Fundamental Physics III

Richard Brito
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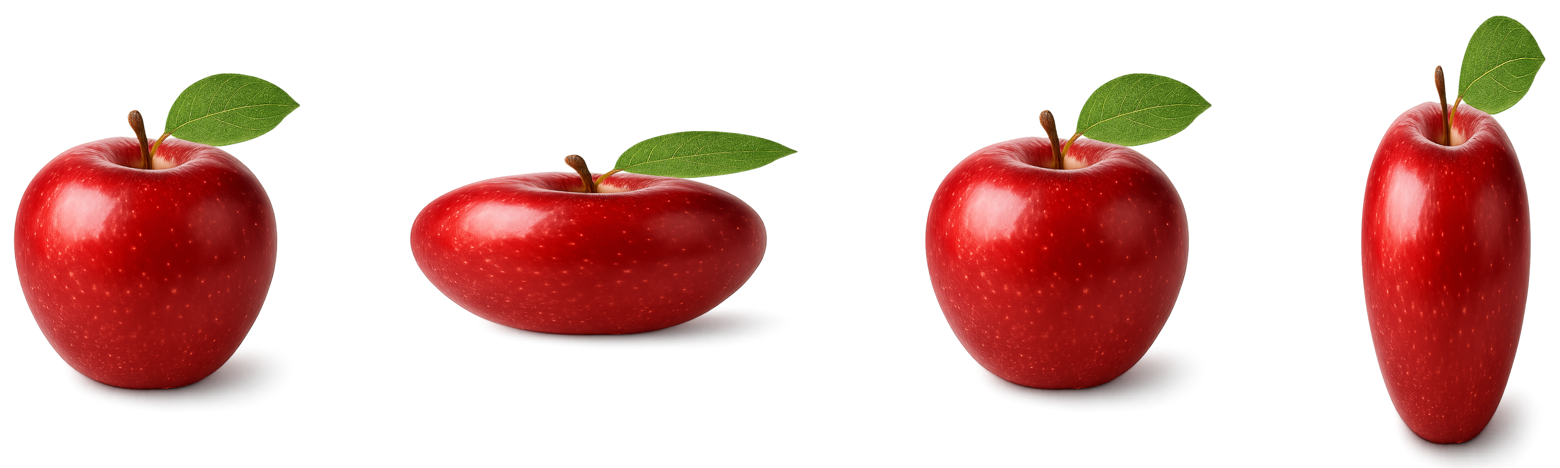
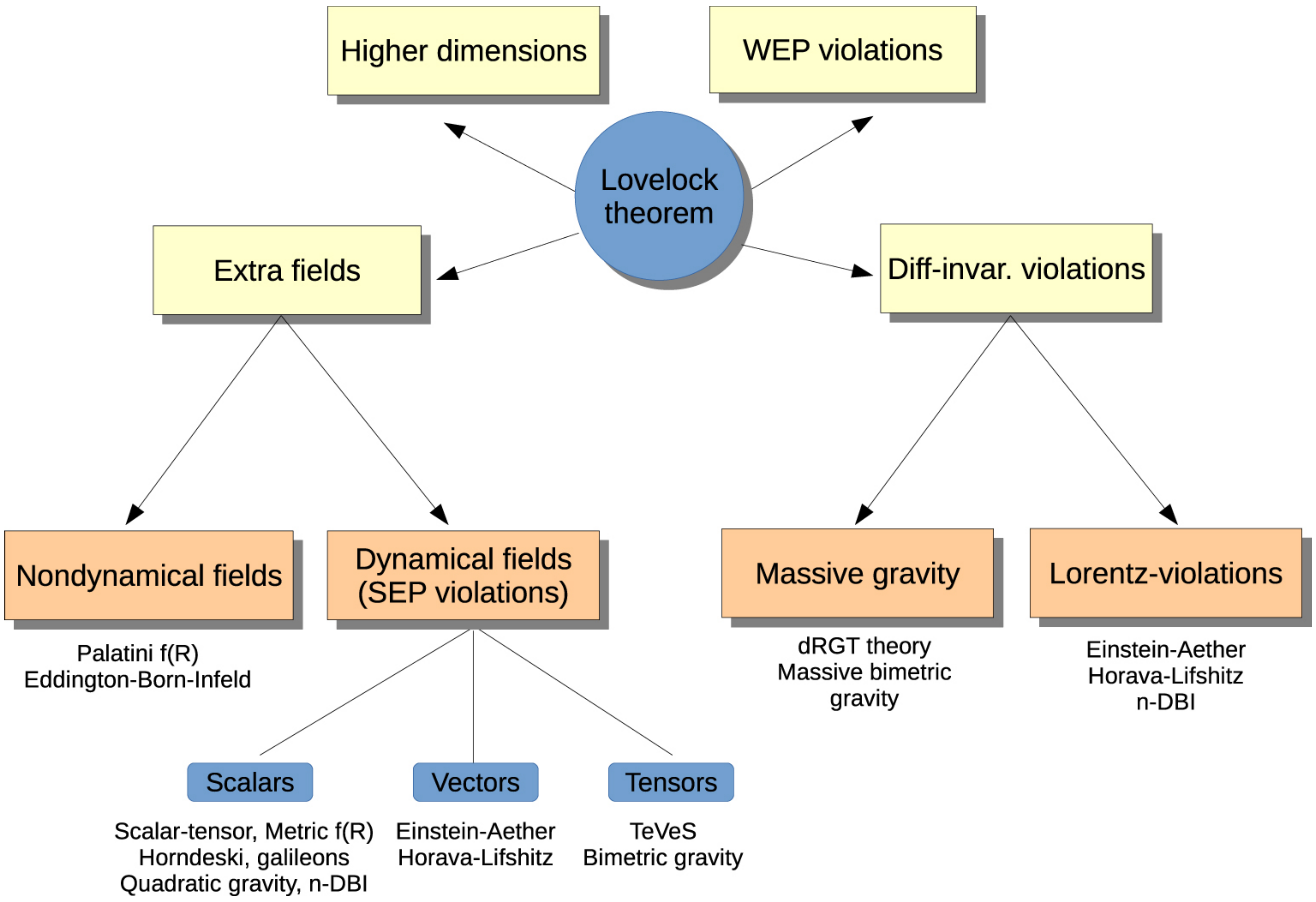
Outline

- Alternative theories of gravity **(FP I)**
- Testing beyond GR theories with EMRIs **(FP I)**
- Tests of the no-hair theorem and nature of compact objects **(FPII)**
- Parametrised tests of GR **(FPIII, this lecture)**
- *Beyond vacuum GR*: environmental effects & searches for dark matter **(FP III, this lecture)**



Parametrised tests of GR



From: E. Berti *et al*, Class. Quantum Grav. 32 243001 (2015)

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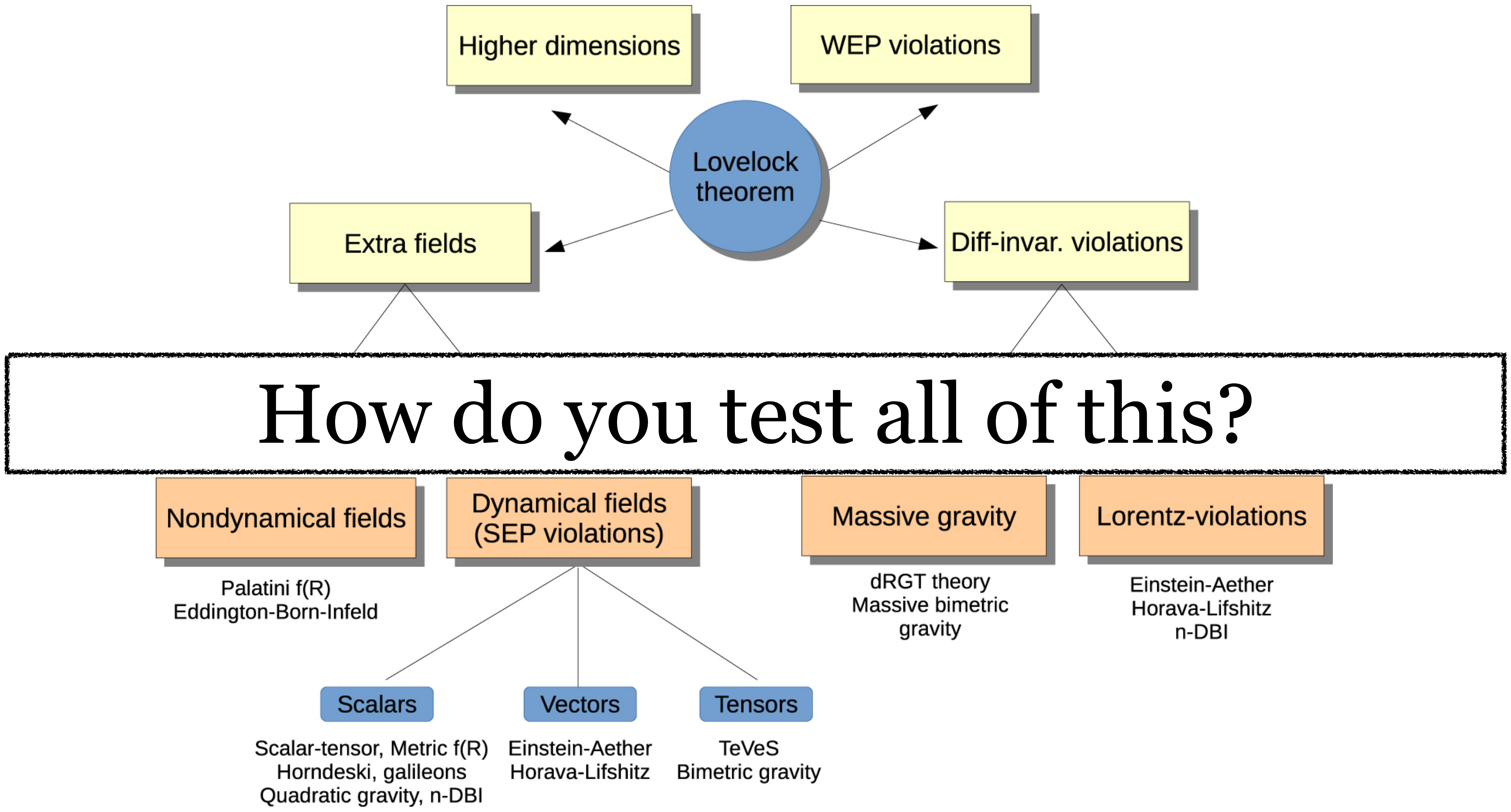

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How to test gravity?

Theory-specific

- Pick a theory, make predictions and test it.
- **Advantages:**
  - Might get stronger constraints on the parameter(s) of interest.
  - Possibility of considering clear smoking-gun deviations from GR.
  - Easy to combine information from different tests.
- **Disadvantages:**
  - Too many theories; no real motivation to choose one over others
  - Challenging to build complete waveform models.

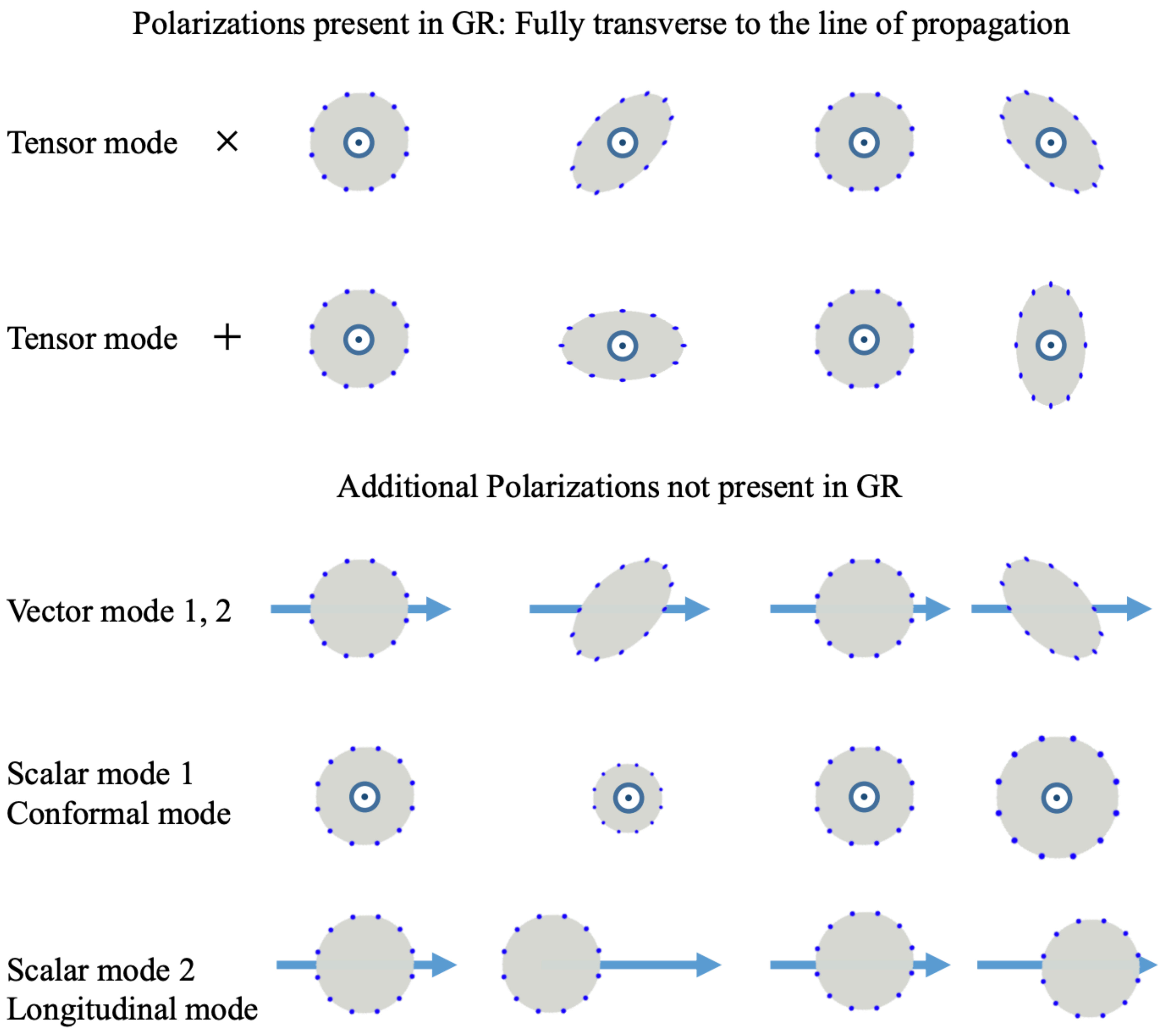
Theory-agnostic

- Consider model-independent deviations from GR (ppN, ppE, EFT approach…).
- **Advantages:**
  - "Easy" to model/implement.
  - *Ideally:* generic enough to encompass several theories/effects
- **Disadvantages:**
  - Too many parameters.
  - For most cases only focuses on part of the waveform.
  - Doesn't always capture all effects

Beyond GR phenomenology

LISA FP WG, Living Rev. Rel. (2022) 25:4

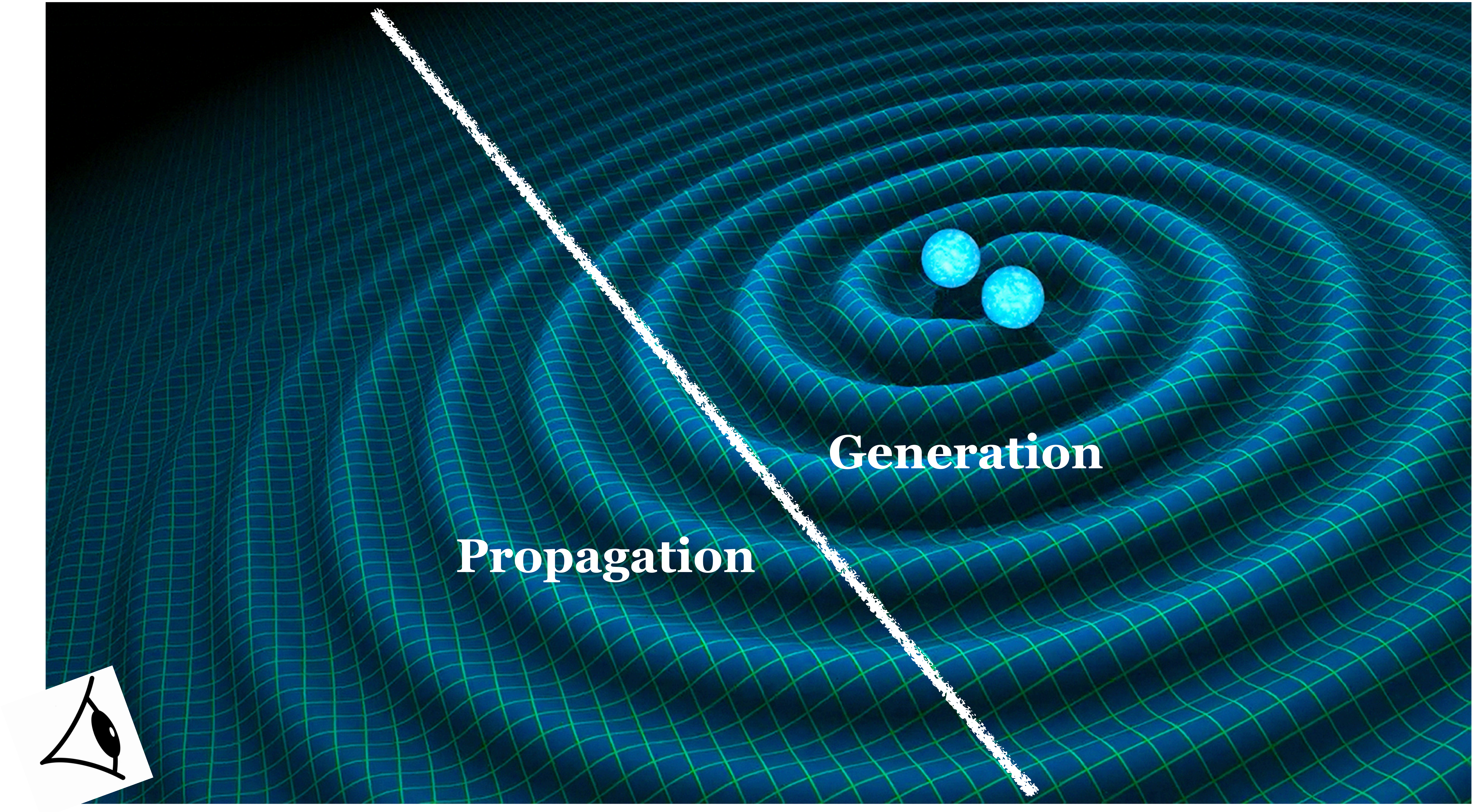
- Additional polarizations (up to 6 independent polarizations for a metric theory)
- Additional channels for energy loss, e.g. extra scalar field
- Modified graviton dispersion relation (graviton mass; Lorentz violations)
- Parity violations (amplitude birefringence)
- Hairy BHs
- Different BH ringdown and new families of QNMs
- Non-zero tidal Love numbers
- …



From: C. Rham, Living Rev. Relativity 17 (2014), 7

Propagation vs Generation

Slide Credit: Nico Yunes



Generation example: dipole radiation

Conservation laws ***do not allow dipole radiation in GR***, but in some beyond GR theories you can have dipole radiation (e.g. scalar-tensor theories)

$$E_{\rm orb} \approx -\,Gm_1m_2/(2r_{\rm orb})\,,\; r_{\rm orb} = (GM/\omega_{\rm orb}^2)^{1/3}$$

$$\dot{E}_{\rm orb} = -\,\left(P_{\rm GW} + P_{\rm nGR}\right)$$

$$P_{\rm GW} \approx \frac{1}{5}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij}\right\rangle \sim \left(\frac{v}{c}\right)^{10}\,, \qquad P_{\rm nGR} \approx \frac{2}{3}\left\langle \ddot{D}_i\ddot{D}^i\right\rangle \sim \left(\frac{v}{c}\right)^{8}$$

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Dipolar radiation forces binary to inspiral faster (notice also $P_{\rm nGR} \gg P_{\rm GW}$ for $v/c \ll 1$)

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GW phase is sensitive to rate of inspiral:

$$\frac{df}{dt} = \left(\frac{dE_{\rm orb}}{df}\right)^{-1}\dot{E}_{\rm orb}\,, \qquad \Psi_{\rm GW} \sim \int^t\int^{t'}\frac{df}{dt''}dt''dt' \sim \left(\pi \mathcal{M} f\right)^{-5/3} + \beta_{\rm nGR}\left(\pi \mathcal{M} f\right)^{-7/3}$$

Propagation example: massive graviton

"Massive" tensor propagating in FLRW cosmological background:

$$h_A'' + 2\mathscr{H} h_A' + \left[c^2k^2 + a^2m_g^2\right] h_A = 0\,, \quad A = +/\times\;, \quad m_g \to 0 \implies \text{GR}$$

$$\frac{v_g}{c} \approx 1 - \frac{1}{2}\left(\frac{c}{\lambda_g f}\right)^2, \qquad \lambda_g = \frac{h}{m_g c}$$

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Modified dispersion relation affects GW phase [C. Will '97]:

$$\delta\Psi_{\mathrm{GW}}^{\mathrm{mg}} \approx \beta_{\mathrm{mg}} \left(\pi \mathscr{M} f\right)^{-1}, \qquad \beta_{\mathrm{mg}} = \frac{\pi^2 D_L \mathscr{M}}{\lambda_g^2}$$

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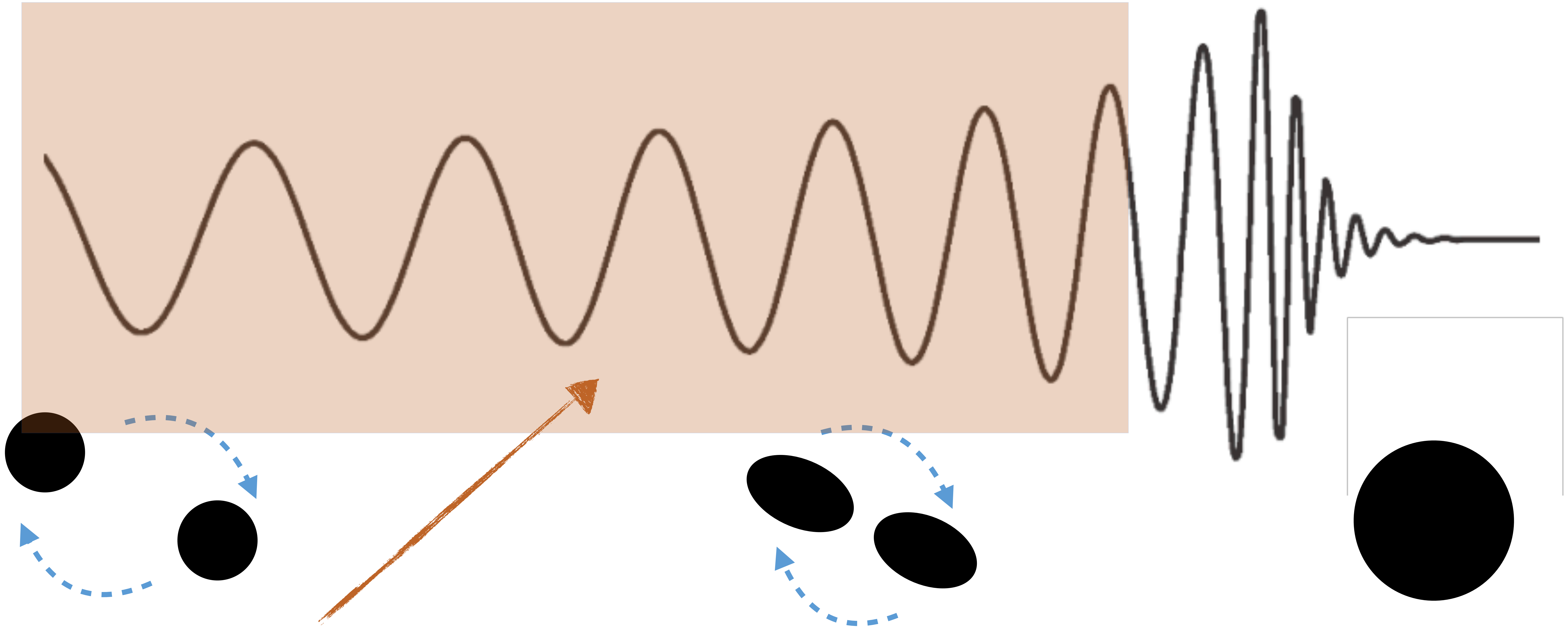
Effect stronger the larger the luminosity distance D_L .

Parametrized post-Einsteinian (ppE) framework

Yunes & Pretorius, PRD 2009



$$\tilde{h}(f) \approx \mathscr{A} e^{i[2\pi f t_c - \varphi_c - \pi/4 + \Psi(f)]}, \quad \Psi(f) = \frac{3}{128} \left(\mathscr{M} \pi f\right)^{-5/3} \left[\cdots + \psi_{-1} x^{-1} + \psi_0 + \psi_1 x + \psi_{3/2} x^{3/2} + \psi_2 x^2 + \cdots\right]$$

$$x \equiv v^2 = (\pi f M)^{2/3}$$

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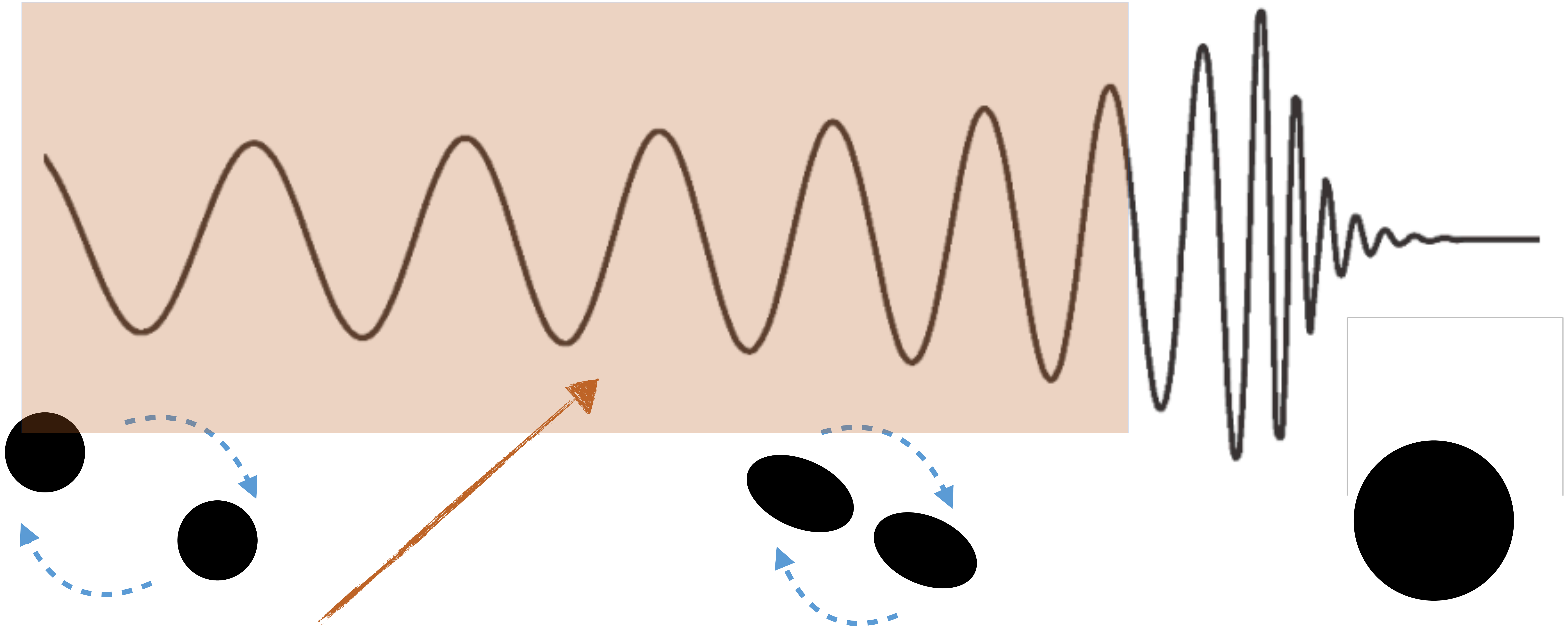

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Necessarily new physics
(e.g. dipole emission) or
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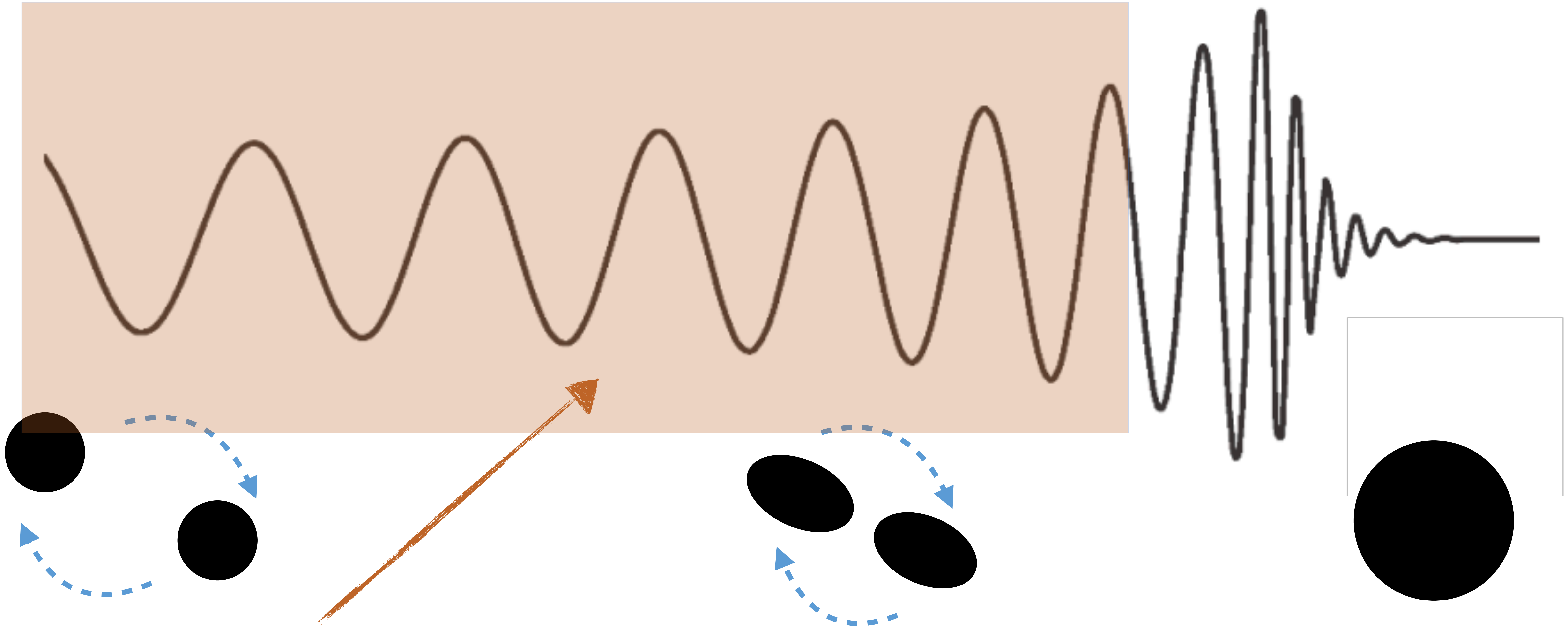

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Necessarily new physics (e.g. dipole emission) or non-vacuum environments

$$x \equiv v^2 = (\pi f M)^{2/3}$$

$$\tilde{h}(f) = \tilde{h}_{GR}(f)\left[1 + \alpha_i(\pi\mathscr{M}f)^{a_i}\right] e^{i\beta_j(\pi\mathscr{M}f)^{b_j/3}}$$

Mapping ppE to different theories

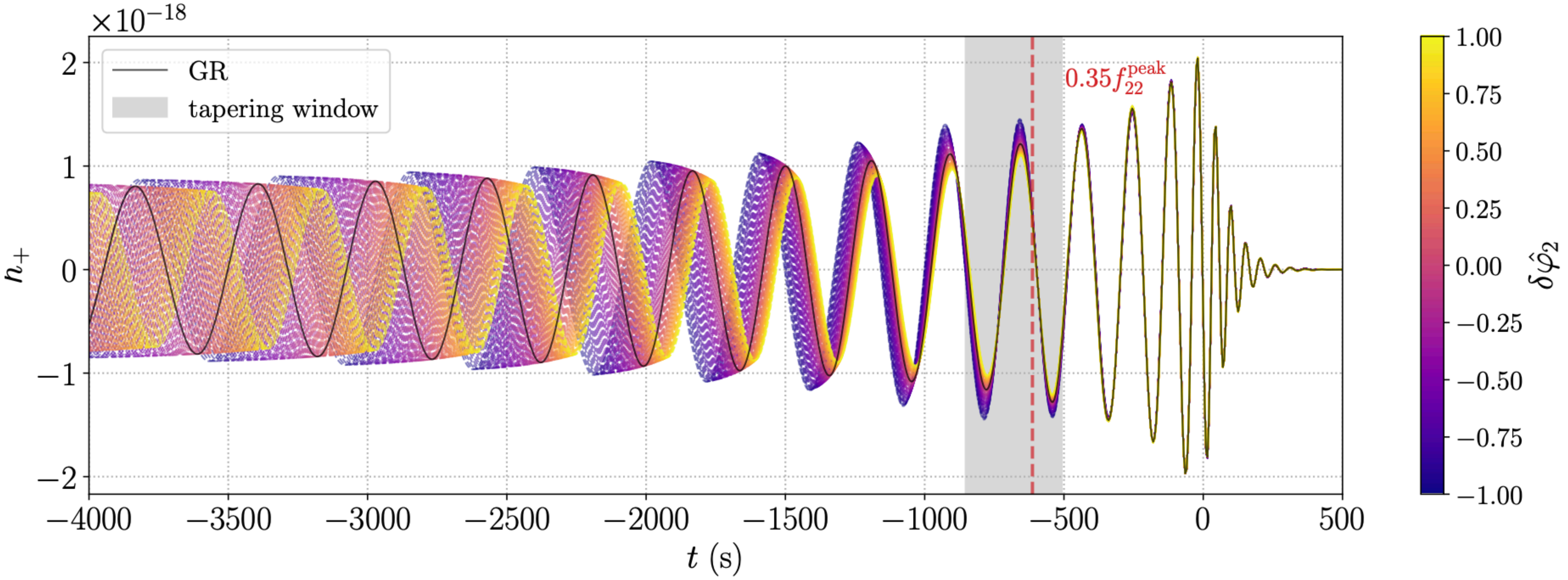
$$\tilde{h}(f) = \tilde{h}_{GR}(f) e^{i\beta(\pi \mathcal{M} f)^{b/3}}$$

Theoretical Effect	Theoretical Mechanism	Theories	ppE b	Order	Mapping
Scalar Dipolar Radiation	Scalar Monopole Activation	EdGB [143, 145, 152, 153]	-7	-1PN	$\beta_{\rm EdGB}$ [143]
	BH Hair Growth	Scalar-Tensor Theories [60, 154]	-7	-1PN	$\beta_{\rm ST}$ [60, 154]
Anomalous Acceleration	Extra Dim. Mass Leakage	RS-II Braneworld [155, 156]	-13	-4PN	$\beta_{\rm ED}$ [144]
	Time-Variation of G	Phenomenological [140, 157]	-13	-4PN	$\beta_{\dot{G}}$ [140]
Scalar Quadrupolar Radiation Scalar Dipole Force Quadrupole Moment Deformation	Scalar Dipole Activation due to Grav. Parity Violation	dCS [143, 158]	-1	$+2$PN	$\beta_{\rm dCS}$ [149]
Scalar/Vector Dipolar Radiation Modified Quadrupolar Radiation	Vector Field Activation due to Lorentz Violation	EA [111, 112], Khronometric [113, 114]	-7 -5	-1PN 0PN	$\beta_{\rm Æ}^{(-1)}$, $\beta_{\rm KG}^{(-1)}$ [115] $\beta_{\rm Æ}^{(0)}$, $\beta_{\rm KG}^{(0)}$ [115]
Modified Dispersion Relation	GW Propagation	Massive Gravity [159–162]	-3	$+1$PN	$\beta_{\rm MDR}$ [148, 159]
		Double Special Relativity [163–166]	$+6$	$+5.5$PN	
		Extra Dim. [167], Horava-Lifshitz [168–170]	$+9$	$+7$PN	
		gravitational SME ($d = 4$) [82]	$+3$	$+4$PN	
		gravitational SME ($d = 5$) [82]	$+6$	$+5.5$PN	
		gravitational SME ($d = 6$) [82]	$+9$	$+7$PN	
		Multifractional Spacetime [171–173]	3–6	4–5.5PN	

From: N. Yunes, K. Yagi & F. Pretorius, PRD94 (2016) 084002

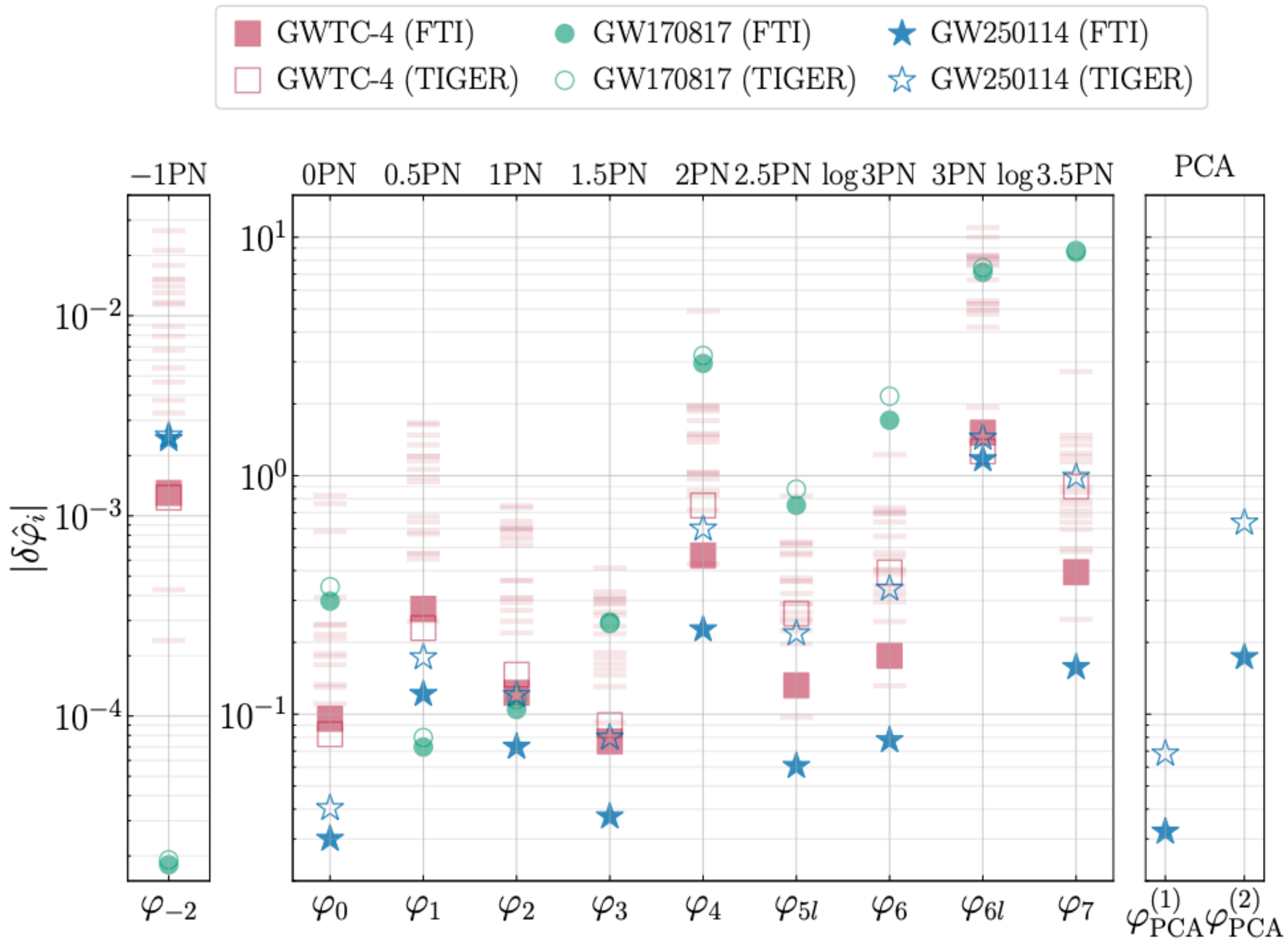
Impact on waveform: example

Note: example shows impact on waveform when adding a "non-GR" 1PN correction (such as massive graviton effect) to inspiral



From: Piarulli+, 2510.06330

Parametrized tests: LVK Constraints



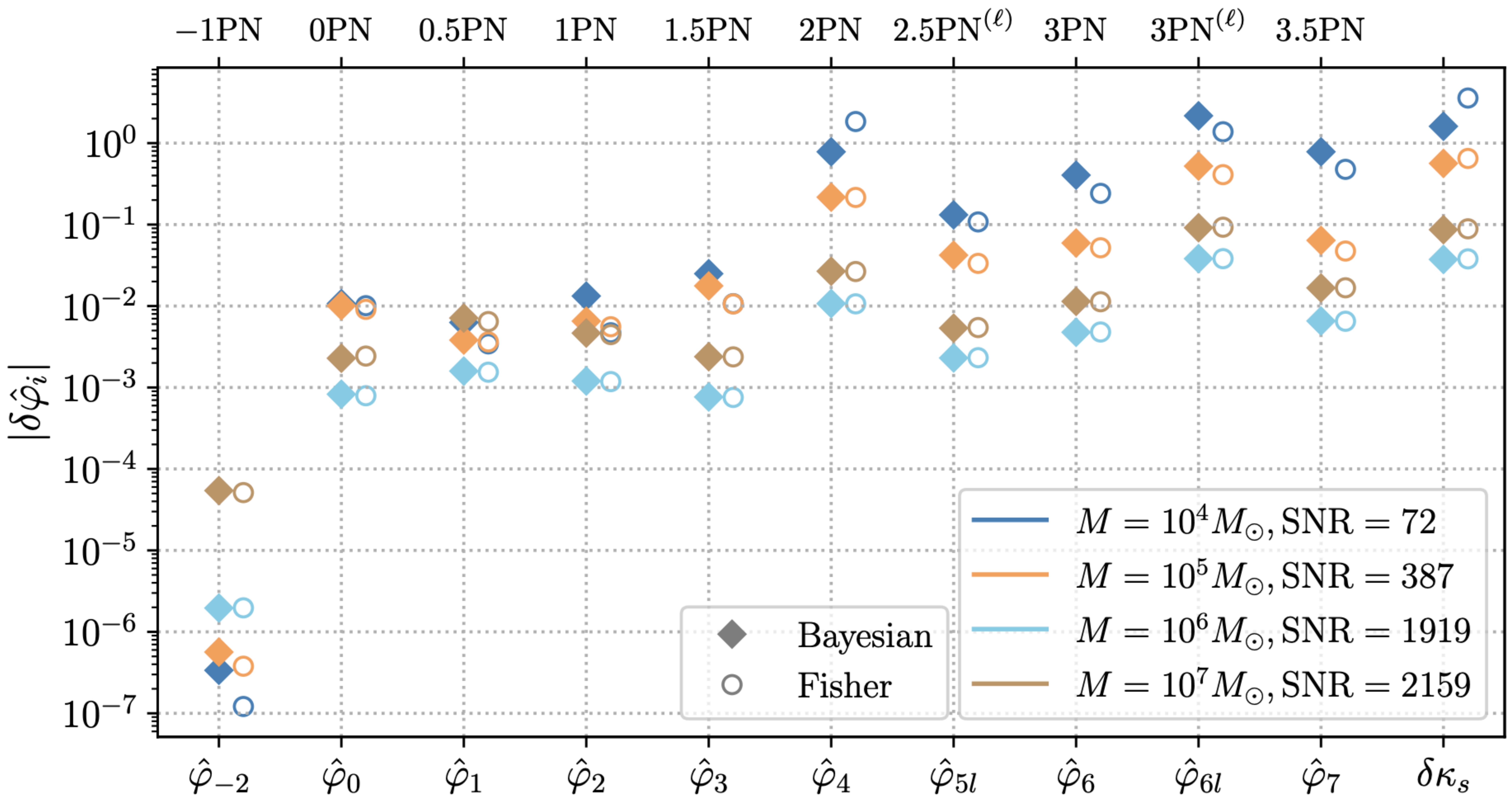
$$\Psi_i \to (1 + \delta\varphi_i)\Psi_i^{\mathrm{GR}}$$

Note 1: for -1PN and 0.5PN should be interpreted as absolute deviations (since those terms are identically zero in GR)

Note 2: analysis varies one $\delta\varphi_i$ varied at a time

From: LVK Collaboration, 2509.08099

LISA prospective constraints



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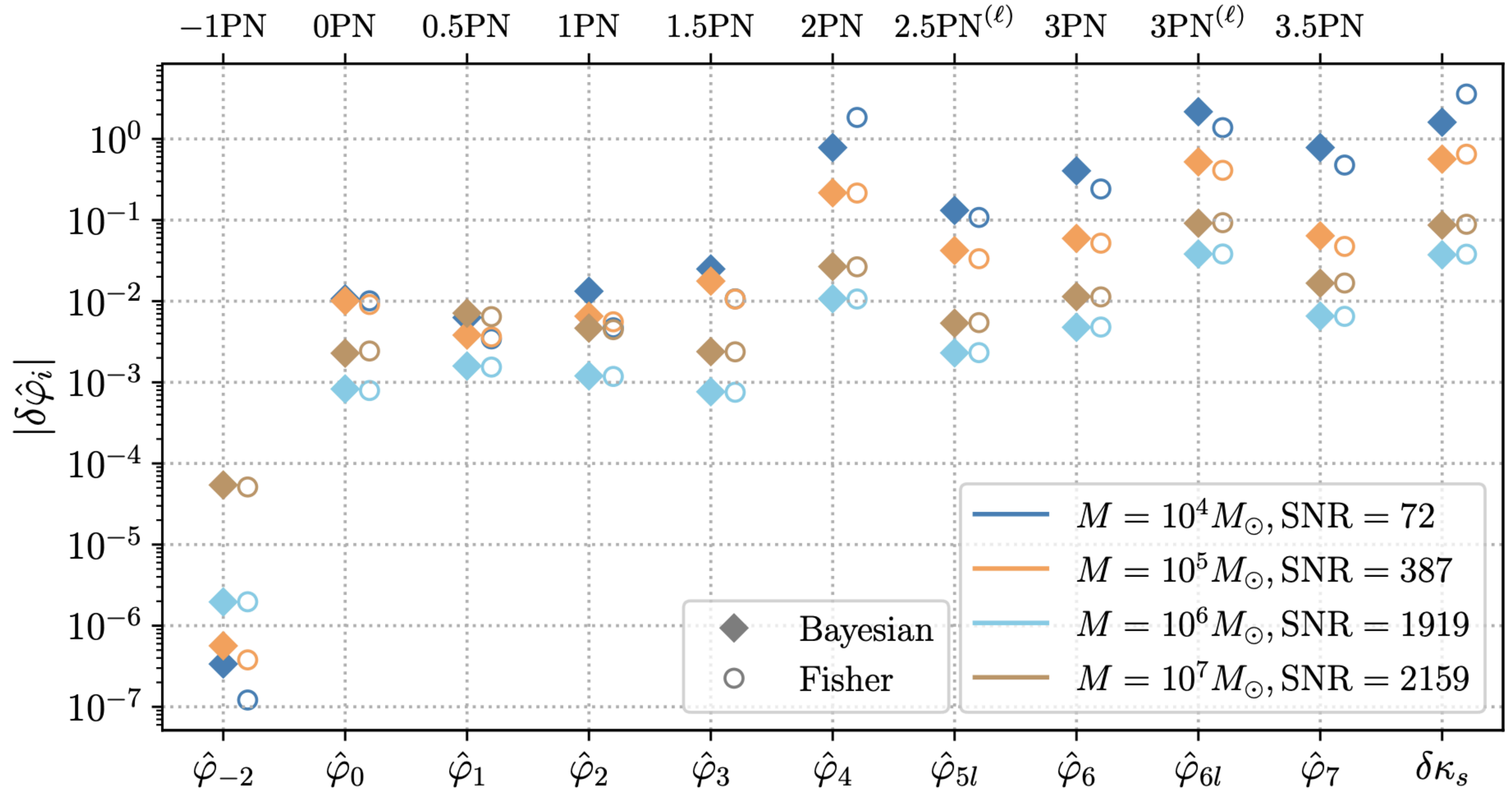
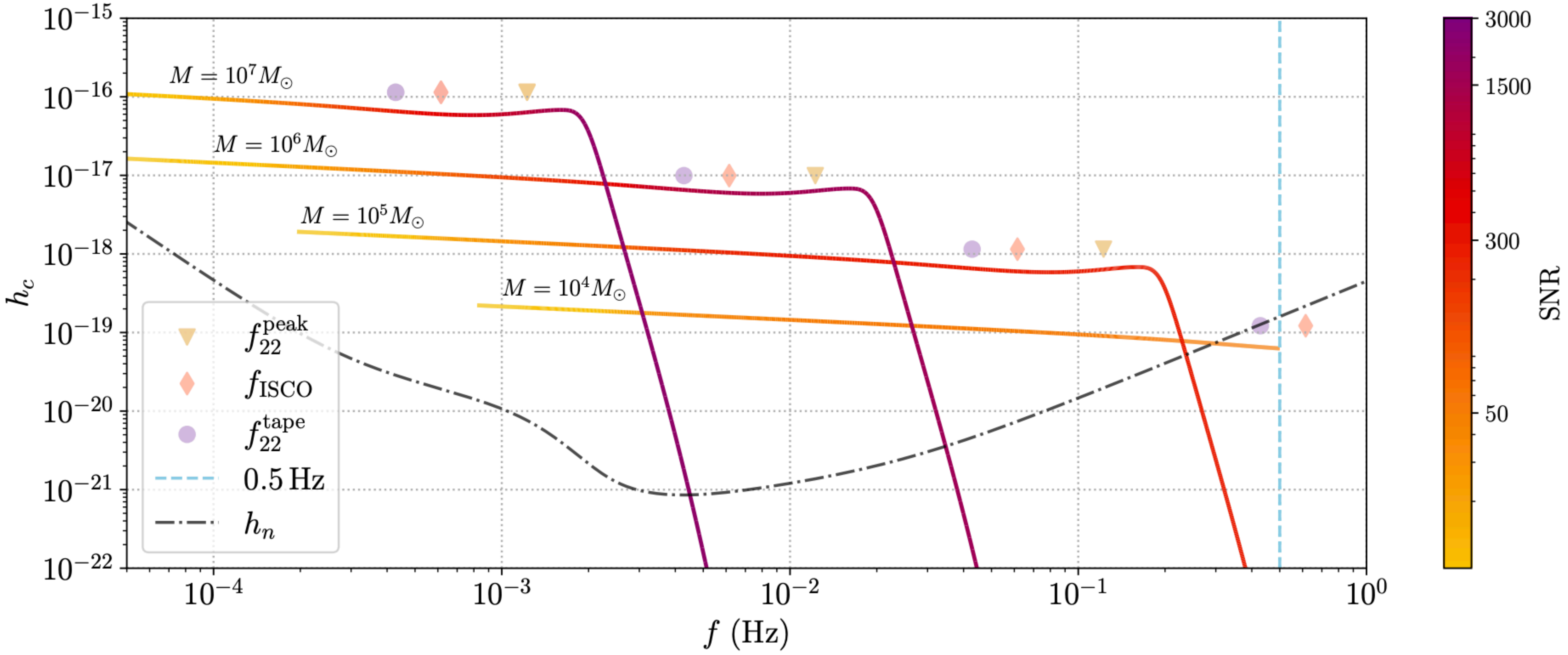


Inspiral dominated events $M = 10^{4-5} M_\odot$ binaries better at constraining lower PN terms

Merger dominated events $M = 10^{6-7} M_\odot$ binaries better at constraining high PN terms



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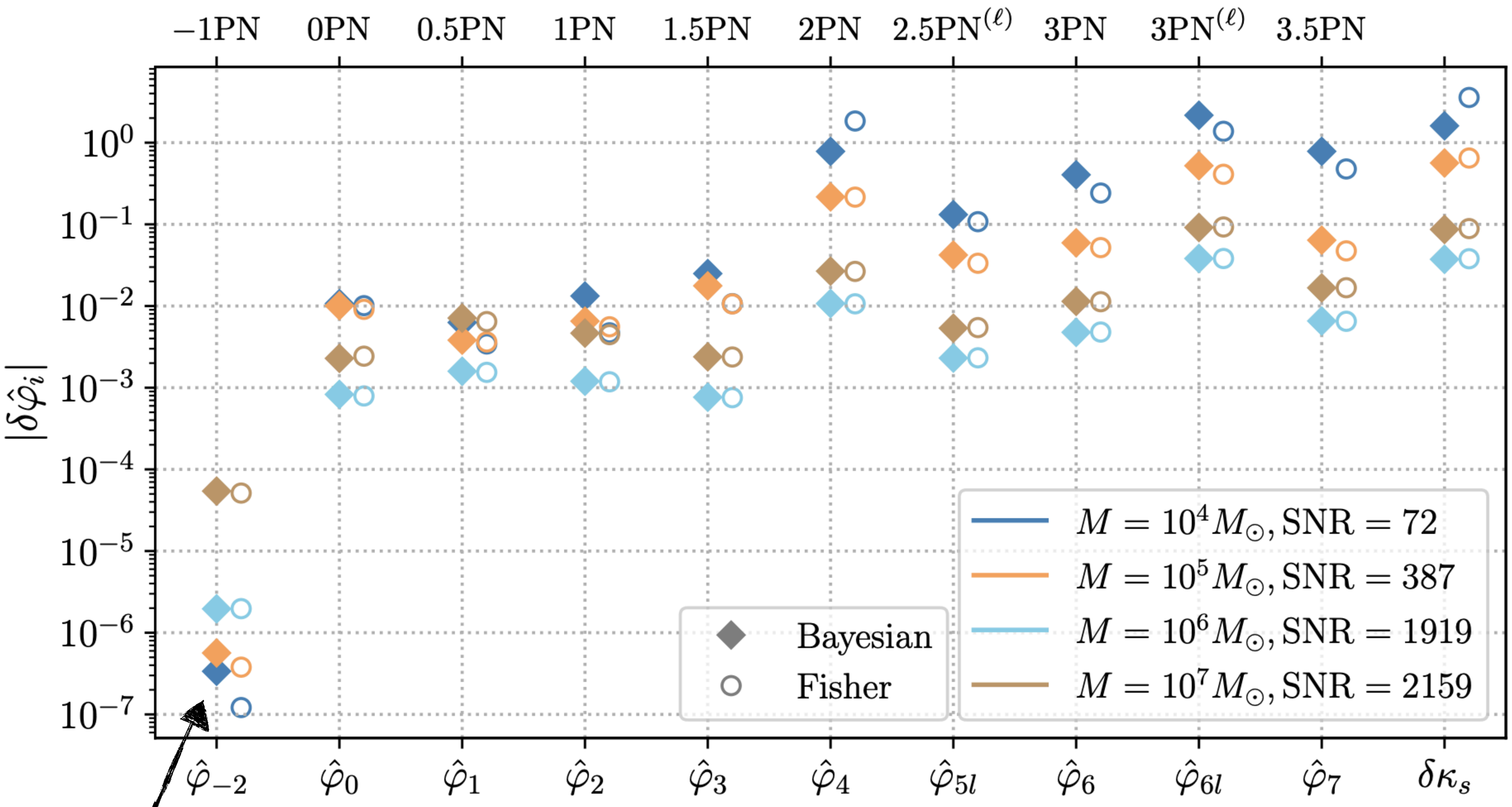
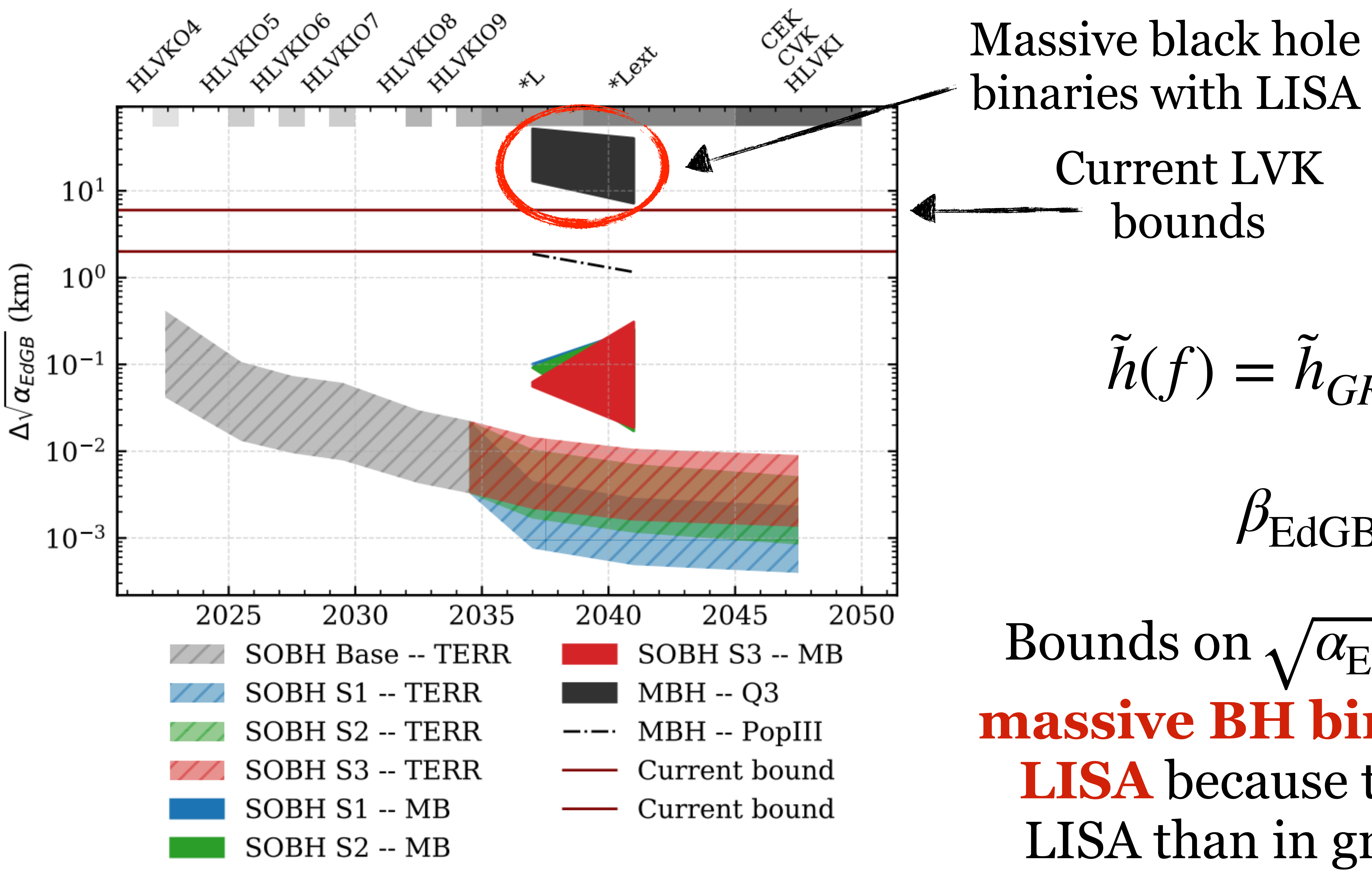


~4 orders of magnitude **better** than current LVK (with BBHs)

From: M. Piarulli+, 2510.06330

But… mapping to specific theory



$$\tilde{h}(f) = \tilde{h}_{GR}(f) e^{i\beta_{\rm EdGB}(\pi \mathcal{M} f)^{-7/3}}$$

$$\beta_{\rm EdGB} \propto \alpha^2_{\rm EdGB}/M^4$$

Bounds on $\sqrt{\alpha_{\rm EdGB}}$ are **weaker with massive BH binaries detected with LISA** because total masses higher in LISA than in ground-base detectors.

From: Perkins, Yunes & Berti, PRD103, 044024 (2021)

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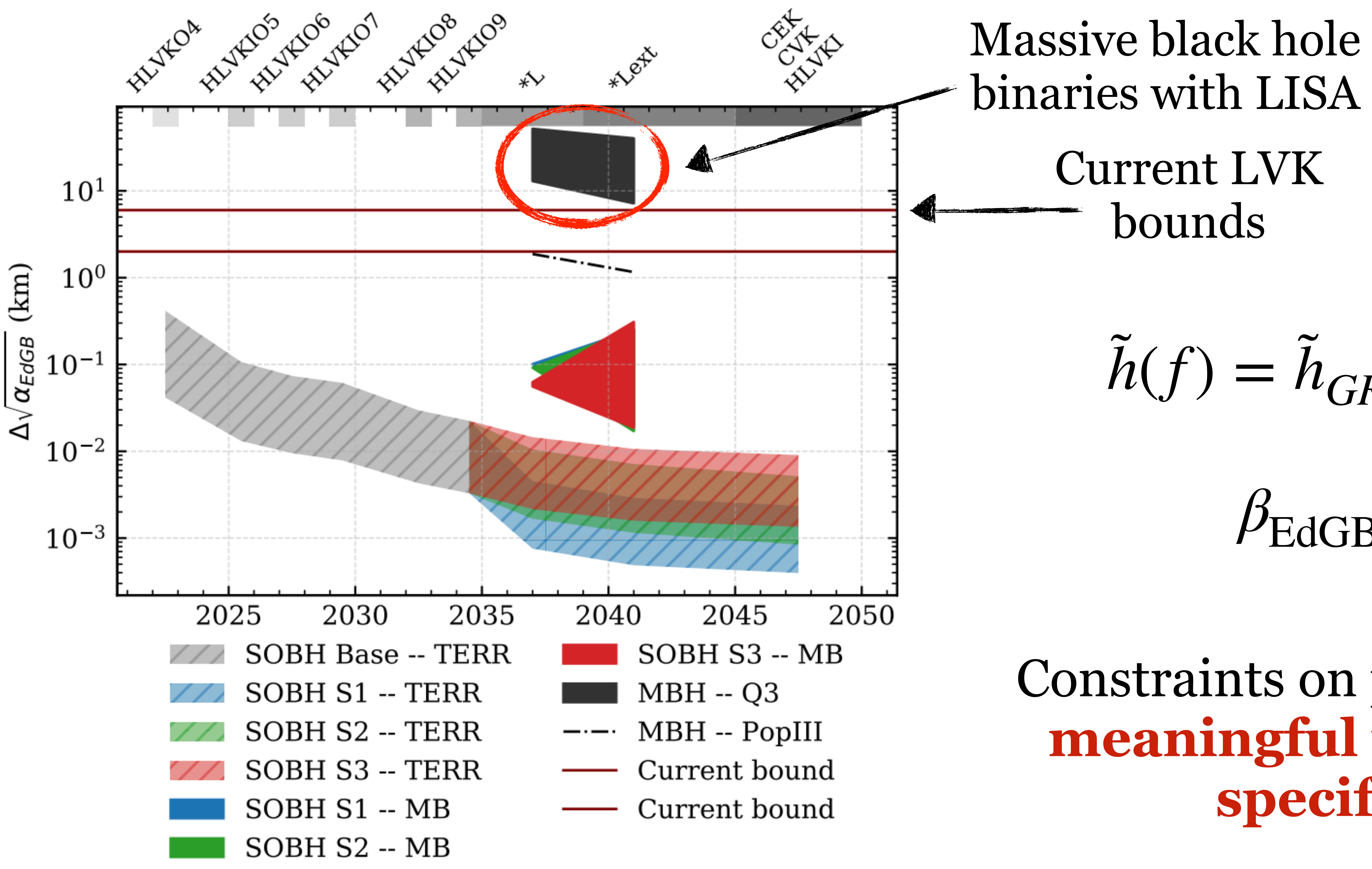

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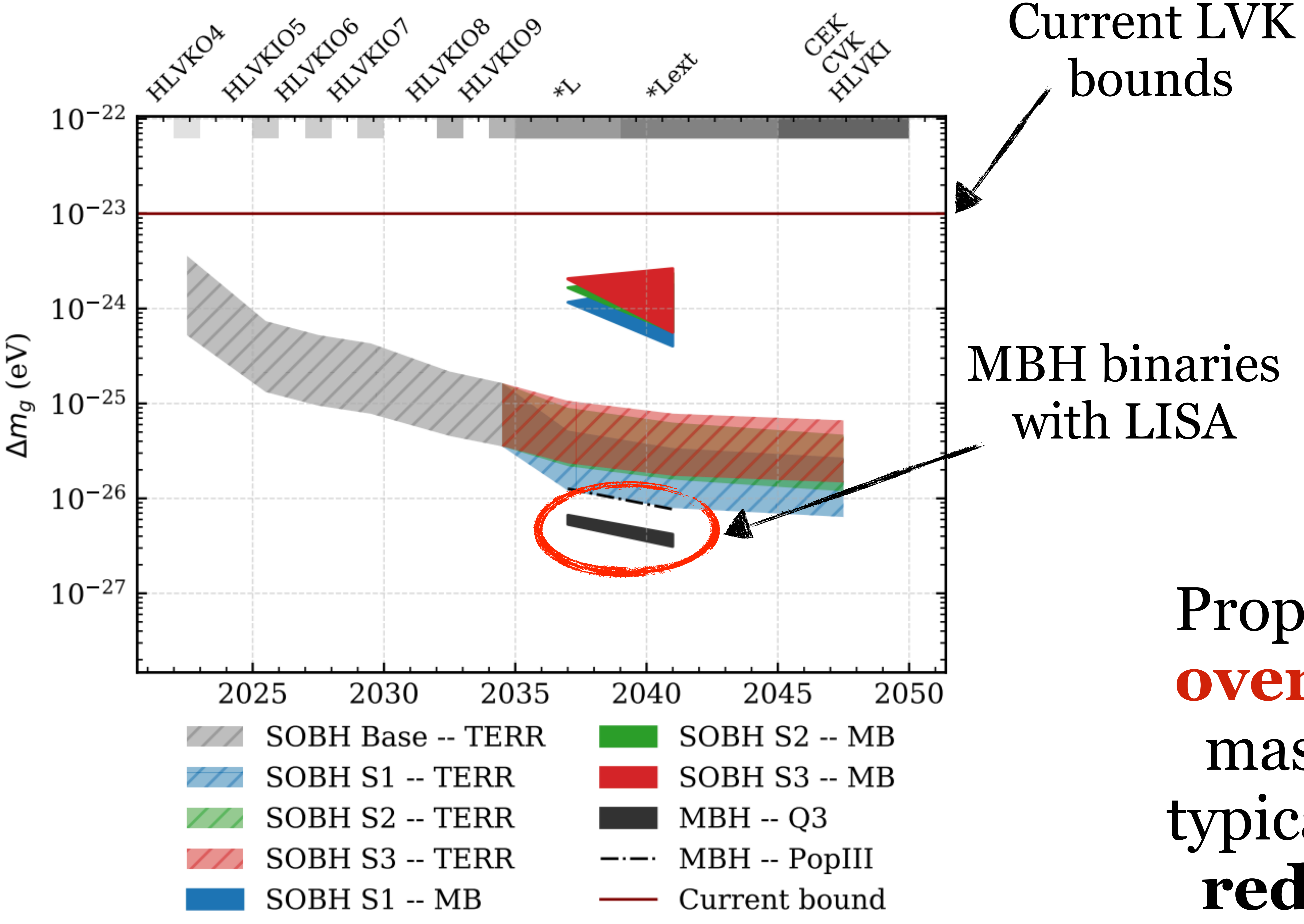
Constraints on ppE parameters **only meaningful when mapping to specific theories**.

From: Perkins, Yunes & Berti, PRD103, 044024 (2021)

Constraints on graviton mass with LISA



$$\tilde{h}(f) = \tilde{h}_{GR}(f)e^{i\beta_{\rm EdGB}(\pi\mathcal{M}f)^{-1}}$$

$$\beta_{\rm mg} = \frac{\pi^2 D_L \mathcal{M}}{\lambda_g^2}$$

Propagation effects **build up over large distances**: LISA massive black hole binaries typically **observed at larger redshifts** than LVK events

From: Perkins, Yunes & Berti, PRD103, 044024 (2021)

Beyond *vacuum* GR

Black holes **do not live in vacuum**.

Can environmental effects (accretion disks, third bodies, dark matter, etc…) spoil precision GW physics?

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[4]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada.*
[5]*Institute for Theory and Computation, Harvard-Smithsonian CfA, 60 Garden Street, Cambridge MA 02138, USA*

(Dated: May 27, 2014)

No, within a broad class of scenarios. Gravitational-wave (GW) astronomy will open a new window on compact objects such as neutron stars and black holes (BHs). It is often stated that large signal-to-noise detections

Barausse, Cardoso & Pani, Phys.Rev. D89 (2014) 104059

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Beyond *vacuum* GR

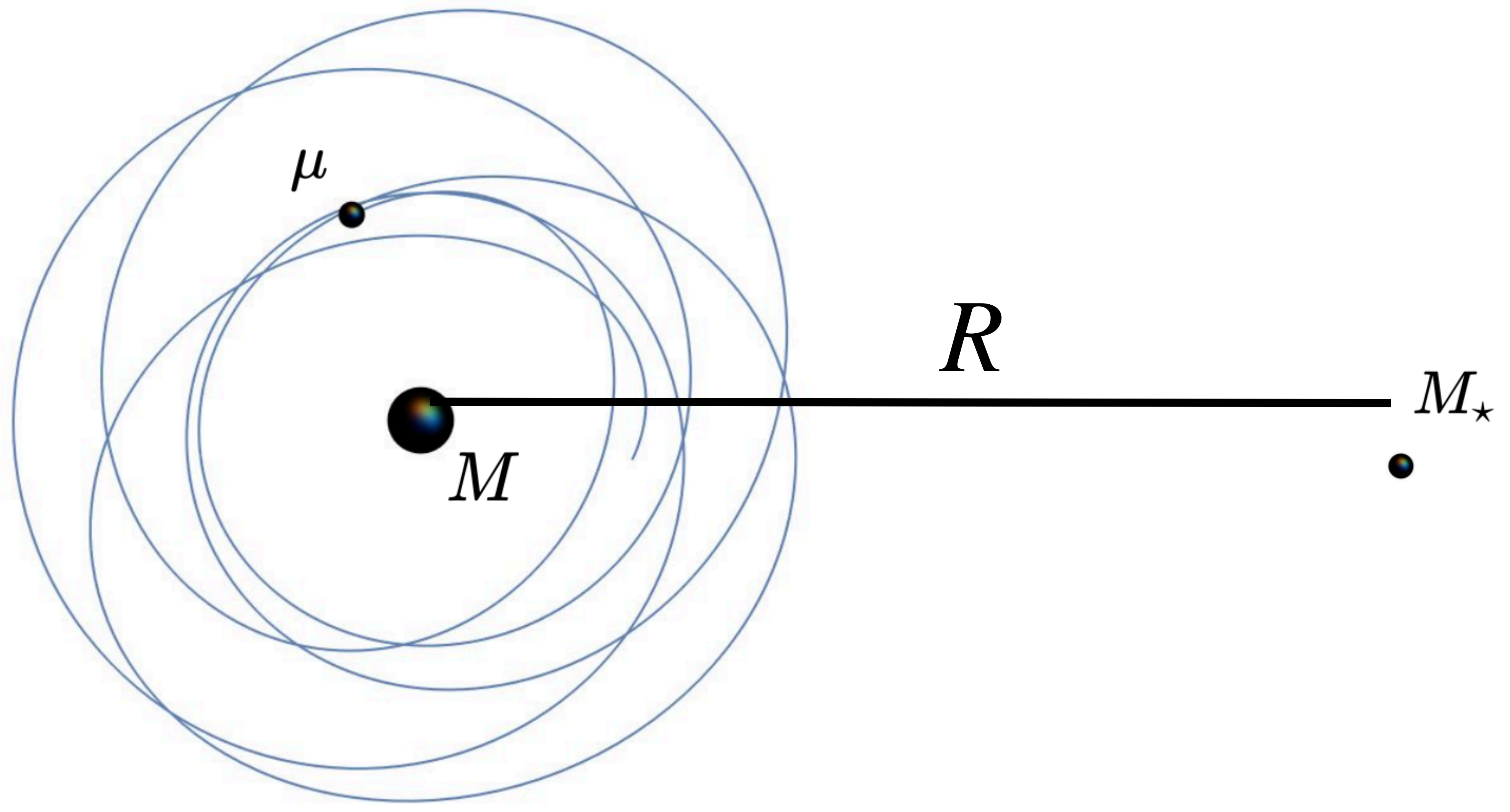
But that doesn't mean environmental effects will not be detectable.

General conclusion from Barausse+ '14:

- Environmental effects ***likely*** to be uncommon/too small for nearly equal mass massive black hole binaries detected in LISA band (but could be important in PTA band for example).
- For E/IMRIs ***could*** be detectable in ***some*** events. Large number of cycles in E/IMRIs helps detecting small effects.

“Non-exotic” black hole environments

Perturbations from third bodies
(main effect: **tidal resonances**)

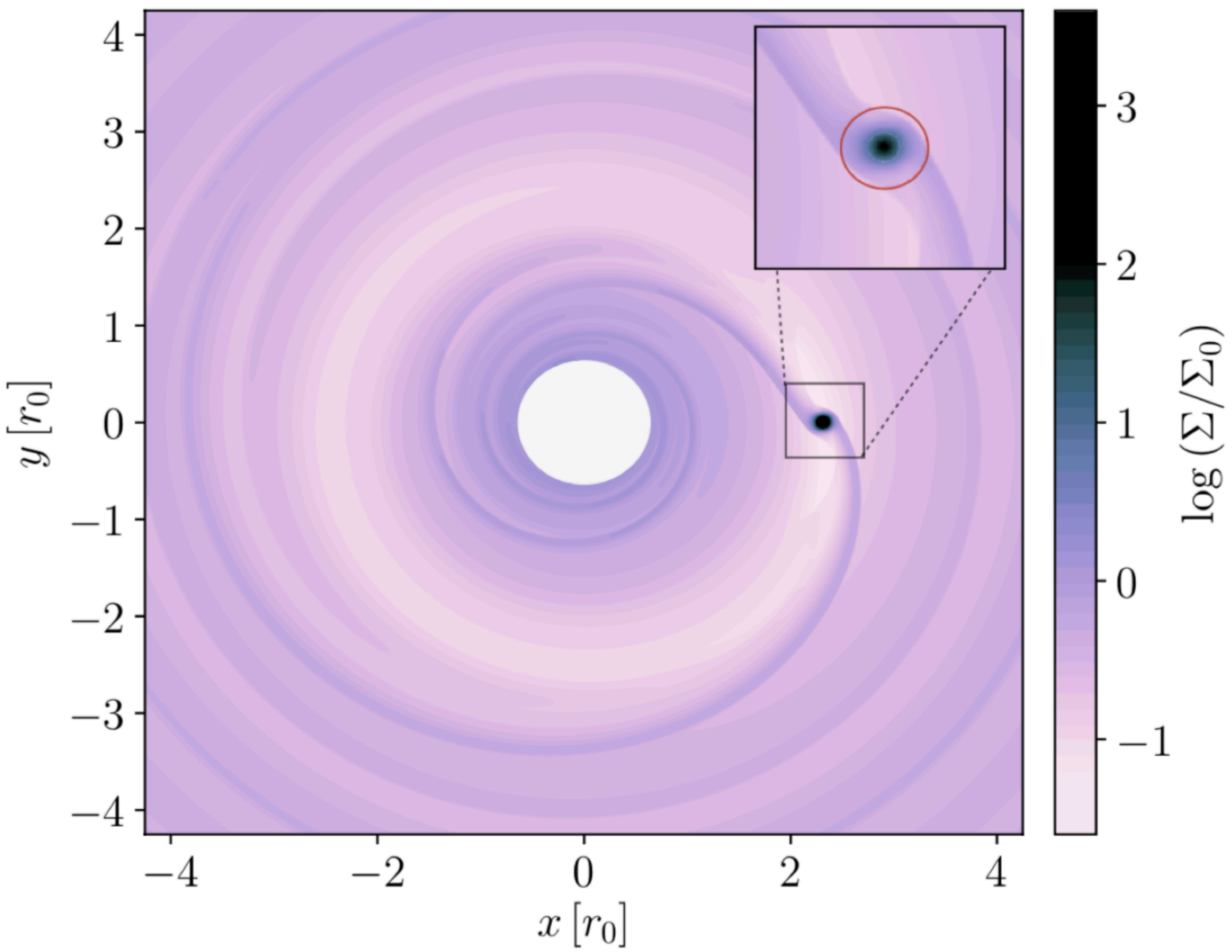


$$\Delta\Psi_{\rm GW} = 1.4\,{\rm rad}\left(\frac{\mu}{10M_\odot}\right)^{-1/2}\left(\frac{M}{4\times10^6M_\odot}\right)^{7/2}\left(\frac{M}{10M_\odot}\right)\left(\frac{R}{4.3{\rm AU}}\right)^{-3}$$

Credit: Bonga, Yang & Hughes ’19, PRL123, 101103

Accretion disks
(main effect: **planetary migration**)

$$\dot{L}_{\rm orb} = -\dot{L}_{\rm GW} - \dot{L}_{\rm mig}$$



Credit: Derdzinski+ ’21, MNRAS501, 3540

“Exotic” black hole environments

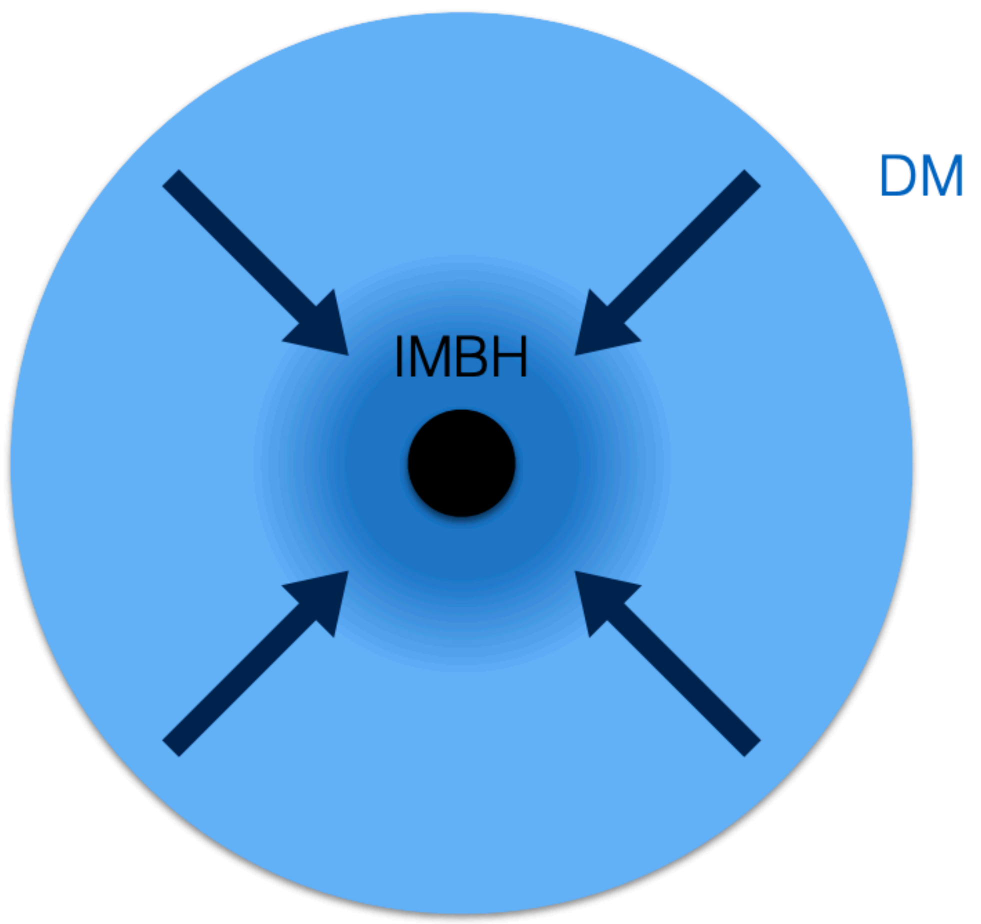
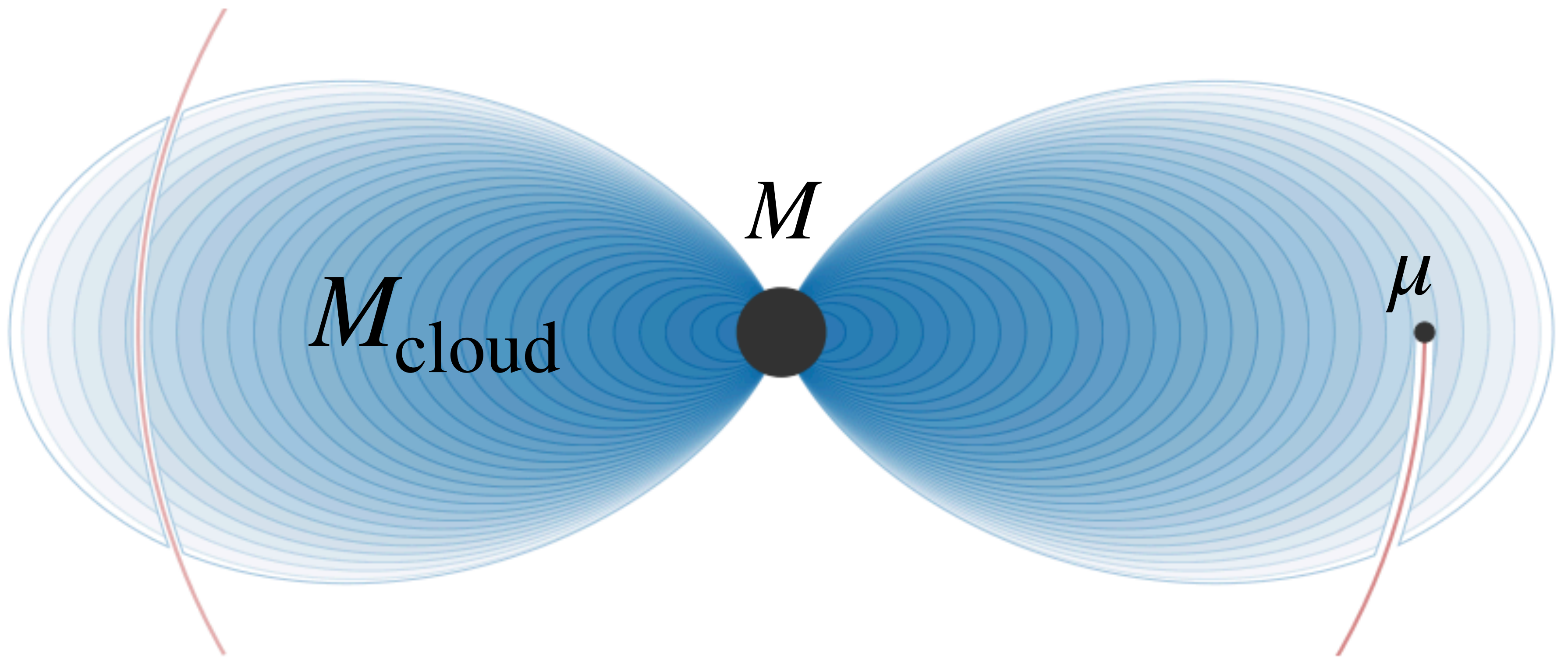


Image: Bradley J. Kavanagh

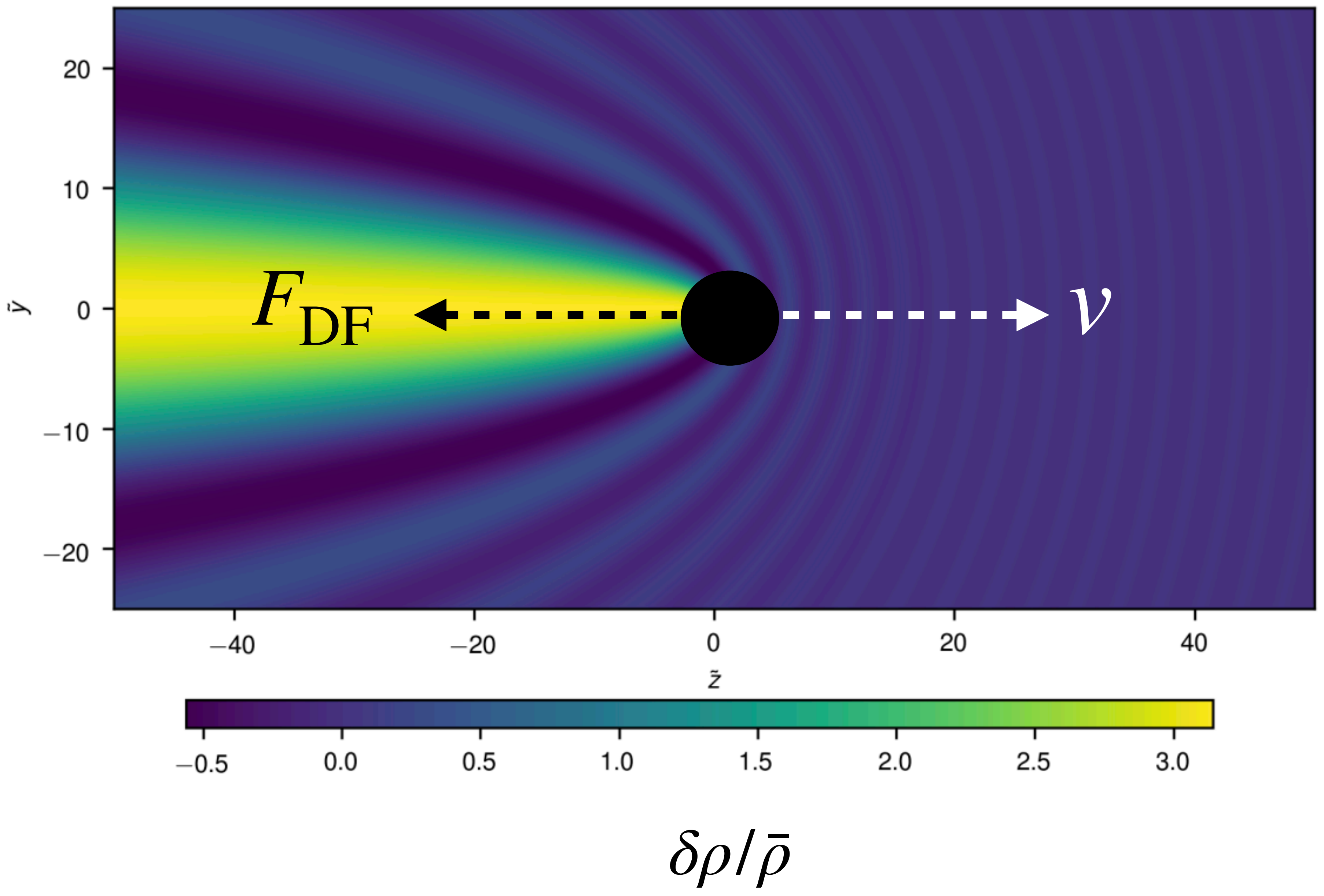


From: Baumann+, PRD105, 115036 (2022)

- **Dark matter spikes** formed through the adiabatic growth of a massive BH at the center of a dark matter halo [Gondolo & Silk ‘99]
- **Boson clouds** (a.k.a. gravitational atoms) formed through superradiant instabilities [Review on superradiance: RB, Cardoso & Pani ’20]

Dynamical friction

S. Chandrasekhar, ’43



Adapted from: Lancaster+, JCAP01(2020)001

Physical intuition: as the object moves through the environment, its gravitational potential induces density perturbations whose backreaction on the object cause it to slow down

For non-relativistic velocities (assumes asymptotically uniform medium):

$$\dot{E}_{\rm DF} \approx \frac{4\pi\mu^2\rho_{\rm DM}}{v}\mathcal{I}(v)$$

$$\frac{\dot{r}_{\rm DF}}{\dot{r}_{\rm GW}} \sim 3\times\left(\frac{M}{10^5 M_\odot}\right)^2\left(\frac{r_{\rm orb}}{100M}\right)^{11/2}\left(\frac{\rho_{\rm DM}}{\rm kg/m^3}\right)\mathcal{I}(v)$$

$\mathcal{I}(v)$ - slowly-varying function of v

$\dot{r}_{\rm GW} \equiv \dot{E}_{\rm GW}/(dE_{\rm orb}/dr)$

Accretion

- Form a small black hole with mass μ moving with constant velocity v in a *collisionless* dark matter medium:

$$\dot{m}_p \approx \sigma \rho_{\rm DM} v \approx \pi r_c^2 \rho_{\rm DM} v \approx \frac{4\pi\mu^2}{v}\rho_{\rm DM}$$

$\sigma \approx \pi r_c^2$ - cross section

$r_c \approx \frac{2\mu}{v^2}$ - "effective capture radius"

Accretion acts on the orbit as an ***effective force*** that slows down moving object (ignoring other effects for the sake of simplicity):

$$L_{\rm orb} = m_p r^2 \Omega_{\rm orb} \approx m_p r^2 \sqrt{M/r^3} \qquad \dot{L}_{\rm orb} = 0 \implies \dot{r} \approx -2\, r\, \dot{m}_p / m_p$$

$$\frac{\dot{r}_{\rm acc}}{\dot{r}_{\rm RR}} \sim 3 \times 10^{-2} \left(\frac{M}{10^5 M_\odot}\right)^2 \left(\frac{r_{\rm orb}}{100M}\right)^{9/2} \left(\frac{\rho_{\rm DM}}{\rm kg/m^3}\right)$$

Impact of environment in waveform

- Orbit is affected by GW radiation-reaction, dynamical friction and accretion :

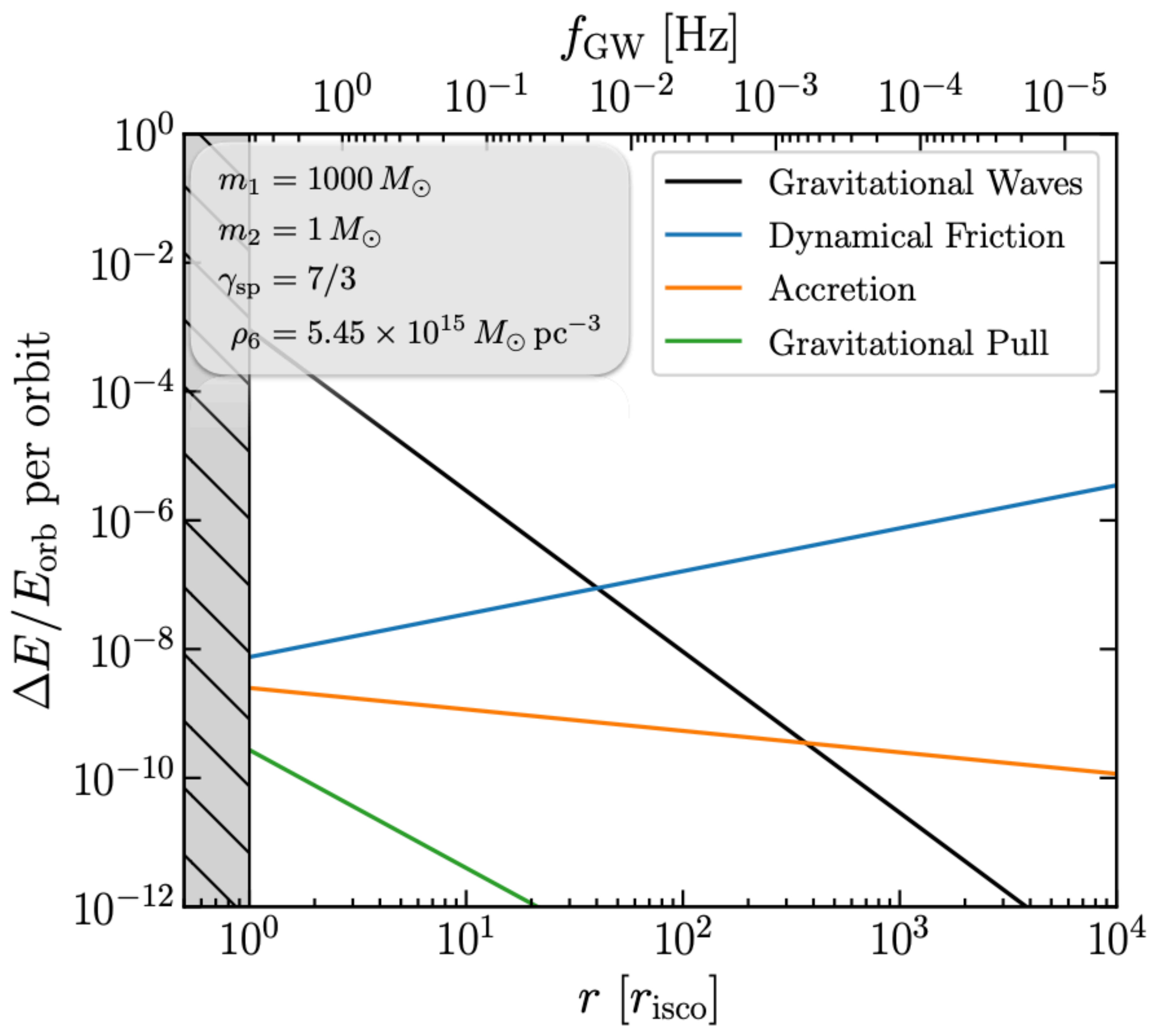
$$\dot{r} = -\dot{r}_{\rm GW} - \dot{r}_{\rm DF} - \dot{r}_{\rm acc}$$

$$E_{\rm orb} = \frac{m^2}{2}v^2 + \Phi_{\rm grav}\,, \quad \Phi_{\rm grav} = -\frac{M}{r} + \Phi_{\rm DM}(r)$$

$$\Omega_{\rm orb} = \sqrt{\Phi'_{\rm grav}(r)/r}$$

- Assuming density profile of the form $\rho_{\rm DM} \approx \rho_0\,(R/r)^\gamma$, impact on GW phase :

$x = (\pi f M)^{2/3}$



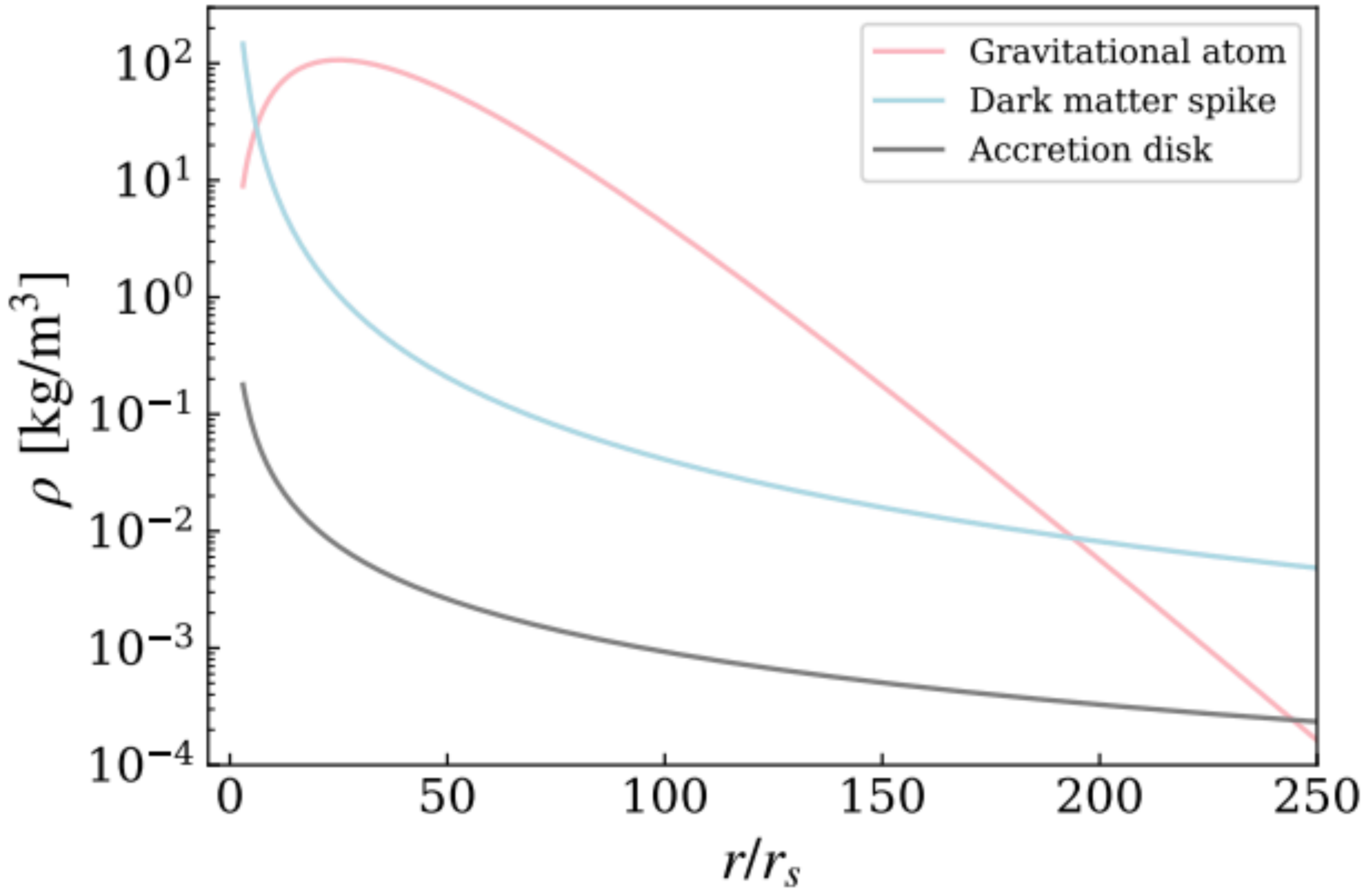
Credit: Bradley J. Kavanagh

$$\delta\Psi_{\rm DF} \propto -\rho_0 x^{\gamma-11/2} \qquad \delta\Psi_{\rm acc} \propto -\rho_0 x^{\gamma-9/2} \qquad \delta\Psi_{\rm grav\,pull} \propto \rho_0 x^{\gamma-3}$$

$(-5.5+\gamma)$ PN $\qquad (-4.5+\gamma)$ PN $\qquad (-3+\gamma)$ PN

Distinguishing black hole environments

From: P. Cole + '23, Nat. Astron. 7, 943-950



(Non-spinning) EMRI with
$M = 10^5 M_\odot$, $\mu = 10 M_\odot$, SNR = 15

Hints that different environments can be distinguished.

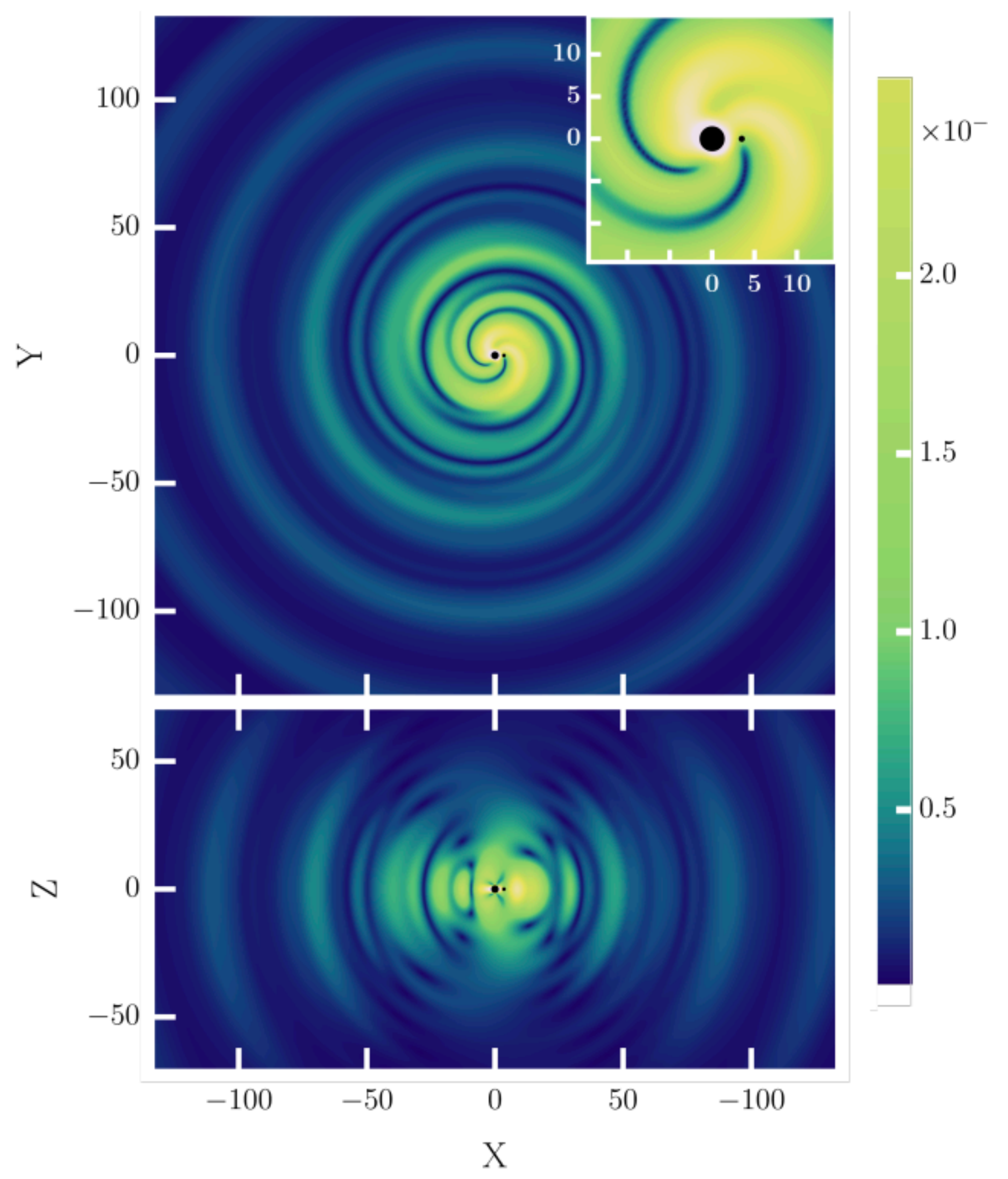
Caveats: study used "Newtonian" order waveforms, neglected spins, circular equatorial orbits...

	Dark dress signal	Accretion disk signal	Gravitational atom signal
Vacuum template	34	6	39
Dark dress template	-	3	39
Accretion disk template	17	-	33
Gravitational atom template	24	6	-

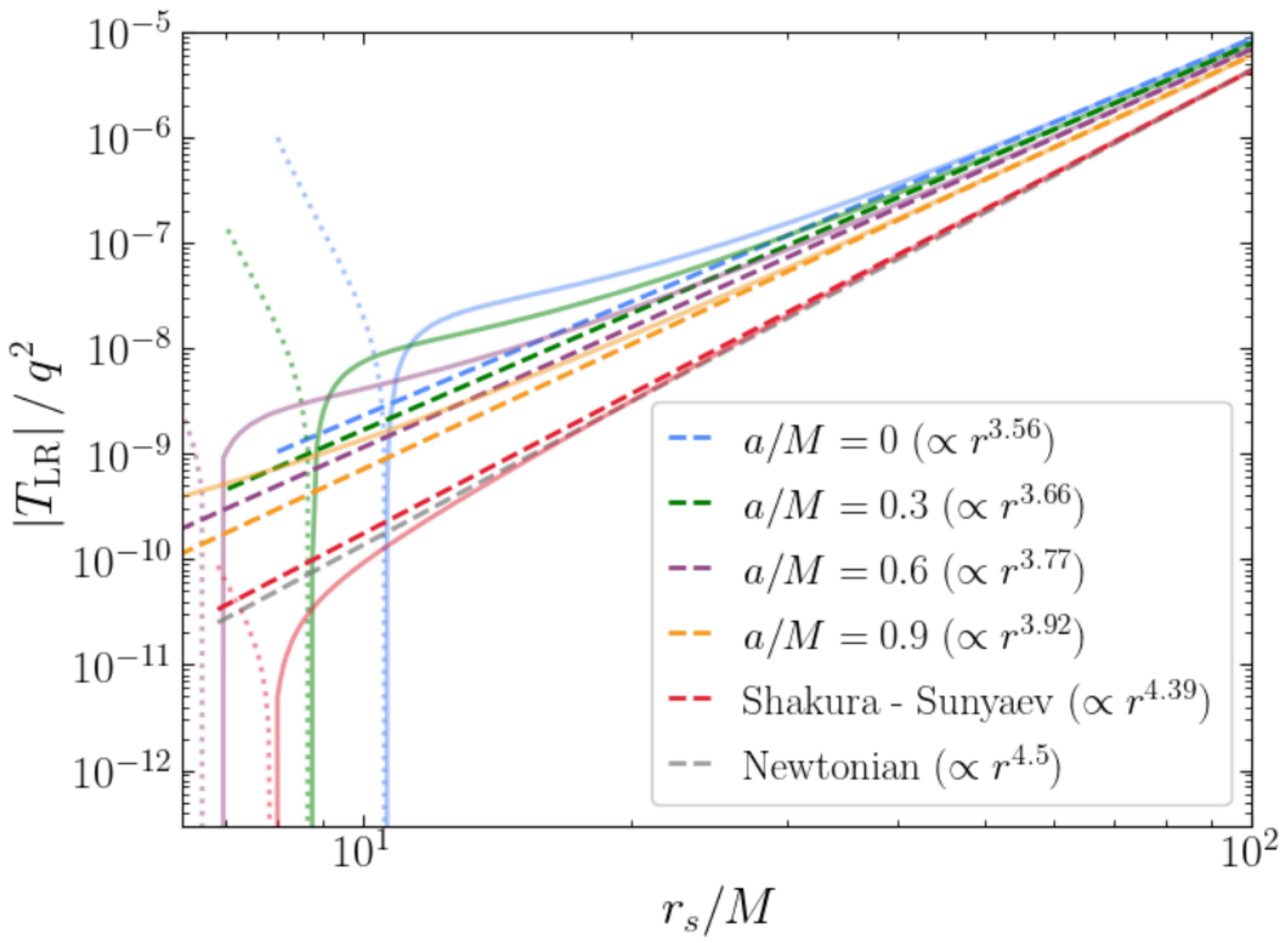
TABLE II. Logarithm of the Bayes factors, $\log_{10} \mathcal{B}$, comparing the evidence for the correct template that fits the signal, with an incorrect template.

Environmental effects: current effort

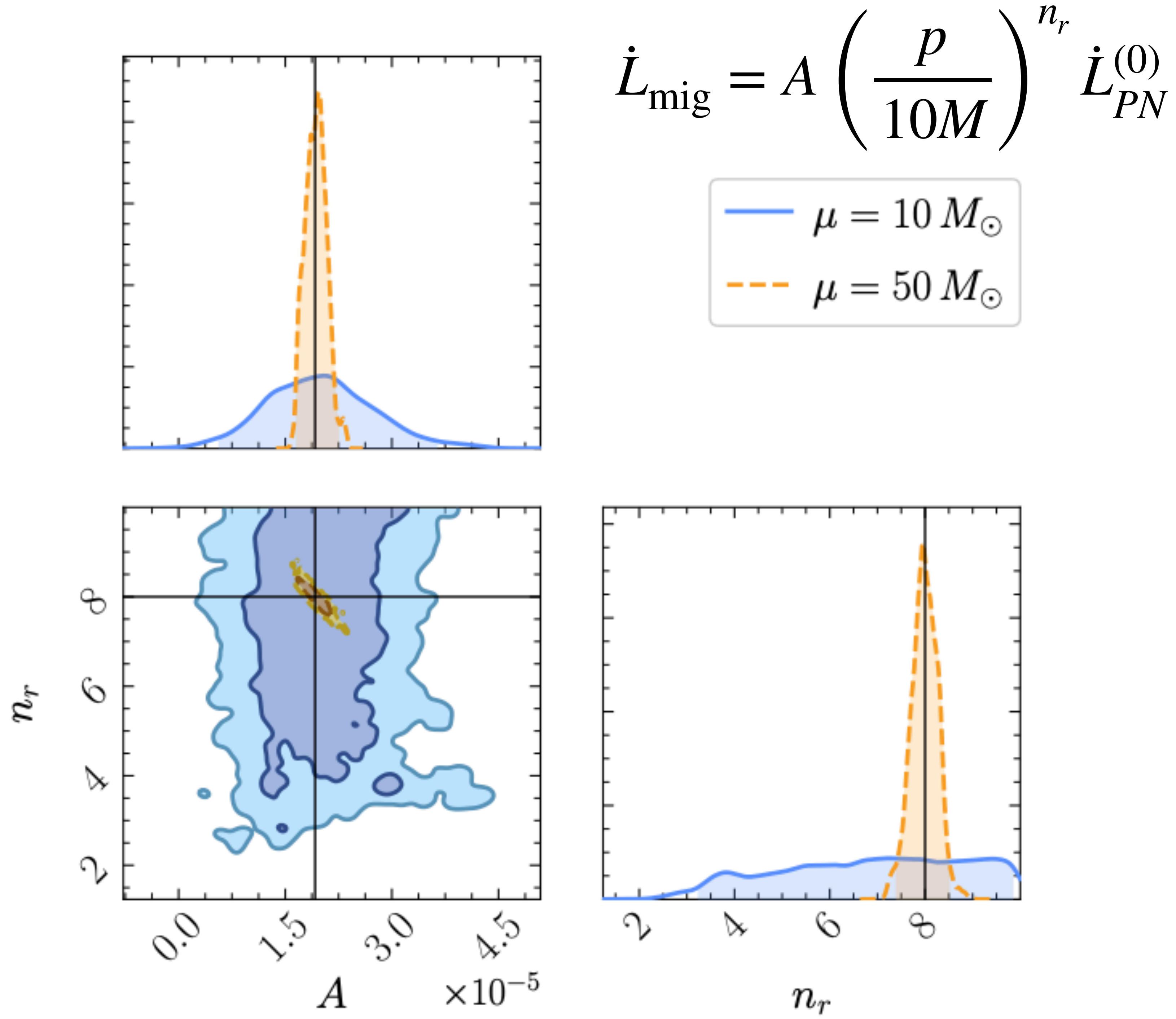
- Compute environmental effects in a **fully relativistic setup** using BH perturbation theory [Cardoso+ '22; RB & Shah '23; Duque+ '24; Conor+ '25; Li+ '25; Duque+ '25; Hegade+ '25; ...]
- **Challenges to be worked on:** How well do we need to model environmental effects? Degeneracies of environments with beyond GR effects? Beyond GR and environmental effects in the context of the global fit?



Dyson+, 2501.09806



Duque+, 2510.02433



Khalvati+, 2410.17310

The next decimal place

“But the history of science shews that even during that phase of her progress in which she devotes herself to ***improving the accuracy of the numerical measurement of quantities with which she has long been familiar***, she is preparing the materials for the ***subjugation of new regions, which would have remained unknown*** if she had contented with the rough guide of the earlier pioneers.” -

James Clerk Maxwell, Introductory Lecture on Experimental Physics, October 25th, 1871