Gravitational waves from quasielliptic compact binary systems in massless scalar-tensor theories

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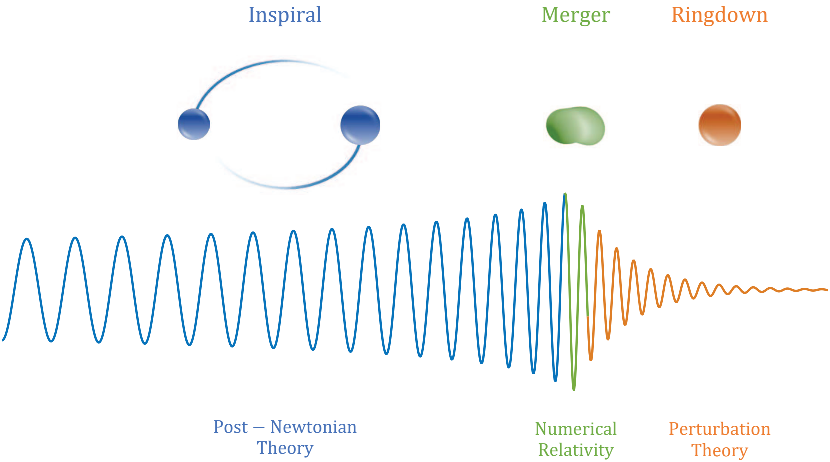
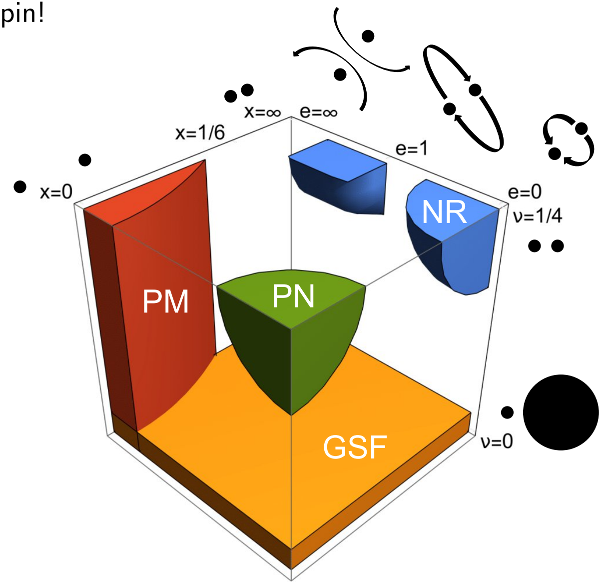
The three stages of a binary



[Antelis & Moreno (2017), arXiv:1610.03567]

Different techniques for different regions of parameter space

Ignoring spin!



Post-Newtonian results: what are they used for?

Post-Newtonian dynamics and waveforms are used:

- alone (in time or frequency domain)
- resummed (e.g. Padé resummations)
- inform EOB models (SEOB and TEOB)
- enter phenomenological waveform models (IMRPhenom)
- hybridized with NR
- hybridized with GSF

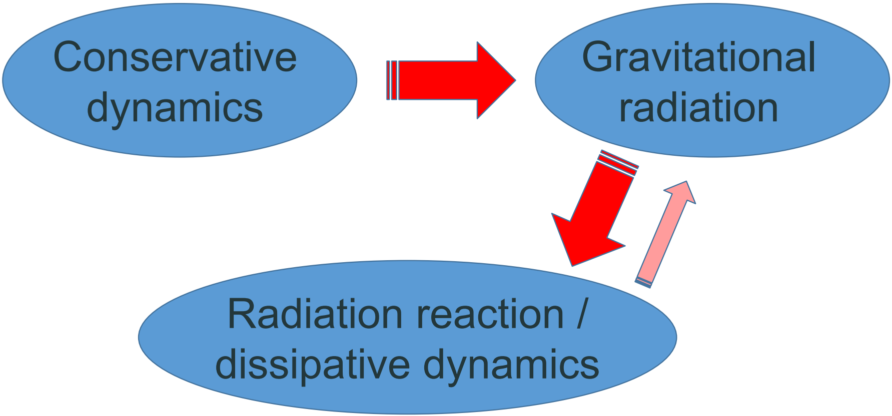
Advantages:

- first-principle method
- fully analytical
- fast to evaluate
- helps understand physics

Disadvantages:

- only valid in inspiral phase
- slow and oscillating convergence
- degrades for high eccentricity
- degrades for high mass-ratios

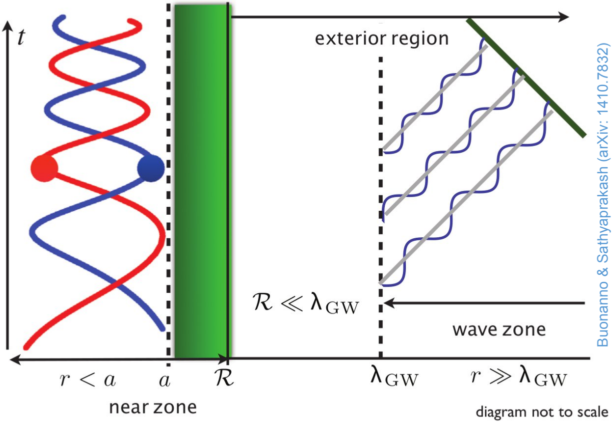
The three sectors of a PN computation



$$\boldsymbol{a} = \boldsymbol{a}_{\mathrm{N}} + \boldsymbol{a}_{1\mathrm{PN}} + \boldsymbol{a}_{2\mathrm{PN}} + \boldsymbol{a}_{2.5\mathrm{PN}} + \dots$$

$$F = F_{\mathrm{N}} + F_{1\mathrm{PN}} + F_{2\mathrm{PN}} + F_{2.5\mathrm{PN}} + \dots$$

Relating near-zone and exterior vacuum zone

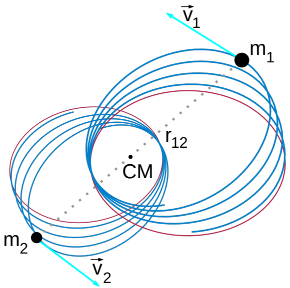


- In NZ, obtain PN expansion of metric [up to homogeneous solution]
- In FZ, obtain PM expansion of metric [up to homogeneous solution]
- Both homogeneous solutions obtained by imposing asymptotic matching in buffer zone

Equations of motion at 4.5PN in GR

[Blanchet, Faye, DT 2024]

The 4.5PN equations of motion [2407.18295]



$$\frac{\mathrm{d}v_{12}^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3} + \ldots\right]n_{12}^i + \ldots\right\}}^{\text{1PN}} + \overbrace{\frac{1}{c^4}\left[\ldots\right]}^{\text{2PN}}$$

$$+ \underbrace{\frac{1}{c^5}\left[\ldots\right]}_{\substack{\text{2.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^6}\left[\ldots\right]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}\left[\ldots\right]}_{\substack{\text{3.5PN}\\ \text{radiation reaction}}} + \underbrace{\frac{1}{c^8}\left[\ldots\right]}_{\text{4PN}} + \underbrace{\frac{1}{c^9}\left[\ldots\right]}_{\substack{\text{4.5PN}\\ \text{radiation reaction}}} + \mathcal{O}\left(\frac{1}{c^{10}}\right)$$

Acceleration in terms of multipolar moments [2407.18295]

Acceleration in terms of the $(\mathrm{M}_L, \mathrm{S}_L)$ at 4.5PN

$$
\begin{aligned}
a^i_{2.5\mathsf{PN}\,1} &= -\frac{2G}{5c^5} y_1^a \mathrm{M}^{(5)}_{ia} \\
a^i_{3.5\mathsf{PN}\,1} &= \frac{G}{c^7}\Big\{ -\frac{11}{105} y_1^b \mathrm{M}^{(7)}_{ib} y_1^2 + \frac{17}{105} y_1^{iab} \mathrm{M}^{(7)}_{ab} - \frac{8}{15} y_1^b \mathrm{M}^{(6)}_{ib} (v_1 y_1) \\
&\qquad + \mathrm{M}^{(6)}_{ab} \Big(\frac{8}{15} y_1^{bi} v_1^a + \frac{3}{5} v_1^i y_1^{ab} \Big) - \frac{2}{5} y_1^b \mathrm{M}^{(5)}_{ib} v_1^2 \\
&\qquad + \frac{G\mathrm{M}^{(5)}_{ia}}{r_{12}} \Big(\frac{7}{5} m_2 n_{12}^a r_{12} + \frac{1}{5} m_2 y_1^a \Big) \\
&\qquad + \mathrm{M}^{(5)}_{ab} \Big[\frac{8}{5} v_1^{bi} y_1^a + \frac{G}{r_{12}} \Big(\frac{1}{5} n_{12}^{bi} m_2 y_1^a - \frac{m_2 n_{12}^i}{r_{12}} y_1^{ab} \Big) \Big] \\
&\qquad + \frac{1}{63} \mathrm{M}^{(7)}_{iab} y_1^{ab} - \frac{16}{45} \varepsilon_{ibj} \mathrm{S}^{(6)}_{aj} y_1^{ab} - \frac{16}{45} \varepsilon_{ibj} v_1^a y_1^b \mathrm{S}^{(5)}_{aj} \\
&\qquad - \frac{32}{45} \varepsilon_{iaj} v_1^a y_1^b \mathrm{S}^{(5)}_{bj} + \frac{16}{45} \varepsilon_{abj} v_1^a y_1^b \mathrm{S}^{(5)}_{ij} \Big\} \\
a^i_{4.5\mathsf{PN}\,1} &= \text{(very long !)}
\end{aligned}
$$

Replace $(\mathrm{M}_L, \mathrm{S}_L) \Rightarrow$ acceleration in terms of $(\mathbf{y}_1, \mathbf{y}_2, \mathbf{v}_1, \mathbf{v}_2)$

Flux balance laws [2407.18295]

Poincaré invariants $(E_{\rm cons}, \boldsymbol{J}_{\rm cons}, \boldsymbol{P}_{\rm cons}, \boldsymbol{G}_{\rm cons})$ conserved by the conservative acceleration ⇒ needed up to 2PN

Fluxes at infinity $(\mathcal{F}_E, \mathcal{F}^i_{\boldsymbol{J}}, \mathcal{F}^i_{\boldsymbol{P}}, \mathcal{F}^i_{\boldsymbol{G}})$ known at relative 2PN [absolute 4.5PN]

We proved all 4 balance laws with relative 2PN accuracy:

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_{\rm cons} + E_{\rm RR}) = -\mathcal{F}_E \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{J}_{\rm cons} + \boldsymbol{J}_{\rm RR}) = -\mathcal{F}_{\boldsymbol{J}}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{P}_{\rm cons} + \boldsymbol{P}_{\rm RR}) = -\mathcal{F}_{\boldsymbol{P}} \qquad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{G}_{\rm cons} + \boldsymbol{G}_{\rm RR}) = \boldsymbol{P} - \mathcal{F}_{\boldsymbol{G}}$$

The $H_{\rm RR}$ are Schott terms. In practice, we first compute

$$\frac{\mathrm{d}H_{\rm cons}}{\mathrm{d}t} + \mathcal{F}_H = \text{(expression)} = -\frac{\mathrm{d}H_{\rm RR}}{\mathrm{d}t}$$

where the fact that (expression) can be written as a total derivative is highly non-trivial and is the core of the proof.

Defining the center-of-mass frame [2407.18295]

Integrating the flux balance equations yields

$$P^i(t) = P_0^i - \int_{t_0}^{t} \mathrm{d}t' \mathcal{F}_{\boldsymbol{P}}(t')$$

$$G^i(t) = G_0^i + P_0^i(t - t_0) - \int_{t_0}^{t} \mathrm{d}t' \mathcal{F}_{\boldsymbol{G}}(t') - \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t'' \mathcal{F}_{\boldsymbol{P}}(t'')$$

where $t_0 =$ initial time, before emission of GWs

Apply Lorentz boost $\Rightarrow$ rest frame of initial system: $P_0^i = 0$ and $G_0^i = 0$

Send $t_0 \to -\infty$. The conditions to be in the CM frame are:

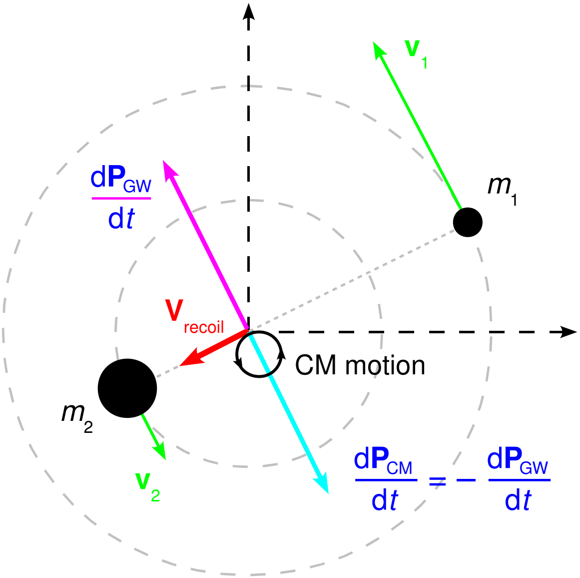
$$G^i(t) + \Gamma^i(t) = 0 \quad \Longrightarrow \quad P^i(t) + \Pi^i(t) = 0$$

where

$$\Pi^i(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \mathcal{F}_{\boldsymbol{P}}(t')$$

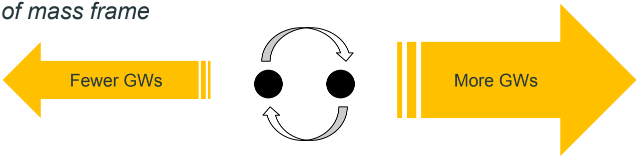
$$\Gamma^i(t) = \int_{-\infty}^{t} \mathrm{d}t' \, \mathcal{F}_{\boldsymbol{G}}(t') + \int_{-\infty}^{t} \mathrm{d}t' \, \Pi^i(t')$$

Gravitational recoin: circular orbits

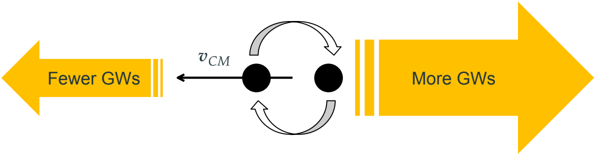


Gravitational recoil: secular effect for eccentric orbits

In the center of mass frame



Asymmetric binary



Gravitational recoil

GWs that escaped to infinity contribute to the center-of-mass frame

Passage to the CM frame [2407.18295]

Solving iteratively for the y_1^i in $G^i + \Gamma^i = 0$, we find

$$y_1^i = \underbrace{x^i\Big(X_2 + \nu\Delta\,\mathcal{P}\Big) + \nu\Delta\,\mathcal{Q}\,v^i}_{\substack{\text{matter contribution} \\ \text{obtained by solving for } y_1^i \text{ in } G^i = 0}} + \underbrace{\mathcal{R}^i}_{\text{radiation contribution}}$$

where

$$\mathcal{R}^i = \underbrace{-\frac{\Gamma^i}{m}}_{3.5\text{PN}} + \underbrace{\frac{\nu}{mc^2}\left[\left(\frac{v^2}{2} - \frac{Gm}{r}\right)\Gamma^i + v^j\Big(\Pi^j + \mathcal{F}_G^j\Big)x^i\right]}_{4.5\text{PN}} + \mathcal{O}(11)$$

The equations of motion in the CM frame [2407.18295]

In the CM frame, we find

$$a^i_{\mathrm{RR}} = a^i_{2.5\mathrm{PN}} + a^i_{3.5\mathrm{PN}} + a^i_{4.5\mathrm{PN}}\Big|_{\mathrm{mat}} + a^i_{4.5\mathrm{PN}}\Big|_{\mathrm{rad}}$$

where

$$a^i_{2.5\mathrm{PN}} = \frac{8G^2m^2\nu}{c^5r^3}\left[v^i\left(\frac{2Gm}{5r} + 3\dot{r}^2 - \frac{6}{5}v^2\right) + n^i\dot{r}\left(\frac{2Gm}{15r} - 5\dot{r}^2 + \frac{18}{5}v^2\right)\right]$$

$$a^i_{3.5\mathrm{PN}} = (\ldots)$$

$$a^i_{4.5\mathrm{PN}}\Big|_{\mathrm{mat}} = (\ldots)$$

and the new non-local contribution reads:

$$a^i_{4.5\mathrm{PN}}\Big|_{\mathrm{rad}} = \frac{G\Delta}{r^2c^2}\left(2n^iv^j + n^jv^i\right)\left[\Pi^j + \mathcal{F}^j_{\boldsymbol{G}}\right].$$

We thus disagree with [gr-qc/9703075] and [2302.11016] who have not taken these nonlocal effects into account.

Post-Newtonian methods applied to scalar-tensor theory

Generalized Fierz-Pauli-Brans-Dicke theory

Action defined in Jordan frame : $S = S_{\rm ST}[g_{\alpha\beta}, \phi] + S_{\rm m}[g_{\alpha\beta}, \mathfrak{m}]$ where

$$S_{\rm ST} = \frac{c^3}{16\pi G} \int {\rm d}^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right]$$

The effective matter action for two point particles reads

$$S_{\rm m} = -c m_1(\phi) \sqrt{-(g_{\alpha\beta})_1 {\rm d}y_1^\alpha y_1^\alpha} + (1 \leftrightarrow 2)$$

The weak equivalence principle is broken: the inertial mass of a neutron star, when idealized as a point particle, depends on the local value of the scalar field [Eardley, PRD 12, 3072 (1975)].

Conformal transformation to Einstein frame

Perturbation around flat space and a constant scalar background

$$h^{\mu\nu} \equiv \sqrt{-\det[(\phi/\phi_0) g_{\alpha\beta}]} \times \frac{g^{\mu\nu}}{\phi/\phi_0} - \eta^{\mu\nu}$$

$$\psi \equiv \phi/\phi_0 - 1$$

Expansions of the functions appearing in ST theory

The ω function is expanded as

$$\omega(\phi) = \omega_0 + (\phi - \phi_0)\omega_0' + \dots$$

For $A \in \{1, 2\}$, the mass function is expanded as

$$m_A(\psi) = m_A \left(1 + s_A\psi + \frac{s_A^2 - s_A + s_A'}{2}\psi^2 + \dots\right)$$

where sensitivities are defined as

$$s_A = \frac{\mathrm{d}\ln m_A(\phi)}{\mathrm{d}\ln\phi}\,, \qquad s_A' = \frac{\mathrm{d}^2\ln m_A(\phi)}{\mathrm{d}\ln\phi^2}\,, \qquad \dots$$

Weakly gravitating stars: $s_A \ll 1$; NS: $s_A \approx 0.2$; BH: $s_A = 1/2$.

Various parameters you might see are just complicated combinations of these parameters: $\tilde{G}, \alpha, \bar{\gamma}, \zeta, \lambda_A, \bar{\beta}_A, \bar{\chi}_A, \bar{\kappa}_A, \bar{\delta}_A, \dots$

Why study this theory ?

To look for deviation to GR ? Not really, ST theory is strongly constrained by binary pulsars [2407.16540] and solar system tests ...

- Simplest motivated deviation to GR — just add a non-minimally coupled scalar field!
- Technology developed for it useful for more complicated, less contrained theories — e.g. scalar Gauss-Bonnet
- Templates useful to search for strong deviations from GR in LVK/ET/LISA data — we only used GR templates for searches, might have missed very exotic signal
- Good toy model for studying GR — dipolar *vs* quadrupolar radiation, no gauge problems with scalar field

The quasi-Keplerian parametrization at 2PN for scalar tensor theories

[DT 2024a]

The Kepler solution

Relative 2 body acceleration in CM frame:

$$a^i = a_1^i - a_2^i = -\frac{G_{\mathsf{eff}}\, m n^i}{r^2}$$

In the bound case, we know that the orbit is an ellipse:

$$r = \frac{a(1-e^2)}{1 + e\cos(\phi - \phi_{\rm peri})}$$

where a is the semimajor axis and e the eccentricity ($e < 1$ for bound orbits), given in terms of the energy ($E < 0$) and angular momentum J:

$$a = -\frac{Gm}{2E} \qquad \text{and} \qquad e = \sqrt{1 + \frac{2EJ}{G^2 m^2}}$$

The Kepler solution

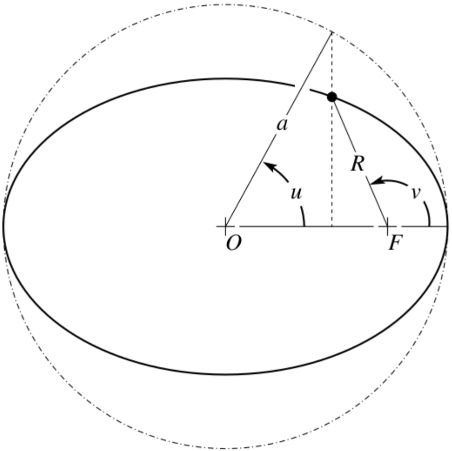
To describe the time evolution, it is however more practical to use the following set of three equations

$$r = a(1 - e\cos u)$$
$$\ell = n(t - t_0) = u - e\sin(u)$$
$$\phi - \phi_0 = v(u)$$

where we have introduced

- the *eccentric anomaly* u, which acts as an affine parameter
- the *true anomaly* $v(u) \equiv 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right]$
- the *mean motion* $n \equiv 2\pi/P$, where P is a time period
- the *mean anomaly* $\ell = n(t - t_0)$, which increases linearly with time and goes from 0 to 2π over one orbit

The Kepler solution



$$r = a(1 - e\cos u)$$
$$\ell = n(t - t_0) = u - e\sin(u)$$
$$\phi - \phi_0 = v$$
$$v = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{u}{2}\right)\right]$$

Figure from [gr-qc/0407049]

The quasi-Keplerian solution at 1PN order

What happens if we now want to solve the equations of motion for the 1PN acceleration ? $a^i = -\frac{G_{12} m n^i}{r^2} + \frac{1}{c^2}\,(\text{many terms})^i$

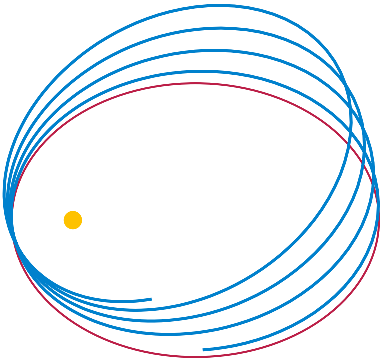
Damour & Deruelle [Ann.IHP.Phys.Th. 43, 1 (1985), p.107] showed that the equations of motion then reads

$$
\begin{aligned}
r &= a_r (1 - e_r \cos u) \\
\phi - \phi_0 &= K v \\
n(t - t_0) &= u - e_t \sin(u) \\
v(u) &= 2 \arctan\left[\sqrt{\frac{1 + e_\phi}{1 - e_\phi}} \tan\left(\frac{u}{2}\right)\right]
\end{aligned}
$$

which is the same equation as before, except:

- there are now three eccentricities e_r, e_t, e_ϕ
- pericenter precession appears via the factor $K = 1 + k$ (with $k \ll 1$)
- a_r and n acquire post-Newtonian corrections

Doubly periodic structure of QK motion



The time between two periastrons is the *radial period* denote P, so the mean motion $n = 2\pi/P$ is the *radial frequency*.

The time for the angular coordinate ϕ to go from 0 to 2π is P/K, so $\omega = nK$ is the *angular frequency*

Thus, $K = 1 + k$ with $k \ll 1$ is a measure of the pericenter precession

The quasi-Keplerian solution at 2PN order

Damour & Schäfer [Nuovo Cim.B 101 (1988) 127] showed that the QK parametrization reads at 2PN

$$
\begin{aligned}
r &= a_r(1 - e_r \cos u) \\
\phi - \phi_0 &= K\Big[v + f_\phi \sin(2v) + g_\phi \sin(3v)\Big] \\
n(t - t_0) &= u - e_t \sin(u) + f_t \sin(v) + g_t(v - u) \\
v(u) &= 2 \arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}} \tan\left(\frac{u}{2}\right)\right]
\end{aligned}
$$

Here, the new parameters f_ϕ, g_ϕ, f_t and g_t are all of order $\mathcal{O}(1/c^4)$, while all other parameters acquire 2PN corrections.

But how do we determine the values of these parameters ?

Determining the QK parameters

Assume we are working in *some theory of gravity* [e.g. GR or ST theory], and that we have determined (in a PN sense):

$$E = f(r, \dot{r}, \dot{\phi}) \qquad \text{and} \qquad J = g(r, \dot{r}, \dot{\phi})$$

For many theories of gravity, we can invert this as

$$\dot{r}^2 = A + \frac{B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3} + \frac{D_2}{r^4} + \frac{D_3}{r^5} + \mathcal{O}\left(\frac{1}{c^6}\right)$$
$$\dot{\phi} = \frac{F}{r^2} + \frac{I_1}{r^3} + \frac{I_2}{r^4} + \frac{I_3}{r^5} + \mathcal{O}\left(\frac{1}{c^6}\right)$$

where A, B, C and F are of order 1, but D_1 and D_2 are 1PN and the others 2PN. **All these parameters are functions of E and J.**

N.B.: this polynomial structure is spoiled by tails in the EOM at 3PN in ST theory and 4PN in GR

Determining the QK parameters [2401.06844]

I obtained expressions for QK parameters (a_r, e_t, g_t, ...) [technical!], e.g.:

$$a_r = -\frac{B}{A} + \frac{D_1}{2C} + \frac{2BD_1^2 - 2BCD_2 + 4B^2D_3 - ACD_3}{2C^3} + \mathcal{O}\left(\frac{1}{c^4}\right)$$

Expression of A, B, ... depends of the theory. For example, in ST theory:

$$B = \tilde{G}\alpha m \left\{1 + \varepsilon\left[3 + \bar{\gamma} - \frac{7}{2}\nu\right] + \varepsilon^2\left[\frac{9}{4} + \frac{3}{4}\bar{\gamma} + \nu\Big(- 12 - \frac{15}{4}\bar{\gamma}\Big) + \frac{21}{4}\nu^2\right]\right\}$$

where $\varepsilon = -2E/(m\nu c^2) > 0$ and $\varepsilon = \mathcal{O}(1/c^2)$.

In you favorite theory:

1. **determine** $E = f(r, \dot{r}, \dot{\phi})$ **and** $J = \tilde{f}(r, \dot{r}, \dot{\phi})$
2. **invert in PN sense to obtain** $\dot{r} = g(E, J, r)$ **and** $\dot{\phi} = \tilde{g}(E, J, R)$
3. **read off** A**,** B**,** C**, ...**
4. **use results of [2401.06844] to obtain QK parametrization**

Peters's formula for ST theories at -1PN order [2401.06844]

In GR, Peters obtained from flux-balance arguments at Newtonian order that (a, e) secularly co-evolve as [PhysRev.136.B1224]

$$a = \frac{c_0' e^{\frac{12}{19}}}{1 - e^2} \left(1 + \frac{121}{304} e^2\right)^{\frac{870}{2299}} \qquad \text{(in GR)}$$

where c_0 is a constant depending on the orbit.

In ST, from the energy for an elliptic orbit and the leading -1PN dipole formula for the fluxes of energy and angular momentum,

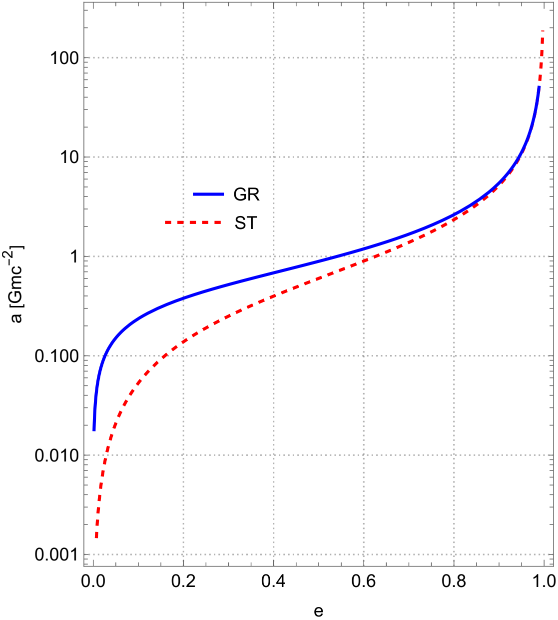
$$\mathcal{F}^s = \frac{G\phi_0(3 + 2\omega_0)}{3c^3} \mathrm{I}_a^{(2)} \mathrm{I}_a^{(2)}\,, \qquad \mathcal{G}_i^s = \frac{G\phi_0(3 + 2\omega_0)}{3c^3} \epsilon_{iab} \mathrm{I}_a^{(1)} \mathrm{I}_b^{(2)}\,,$$

I obtain at leading-order

$$a = \frac{c_0 e^{4/3}}{1 - e^2} \qquad \text{(in ST)}$$

Remarkably, it does not depend on the ST parameters!

Peters's formula for ST theories at -1PN order [2401.06844]



Fluxes of energy and angular momentum at 1.5PN

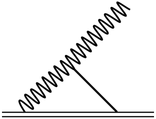
[DT 2024b]

The fluxes: tails, memory, and more [2410.12898]

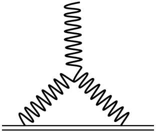
The fluxes are divided into a scalar and tensor contribution:
$\mathcal{F}^{\text{tot}} = \mathcal{F} + \mathcal{F}^s$ and $\mathcal{G}^{\text{tot}} = \mathcal{G} + \mathcal{G}^s$ [see also [2407.10908] for expressions in terms of orbital variables, without specifying the motion]

There are contributions from:

- tails



- memory



- hereditary terms $\mathbf{\Pi}^s$ arising from passage to CM frame
- instantaneous terms

We thus have

$$\mathcal{F} = \mathcal{F}^{\text{inst}} + \mathcal{F}^{\text{tail}} \qquad \mathcal{F}^s = \mathcal{F}^{s,\text{inst}} + \mathcal{F}^{s,\text{tail}},$$
$$\mathcal{G}_i = \mathcal{G}_i^{\text{inst}} + \mathcal{G}_i^{\text{tail}} + \mathcal{G}_i^{\text{mem}} \qquad \mathcal{G}_i^s = \mathcal{G}_i^{s,\text{inst}} + \mathcal{G}_i^{s,\mathbf{\Pi}^s} + \mathcal{G}_i^{s,\text{tail}}$$

Fourier expansion of moments at Newtonian order [2410.12898]

At Newtonian order, I_L^s is periodic in $\ell = n(t - t_0)$, so we can decompose it as a Fourier *series*:

$$\mathrm{I}_L^s(t) = \sum_{p \in \mathbb{Z}} {}_p\widetilde{\mathrm{I}}_L^s \, e^{\mathrm{i}p\ell}$$

The coefficients are given by

$${}_p\widetilde{\mathrm{I}}_L^s = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\ell \, \mathrm{I}_L^s(t) e^{-\mathrm{i}p\ell}$$

Changing variables to the eccentric anomaly u [using Kepler's equation $\ell = u - e_t \sin(u)$], we find that all integrals reduce to Bessel functions:

$$J_p(x) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}u \, e^{-\mathrm{i}(pu - x \sin u)}$$

For example, the scalar dipole reads at Newtonian order:

$${}_p\widetilde{\mathrm{I}}_x^s \propto \frac{1}{p} J_p(ep) \qquad {}_p\widetilde{\mathrm{I}}_y^s \propto -\frac{\mathrm{i}\sqrt{1-e^2}}{ep} J_p(ep) \qquad {}_p\widetilde{\mathrm{I}}_z^s = 0$$

Computing the tails [2410.12898]

The leading tail term reads:

$$\mathcal{F}^{s,\text{tail}}_{\mathrm{M}\times\mathrm{I}_i} = \frac{4G^2\mathrm{M}(3+2\omega_0)}{3c^6} \overset{(2)}{\mathrm{I}}{}^s_i \int_0^\infty \mathrm{d}\tau \, \overset{(4)}{\mathrm{I}}{}^s_i(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1\right]$$

Replace moments by Fourier decomposition $\mathrm{I}^s_i(t) = \mathcal{I}^s_1 \sum_{p\in\mathbb{Z}} {}_p\widehat{\mathrm{I}}^s_L \, e^{\mathrm{i}p\ell}$ and use the integration formula

$$\int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2b_0}\right) e^{-ip\omega\tau} = \frac{\mathrm{i}}{p\omega}\Big[\ln\Big(\frac{2|p|b_0}{c}\Big) + \gamma_\mathrm{E} + \mathrm{i}\frac{\pi}{2}\mathrm{sg}(p)\Big]$$

Obtain after orbit averaging:

$$\left\langle \mathcal{F}^{s,\text{tail}}_{\mathrm{M}\times\mathrm{I}_i} \right\rangle \propto \nu^2 x^{11/2} \varphi^s_1(e_t)$$

where the enhancement function $\varphi^s_1(e_t) = 1 + 7e_t^2 + \mathcal{O}(e_t)$ and reads

$$\varphi_1(e_t) = 2\sum_{p=1}^{\infty} p^5 \sum_{\substack{m\in\{-1,1\}\\ s\in\{-1,1\}}} \left({}^m_p\widehat{\mathrm{I}}^s_i\right)\left({}^m_p\widehat{\mathrm{I}}^s_i\right)^*$$

Memory and Π^s contributions

For $\mathcal{G}_i^{\rm mem}$ and $\mathcal{G}_i^{s,\mathbf{\Pi}^s}$:

- separate oscillatory “AC” terms and nonoscillatory “DC” terms
- for the AC terms, replace moments by their Fourier decomposition, and compute integrals straightforwardly
- for DC terms, integrate wrt time using the expressions $a(t)$ and $e(t)$ [Peters's formula]

We find that $\left\langle \mathcal{G}_i^{\rm mem} \right\rangle = 0$ and $\left\langle \mathcal{G}_i^{s,\mathbf{\Pi}^s} \right\rangle = 0$

From fluxes to evolution of orbital elements [2410.12898]

Consider an orbital element $\xi \in \{x, e_t, ...\}$: It will satisfy:

$$\left\langle \frac{\mathrm{d}\xi}{\mathrm{d}t} \right\rangle = \frac{\partial \xi}{\partial E} \left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle + \frac{\partial \xi}{\partial J} \left\langle \frac{\mathrm{d}J}{\mathrm{d}t} \right\rangle$$

The quasi-Keplerian parametrization gives us $\xi(E, J)$ so we can compute the partial derivatives $\partial\xi/\partial E$ and $\partial\xi/\partial J$.

We use the flux balance laws to write

$$\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = -\left\langle \mathcal{F} \right\rangle - \left\langle \mathcal{F}^s \right\rangle$$
$$\left\langle \frac{\mathrm{d}J}{\mathrm{d}t} \right\rangle = -\left\langle \mathcal{G} \right\rangle - \left\langle \mathcal{G}^s \right\rangle$$

and we have now computed the orbit-averaged fluxes!

Since all the orbital parameters are related by the QK parametrization, we just need to evolve a pair, e.g. (x, e_t).

Evolution of (x, e_t) [2410.12898]

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle = \frac{2c^3\zeta\nu x^4}{3\tilde{G}\alpha m}\Bigg\{ \frac{4\mathcal{S}_-^2\left(1+\frac{1}{2}e_t^2\right)}{(1-e_t^2)^{5/2}} + \frac{x}{15(1-e_t^2)^{7/2}}\left(\mathfrak{x}_1 + e_t^2\mathfrak{x}_2 + e_t^4\mathfrak{x}_3\right)$$
$$+ 8\pi\left(1+\frac{1}{2}\bar{\gamma}\right)\mathcal{S}_-^2\,\varphi_1^s(e_t)\,x^{3/2}$$
$$+ x^2\left(\frac{\mathfrak{x}_4 + e_t^2\mathfrak{x}_5 + e_t^4\mathfrak{x}_6 + e_t^6\mathfrak{x}_7}{(1-e_t^2)^{9/2}} + \frac{\mathfrak{x}_8 + e_t^2\mathfrak{x}_9 + e_t^4\mathfrak{x}_{10}}{(1-e_t^2)^4}\right)$$
$$+ 4\pi\left(1+\frac{1}{2}\bar{\gamma}\right)x^{5/2}\Bigg(\mathcal{X}_{11}\varphi_2(e_t) + \mathcal{X}_{12}\varphi_2^s(e_t) + \mathcal{X}_{13}\alpha_1^s(e_t) + \mathcal{X}_{14}\theta_1^s(e_t)$$
$$+ \left(\mathcal{X}_{15} + e_t^2\mathcal{X}_{16}\right)\frac{\varphi_1^s(e_t)}{1-e_t^2} + \mathcal{X}_{17}\frac{\tilde{\varphi}_1^s(e_t)}{(1-e_t^2)^{3/2}} + \mathcal{X}_{18}e_t^2\varphi_0^s(e_t)\Bigg) + \mathcal{O}(x^3)\Bigg\},$$

$$\left\langle \frac{\mathrm{d}e_t}{\mathrm{d}t} \right\rangle = -\frac{c^3\zeta\nu x^3 e_t}{\tilde{G}\alpha m}\Bigg\{ \frac{2\mathcal{S}_-^2}{(1-e_t^2)^{3/2}} + \frac{x}{15(1-e_t^2)^{5/2}}\left(\mathfrak{E}_1 + e_t^2\mathfrak{E}_2\right)$$
$$+ \frac{8\pi}{3}\left(1+\frac{1}{2}\bar{\gamma}\right)\mathcal{S}_-^2\frac{1-e_t^2}{e_t^2}\left(\varphi_1^s(e_t) - \frac{\tilde{\varphi}_1^s}{\sqrt{1-e_t^2}}\right)x^{3/2} + x^2\left(\frac{\mathfrak{E}_3 + e_t^2\mathfrak{E}_4 + e_t^4\mathfrak{E}_5}{(1-e_t^2)^{7/2}} + \frac{\mathfrak{E}_6 + e_t^2\mathfrak{E}_7}{(1-e_t^2)^3}\right)$$
$$+ 4\pi\left(1+\frac{1}{2}\bar{\gamma}\right)x^{5/2}\Bigg[\mathfrak{E}_8\frac{1-e_t^2}{e_t^2}\left(\varphi_2(e_t) - \frac{\tilde{\varphi}_2}{\sqrt{1-e_t^2}}\right) + \mathfrak{E}_9\frac{1-e_t^2}{e_t^2}\left(\varphi_2^s(e_t) - \frac{\tilde{\varphi}_2^s}{\sqrt{1-e_t^2}}\right)$$
$$+ \mathfrak{E}_{10}\frac{1-e_t^2}{e_t^2}\left(\alpha_1^s(e_t) - \frac{\tilde{\alpha}_2^s}{\sqrt{1-e_t^2}}\right) + \mathfrak{E}_{11}\frac{1-e_t^2}{e_t^2}\left(\theta_1^s(e_t) - \frac{\tilde{\theta}_1^s}{\sqrt{1-e_t^2}}\right)$$
$$+ \frac{\mathfrak{E}_{12}}{e_t^2}\left(\varphi_1^s(e_t) - \frac{\tilde{\varphi}_1^s}{\sqrt{1-e_t^2}}\right) + \mathfrak{E}_{13}\varphi_1^s(e_t) + \frac{\mathfrak{E}_{14}}{\sqrt{1-e_t^2}}\tilde{\varphi}_1^s(e_t)$$
$$+ \mathfrak{E}_{15}(1-e_t^2)\varphi_0^s(e_t)\Bigg] + \mathcal{O}(x^3)\Bigg\},$$

Conclusion

Recapitulation

In general relativity

- equations of motion at 4.5PN
- new hereditary terms at 4.5PN !

In scalar-tensor theory

- fluxes and waveform for quasicircular orbits
- quasi-Keplerian parametrization at 2PN for quasi-elliptic orbits
- fluxes and orbital element evolution at 1.5PN [$\mathrm{N}^{2.5}$LO]

Backup slides

Action defined in Jordan frame : $S = S_{\mathrm{ST}}[g_{\alpha\beta}, \phi] + S_{\mathrm{m}}[g_{\alpha\beta}, \mathfrak{m}]$ which reads

$$S_{\mathrm{ST}} = \frac{c^3}{16\pi G} \int \mathrm{d}^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \right]$$

For the post-Newtonian setup, better to work in Einstein frame. Define

$$\varphi = \frac{\phi}{\phi_0} \qquad \text{and} \qquad \tilde{g}_{\mu\nu} = \frac{\phi}{\phi_0} g_{\mu\nu} \qquad \text{where} \qquad \phi \underset{r\to\infty}{\longrightarrow} \phi_0$$

The action in Einstein frame then reads

$$S = \frac{c^3 \phi_0}{16\pi G} \int \mathrm{d}^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \frac{3 + 2\omega(\phi)}{2\varphi^2} \tilde{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right] + S_{\mathrm{m}}[\varphi^{-1} \tilde{g}_{\alpha\beta}, \mathfrak{m}]$$

Equivalence to DEF gravity

Our Einstein frame action

$$S = \frac{c^3\phi_0}{16\pi G}\int \mathrm{d}^4x\sqrt{-\tilde{g}}\Big[\tilde{R} - \frac{3+2\omega(\phi_0\varphi)}{2\varphi^2}\tilde{g}^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi\Big] + S_\mathrm{m}[\varphi^{-1}\tilde{g}_{\alpha\beta}, \mathfrak{m}]$$

is equivalent to Damour & Esposito-Farèse (DEF) gravity [gr-qc/9602056]:

$$S_\mathrm{DEF} = \frac{c^3}{16\pi G_*}\int \mathrm{d}^4x\sqrt{-g_*}\left[R_* - 2g_*^{\alpha\beta}\partial_\alpha\bar{\varphi}_*\partial_\beta\bar{\varphi}_*\right] + \mathcal{S}_\mathrm{m}\left[\mathcal{A}(\bar{\varphi}_*)g^*_{\alpha\beta}, \mathfrak{m}\right]$$

where $G_* = G/\phi_0$, $\bar{g}_{\mu\nu} = \tilde{g}_{\mu\nu}$ and $\bar{\varphi} = \mathcal{T}(\phi)$, where

$$\mathcal{T}(x) = \frac{1}{2}\int^x \mathrm{d}y\sqrt{\frac{3+2\omega(y)}{2y^2}}$$

Field equations

The field equations should be expressed using the Landau & Lifschitz formulation. Perturbation of (conformal inverse) metric around Minkowski:

$$\boxed{h^{\mu\nu} = \sqrt{\tilde{g}}\tilde{g}^{\mu\nu} - \eta^{\mu\nu}}$$

[At linear level, this is equivalent the "trace reversed metric"]

Perturb (normalized) scalar field around background value:

$$\varphi = 1 + \psi$$

Restriction to harmonic gauge $\partial_\mu h^{\mu\nu} = 0$, the field equations read:

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4\phi_0}\left[\varphi(-g)T^{\mu\nu} + \frac{c^4\phi_0}{16\pi G}\Lambda^{\mu\nu}[h,\psi]\right]$$

$$\Box\psi = \frac{8\pi G}{c^4\phi_0}\left[\frac{\varphi\sqrt{-g}}{[3+2\omega(\phi_0\phi)]}\left(T - 2\varphi\frac{\partial T}{\partial\varphi}\right) + \frac{c^4\phi_0}{8\pi G}\Lambda_s[h,\psi]\right]$$

where the non-linear couplings are described by $\Lambda^{\mu\nu}[h,\psi]$ and $\Lambda_s[h,\psi]$

Fluxes at Newtonian order

At Newtonian order [reminder: the leading order is -1PN], the flux is *instantaneous*, i.e. no tails or memory. The QK representation allows us to write the fluxes only in terms of the eccentric anomaly:

$$\mathcal{F} = f[r, \phi, \dot{r}, \dot{\phi}] = g[r, \phi] = h[u]$$

After some trigonometry, we find that the structure is in fact

$$\mathcal{F} = \sum_k \left[\frac{\alpha_k}{[1 - e_t \cos(u)]^k} + \frac{\beta_k \sin(u)}{[1 - e_t \cos(u)]^k} \right]$$

The orbit averaged flux reads:

$$\langle \mathcal{F} \rangle = \frac{1}{P} \int_0^P \mathrm{d}t \mathcal{F} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\ell \, \mathcal{F} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}u \frac{\mathrm{d}\ell}{\mathrm{d}u} \mathcal{F}$$

where $\mathrm{d}\ell / \mathrm{d}u = 1 - e_t \cos(u)$. We can then use:

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{\mathrm{d}u}{[1 - e_t \cos(u)]^n} = \frac{P_{n-1}(1/\sqrt{1 - e_t^2})}{(1 - e_t^2)^{n/2}}$$

Linearized metric in exterior vacuum

N.B. I will focus on the scalar field for pedagogy

In the exterior vacuum zone, we formally perform a multipolar post-Minkowskian expansion $\psi = G\psi_1 + G^2\psi_2 + \dots$

At linear level, the scalar field equation reads $\Box\psi_1 = 0$, so we can express it as a multipolar expansion [Thorne 1980]:

$$\psi_1 = -\frac{2}{c^2}\sum_{\ell\geqslant 0}\frac{(-)^\ell}{\ell!}\partial_L\left[r^{-1}\mathrm{I}^s_L\right]$$

The "source moments" can be matched to a near-zone, post-Newtonian ($v \ll c$) computation involving the matter, such that they can be expressed as functions of the phase space variable of the compact binary system

$$\mathrm{I}^s_L[\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{v}_1, \boldsymbol{v}_2]$$

Multipolar moments: an example

For example, we have [2201.10924]

$$I_i^s = -\frac{m_1(1-2s_1)y_1^i}{\phi_0(3+\omega_0)} - \frac{m_2(1-2s_2)y_2^i}{\phi_0(3+\omega_0)} + \mathcal{O}\left(\frac{1}{c^2}\right)$$

where various ST parameters come from

$$\omega(\phi) = \omega_0 + (\phi - \phi_0)\omega_0' + ...$$

and [for $A \in \{1,2\}$] :

$$m_A(\psi) = m_A\left(m_A + s_A\psi + ...\right)$$

Note that the weak equivalence principle is broken so the inertial mass of a star (seen as a point-particle) can depend on the local value of the scalar field, hence the need to introduce sensitivities, e.g.

$$s_A = \frac{\mathrm{d}\ln m_A(\phi)}{\mathrm{d}\ln\phi}$$

MPM metric in exterior vacuum

Now that the linear metric is entirely determined, we go back to the MPM expansion: $\psi = G\psi_1 + G^2\psi_2 + ...$ and inject it into our full vacuum field equation

$$\Box\psi = \Lambda_s[h, \psi]$$

where $\Lambda_s[h, \psi]$ is at least quadratic in the fields. Thus, we contruct the MPM metric by iterating:

$$\Box\psi_n = \Lambda_s^{(n)}[h_1, ..., h_{n-1}; \psi_1, ..., \psi_{n-1}]$$

This generates nonlocal effects such as tail, the quadratic memory, etc. !

Radiative moments

Once the MPM metric constructed, we can discard all subdominant terms in the $R \to \infty$ limit. We thus recover an (asymptotically) multipolar structure:

$$\psi \sim \frac{1}{r} \sum \hat{n}_L \mathcal{U}_L^s$$

We recover the tail terms of GR , but also find new ST tail terms and a new ST memory term:

$$\mathcal{U}_{ij} = \overset{(2)}{\mathrm{I}}_{ij} + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} \mathrm{d}\tau \, \overset{(4)}{\mathrm{I}}_{ij}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + \frac{11}{12} \right] + \frac{G(3+2\omega_0)}{3c^3} \int_0^{+\infty} \mathrm{d}\tau \left[\overset{(2)}{\mathrm{I}}{}^s_{\langle i} \overset{(2)}{\mathrm{I}}{}^s_{j\rangle} \right](u-\tau) + (\text{inst}) + \mathcal{O}\left(\frac{1}{c^4}\right)$$

$$\mathcal{U}_i^s = \overset{(1)}{\mathrm{I}}{}^s_i + \frac{2GM}{\phi_0 c^3} \int_0^{+\infty} \mathrm{d}\tau \, \overset{(3)}{\mathrm{I}}{}^s_i(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1 \right] + (\text{inst}) + \mathcal{O}\left(\frac{1}{c^6}\right)$$

Fluxes at infinity

Now that we know that asymptotic structure of the scalar waves [idem for GWs] at $\mathcal{I}^+$, we can deduce the fluxes of energy and angular momentum that they carry [2401.06844]:

$$\mathcal{F}^s = \frac{c^3 R^2 (3+2\omega_0)\phi_0}{16\pi G} \int \mathrm{d}^2\Omega \dot{\psi}^2$$
$$= \sum_{\ell=0}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}\ell!(2\ell+1)!!} \dot{\mathcal{U}}^s_L \dot{\mathcal{U}}^s_L$$

$$\mathcal{G}^s_i = \frac{c^3 R^3 (3+2\omega_0)\phi_0}{16\pi G} \int \mathrm{d}^2\Omega \dot{\psi} \epsilon_{iab} n_a \partial_b \psi$$
$$= \sum_{\ell=1}^{\infty} \frac{G\phi_0(3+2\omega_0)}{c^{2\ell+1}(\ell-1)!(2\ell+1)!!} \epsilon_{iab} \mathcal{U}^s_{aL-1} \dot{\mathcal{U}}^s_{bL-1}$$

where we have used $\psi \sim \frac{1}{r} \sum \hat{n}_L \mathcal{U}^s_L(t-r/c)$.

Quasicircular orbits

Why are we interested in the fluxes ? Consider the case of a quasicircular orbit. First, the angular momentum flux is related to the energy flux by $\mathcal{F} = \omega \mathcal{G}$, so we only consider the energy balance law:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\mathcal{F} - \mathcal{F}^s$$

In the COM frame, the only dynamical variables are $r = |\boldsymbol{y}_1 - \boldsymbol{y}_2|$, $\boldsymbol{n} = (\boldsymbol{y}_1 - \boldsymbol{y}_2)/r$ and $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$.

The fluxes depend on them only through $r \approx (Gm/\omega^2)^{2/3}$, $v^2 \approx (Gm\omega)^{2/3}$ and $\boldsymbol{n} \cdot \boldsymbol{v} \approx 0$, where ω is the orbital frequency.

Thus, the energy balance equation reduces to an equation of the type:

$$\boxed{\frac{\mathrm{d}\omega}{\mathrm{d}t} = f(\omega)}$$

This immediately yields the phase and frequency evolution !

How to treat nonlocal terms ?

Starting at 0.5PN order, nonlocal tail terms appear, such as:

$$\mathcal{F}^{s,\mathrm{tail}} = \frac{4G^2\mathrm{M}(3+2\omega_0)}{3c^6} \overset{(2)}{\mathrm{I}}{}^{s}_{i} \int_0^{\infty} \mathrm{d}\tau \, \overset{(4)}{\mathrm{I}}{}^{s}_{i}(u-\tau) \left[\ln\left(\frac{c\tau}{2b_0}\right) + 1\right]$$

For a quasi-circular orbit,

$$I^i_s(t) = \sum_{k=-k_{\mathrm{max}}}^{k_{\mathrm{max}}} \alpha_k e^{ik\omega t}$$

where $k \neq 0$, so we trivially use the formula

$$\int_0^{+\infty} \mathrm{d}\tau \ln\left(\frac{c\tau}{2b_0}\right) e^{-ik\omega\tau} = \frac{\mathrm{i}}{k\omega}\Big[\ln\Big(\frac{2|n|b_0}{c}\Big) + \gamma_{\mathrm{E}} + \mathrm{i}\frac{\pi}{2}\mathrm{sg}(n)\Big]$$

to compute the flux. **However, for an (quasi-)elliptic orbit, the time dependence of the dipole is much more complicated!**

⇒ the solution is to compute expand the moments in *Fourier series*

Fluxes associated to tails

Injecting the Fourier expansion into the tail integrals that enter the flux, using standard integrals and orbit averaging, I find for example:

$$\begin{aligned}\left\langle \mathcal{F}^{s,\mathrm{tail}} \right\rangle &= \frac{4\pi G^2 \mathrm{M}(3+2\omega_0)}{c^6} \\ &\times \left\{ \frac{n^5}{3} \sum_{p=1}^{\infty} \left[p^5 \sum_{\substack{m\in\{-1,1\} \\ s\in\{-1,1\}}} \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right) \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right)^{*} + 5kp^4 \sum_{m\in\{-1,1\}} m \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right) \left({}^{m}_{p}\widetilde{\mathrm{I}}^{s}_{i} \right)^{*} \right] \right. \\ &\qquad \left. + \frac{n^7}{30c^2} \sum_{p=1}^{\infty} p^7 \left({}_{p}\widetilde{\mathrm{I}}^{s}_{ij} \right) \left({}_{p}\widetilde{\mathrm{I}}^{s}_{ij} \right)^{*} + \frac{n^3}{\phi_0^2 c^2} \sum_{p=1}^{\infty} p^3 \left({}_{p}\widetilde{E}^{s} \right) \left({}_{p}\widetilde{E}^{s} \right)^{*} \right\}\end{aligned}$$

where the Fourier coefficients are given as functions of x et e_t.

Fluxes associated to tails in terms of enhancement functions

Finally we can express the orbit average flux as

$$\left\langle \mathcal{F}^{s,\text{tail}} \right\rangle = \frac{c^5 x^5 \nu^2 \zeta}{3\tilde{G}\alpha} \times 4\pi(1+\bar{\gamma}/2)\sqrt{x}\Bigg\{ 2\mathcal{S}_-^2 \varphi_1^s(e_t) + x\left[\mathcal{C}_1 \varphi_2^s(e_t) + \frac{41}{15}\mathcal{S}_-^2 \nu \theta_1^s(e_t) + \mathcal{C}_2 \alpha_1^s(e_t) + \mathcal{C}_3 x \varphi_0^s(e_t) \right] \Bigg\}$$

$$\left\langle \mathcal{G}^{s,\text{tail}} \right\rangle = \frac{c^2 x^{7/2} \nu^2 \zeta}{3} \times 4\pi(1+\bar{\gamma}/2)\sqrt{x}\Bigg\{ 2\mathcal{S}_-^2 \tilde{\varphi}_1^s(e_t) + x\left[\mathcal{D}_1 \tilde{\varphi}_2^s(e_t) + \frac{41}{30}\mathcal{S}_-^2 \nu \tilde{\theta}_1^s(e_t) + \mathcal{D}_2 \tilde{\alpha}_1^s(e_t) \right] \Bigg\}$$

where $\varphi_s^s(e_t)$, etc., are *enhancement functions* of the eccentricity whose limit as $e_t \to 1$ is 1 [except for $\alpha_1^s(e_t)$ and $\alpha_2^s(e_t)$, for which it is zero]. Thus, we explicitly recover the expression for circular orbits of [2201.10924].

Expression of the enhancement functions

Typically, an enhancement function is exactly defined in terms of the Fourier coefficients, e.g.

$$\varphi_1^s(e_t) = 2\sum_{p=1}^{\infty} \sum_{\substack{m\in\{-1,1\} \\ s\in\{-1,1\}}} p^5 \left({}^{m}_{p}\widehat{\mathrm{I}}_i^{s,00}\right)\left({}^{m}_{p}\widehat{\mathrm{I}}_i^{s,00}\right)^*$$

This is simply a function over $e_t \in [0,1]$ which can be computed numerically, but we can also perform a $e_t \to 0$ expansion:

$$\varphi_1^s(e_t) = 1 + 7e_t^2 + \frac{717}{32}e_t^4 + \frac{7435}{144}e_t^6 + \frac{7305575}{73728}e_t^8 + \frac{103947697}{614400}e_t^{10} + \mathcal{O}(e_t^{12})$$

For increased accuracy, it is possible to resum these by factorizing by $(1-e_t^2)^{-n/2}$ for some n [2308.13606]. Other more complex resummation methods exist as well [1607.05409].

Memory contribution to the flux

The angular momentum flux also has a *memory-like* nonlocal term:

$$\mathcal{G}_i^{\mathrm{mem}} = -\frac{2G^2\phi_0(3+2\omega_0)}{15c^8}\epsilon_{ik(j}\overset{(3)}{\mathrm{I}}_{a)k}\int_0^{\infty}\mathrm{d}\tau\left[\overset{(2)}{\mathrm{I}}{}^{s}_{a}\,\overset{(2)}{\mathrm{I}}{}^{s}_{j}\right](u-\tau)$$

Replacing the moments by their Fourier decomposition leads to an integrand of the type $\sum_k \alpha_k e^{ik\ell}$.

The $k \neq 0$ terms correspond to the "AC' contribution". They are trivial to integrate and can to be shown to vanish upon orbit averaging.

The $k = 0$ is the "DC term", and reads:

$$\mathcal{G}_{\mathrm{DC}}^{\mathrm{mem}} = \frac{4G^2G^3m^5\nu^2}{105c^{10}}\mathrm{I}_{xy}^{(3)}(u)\int_0^{\infty}\mathrm{d}\tau\,\left[\frac{e^2(13+2e^2)}{a^5(1-e^2)^{7/2}}\right](t_{\mathrm{ret}}-\tau)$$

The DC term is finite and essentially constant, so the DC flux vanishes upon orbit averaging.

Thus, $\boxed{\langle\mathcal{G}^{\mathrm{mem}}\rangle = 0}$

Instantaneous' flux at 1.5PN

Finally, we can compute the intanteneous flux at 1.5PN with the 2PN QK parametrization in the same way as before ... but there is a complicating when trying to go the the CoM frame !

Indeed, the linear momentum P^i and CoM position G^i of the matter content satisfy the balance equations:

$$\frac{\mathrm{d}G_i}{\mathrm{d}t} = P_i - \mathcal{F}^i_{s,\boldsymbol{G}} - \mathcal{F}^i_{\boldsymbol{G}} \qquad \text{and} \qquad \frac{\mathrm{d}P_i}{\mathrm{d}t} = -\mathcal{F}^i_{s,\boldsymbol{P}} - \mathcal{F}^i_{\boldsymbol{P}}$$

where the fluxes enter at 2.5PN order. But COM frame is defined not only for the matter content, but also for the GWs ! Thus is is defined by

$$G_i + \int_{-\infty}^{t} \mathrm{d}t' \int_{-\infty}^{t'} \mathrm{d}t'' \left[\mathcal{F}^i_{s,\boldsymbol{G}} + \mathcal{F}^i_{\boldsymbol{G}}\right](t) + \int_{-\infty}^{t} \mathrm{d}t' \left[\mathcal{F}^i_{s,\boldsymbol{P}} + \mathcal{F}^i_{\boldsymbol{P}}\right](t') = 0$$

which reduces to $G_i = 0$ only at 2PN order. Thus, we have an extra nonlocal term to deal with!

$\Rightarrow$ under investigation

Evolution equations at 2.5PN order

With the 1.5PN fluxes thus derived, I yet have to obtain:

$$\left\langle \frac{\mathrm{d}x}{\mathrm{d}t} \right\rangle = f(x, e_t) \qquad \text{and} \qquad \left\langle \frac{\mathrm{d}e_t}{\mathrm{d}t} \right\rangle = g(x, e_t)$$

This is the generalization to elliptic orbits of the "chirp" for circular orbits, i.e. one of the main observable in a gravitational wave !