent Vennin

Laboratoire Astroparticule et Cosmologie, CNRS Université de Paris,
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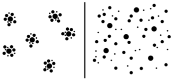
Chiara Animali at TUG 2023

Are primordial black holes clustered at formation?

- Two-point correlation function of overdensities

$$1+\xi(r) = \frac{p\left(\zeta(0)>\zeta_c, \zeta(r)>\zeta_c\right)}{p^2\left(\zeta(0)>\zeta_c\right)} \equiv \frac{P_2}{P_1^2}$$

$$\xi(r) \equiv \xi_{\rm PBH}(r) \equiv \xi_{\rm red}(r) \qquad \text{for} \qquad r \gtrsim CR$$



clustered vs non-clustered spatial distribution

- If ζ is a gaussian field: Poissonian distribution   Y. Ali-Hamoud [2018], PRL 121,081304

- Small perturbative non gaussianities: generation of little amount of clustering at formation

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- Non perturbative non gaussianities?

$$P_1 = \int_{\zeta_c} \mathrm{d}\zeta\, P(\zeta) \qquad P_2 = \int_{\zeta_c} \mathrm{d}\zeta_1 \mathrm{d}\zeta_2\, P(\zeta_1, \zeta_2)$$

C.A., V. Vennin
In preparation

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Small-scale clustering of primordial black holes: Cloud-in-cloud and exclusion effects

Pierre Auclair and Baptiste Blachier
Phys. Rev. D **109**, 123538 – Published 24 June 2024

Chiara Animali at TUG 2023

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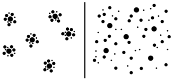

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Journal of Cosmology and Astroparticle Physics

PAPER

Clustering of primordial black holes from quantum diffusion during inflation

Chiara Animali[1] and Vincent Vennin[1]

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Motivations

Motivations for PBHs

- May constitute part of Dark Matter
- May seed the formation of SMBHs
- May lead to some of the mergers seen by LVK

Constraints

- Microlensing
- Cosmic Microwave Background
- Limits to their merger rates (GW)

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Motivations to study clustering

Sizeable clustering may:

- change past and present merger rate of PBH binaries
- modify the formation of cosmological structures
- relax bounds set by CMB and microlensing

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Motivations to study clustering

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Evolution of PBH clustering involves complicated non-linear dynamics, the initial amount of clustering can produce drastic effects on the subsequent evolution.

Different approaches to study spatial clustering

Most approaches used in the literature rely on large-scale structure formalism applied in the context of galaxy and halo formation. We are mostly interested in the two-point correlation function.

- Poisson model and bias theory
- Press-Schechter formalism
- Excursion-set formalism
- Peak theory

Different approaches to study spatial clustering

Most approaches used in the literature rely on large-scale structure formalism applied in the context of galaxy and halo formation. We are mostly interested in the two-point correlation function.

- Poisson model and bias theory
- Press-Schechter formalism
- Excursion-set formalism
- Peak theory

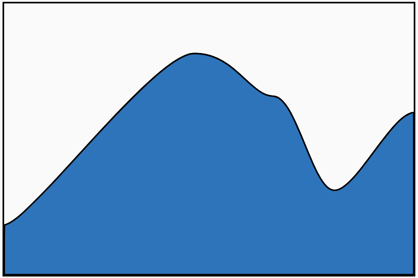
Purpose of the presentation:

- Explicit some of the key differences between the different approaches above.
- Derive explicit expressions for the initial two-point statistics of PBHs...
- using the excursion-set formalism...
- in order to account for "cloud-in-cloud"...
- and exclusion effects at short separation scales

Poisson model and bias theory

One postulates the existence of a field $\rho_{\mathrm{PBH}}(\vec{x})$:

- Probability to form a PBH is $\rho_{\mathrm{PBH}}\delta V$
- Average density of PBHs is $n = \langle \rho_{\mathrm{PBH}} \rangle$



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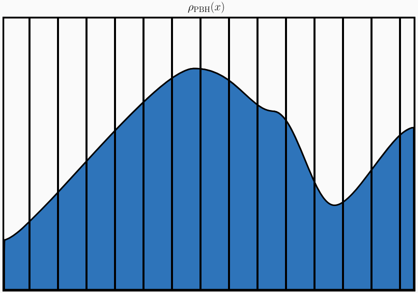

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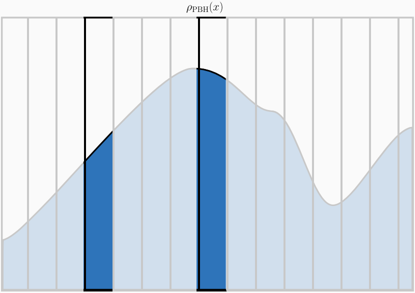
$$\delta P = \rho_{\mathrm{PBH}}(\vec{x}_1)\delta V_1 \rho_{\mathrm{PBH}}(\vec{x}_2)\delta V_2$$



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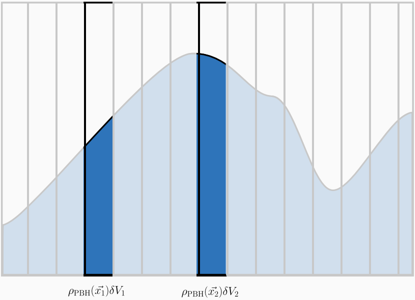
- Auto-correlation function

$$\xi_{\rm PBH}(r) = \left\langle \frac{[\rho_{\rm PBH}(\vec{x}+\vec{r}) - n][\rho_{\rm PBH}(\vec{x}) - n]}{n^2} \right\rangle$$

- Average joint probability to form PBHs

$$\delta P(r) = n^2[1 + \xi_{\rm PBH}(r)]\delta V_1 \delta V_2$$



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Technical challenges

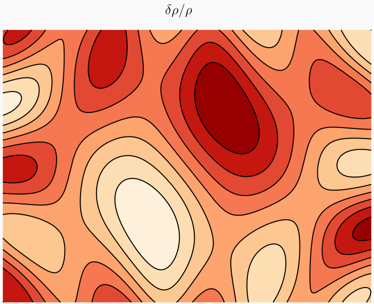
- A perturbation theory for the (over-)density fields δ after inflation
- A bias b to relate ρ_{PBH} with δ

Limitations

- “Cloud-in-cloud”
- Small-scale exclusion effects
- Estimating the bias is difficult

“Press-Schechter”-inspired Ali-Haimoud 2018

- Critical density threshold δ_c



[a] $\delta(\vec{x})$ Gaussian field

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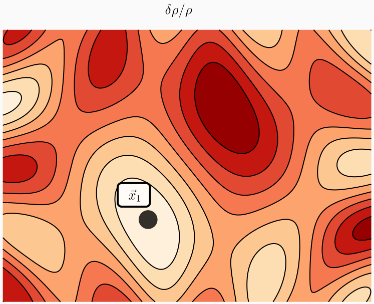
- Critical density threshold δ_c
- Probability to form a PBH around $\vec{x}_1$ [a]

$$P_1 = \int_{\delta_c}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\delta^2}{2\sigma^2}\right) \mathrm{d}\delta = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma}\right)$$



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$$P_2(r) = \iint_{\delta_c}^{\infty} \frac{1}{2\pi\,\det(\Sigma)} \exp\left(-\frac{1}{2}\vec{\delta}^T\Sigma^{-1}\vec{\delta}\right) \mathrm{d}^2\delta$$

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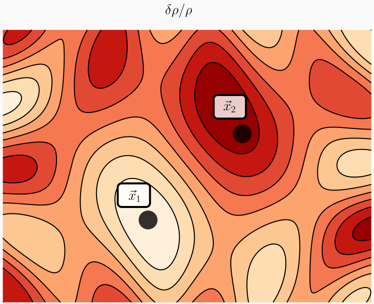

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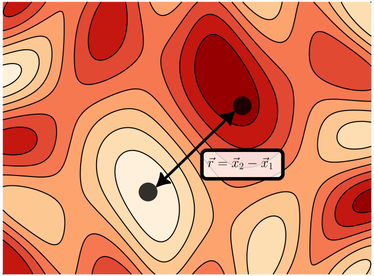
$\delta\rho/\rho$



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Technical challenges

- A perturbation theory for the (over-)density fields δ after inflation
- Estimation of the density threshold δ_c based on numerical relativity

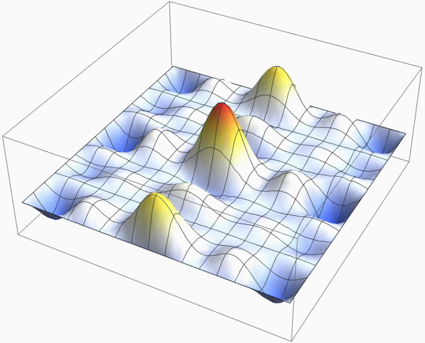
Limitations

- "Cloud-in-cloud"
- Cannot account for small-scale exclusion effects

Baseline to evaluate our work

[a] $\delta(\vec{x})$ Gaussian field

Coarse-graining



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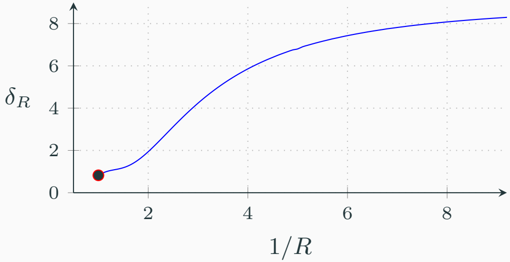

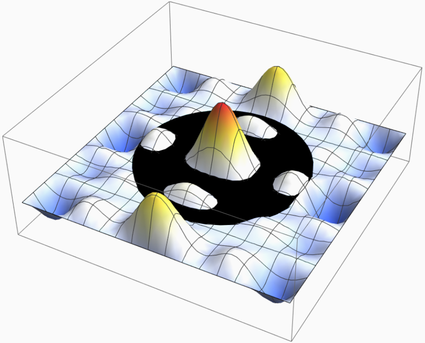

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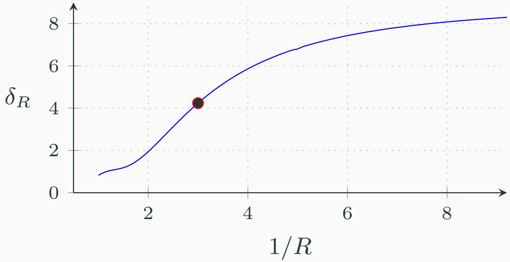

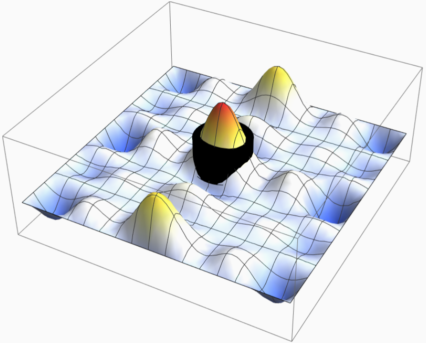

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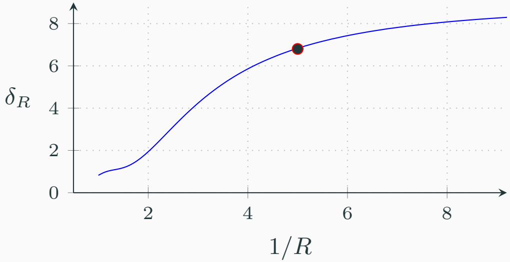

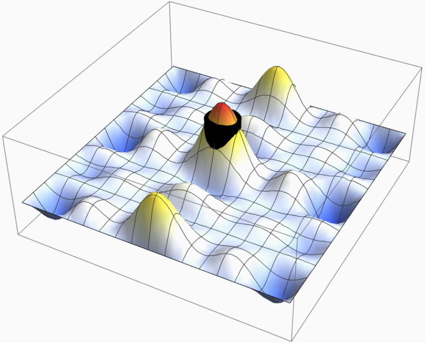

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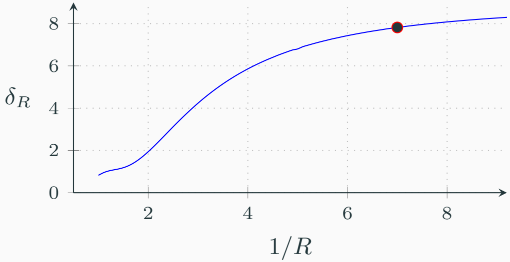

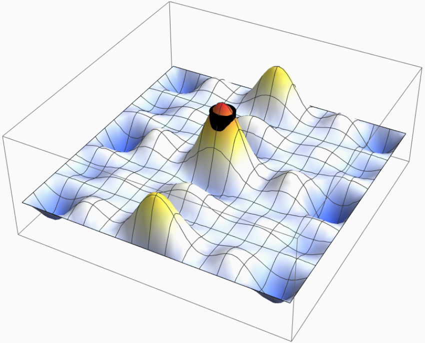

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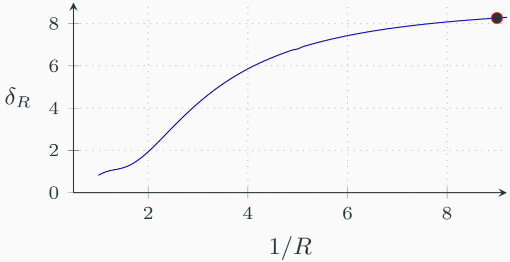
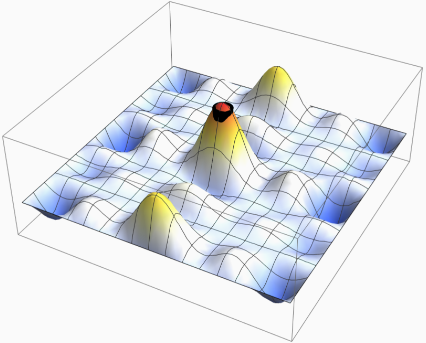
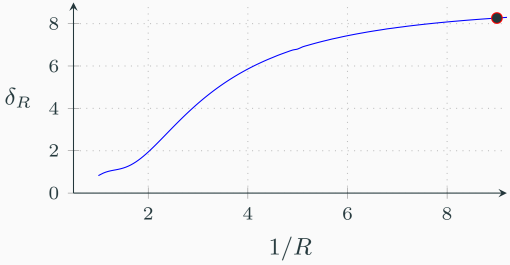
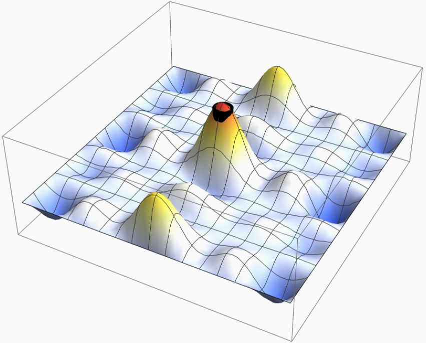








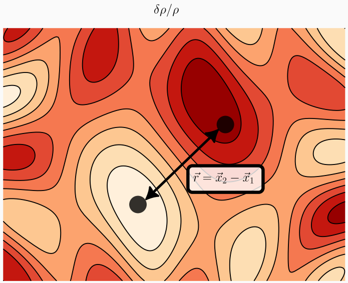
Excursion-set formalism Bond et al. 1991

- Multi-scale analysis $\Leftrightarrow$ Langevin trajectories δ_R
- Gravitationally bound $\Leftrightarrow$ Barrier crossing $\delta_c(R)$
- PBHs $\Leftrightarrow$ first passage time problem $P_{\rm FPT}$

Joint probability to form pairs of PBHs

Setup

- Take a realization of $\delta\rho/\rho$ and two points
- On scales $\gg r$, they see $\approx$ same perturbations
- On scales $\ll r$, they see independent perturbations



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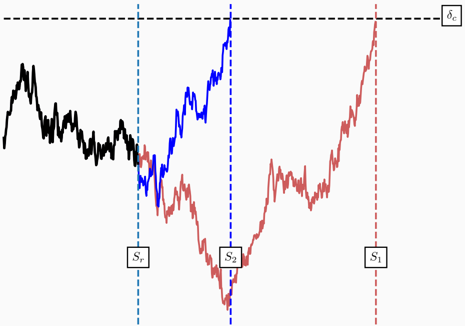

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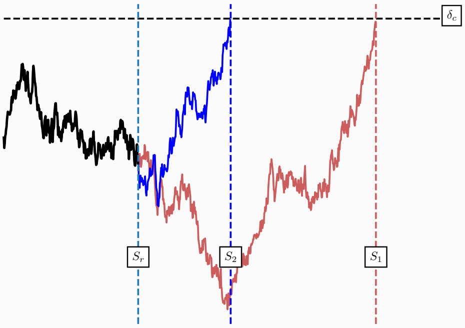

Joint probability to form a pair of PBHs with masses S_1, S_2

$$P_2(S_1, S_2; r) = \int_{-\infty}^{\delta_c(S_r)} \mathrm{d}\delta_r \, P(\delta_r, S_r) P_{\mathrm{FPT}}(S_1|\delta_r, S_r) P_{\mathrm{FPT}}(S_2|\delta_r, S_r)$$

Auto-correlation function (scale-invariant threshold δ_{c})

- Probability to form one PBH

$$P_1 = \mathrm{erfc}\left(\frac{\delta_{\mathrm{c}}}{\sqrt{2}\sigma}\right)$$

Auto-correlation function (scale-invariant threshold $\delta_{\rm c}$)

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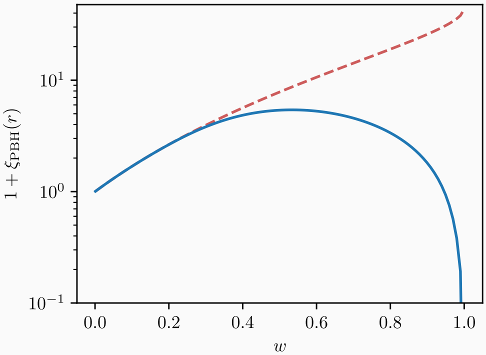
- Two-point correlation function

$$1 + \xi_{\mathrm{PBH}}(r) = \frac{P_2(r)}{P_1^2}$$

- ω measures separations

$$\omega \equiv \frac{S_r}{\sigma^2} \approx \begin{cases} 0 & r \to \infty \\ 1 & r \to 0 \end{cases}$$

Two-point correlation function for $\delta_c = 2\sigma$



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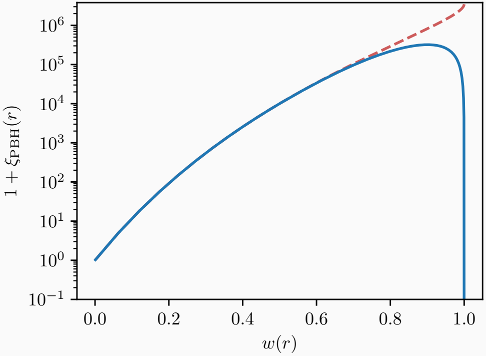
Two-point correlation function for $\delta_c = 5\sigma$



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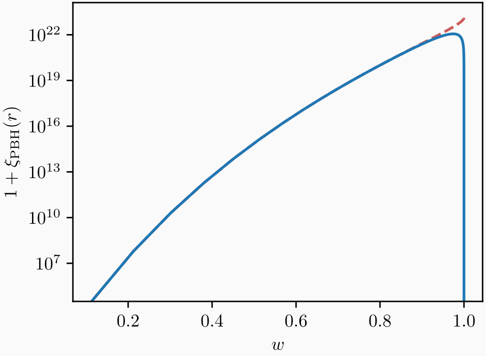
Two-point correlation function for $\delta_c = 10\sigma$



Pairwise correlation functions

Joint probability to form a pair of PBHs with masses S_1, S_2

$$P_2(S_1, S_2; r) = \int_{-\infty}^{\delta_c(S_r)} \mathrm{d}\delta_r \, P(\delta_r, S_r) P_{\mathrm{FPT}}(S_1|\delta_r, S_r) P_{\mathrm{FPT}}(S_2|\delta_r, S_r)$$

Setup

- Take two PBHs with masses M_1, M_2
- Excess probability to find them at distance r

$$1 + \xi_{S_1,S_2}(r) = \frac{P_2(S_1, S_2; r)}{P_1(S_1)P_1(S_2)}$$

- $w_i = S_i/S_r$

Joint probability to form a pair of PBHs with masses S_1, S_2

$$P_2(S_1, S_2; r) = \int_{-\infty}^{\delta_c(S_r)} \mathrm{d}\delta_r \, P(\delta_r, S_r) P_{\mathrm{FPT}}(S_1|\delta_r, S_r) P_{\mathrm{FPT}}(S_2|\delta_r, S_r)$$

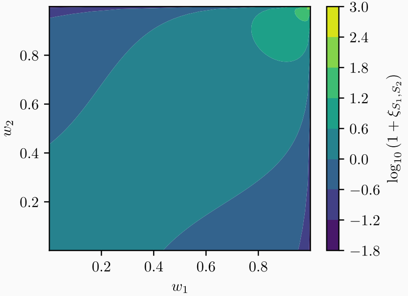
$$\delta_c/\sqrt{2S_r} = 1$$

Setup

- Take two PBHs with masses M_1, M_2
- Excess probability to find them at distance r

$$1 + \xi_{S_1,S_2}(r) = \frac{P_2(S_1, S_2; r)}{P_1(S_1)P_1(S_2)}$$

- $w_i = S_i/S_r$



Pairwise correlation functions

Joint probability to form a pair of PBHs with masses S_1, S_2

$$P_2(S_1, S_2; r) = \int_{-\infty}^{\delta_c(S_r)} \mathrm{d}\delta_r \, P(\delta_r, S_r) P_{\mathrm{FPT}}(S_1|\delta_r, S_r) P_{\mathrm{FPT}}(S_2|\delta_r, S_r)$$

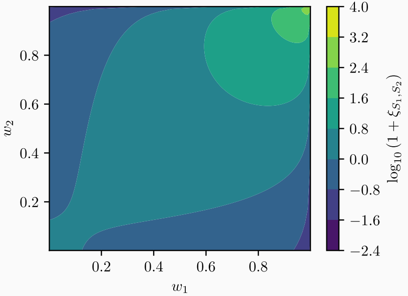
$$\delta_c/\sqrt{2S_r} = 2$$

Setup

- Take two PBHs with masses M_1, M_2
- Excess probability to find them at distance r

$$1 + \xi_{S_1,S_2}(r) = \frac{P_2(S_1, S_2; r)}{P_1(S_1)P_1(S_2)}$$

- $w_i = S_i/S_r$



Pairwise correlation functions

Joint probability to form a pair of PBHs with masses S_1, S_2

$$P_2(S_1, S_2; r) = \int_{-\infty}^{\delta_c(S_r)} \mathrm{d}\delta_r \, P(\delta_r, S_r) P_{\mathrm{FPT}}(S_1|\delta_r, S_r) P_{\mathrm{FPT}}(S_2|\delta_r, S_r)$$

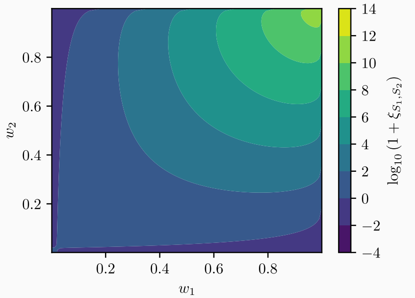
$$\delta_c/\sqrt{2S_r} = 5$$

Setup

- Take two PBHs with masses M_1, M_2
- Excess probability to find them at distance r

$$1 + \xi_{S_1,S_2}(r) = \frac{P_2(S_1, S_2; r)}{P_1(S_1)P_1(S_2)}$$

- $w_i = S_i/S_r$



Joint probability to form a pair of PBHs with masses S_1, S_2

$$P_2(S_1, S_2; r) = \int_{-\infty}^{\delta_c(S_r)} \mathrm{d}\delta_r \, P(\delta_r, S_r) P_{\mathrm{FPT}}(S_1|\delta_r, S_r) P_{\mathrm{FPT}}(S_2|\delta_r, S_r)$$

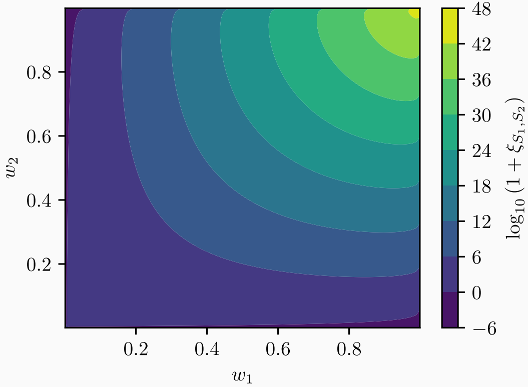
Setup

- Take two PBHs with masses M_1, M_2
- Excess probability to find them at distance r

$$1 + \xi_{S_1,S_2}(r) = \frac{P_2(S_1, S_2; r)}{P_1(S_1) P_1(S_2)}$$

- $w_i = S_i / S_r$

$\delta_c/\sqrt{2S_r} = 10$



Conclusion

Summary

- PBH collapse $\implies$ first-passage time of a Langevin trajectory
- Clustering $\implies$ joint first-passage times of two trajectories with common past
- Effective exclusion effects on small scales $\xi_{\mathrm{PBH}} \to -1$, and cloud-in-cloud
- Access to the probability distribution of the mass-ratio

Ongoing and future works

- From "abstract" $\to$ actual physical scenarii
- From "initial clustering" $\to$ merger rate *with B. Blachier*
- Beyond Gaussian initial conditions (Stochastic inflation) *with C. Animali, B. Blachier, V. Vennin*

Two-point correlation function (fixed δ_c)

Mean number density

$$n = P_1 \equiv \int_0^{\sigma^2} P_{\mathrm{FPT}}(s)\,\mathrm{d}s = \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \ , \quad \nu \equiv \frac{\delta_\mathrm{c}}{\sigma}$$

For a fixed threshold δ_c, the two integrals over S_1 and S_2 in P_2 can be computed analytically.

$$P_2 = \frac{\sqrt{2}\,\mathrm{e}^{-\frac{\nu^2}{2w}}}{\sqrt{\pi w}} \int_0^{\infty} \sinh\left(\frac{\nu}{w}x\right) \mathrm{erfc}^2\left[\frac{x}{\sqrt{2(1-w)}}\right] \mathrm{e}^{-\frac{x^2}{2w}}\,\mathrm{d}x\,.$$

Pairwise correlation function (fixed δ_c)

Using

$$\int_0^\infty \mathrm{d}x\, x^2\, \mathrm{e}^{-\beta x^2} \sinh(\gamma x) = \frac{\sqrt{\pi}(2\beta+\gamma^2)}{8\beta^2\sqrt{\beta}}\, \mathrm{e}^{\frac{\gamma^2}{4\beta}}\, \mathrm{erf}\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{\gamma}{4\beta^2},$$

the cross-correlation writes

$$1+\xi_{S_1,S_2}(r) = \frac{\mathrm{e}^{\lambda^2(w_1+w_2-1)}}{\sqrt{\pi}\lambda} \frac{\sqrt{(1-w_1)(1-w_2)}}{(1-w_1w_2)^2} \left\{1+\sqrt{\pi}\lambda\sqrt{\frac{(1-w_1)(1-w_2)}{1-w_1w_2}}\right.$$
$$\left. \times \left[\frac{1}{2\lambda^2}\frac{1-w_1w_2}{(1-w_1)(1-w_2)}+1\right] \mathrm{e}^{\lambda^2\frac{(1-w_1)(1-w_2)}{1-w_1w_2}}\, \mathrm{erf}\left[\lambda\sqrt{\frac{(1-w_1)(1-w_2)}{1-w_1w_2}}\right]\right\}.$$