LISA Instrumentation III: instrument modelling, TDI, L0-L1

LISA School for Early-career Scientists
11th of October 2025, Les Houches
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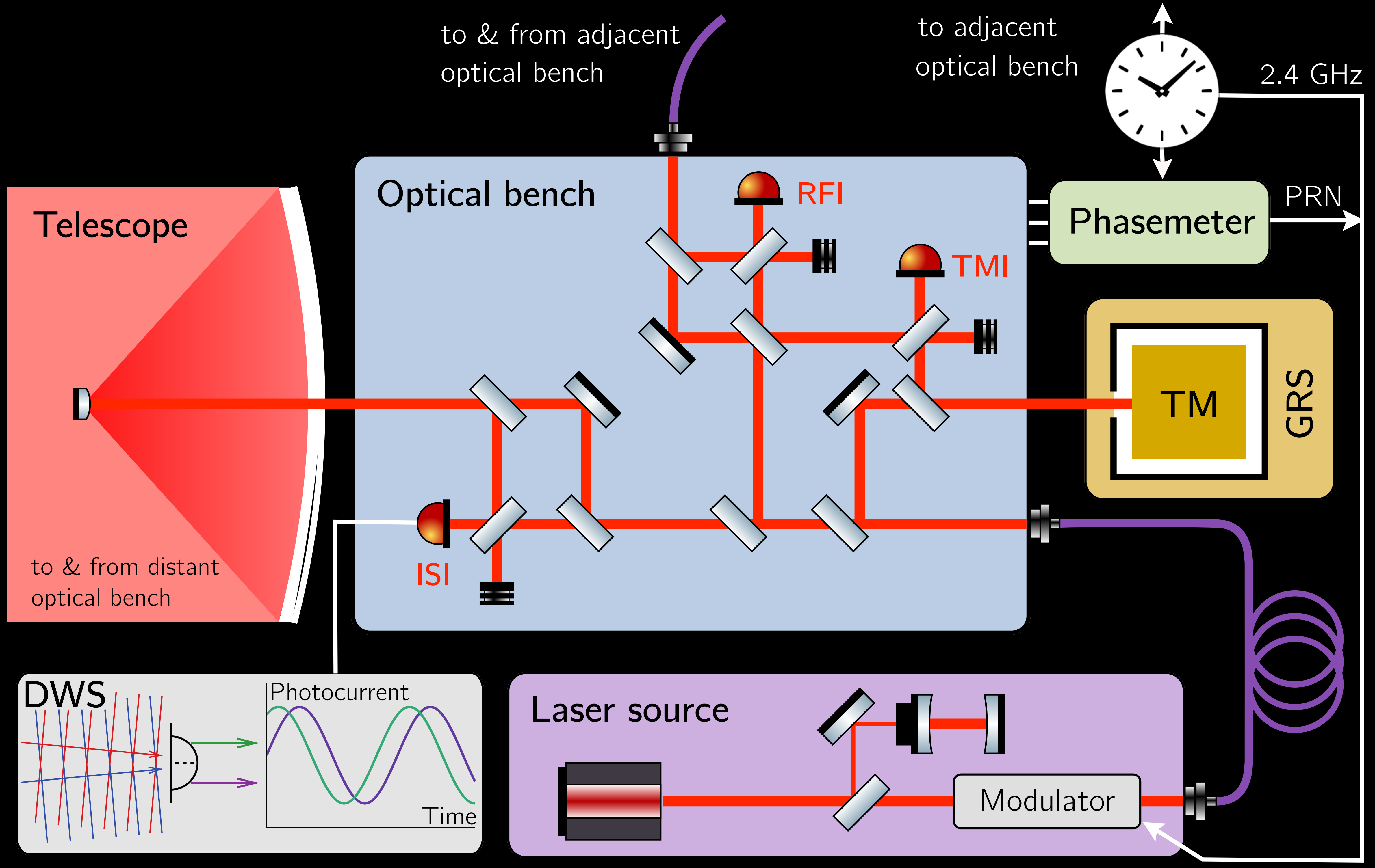
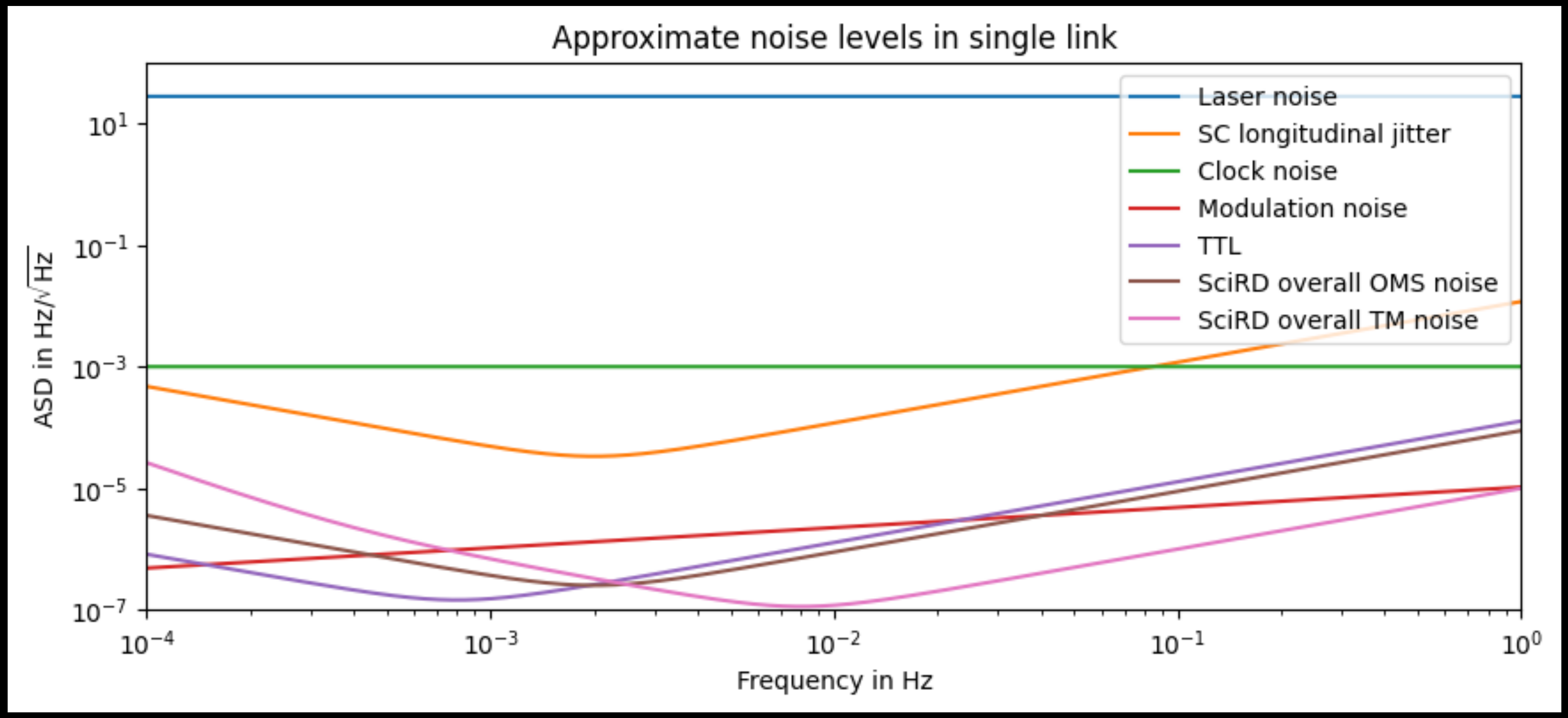
Summary on instrument

- LISA is not exactly ‘LIGO in Space’, but faces unique technical challenges
- LISA data pre-processing (‘L0.5-L1’):
  - Combines $\approx$ 66 main scientific interferometric measurements with ground tracking information and auxiliary sensors
  - Outputs:
    - 3 synchronized scientific variables, 2 ‘Michelson-like’, one ‘null-channel’
    - Other quantities needed for DA: spacecraft positions, time couples, light-travel times, noise estimates
- We will try to summarise how these processing steps work
- Not covered in this talk: countless technical details and engineering!

LISA measurements

- The main LISA measurements:
  - 3 main interferometring signals (SCI, RFI, TMI)
  - Auxilliary sideband beatnotes in the SCI for clock noise exchange
  - Auxilliary sideband beatnotes in the RFI for local frequency distribution correction
  - Absolute ranging via additional pseudo-random noise (PRN) code modulation (+ local codes)
  - Angular jitter correction via differential wavefront sensing (DWS)
- Dominant noise sources to be suppressed:
  - Laser noise
  - S/C longitudinal jitter
  - Clock noise in main phase measurements
  - Timing noise in clock distribution chain
  - and angular jitter (TTL)





 Note: Illustrative, numbers to be seen as placeholders

LISA Measurements: high-level model

Laser Beams

- Electromagnetic field of the laser beams in a fixed point, using the plane wave approximation: $E(\tau) = E_0(\tau)\cos(2\pi\Phi(\tau)) = Re\left[E_0(\tau)e^{i2\pi\Phi(\tau))}\right]$

- $\Phi(\tau)$ is rapidly evolving:

$$\dot{\Phi}(\tau) = \nu(\tau) = \nu_0 + \nu^o(\tau) + \nu^\epsilon(\tau)$$

total instantaneous frequency

central frequency
constant 281.6 THz

frequency offsets
Dopplers, programmed offsets, etc.
$\approx 10\,\mathrm{MHz}$

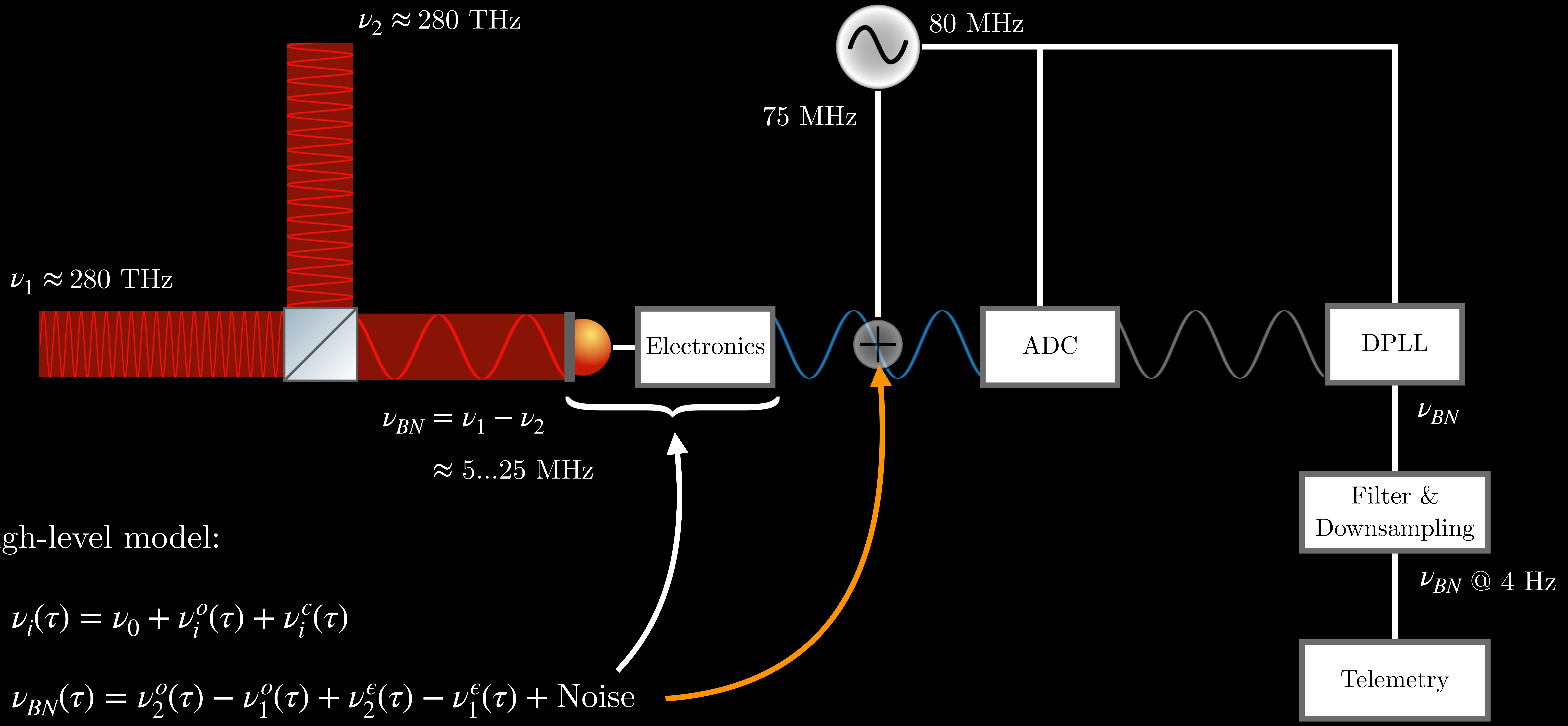
frequency fluctuations
noises, GW, etc.
$\approx 100\,\mathrm{Hz} + 100\,\mathrm{nHz}$

- Phase is the more fundamental quantity

- However: frequency usually easier to work with in practice, almost same information (modulo integration constant)

- Often useful to use complex field amplitude: $E(\tau) \equiv E_0(\tau)e^{i2\pi\Phi(\tau))}$

Measurement chain

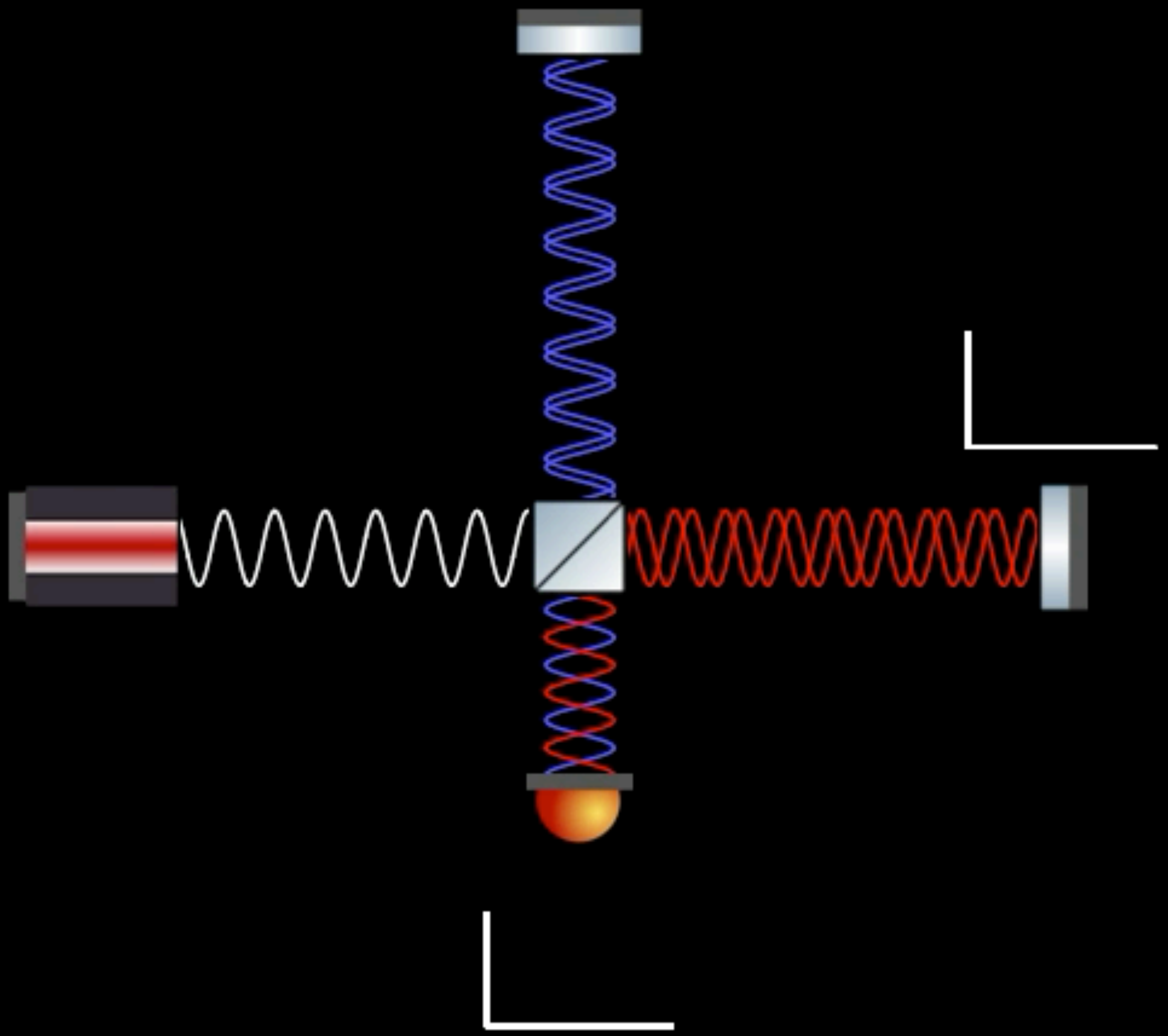


- High-level model:
  - $\nu_i(\tau) = \nu_0 + \nu_i^o(\tau) + \nu_i^\epsilon(\tau)$
  - $\nu_{BN}(\tau) = \nu_2^o(\tau) - \nu_1^o(\tau) + \nu_2^\epsilon(\tau) - \nu_1^\epsilon(\tau) + \text{Noise}$
  - First: focus on in-band laser fluctuations in $\nu_i^\epsilon(\tau)$

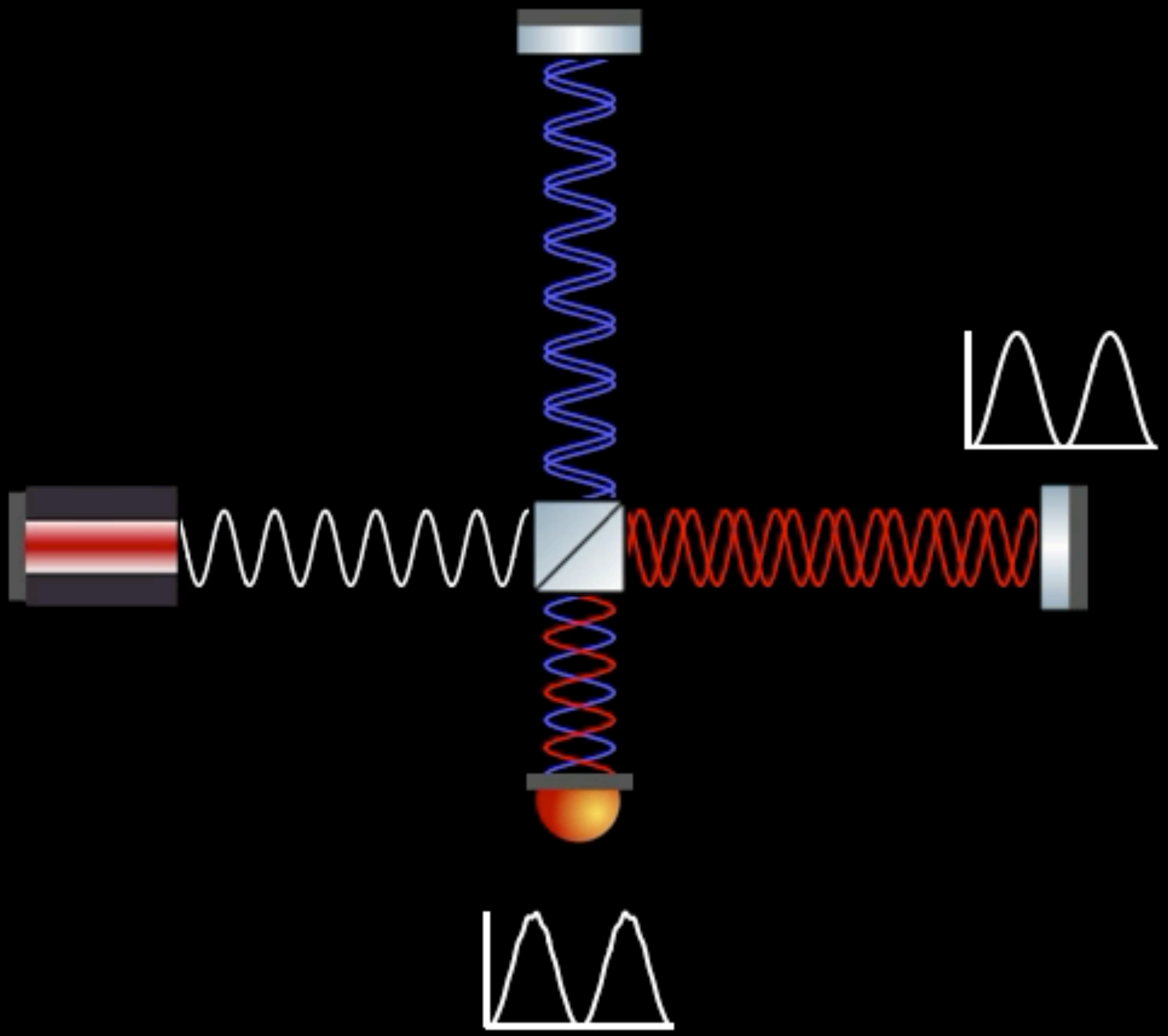
Note: Illustrative, numbers to be seen as placeholders

Laser noise cancellation in LISA

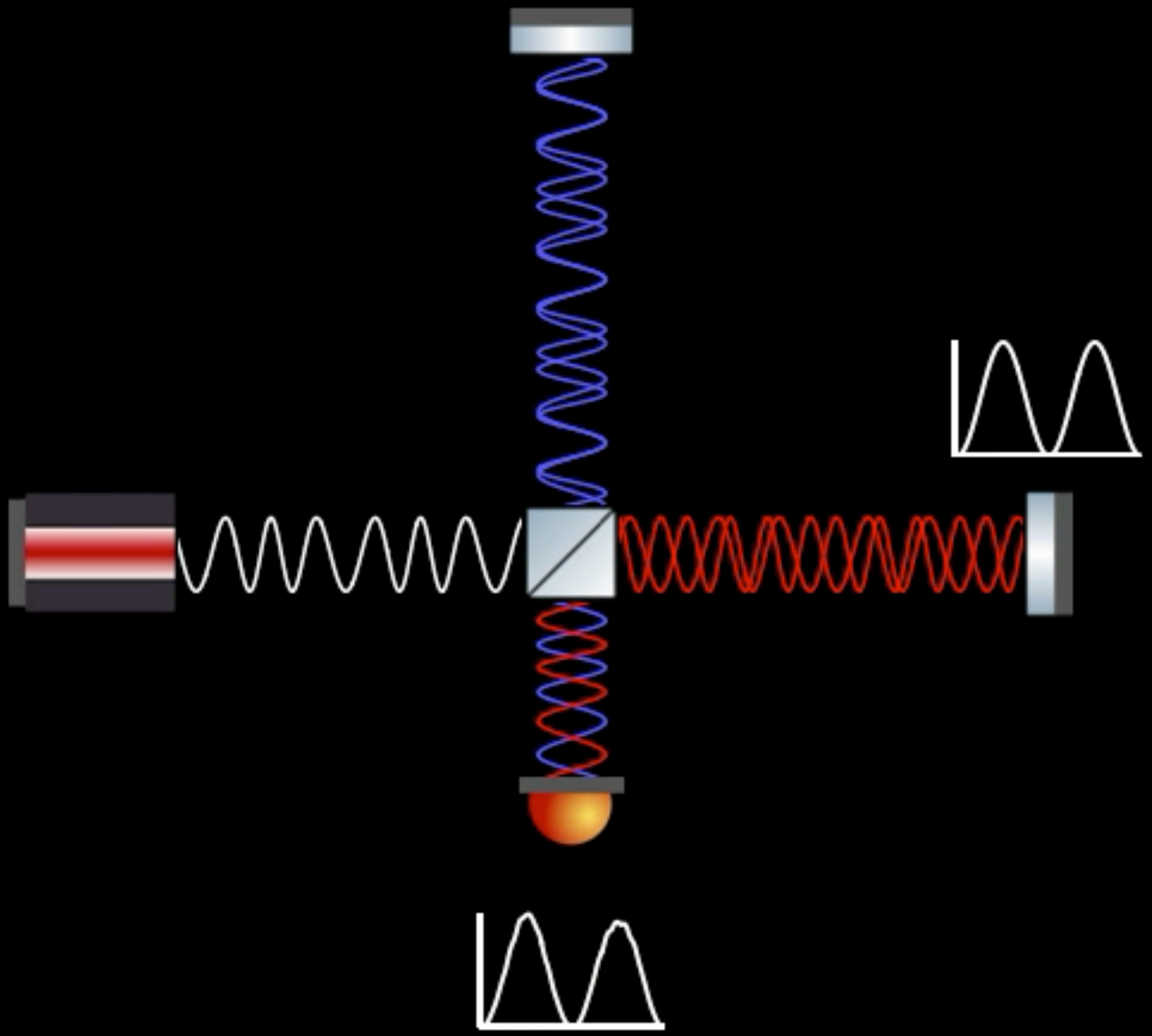
Laser noise cancellation in interferometers



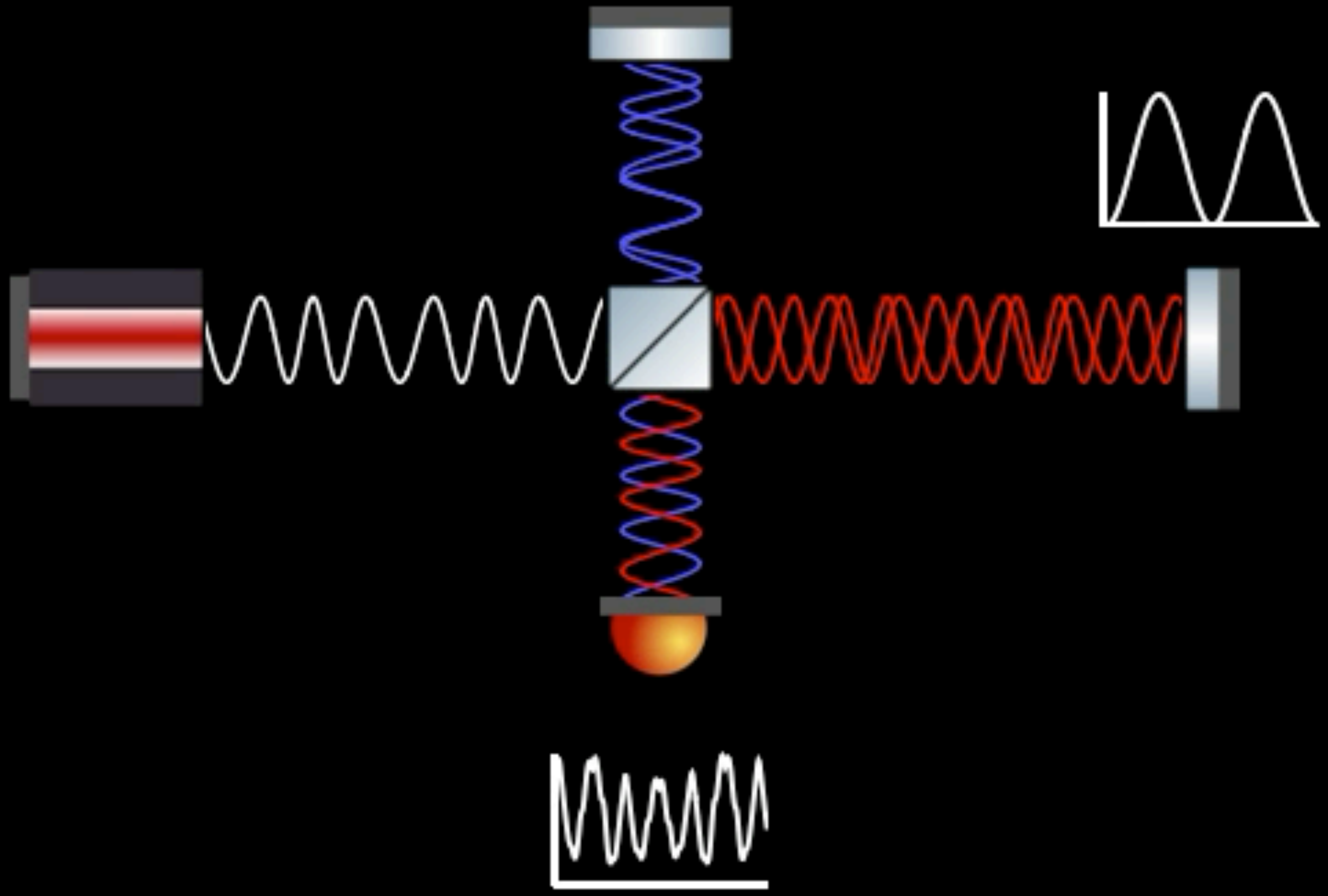
Laser noise cancellation in interferometers



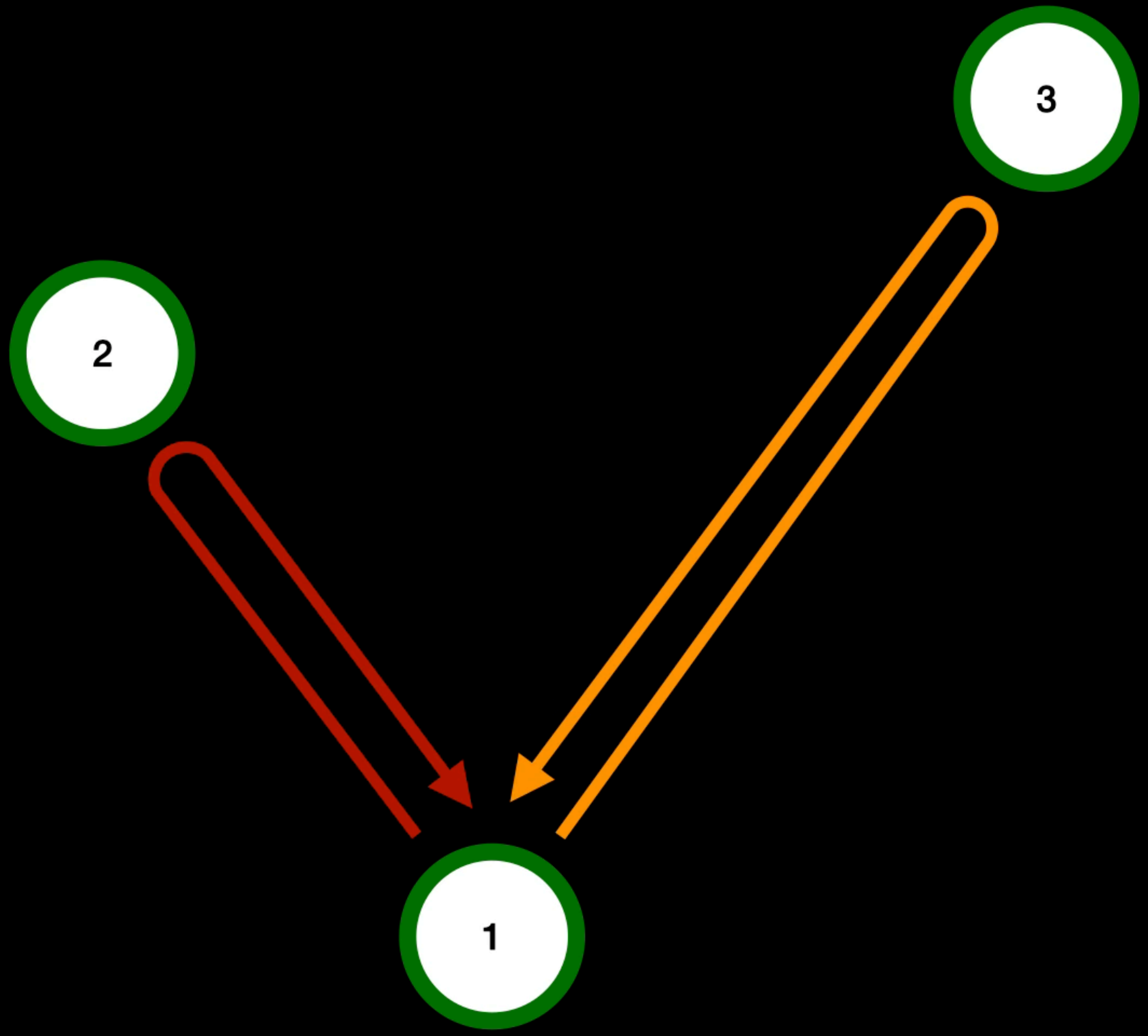
Laser noise cancellation in interferometers

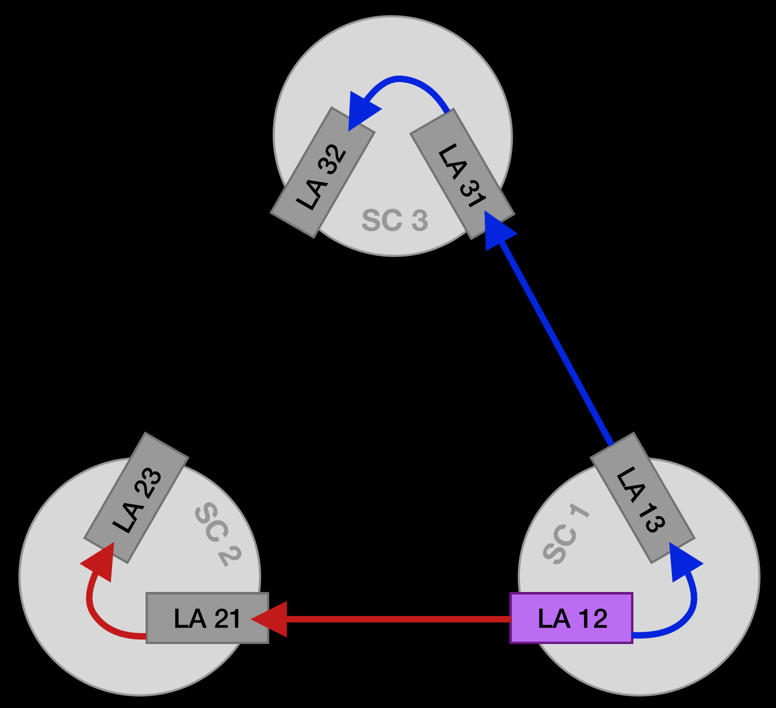


Laser noise cancellation in interferometers

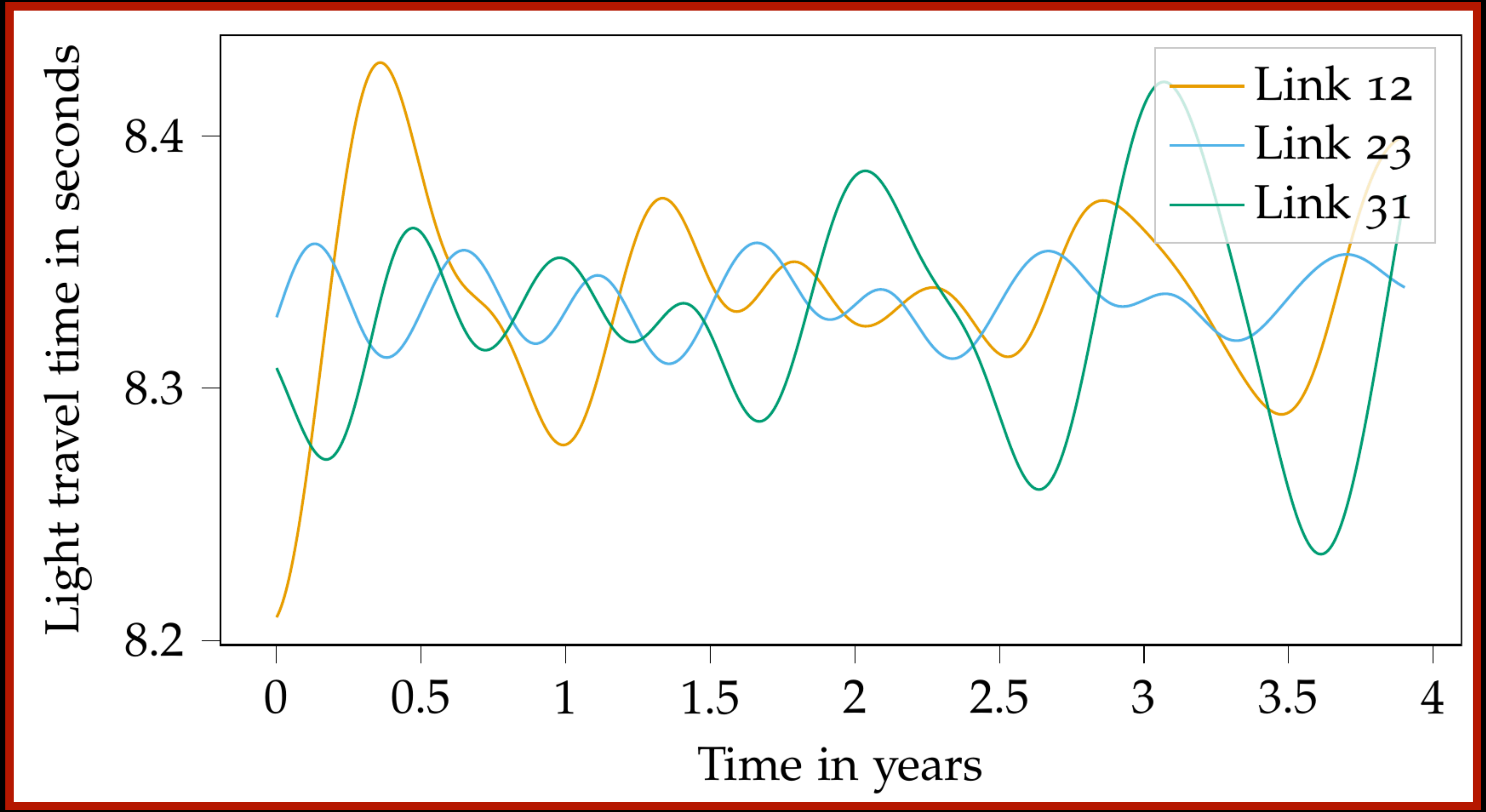


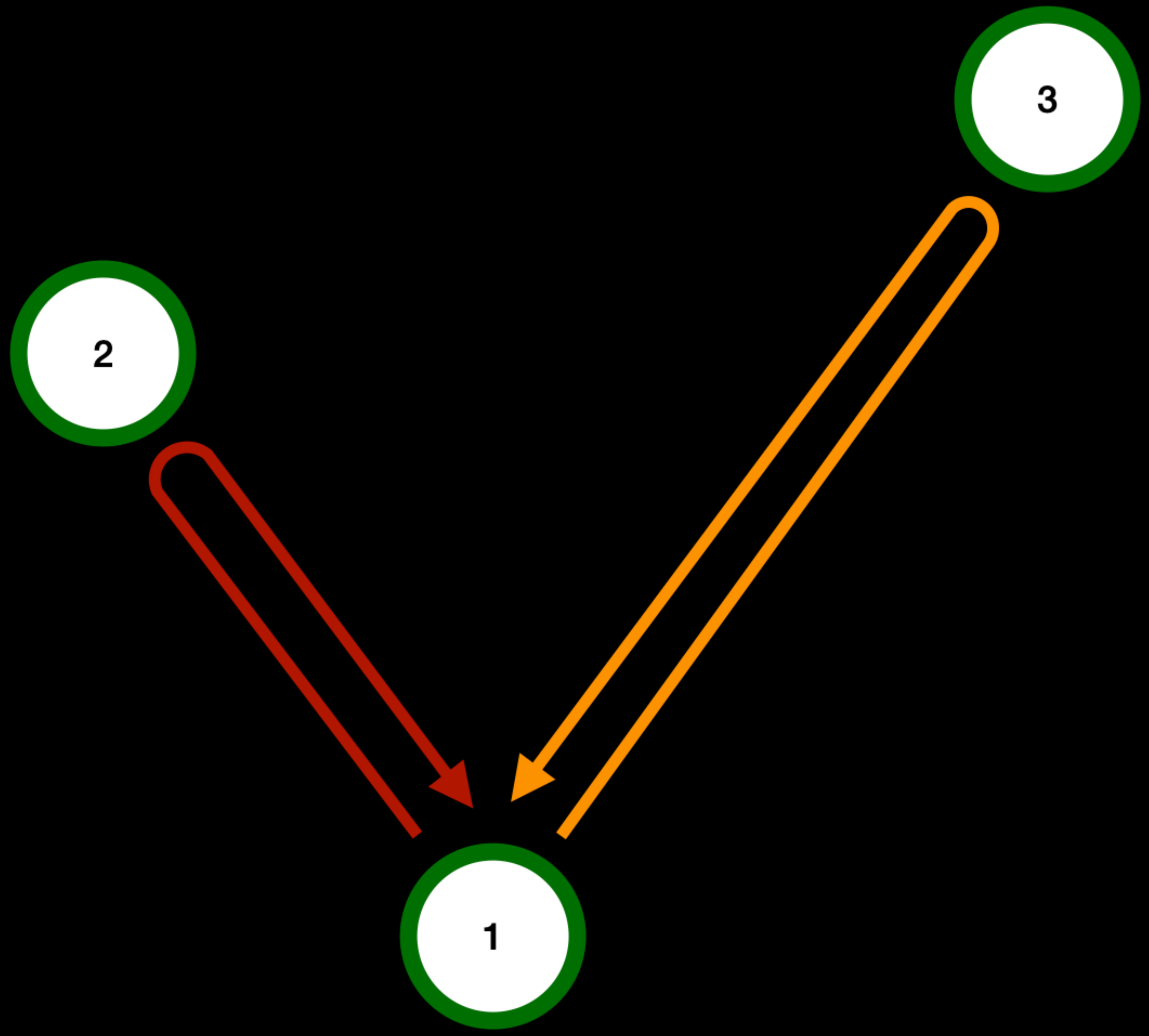
Laser noise cancellation in LISA





Laser noise cancellation in LISA



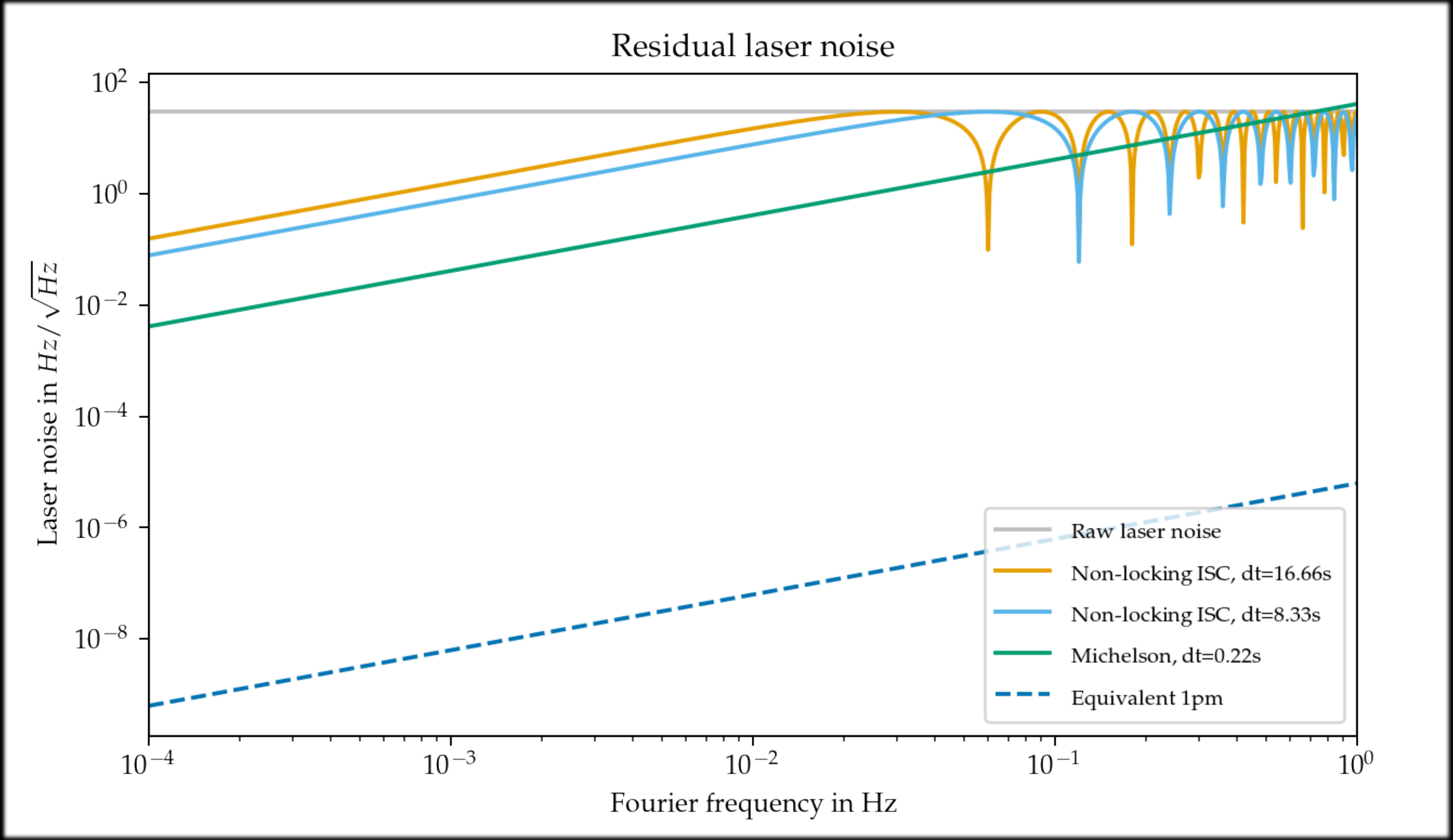


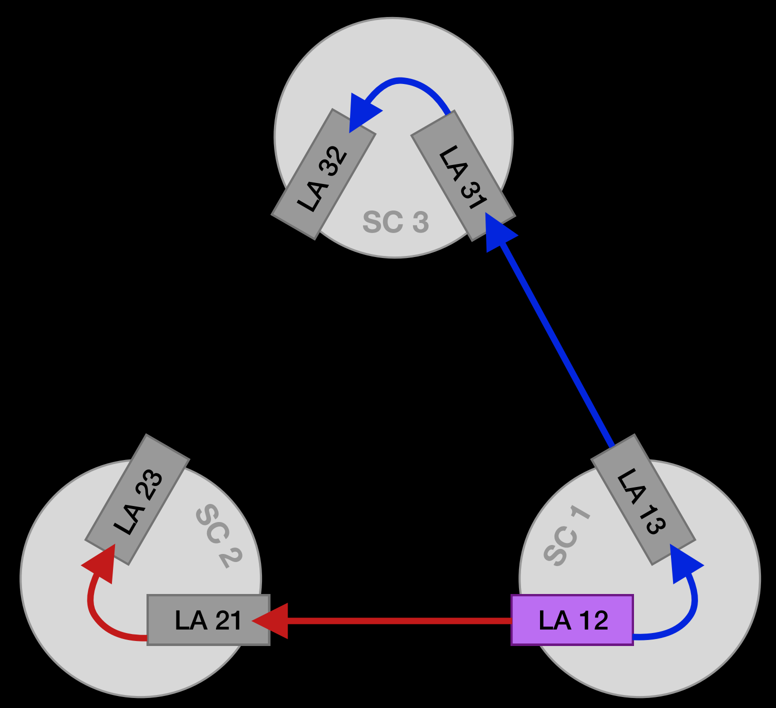
- Laser noise will enter as $\Phi(t - \delta t_1) - \Phi(t - \delta t_2)$, which in the frequency domain becomes (with $\delta t = \delta t_1 - \delta t_2$)

$$S_{\Phi,\mathrm{TDI}} = 4\sin(\pi f \delta t)^2\, S_\Phi \approx (2\pi f)^2\, \delta t^2\, S_\Phi$$

Residual laser noise in LISA

$$S_{\Phi,\mathrm{TDI}} = 4\sin(\pi f \delta t)^2\, S_\Phi \approx (2\pi f)^2\, \delta t^2\, S_\Phi$$

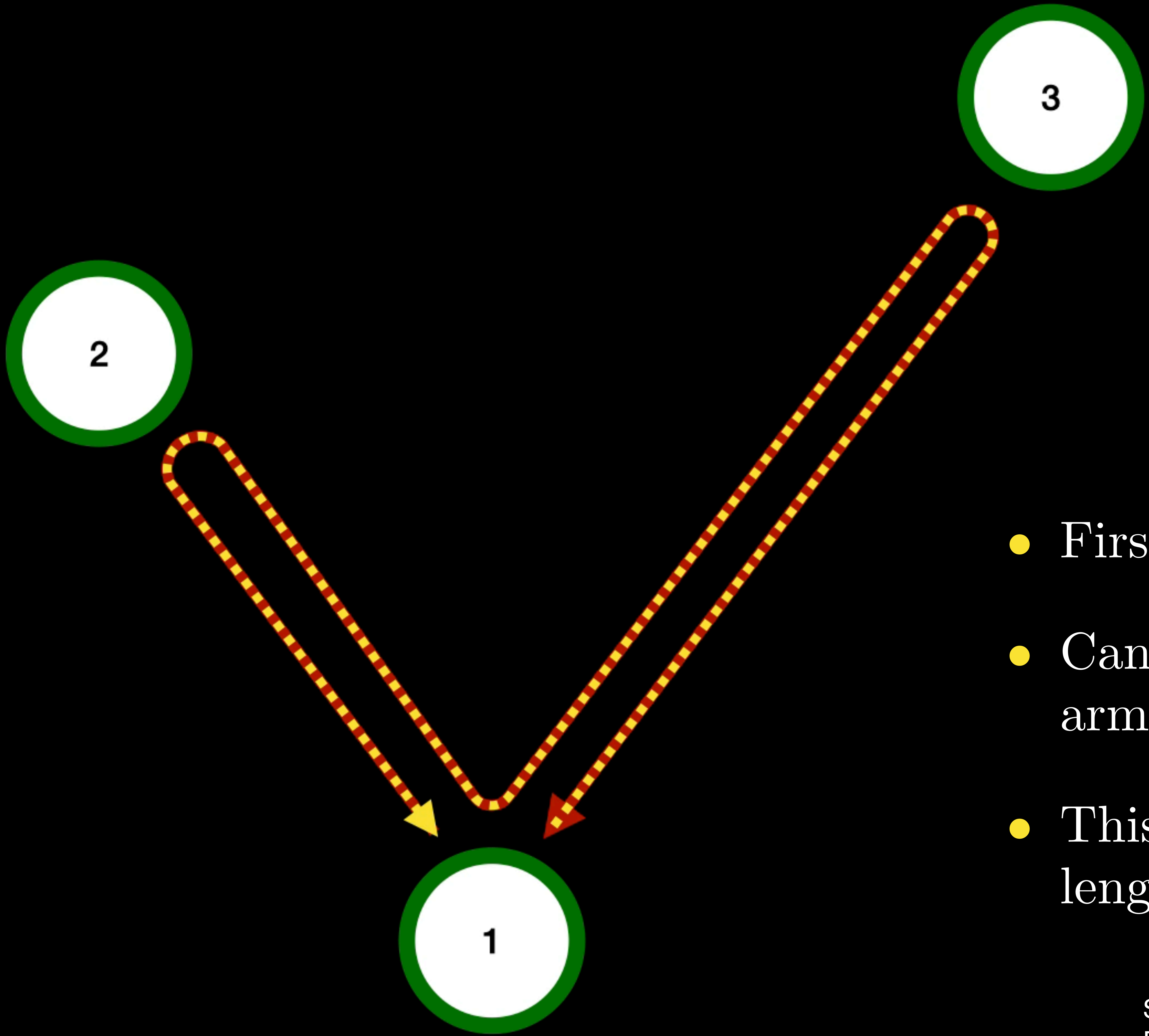




Note: Illustrative, neither laser noise nor actual requirement are white across the band. See https://ntrs.nasa.gov/citations/20240012428 for details on the laser

Time delay interferometry

Time-Delay Interferometry



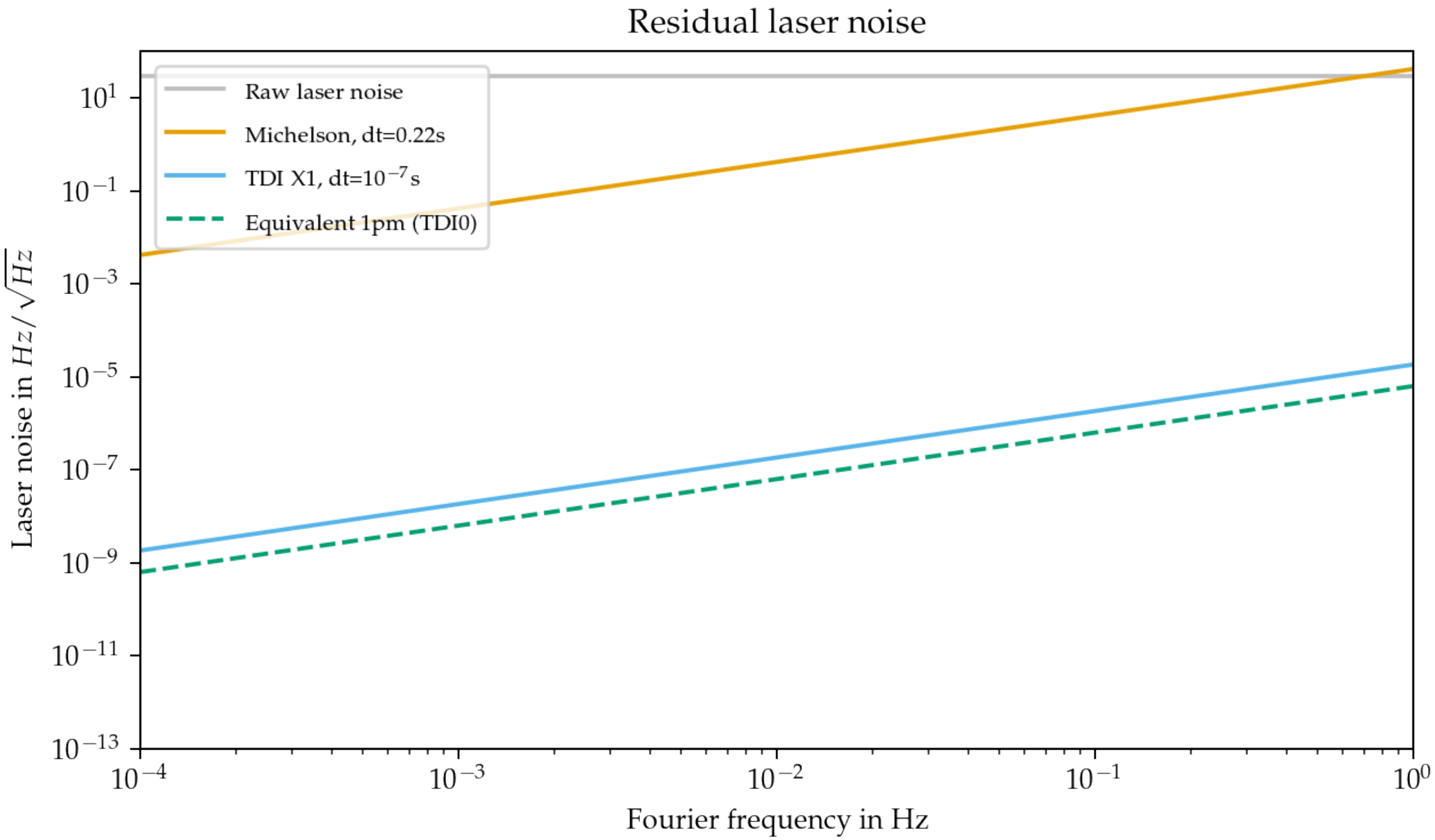
- First proposed in [Tinto et al., 1999]
- Cancel laser noise by constructing equal arm interferometer in post-processing
- This is an example for constant arm lengths (1st generation TDI)

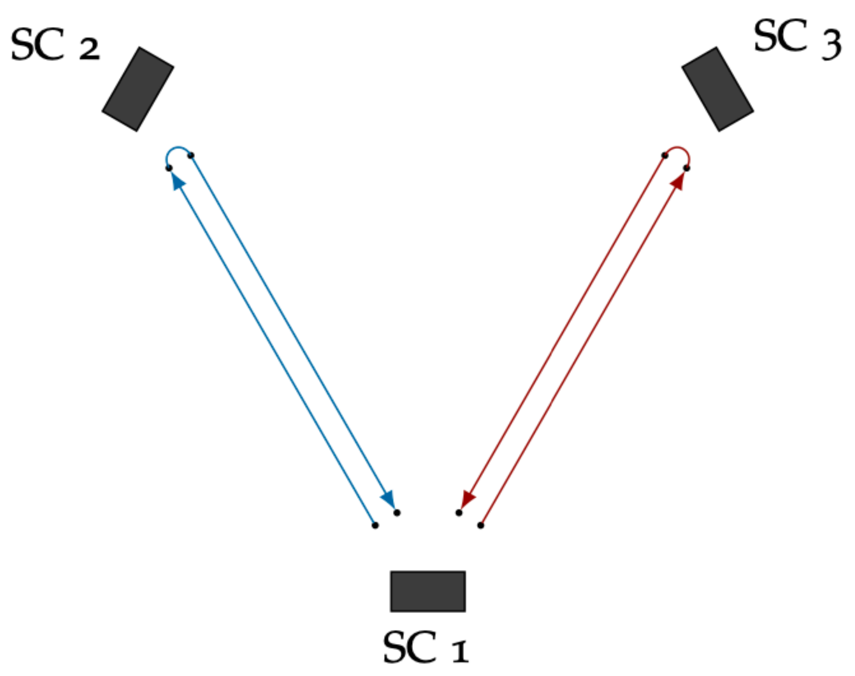
See living review [Tinto & Dhurandhar, 2020] for detailed references on TDI
First reference in history section (not quite TDI): [Faller and Bender, 1984]

Residual laser noise in LISA

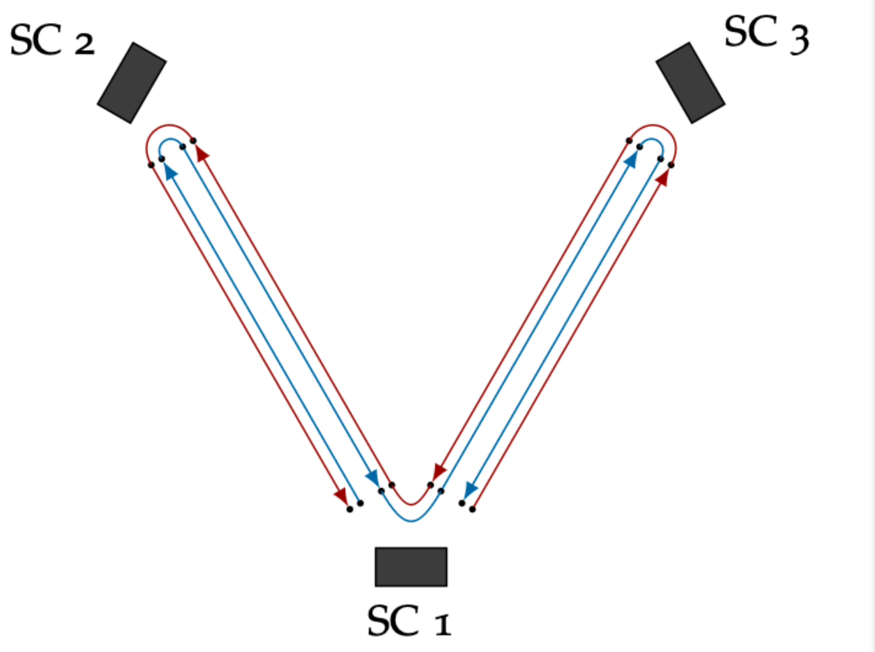
$$S_{\Phi,\mathrm{TDI}} \approx (2\pi f)^2\, \delta t^2\, S_\Phi$$

Average light travel times with ESA orbits: $d_A \approx 8.3\,\mathrm{s}$, $\dot{d}_A \approx 10^{-9}$ and $\ddot{d}_A \approx 10^{-15}\,\mathrm{s}^{-1}$





$\Delta\tau_0 \approx 2(d_{12} - d_{13})$

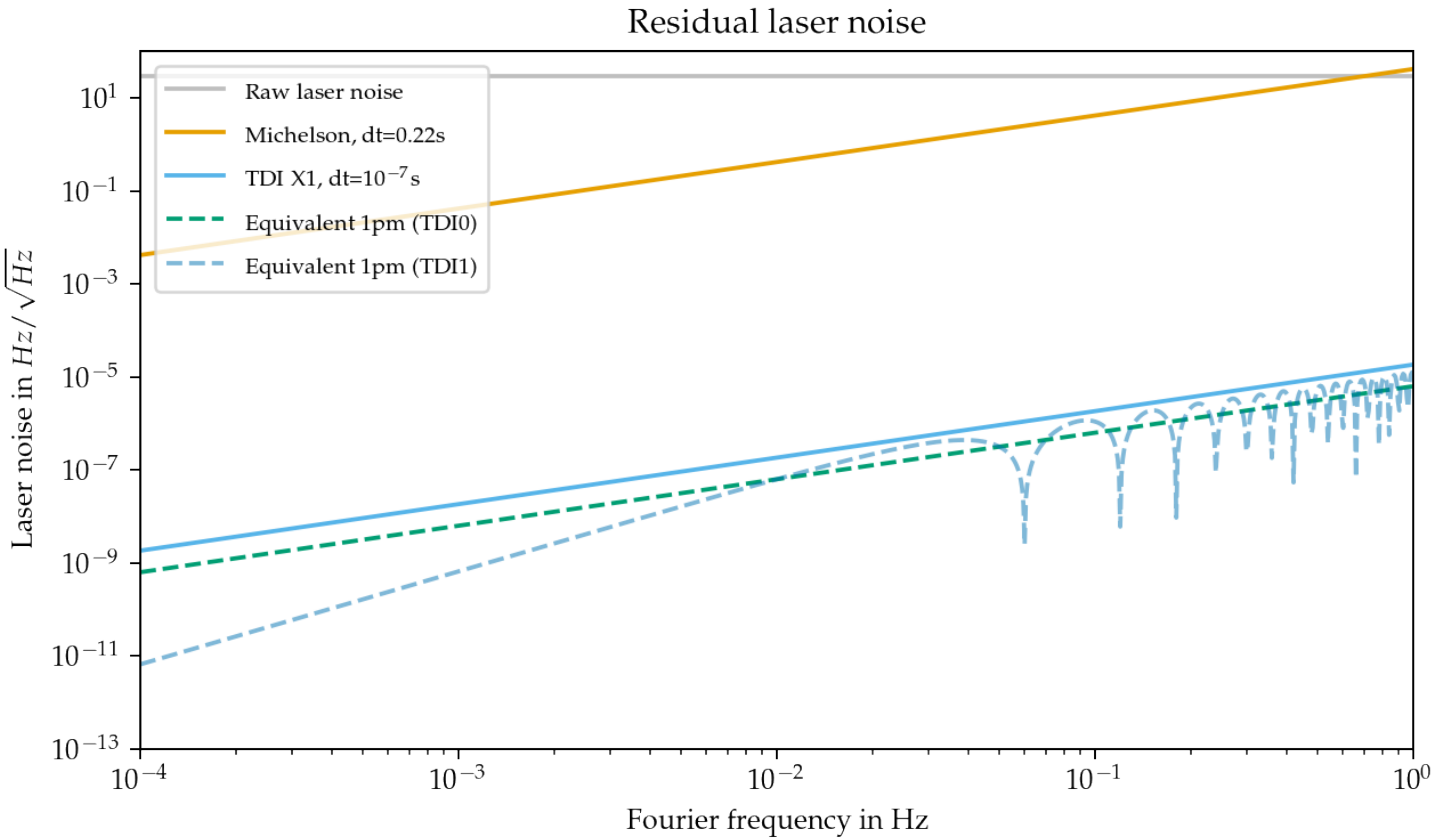


$\Delta\tau_1 \approx 4d(\dot{d}_{31} - \dot{d}_{12})$

Note: Illustrative, neither laser noise nor actual requirement are white across the band

Residual laser noise in LISA

$$S_{\Phi,\mathrm{TDI}} \approx (2\pi f)^2\, \delta t^2\, S_{\Phi}$$



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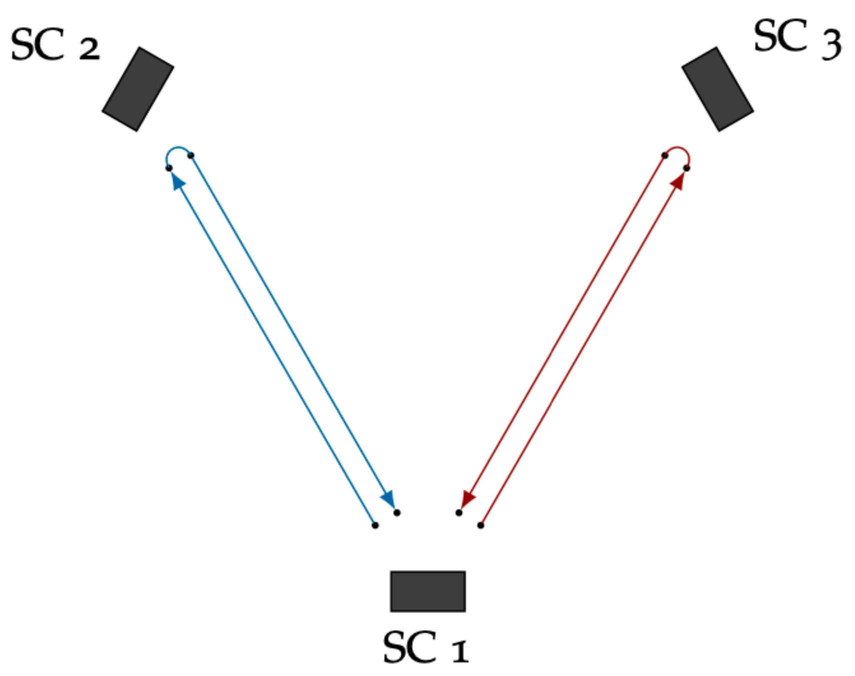
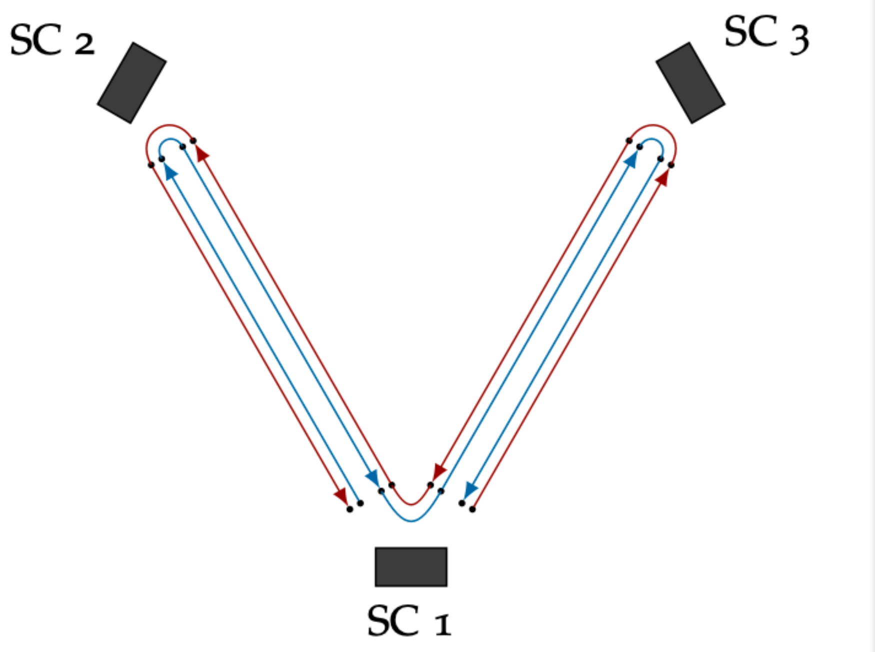


$\Delta\tau_0 \approx 2(d_{12} - d_{13})$



$\Delta\tau_1 \approx 4d(\dot{d}_{31} - \dot{d}_{12})$

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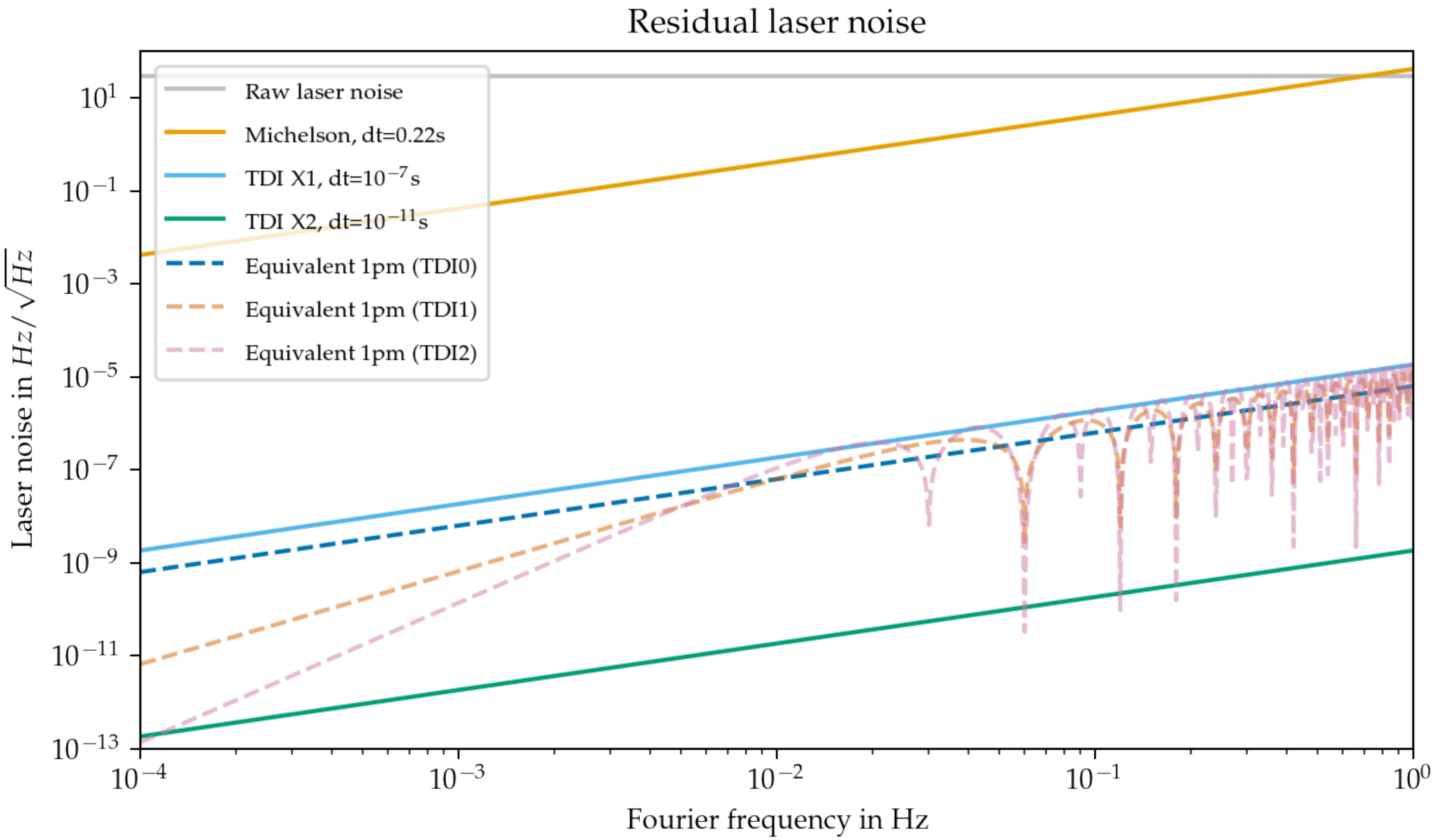
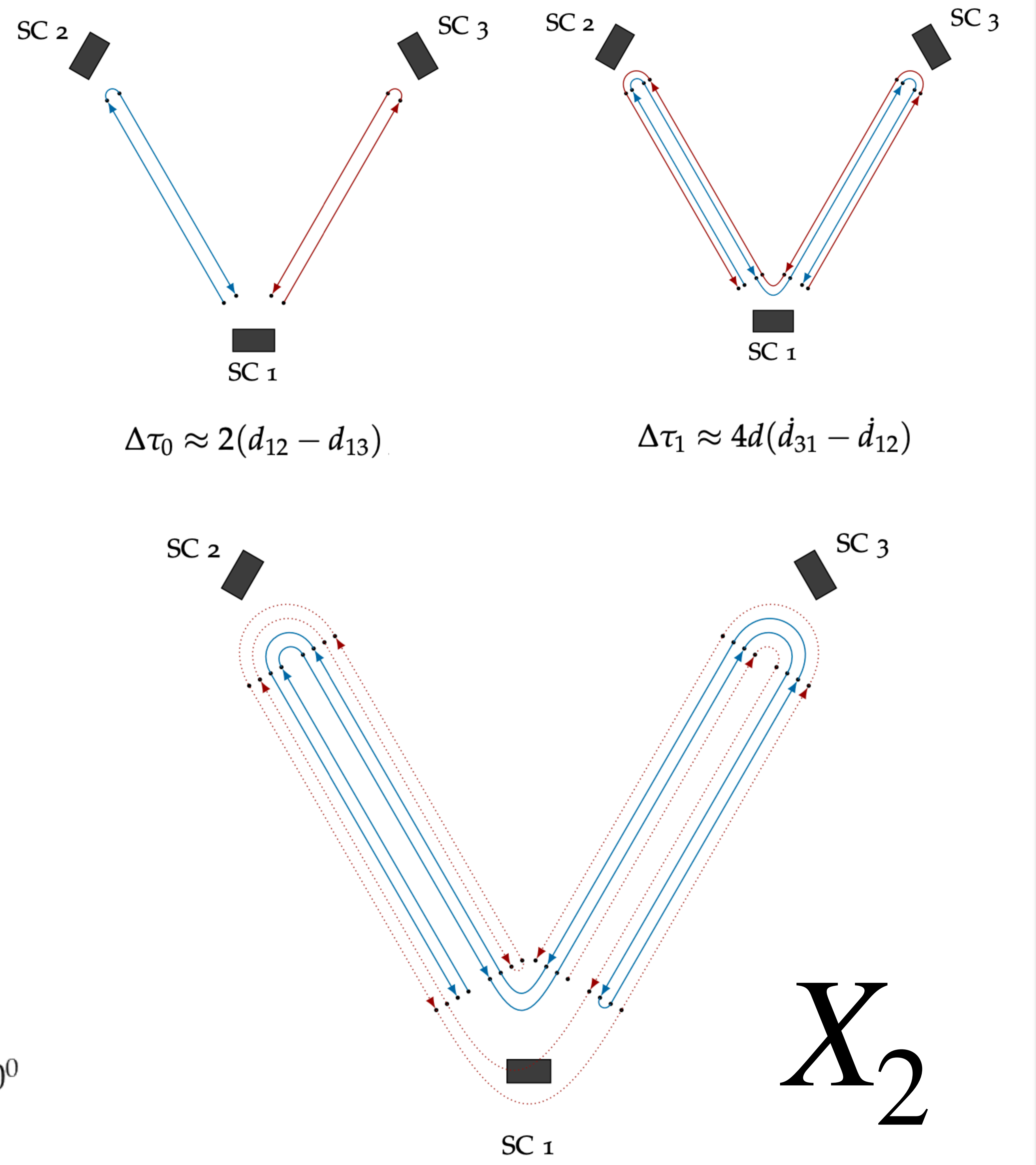
Residual laser noise in LISA

$S_{\Phi,\mathrm{TDI}} \approx (2\pi f)^2\, \delta t^2\, S_{\Phi}$

Average light travel times with ESA orbits: $d_A \approx 8.3\,\mathrm{s}$, $\dot{d}_A \approx 10^{-9}$ and $\ddot{d}_A \approx 10^{-15}\,\mathrm{s}^{-1}$





$\Delta\tau_0 \approx 2(d_{12} - d_{13})$

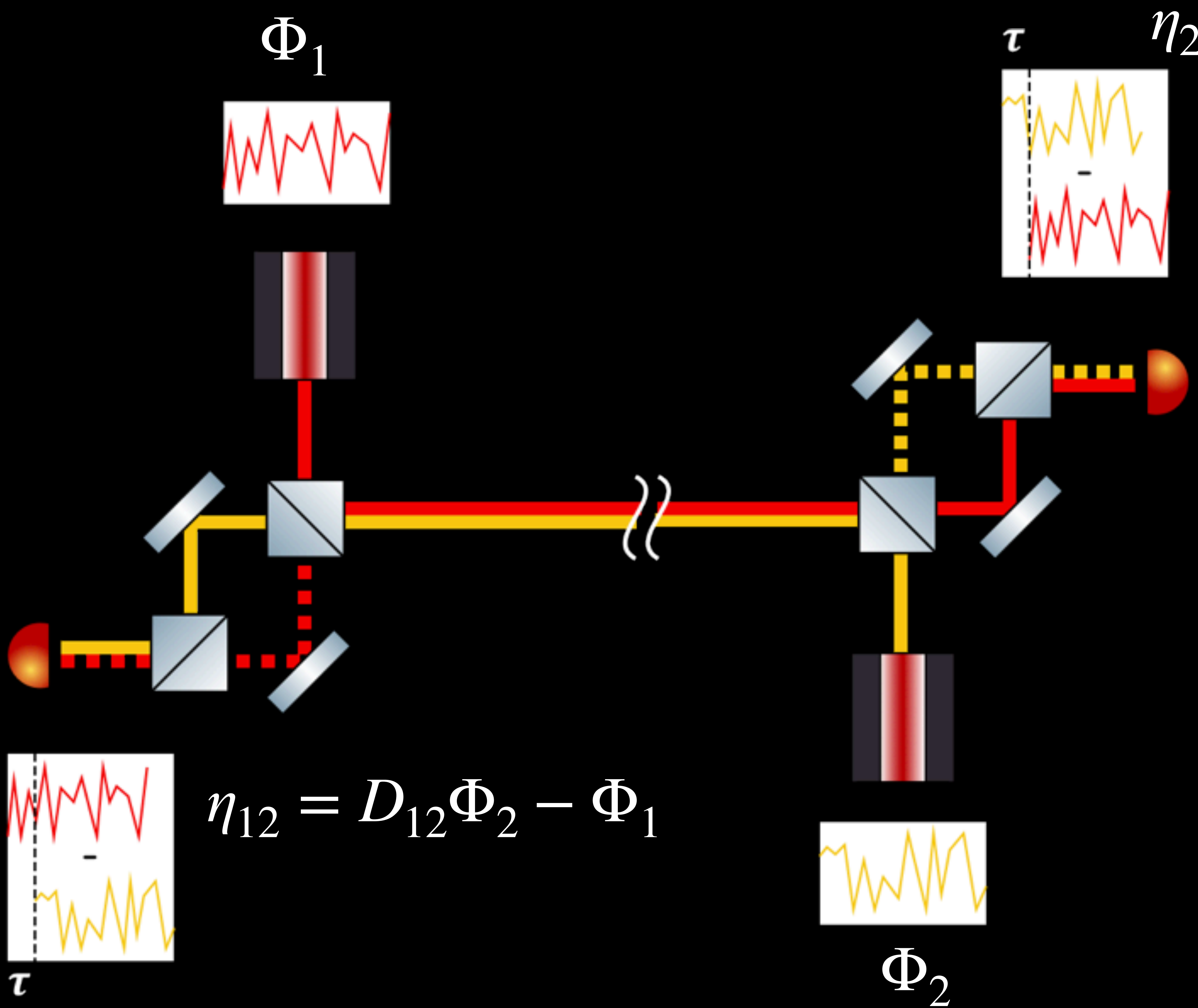
$\Delta\tau_1 \approx 4d(\dot{d}_{31} - \dot{d}_{12})$

$\Delta\tau_2 \approx 8d\left[\left(\dot{d}_{12}^2 - \dot{d}_{31}^2\right) - 2d\left(\ddot{d}_{12} - \ddot{d}_{31}\right)\right]$

Note: Illustrative, neither laser noise nor actual requirement are white across the band

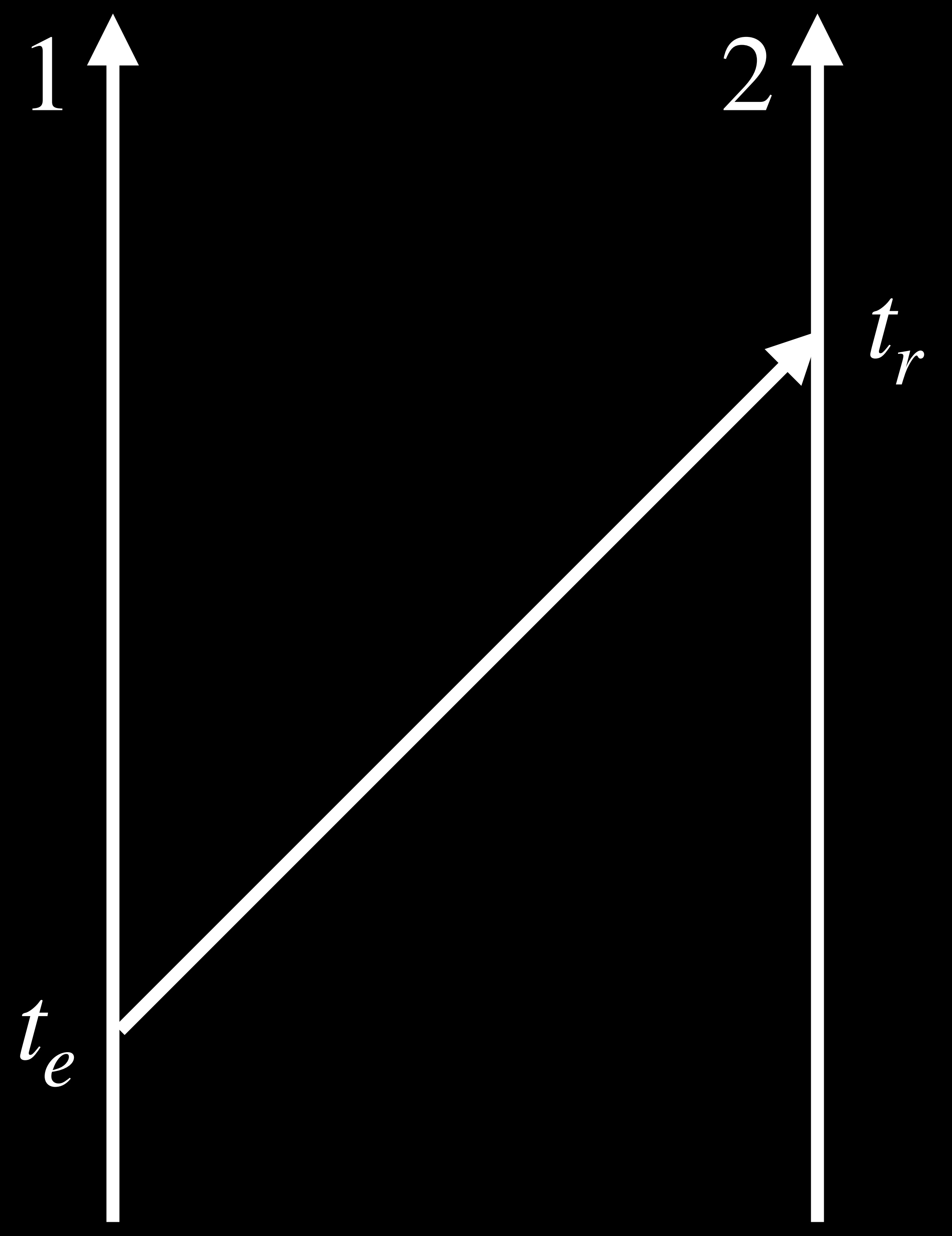
How does TDI work, in practice?

TDI toy model



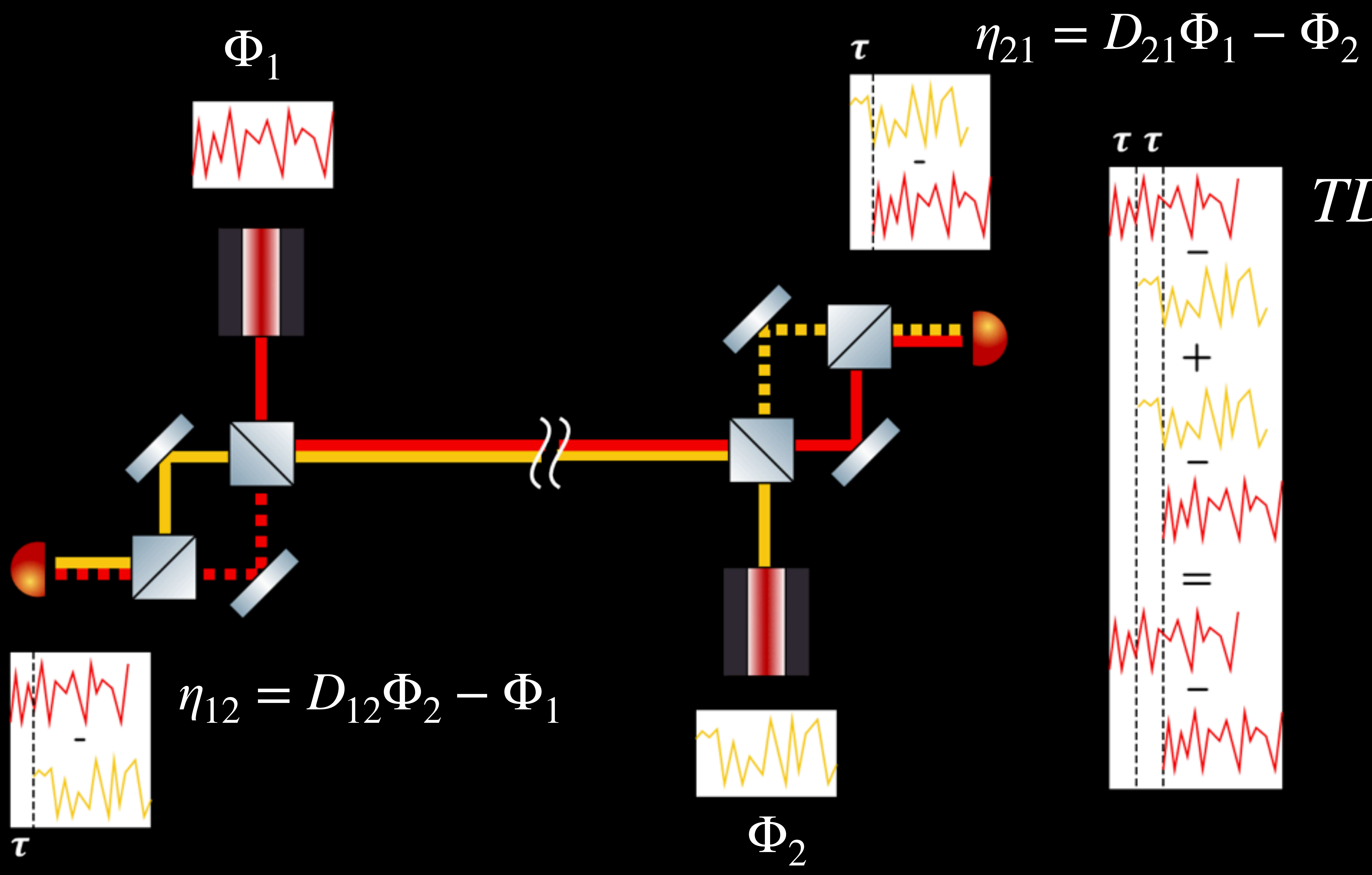
$$\eta_{21} = D_{21}\Phi_1 - \Phi_2$$

$$\eta_{12} = D_{12}\Phi_2 - \Phi_1$$



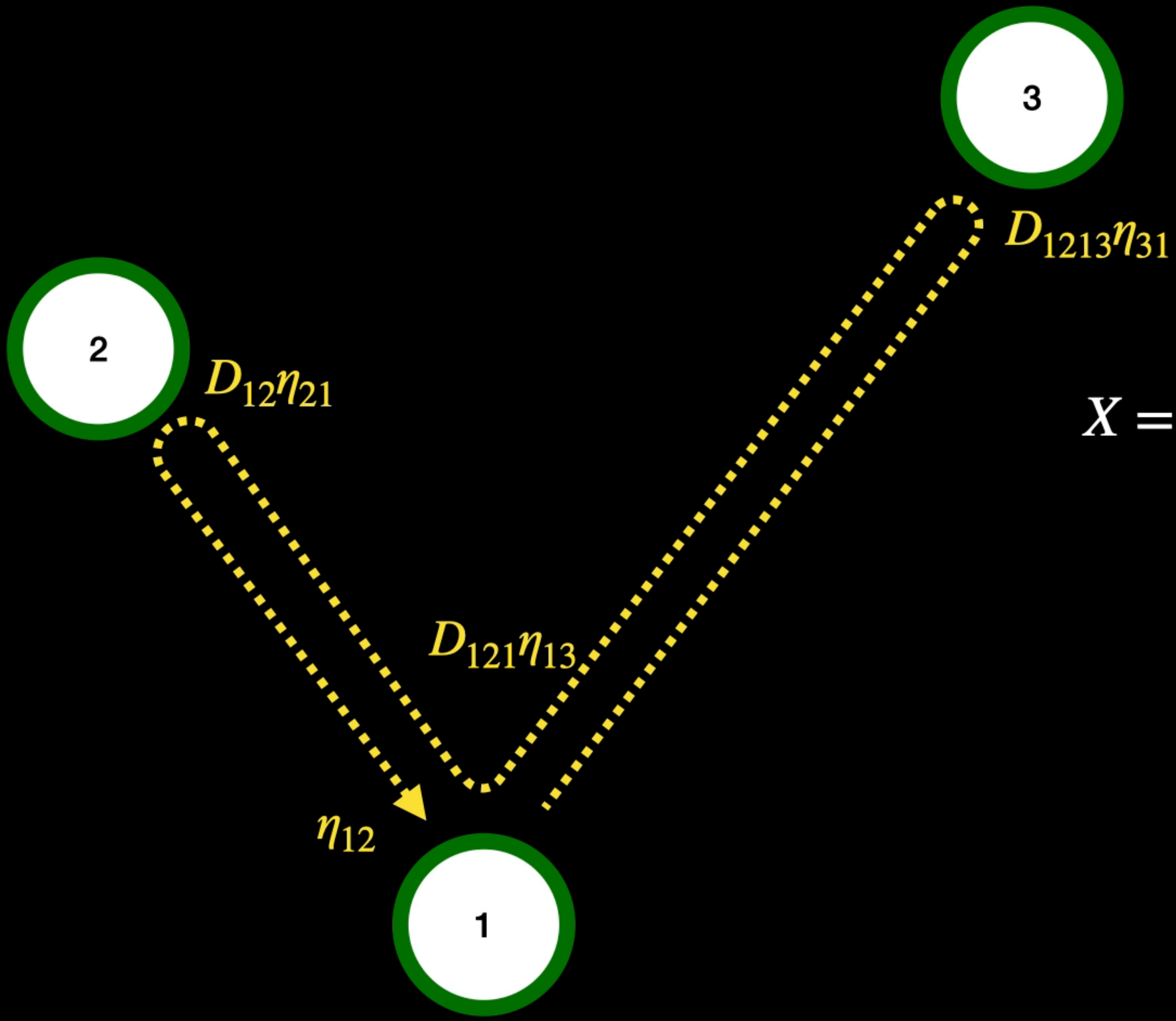
$$\Phi_1(t_e) \equiv D_{21}\Phi_1(t_r)$$
$$= \Phi_1(t_r - d_{21}(t_r))$$

TDI toy model



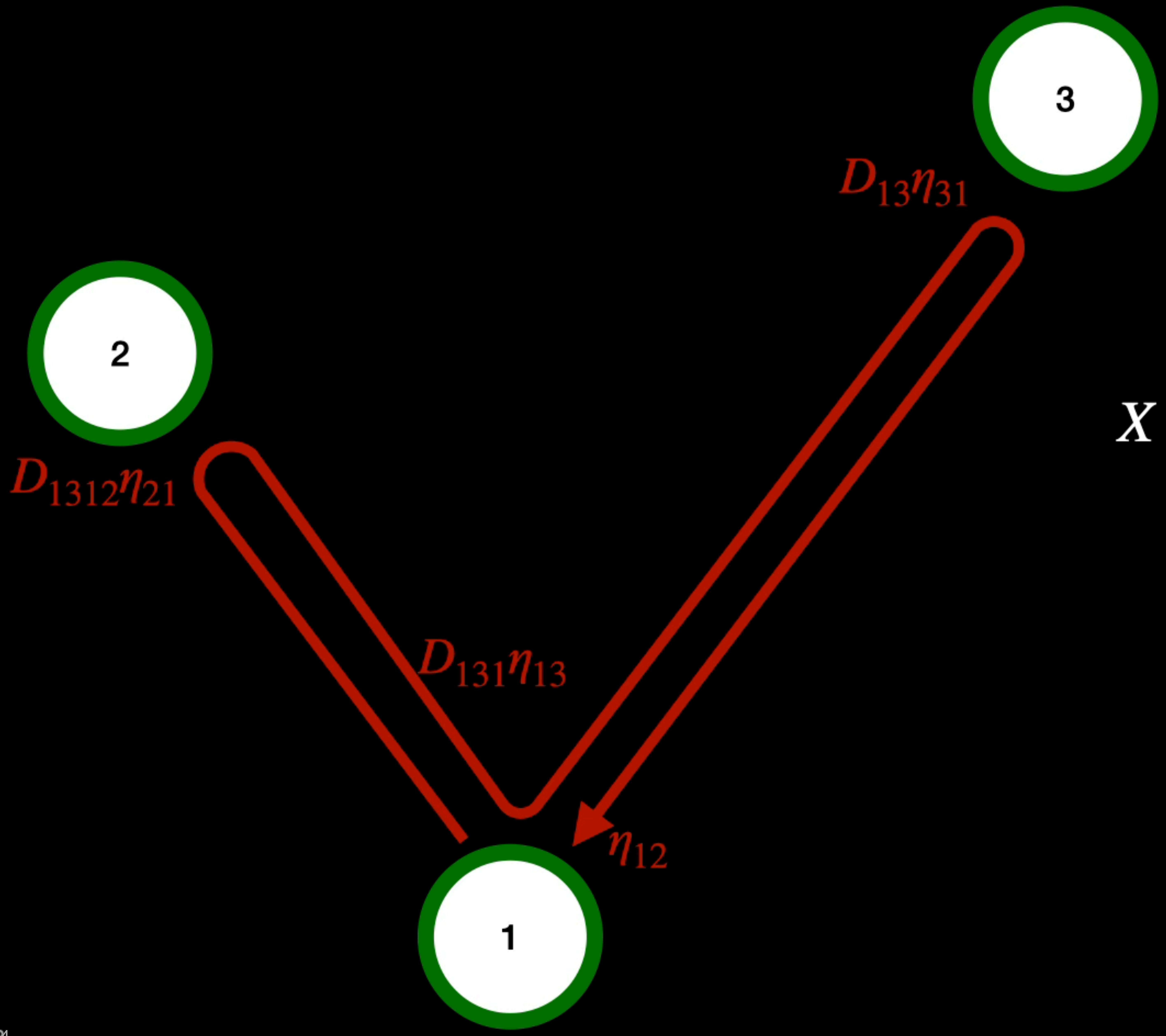
$$TDI = \eta_{12} + \mathbf{D}_{12}\eta_{21}$$
$$= \mathbf{D}_{12}\Phi_2 - \Phi_1$$
$$+\mathbf{D}_{12}(\mathbf{D}_{21}\Phi_1 - \Phi_2)$$
$$= \mathbf{D}_{121}\Phi_1 - \Phi_1$$

Full first generation TDI



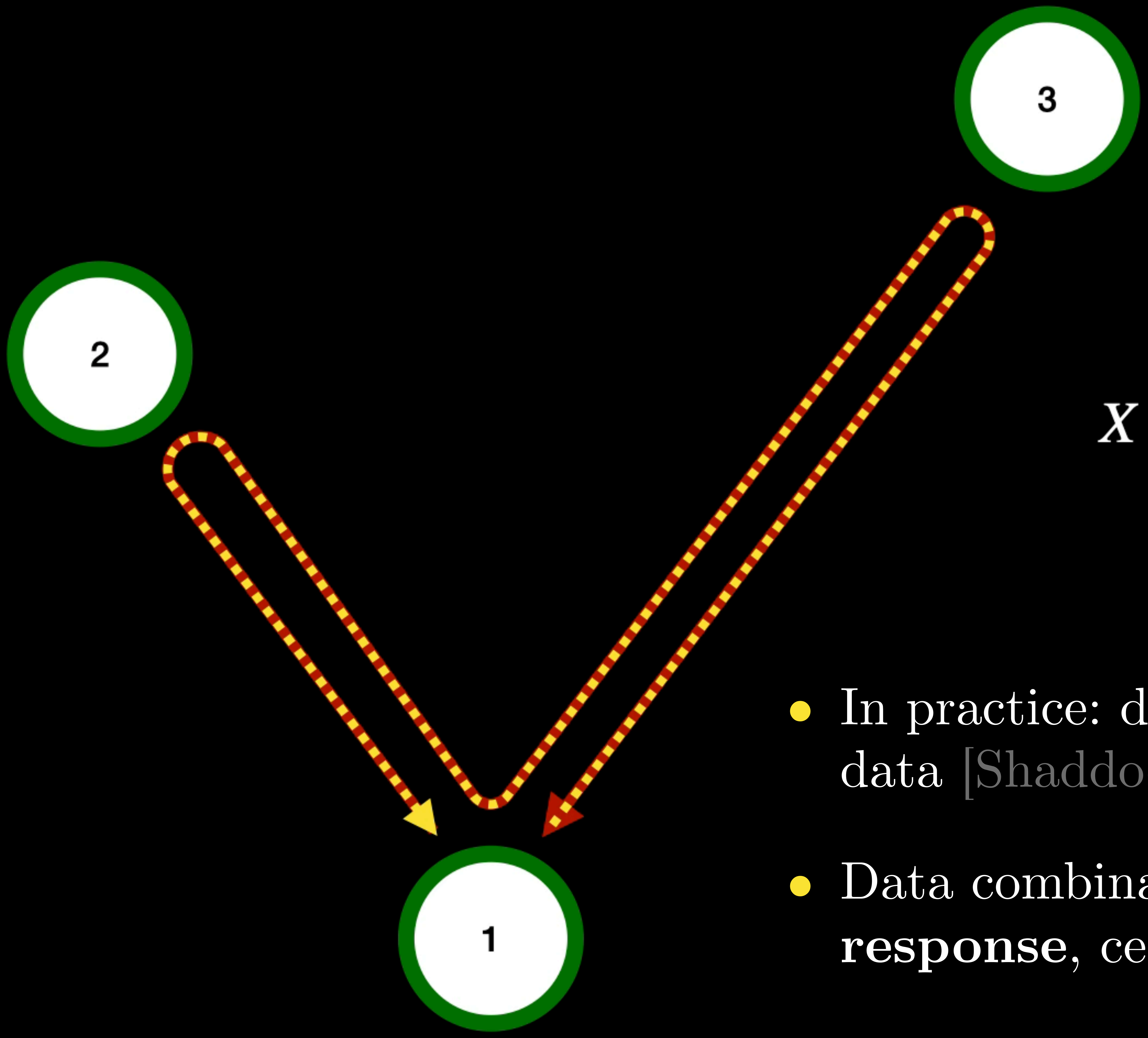
$$X = \eta_{12} + D_{12}\eta_{21} + D_{121}\eta_{13} + D_{1213}\eta_{31}$$

Full first generation TDI



$$X = \eta_{12} + D_{12}\eta_{21} + D_{121}\eta_{13} + D_{1213}\eta_{31}$$
$$- \eta_{12} - D_{13}\eta_{31} - D_{131}\eta_{13} - D_{1312}\eta_{21}$$

Full first generation TDI



$$X = \eta_{12} + D_{12}\eta_{21} + D_{121}\eta_{13} + D_{1213}\eta_{31}$$
$$- \eta_{12} - D_{13}\eta_{31} - D_{131}\eta_{13} - D_{1312}\eta_{21}$$
$$= (D_{12131} - D_{13121})\Phi_1$$

- In practice: delays realised by interpolating 4 Hz data [Shaddock et al., 2004].
- Data combination has **strong impacts on GW response**, central to LISA data analysis

TDI variables, how to find them I: algebraic solution

Algebraic approach to TDI - framework

- Consider the single link equations $\eta_{ij} = D_{ij}p_j - p_i$
- η_{ij}, p_{ij}: time series, living in $\mathcal{T} = \{x : \mathbb{R} \to \mathcal{K}\}$, where $\mathcal{K}$ is a field (usually $\mathbb{R}$)
- $D_{ij} : \mathcal{T} \to \mathcal{T}$: operator/function, mapping time-series into time-series
- Set of our 6 delay operators: $D_{set} = \{D_{ij} \,|\, ij \in \mathcal{I}_2\}$, equipped with natural multiplication through function composition (ie., apply multiple delays in sequence)
- Set of monomials M: the set of all finite (ordered) products of elements in D_{set} (semi-group)
- Set of polynomials of delay operators:

$$P = \left\{ \sum_k c_k M_k \,|\, c_k \in \mathcal{K}, M_k \in M \right\}$$

- Equipped with natural addition and multiplication, forms the **ring** $\mathcal{K}[D_{set}]$
- We can also interpret P as a space of maps $P_i : \mathcal{T} \to \mathcal{T}$, acting on time series as

$$\left(\sum_k c_k M_k \right) x(t) = \sum_k c_k (M_k x(t))$$

Algebraic approach to TDI - formulation

- We can write the single link expression in matrix form:

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \in \mathscr{T}^3, \qquad \vec{\eta} = \begin{pmatrix} \eta_{12} \\ \eta_{23} \\ \eta_{31} \\ \eta_{13} \\ \eta_{32} \\ \eta_{21} \end{pmatrix} \in \mathscr{T}^6, \qquad D = \begin{pmatrix} -1 & D_{12} & 0 \\ 0 & -1 & D_{23} \\ D_{31} & 0 & -1 \\ -1 & 0 & D_{13} \\ 0 & D_{32} & -1 \\ D_{21} & -1 & 0 \end{pmatrix} : \mathscr{T}^3 -> \mathscr{T}^6$$

$$\implies \vec{\eta} = D\vec{p}$$

- For laser noise suppression, construct $\text{TDI} = \sum_{ij \in \mathscr{I}_2} P_{ij}\eta_{ij} = \overrightarrow{P}\vec{\eta} = \overrightarrow{P}D\vec{p} \equiv 0$

- For arbitrary p_i: need $\overrightarrow{P}D \equiv 0 \implies D^T\overrightarrow{P}^T \equiv 0$

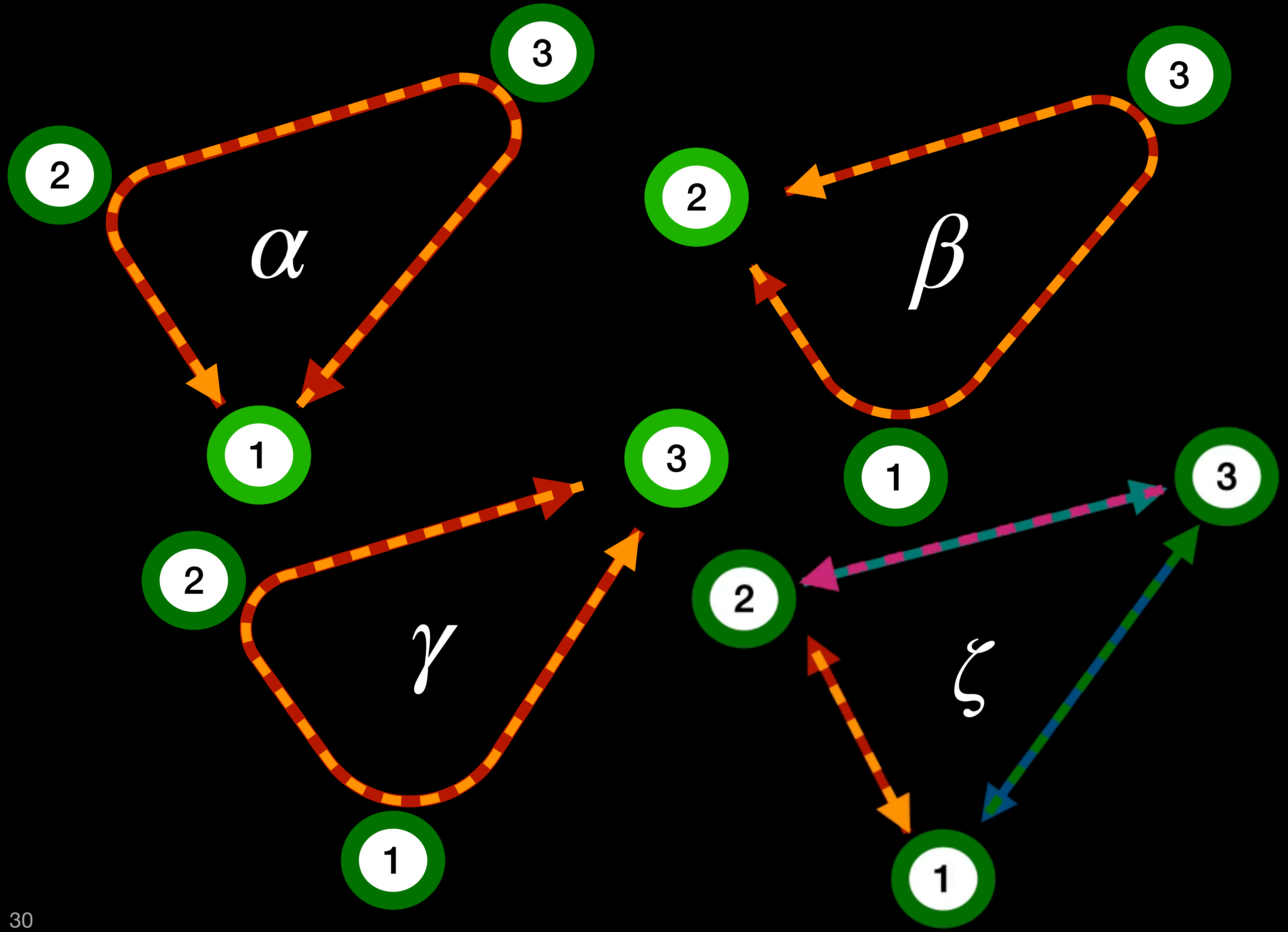
- Now: interpret D^T as a map $D^T : \mathscr{K}[D_{set}]^6 \to \mathscr{K}[D_{set}]^3$ between 'vectors' of polynomials

Algebraic approach to TDI - formulation

- Space of TDI combinations: simply the kernel of $D^T : \mathscr{K}[D_{set}]^6 \to \mathscr{K}[D_{set}]^3$
- Note: $\mathscr{K}[D_{set}]$ is a ring, not a field
- Consequently: $\mathscr{K}[D_{set}]^6$ is not a vector space, but a **module** over that ring.
- TDI solutions form a **sub-module** of $\mathscr{K}[D_{set}]^6$**,** $\approx$ first module of syzygy
- This is a well framed algebraic problem, can be solved for the case of constant (commutative) delays using standard algorithms or specialized software (e.g., Macauly2)
- Number of generators to generate the sub-module depend on assumptions:
  - '0th generation' TDI: All delays equal, $D_{ij} = D$
  - '1st generation' TDI: Static constellation, $D_{ij} = D_{ji}$
  - 'Modified 1st generation'/'1.5th generation' TDI: Rotating but rigid constellation, $D_{ij} \neq D_{ji}$

Which variables to use?

- For a static constellation, all TDI variables can be build from 4 generators [Dhurandhar et al., 2002]
- Only three independent: $(1 - D_{12}D_{23}D_{31})\zeta = (D_{23} - D_{31}D_{12})\alpha + (D_{31} - D_{12}D_{23})\beta + (D_{12} - D_{23}D_{31})\gamma$

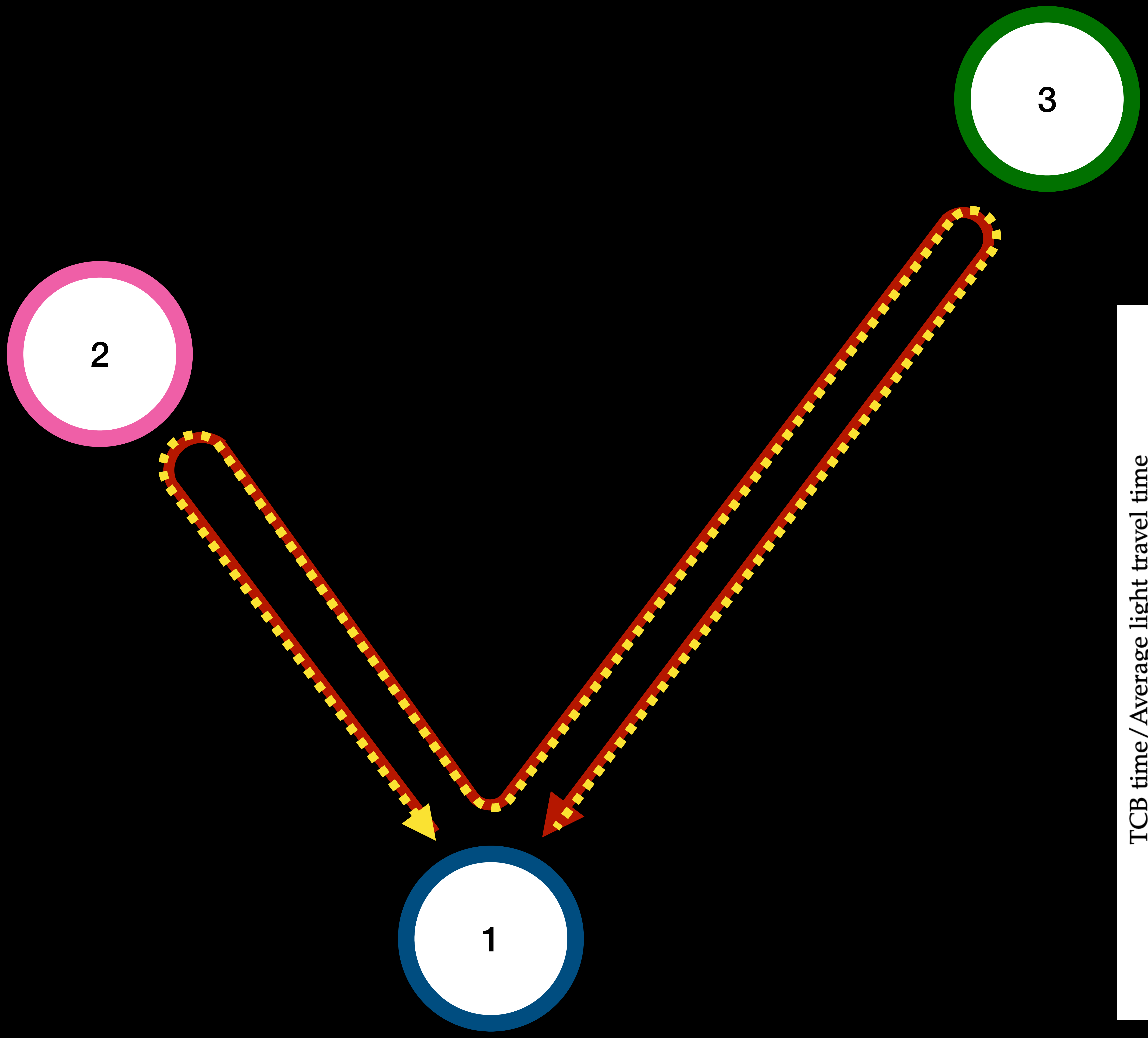


- In 'good' (?) approximation:
  - Only 3 channels independent even in realistic scenarios (time-varying orbits)
  - Popular choice: 3 Michelson combinations (X_2, Y_2, Z_2)
  - Under (strong) assumptions: easy to construct noise- and signal orthogonal (A, E, T)
  - At low frequency: only A,E sensitive to GWs, and $S_h^A \simeq S_h^E \simeq S_h^X$

2nd generation TDI

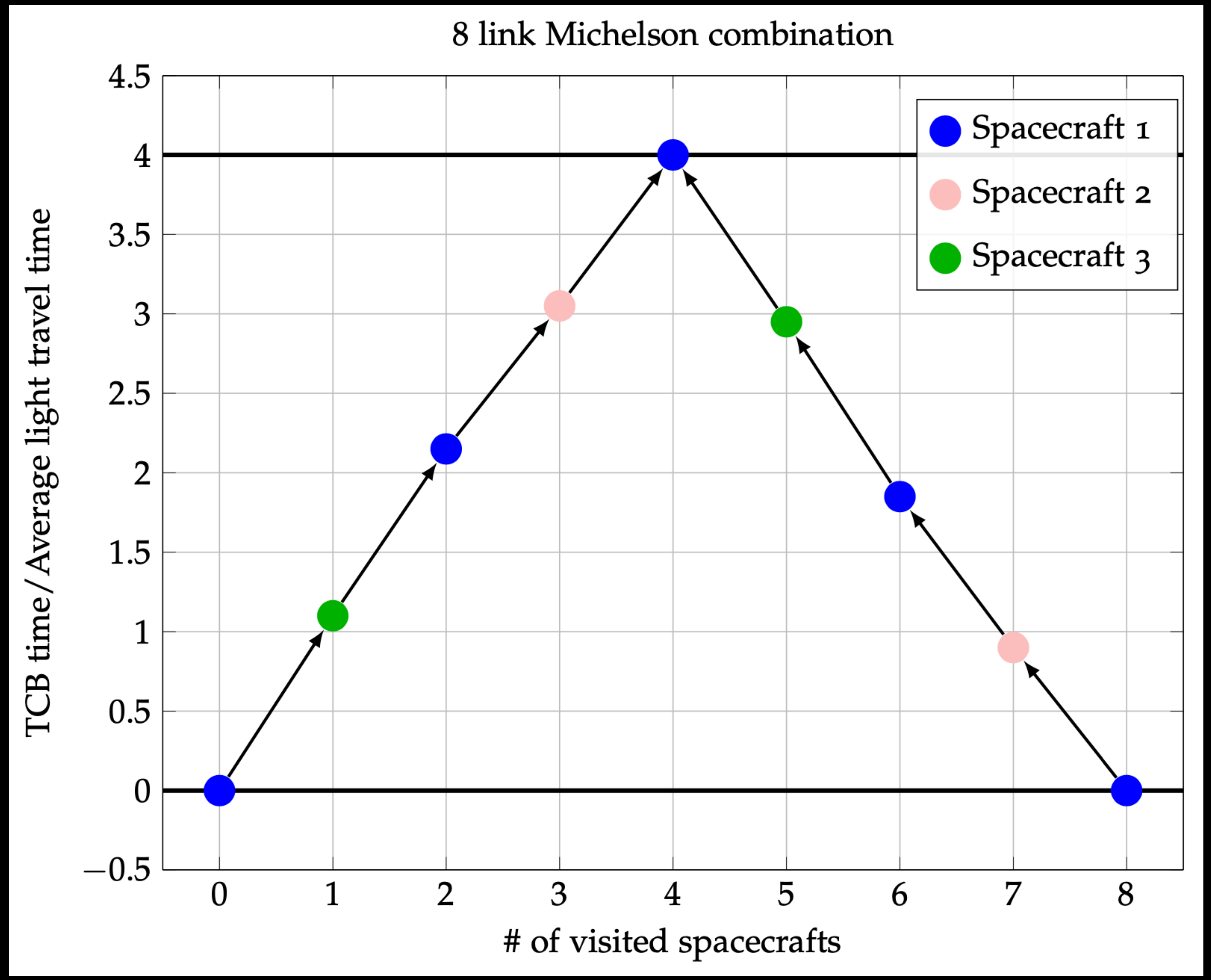
- With time-varying arms, algebraic solution for TDI is not known [Tinto 2021].
- Solutions were first discovered by modifying existing 1st generation variables [Tinto et al.]
- Many more solution found using ‘Geometric TDI’, using linear approximation of light travel times [Vallisneri]
- Recently revisited by different groups (e.g., [Muratore et al.]), who found more combinations using a symmetry in the LISA orbits

Geometric TDI



“13 31 12 21 -13 -31 -12 -21”

“13121 -13121”



Search algorithm for 2nd generation TDI

- Solutions can be found by exhaustive computation of all possible strings:
  - Generate all possible strings up to a certain length,
  - Compute time difference δt between first and last event in chain,
  - Discard combinations for which $\delta t >$ threshold.
- δt can first be evaluated assuming constant arm lengths (1st gen TDI)
  - Simple counting is sufficient (all links have to enter equal times in forward and backward time direction)
  - This allows to discard a vast number of strings.
- For 2nd generation TDI, one can use an analytical expansion $L_{ij}(t) = L_{ij} + \dot{L}_{ij}\,t$ and require cancellation to first order in $\dot{L}_{ij}$
  - Performing this expansion in the TCB [Vallisneri 2005] does not account for certain symmetries, and excludes Sagnac-like combinations
  - [Muratore et al.] showed that additional combinations can be found by doing the expansion in the Fermi-normal frame
  - Alternatively, δt can also be evaluated numerically

TDI Combinations: overview

- Many solutions to the problem of 2nd generation TDI:
  - [Vallisneri] added many 16+ link combinations
  - [Muratore et al.] added additional 12, 14 and 16 link combinations
- Table reduced by various symmetries:
  - Rotation, reflection of constellation
  - String-reversal
  - Time reversal
- 34 core combinations up to 16 links represent 210 distinct variables

Name	Normal string	$\dot{L}$ closed	M.S.	T.S.	Trivial
$\alpha_2 \rightarrow$ C^{12}_1	"1231321 -1321231"		✓	✓	
C^{12}_2	"12321 -1321 131 -1231"			✓	
C^{12}_3	"121 -13 32 -21 13 -323 31 -12 23 -31"		✓	✓	
C^{14}_1	"121321 -13212 231 -1231"				
C^{14}_2	"1213 -3213 32 -2123 3123 -31"			✓	
C^{14}_3	"1213 -32 21 -13 32 -2123 31 -12 23 -31"			✓	
$X_2 \rightarrow$ C^{16}_1	"121313121 -131212131"	✓	✓	✓	
C^{16}_2	"121323121 -132121231"			✓	
C^{16}_3	"123121321 -132121231"			✓	
C^{16}_4	"12121 -13121 13131 -12131"	✓	✓	✓	✓
C^{16}_5	"1213121 -13121 131 -12131"	✓		✓	
C^{16}_6	"1213212 -23121 132 -21231"	✓		✓	
C^{16}_7	"123123 -31321 1313 -32131"	✓		✓	
C^{16}_8	"12313123 -31321 13 -32131"	✓		✓	
C^{16}_9	"12121 -13212 23132 -21231"			✓	
C^{16}_{10}	"1213121 -13212 232 -21231"			✓	
C^{16}_{11}	"12312321 -1321 13 -321231"				
C^{16}_{12}	"1231321 -13123 313 -32131"			✓	
C^{16}_{13}	"12121 -1321 132 -212 231 -1231"			✓	
C^{16}_{14}	"1213121 -13 32 -2123212 23 -31"		✓	✓	
C^{16}_{15}	"12132 -2123 3121 -13212 23 -31"		✓		
C^{16}_{16}	"1213 -3212 232 -2123 3121 -131"		✓	✓	
C^{16}_{17}	"12132 -21321 1312 -2312 23 -31"				
C^{16}_{18}	"1213 -321 1321 -1312 231 -1231"			✓	
C^{16}_{19}	"1232321 -1321 13 -323 31 -1231"			✓	
C^{16}_{20}	"1232321 -1323 31 -121 13 -3231"			✓	
C^{16}_{21}	"12123 -3121 13 -32 213 -3212 23 -31"	✓			
C^{16}_{22}	"1213 -3212 23 -3121 13 -32 2123 -31"	✓	✓		
C^{16}_{23}	"12121 -13 32 -2123 313 -3212 23 -31"			✓	
C^{16}_{24}	"1231 -121 13 -321 1321 -131 12 -231"		✓		✓
C^{16}_{25}	"12321 -1323 31 -12 232 -21 13 -3231"			✓	
C^{16}_{26}	"12121 -13 32 -21 13 -32123 31 -12 23 -31"			✓	
C^{16}_{27}	"12132 -21 13 -32 21 -13123 31 -12 23 -31"			✓	
C^{16}_{28}	"121 -132 21 -13 32 -21 131 -123 31 -12 23 -31"		✓		✓

TDI Combinations: overview

Name	Normal string	$\dot{L}$ closed	M.S.	T.S.	Trivial
C_1^{12}	"1231321 -1321231"		✓	✓	
C_2^{12}	"12321 -1321 131 -1231"			✓	
C_3^{12}	"121 -13 32 -21 13 -323 31 -12 23 -31"		✓	✓	
C_1^{14}	"121321 -13212 231 -1231"				
C_2^{14}	"1213 -3213 32 -2123 3123 -31"			✓	
C_3^{14}	"1213 -32 21 -13 32 -2123 31 -12 23 -31"			✓	
C_1^{16}	"121313121 -131212131"	✓	✓	✓	
C_2^{16}	"121323121 -132121231"			✓	
C_3^{16}	"123121321 -132121231"			✓	
C_4^{16}	"12121 -13121 13131 -12131"	✓	✓	✓	✓
C_5^{16}	"1213121 -13121 131 -12131"	✓		✓	
C_6^{16}	"1213212 -23121 132 -21231"	✓		✓	
C_7^{16}	"123123 -31321 1313 -32131"	✓		✓	
C_8^{16}	"12313123 -31321 13 -32131"	✓		✓	
C_9^{16}	"12121 -13212 23132 -21231"			✓	
C_{10}^{16}	"1213121 -13212 232 -21231"			✓	
C_{11}^{16}	"12312321 -1321 13 -321231"				
C_{12}^{16}	"1231321 -13123 313 -32131"			✓	
C_{13}^{16}	"12121 -1321 132 -212 231 -1231"			✓	
C_{14}^{16}	"1213121 -13 32 -2123212 23 -31"		✓	✓	
C_{15}^{16}	"12132 -2123 3121 -13212 23 -31"		✓		
C_{16}^{16}	"1213 -3212 232 -2123 3121 -131"		✓	✓	
C_{17}^{16}	"12132 -21321 1312 -2312 23 -31"				
C_{18}^{16}	"1213 -321 1321 -1312 231 -1231"			✓	
C_{19}^{16}	"1232321 -1321 13 -323 31 -1231"			✓	
C_{20}^{16}	"1232321 -1323 31 -121 13 -3231"			✓	
C_{21}^{16}	"12123 -3121 13 -32 213 -3212 23 -31"	✓			
C_{22}^{16}	"1213 -3212 23 -3121 13 -32 2123 -31"	✓	✓		
C_{23}^{16}	"12121 -13 32 -2123 313 -3212 23 -31"			✓	
C_{24}^{16}	"1231 -121 13 -321 1321 -131 12 -231"		✓		✓
C_{25}^{16}	"12321 -1323 31 -12 232 -21 13 -3231"			✓	
C_{26}^{16}	"12121 -13 32 -21 13 -32123 31 -12 23 -31"			✓	
C_{27}^{16}	"12132 -21 13 -32 21 -13123 31 -12 23 -31"			✓	
C_{28}^{16}	"121 -132 21 -13 32 -21 131 -123 31 -12 23 -31"		✓		✓

Name	Timeshift	Expression
C_1^{12}	1	$(1-xyz)\alpha$
C_2^{12}	xy^2	$(y-xz)\alpha$
C_3^{12}	yz	$(y-xz)\zeta$
C_1^{14}	xy	$(1-z^2)\,\alpha$
C_2^{14}	yz	$(1-z^2)\,\gamma$
C_3^{14}	y	$(1-z^2)\,\zeta$
C_1^{16}	1	$(1-y^2z^2)\,(\alpha-z\beta-y\gamma+yz\zeta)$
C_2^{16}	1	$(1-xyz^3)\,\alpha-z(1-xyz)\beta$
C_3^{16}	1	$(1-xyz^3)\,\alpha$
C_4^{16}	y^4z^2	$(y-z)(y+z)(\alpha-z\beta-y\gamma+yz\zeta)$
C_5^{16}	y^2	$(1-z^2)\,(\alpha-z\beta-y\gamma+yz\zeta)$
C_6^{16}	xy	$(1-z^2)\,(z\alpha-\beta)$
C_7^{16}	xy^3	$(y-xz)(y\alpha-\gamma)$
C_8^{16}	y	$(1-xyz)(y\alpha-\gamma)$
C_9^{16}	$x^2y^2z^2$	$(xy-z^3)\,\alpha+(z^2-xyz)\,\beta$
C_{10}^{16}	x^2y	$(x-yz^3)\,\alpha+(yz^2-xz)\,\beta$
C_{11}^{16}	y	$(1-x^2z^2)\,\alpha$
C_{12}^{16}	y^2	$(1-xyz)\alpha+(xz-y)\gamma$
C_{13}^{16}	x^2yz	$(xy-z^3)\,\alpha$
C_{14}^{16}	y	$(xyz^2-z)\,\gamma+(1-xyz^3)\,\zeta$
C_{15}^{16}	xz^2	$(xy-z)\gamma+(1-xyz)\zeta$
C_{16}^{16}	yz^2	$(xy-z^3)\,\gamma+(-xyz+z^2)\,\zeta$
C_{17}^{16}	xy^2z^2	$(y-xz)\beta$
C_{18}^{16}	x	$(x-yz)\alpha$
C_{19}^{16}	xy^2	$(y-x^3z)\,\alpha$
C_{20}^{16}	xy^2	$(y-x^3z)\,\alpha+(x^2z-xy)\,\zeta$
C_{21}^{16}	yz^2	$(xz-y)(\gamma-z\zeta)$
C_{22}^{16}	yz^2	$(1-z^2)\,(\gamma-z\zeta)$
C_{23}^{16}	y^2z^2	$(xz^2-yz)\,\gamma+(y-xz^3)\,\zeta$
C_{24}^{16}	xyz^3	$(z^2-y^2)\,\alpha$
C_{25}^{16}	x^2y	$(y-xz)\alpha+(z-xy)\zeta$
C_{26}^{16}	yz	$(y-xz^3)\,\zeta$
C_{27}^{16}	x	$(1-xyz)\zeta$
C_{28}^{16}	y^3z	$(y-z)(y+z)\zeta$

- Going back to the assumption of 3 constant arms, all these variables can be represented in terms of 4 generators $\alpha, \beta, \gamma, \zeta$
- This is often sufficient to describe instrumental noises + GW response

$$x \approx D_{23} \approx D_{32}$$
$$y \approx D_{31} \approx D_{13}$$
$$z \approx D_{12} \approx D_{21}$$

[Hartwig&Muratore2022]

TDI-∞

- Standard TDI formulation: continous time
- Actual data: sampled time series
- TDI-∞: re-formulate TDI on sample-space

$$\mathbf{y} = \begin{bmatrix} y_1(t_1) \\ y_2(t_1) \\ y_1(t_2) \\ \cdots \\ y_1(t_n) \\ y_2(t_n) \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} p(t_1) \\ p(t_2) \\ \cdots \\ p(t_n) \end{bmatrix} \qquad \mathbf{y} = \mathbf{Mp} + \mathbf{h} + \mathbf{n} \rightarrow \mathbf{Mp}$$

  Similar to algebraic approach: define observable as $\mathbf{o} = \mathbf{Ty} = \mathbf{TMp}$, look for solutions of $\mathbf{TM} = 0$

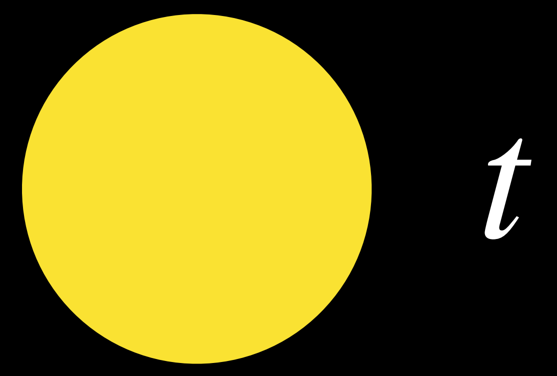
- Difference: problem on regular vector space, ‘standard’ linear algebra!
- Disadvantage: computational complexity (large matrix), interpretability

[Vallisneri et al, 2020]

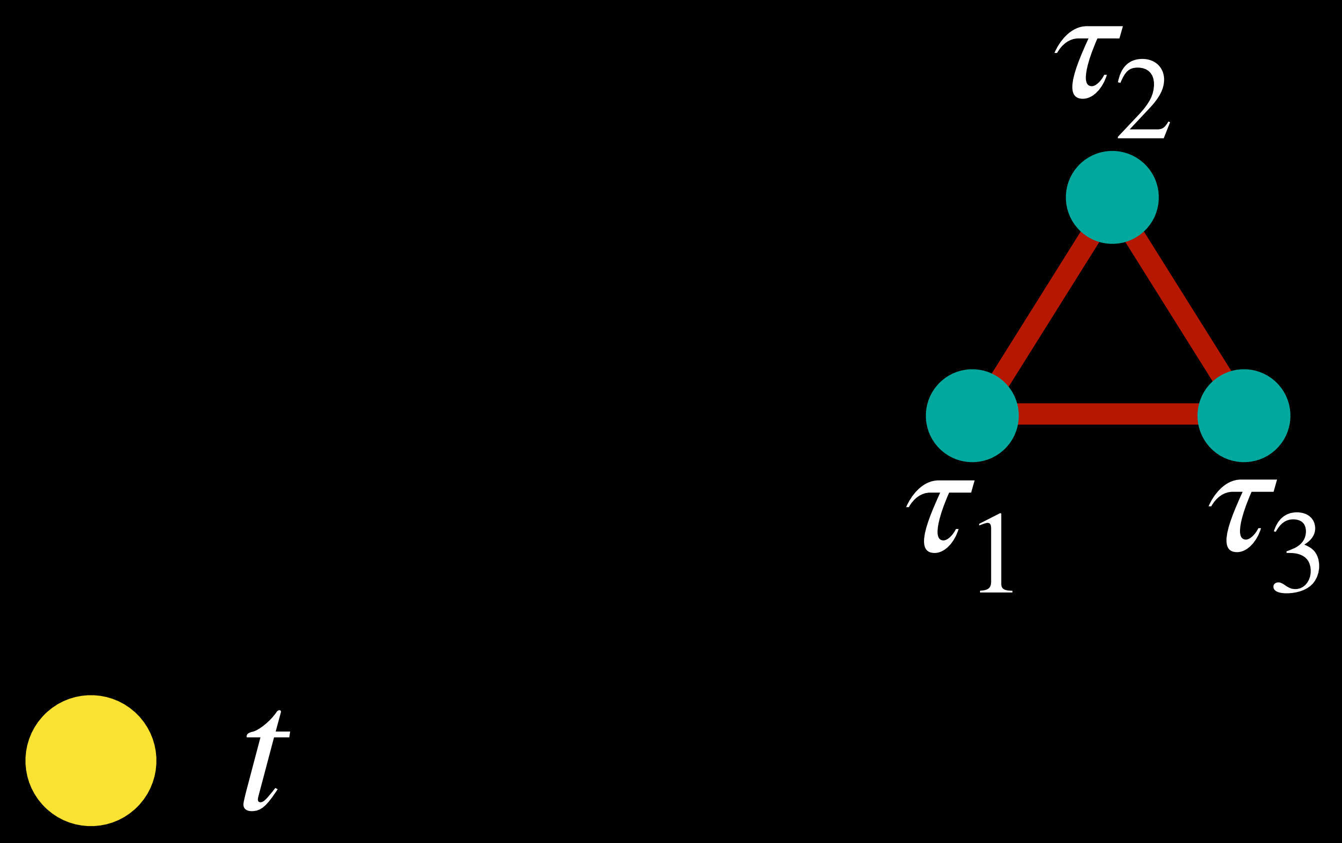
TDI with desynchronized clocks

Timescales in LISA

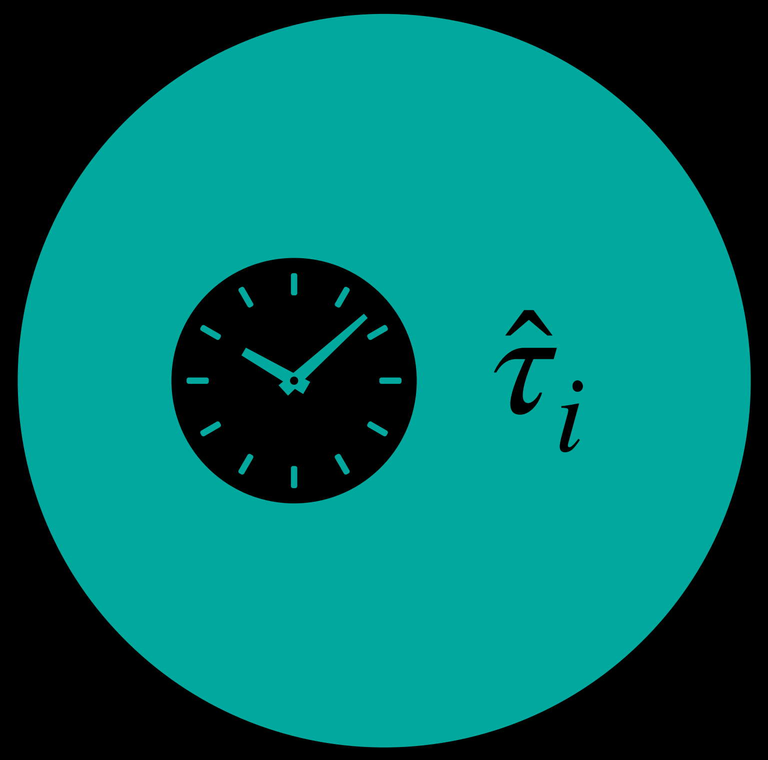
- TCB time t
  - Defined as the time shown of a **perfect** clock sitting at the solar system baricenter
  - Global timescale, used for data analysis + ‘standard’ formulation of TDI



- One proper time τ_i for each spacecraft i ($i = 1{,}2{,}3$)
  - Defined as the time shown of a **perfect** clock sitting in spacecraft i
  - Related to t (and each other) by General Relativity
  - Used for describing physics inside one spacecraft

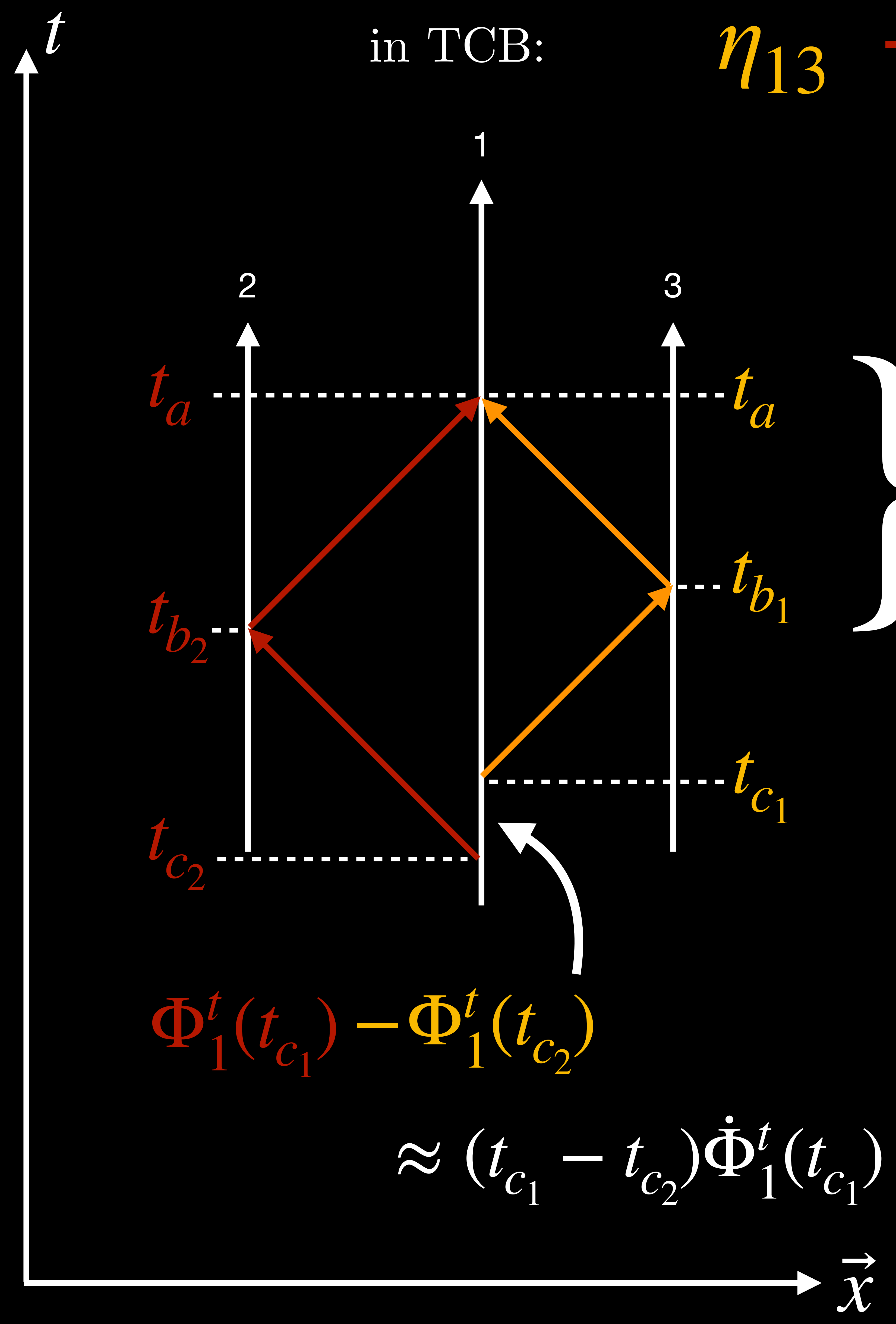


- One onboard clock time $\hat{\tau}_i$ for each spacecraft i ($i = 1{,}2{,}3$)
  - Defined as the time shown of the **actual** clock sitting in spacecraft i
  - Differs from τ_i by instrumental imperfections
  - **Only timescale directly accessible by the satellites**



Geometric TDI with clock times

Measurements in TCB: $\eta_{13} - \eta_{12} + D_{13}\eta_{31} - D_{12}\eta_{21}$



Light travel time in TCB, Computed from S/C position estimates

$$\Phi_1^t(t_{c_1}) - \Phi_1^t(t_{c_2})$$

$$\approx (t_{c_1} - t_{c_2})\dot{\Phi}_1^t(t_{c_1})$$

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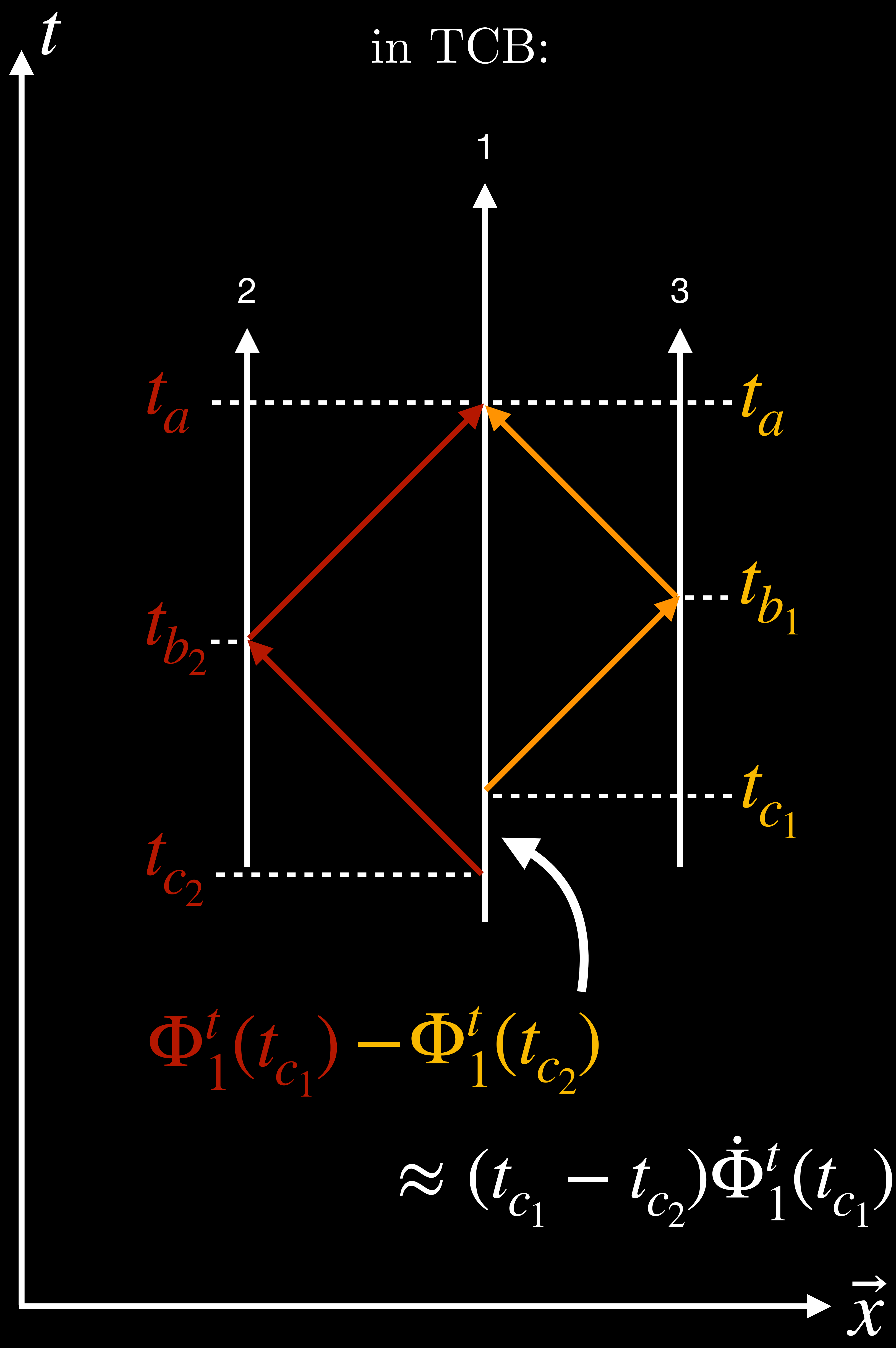
Geometric TDI with clock times

Measurements in TCB:

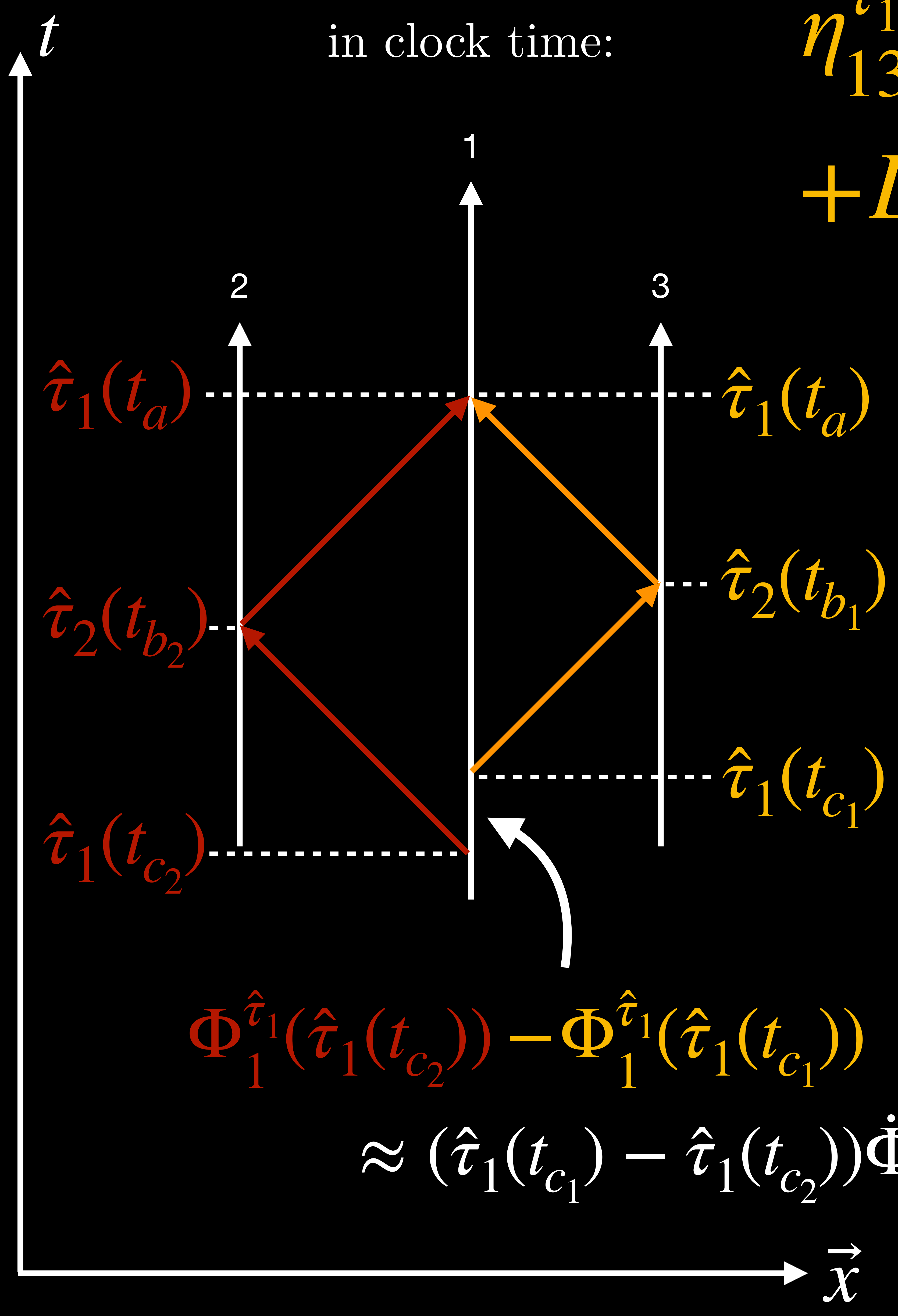


$$\Phi_1^t(t_{c_1}) - \Phi_1^t(t_{c_2})$$

$$\approx (t_{c_1} - t_{c_2})\dot{\Phi}_1^t(t_{c_1})$$

Measurements in clock time:

$$\eta_{13}^{\hat{\tau}_1} - \eta_{12}^{\hat{\tau}_1} + D_{13}^{\hat{\tau}_1}\eta_{31}^{\hat{\tau}_3} - D_{12}^{\hat{\tau}_1}\eta_{21}^{\hat{\tau}_2}$$



Pseudorange, measurable on S/C

$$\Phi_1^{\hat{\tau}_1}(\hat{\tau}_1(t_{c_2})) - \Phi_1^{\hat{\tau}_1}(\hat{\tau}_1(t_{c_1}))$$

$$\approx (\hat{\tau}_1(t_{c_1}) - \hat{\tau}_1(t_{c_2}))\dot{\Phi}_1^{\hat{\tau}_1}(\hat{\tau}_1(t_{c_1}))$$

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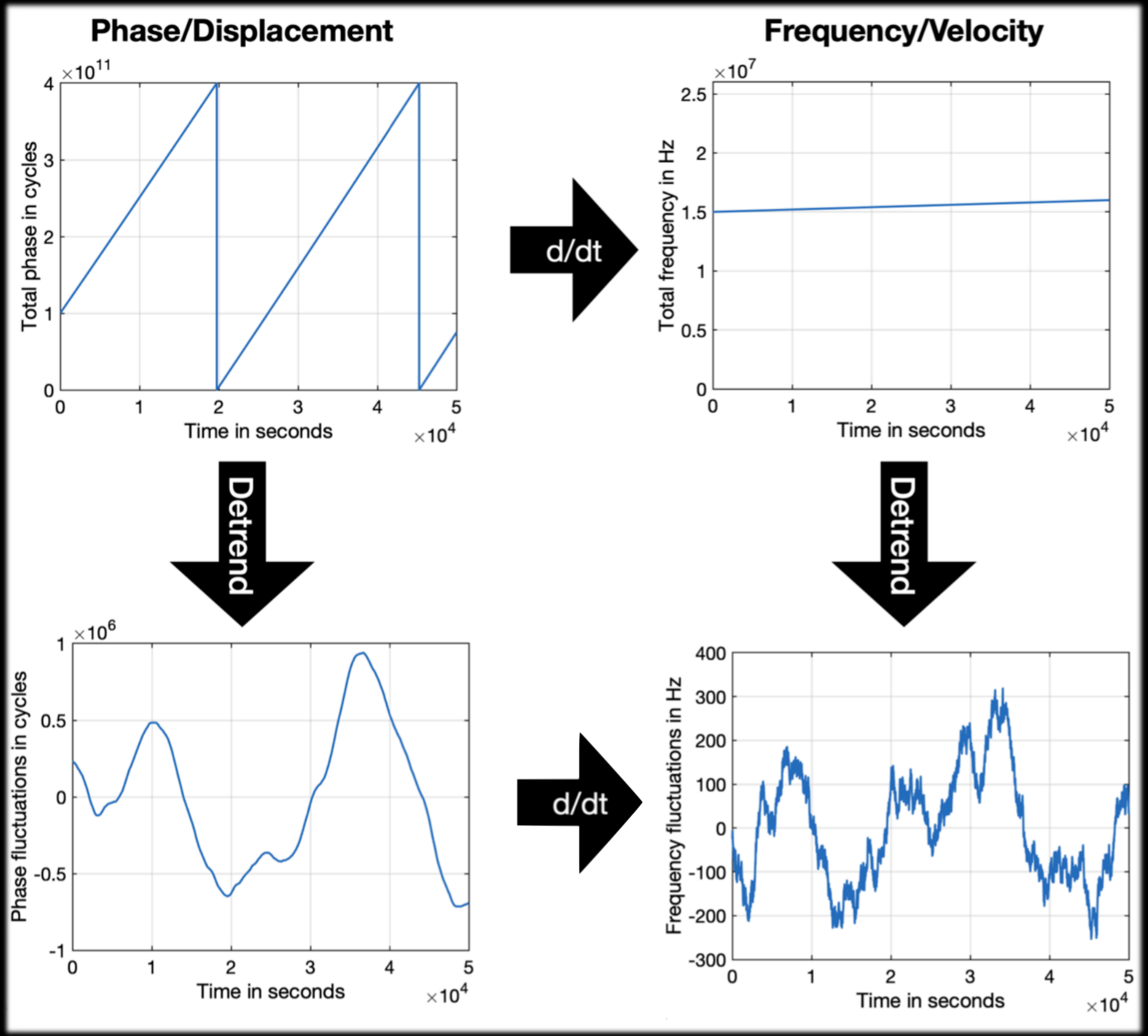
TDI: Phase vs. Frequency

Different units: overview



TDI with frequency

- TDI is usually formulated in terms of phase, where we have

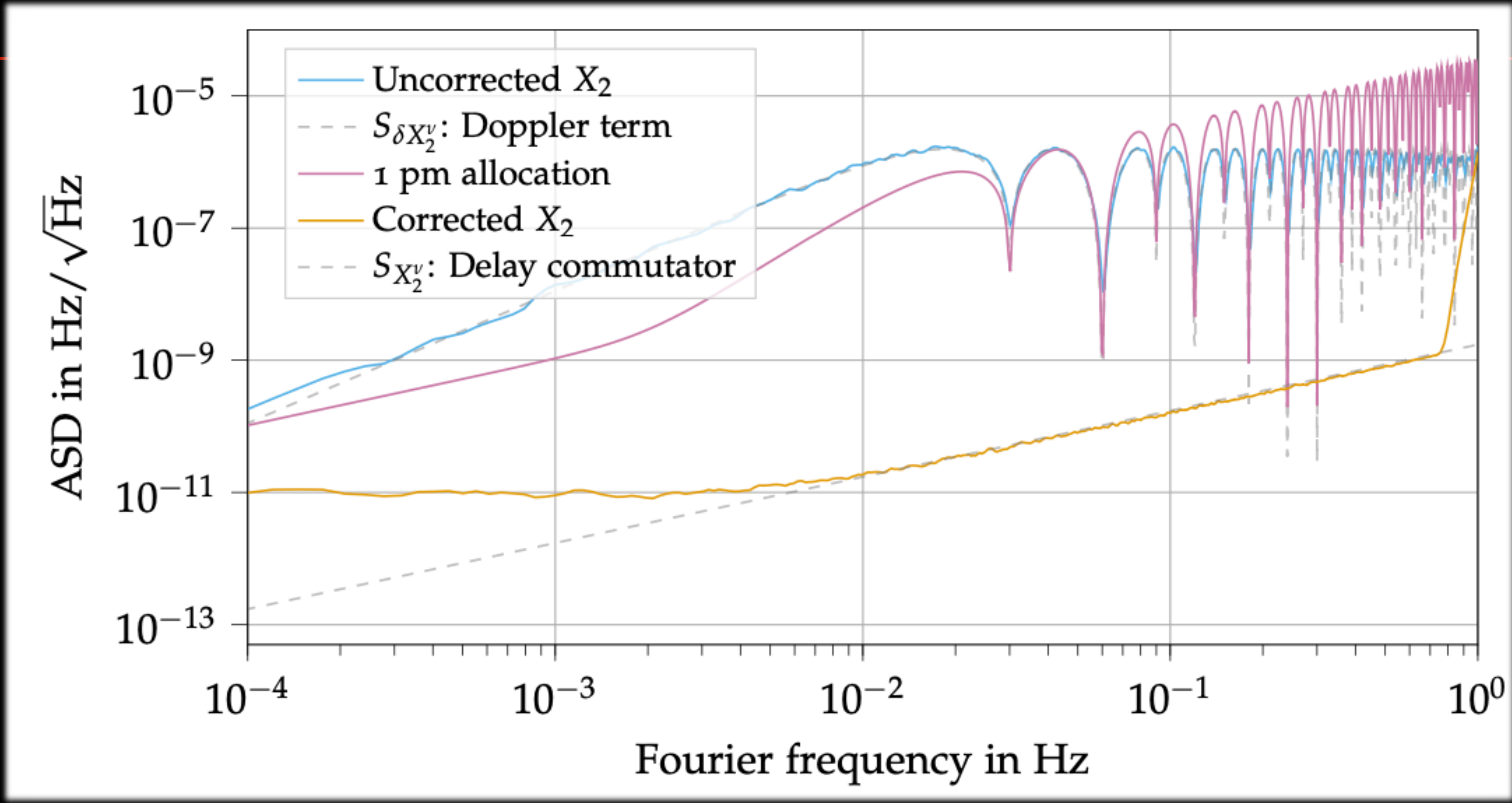
$$\eta_{12}(t) = D_{12}\Phi_2(t) - \Phi_1(t)$$
$$= \Phi_2(t - d_{12}(t)) - \Phi_1(t)$$

- In terms of frequency, we get instead

$$\dot\eta_{12}(t) = (1 - \dot d_{12}(t)) \times \dot\Phi_2(t - d_{12}(t)) - \dot\Phi_1(t)$$
$$\equiv \dot D_{12}\nu_2(t) - \nu_1(t)$$

- Neglecting Doppler shifts causes large residuals

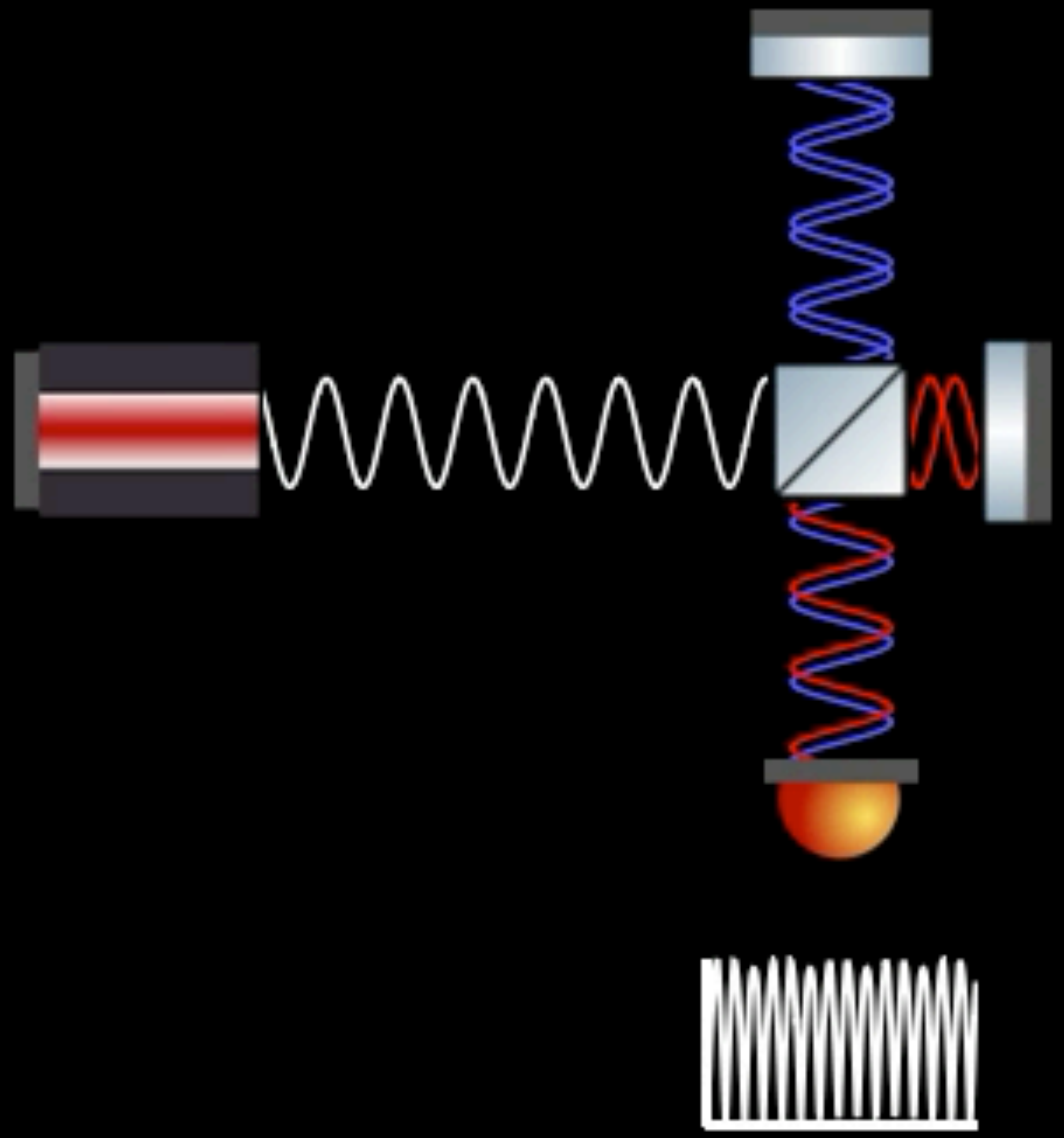
- Solution: replace all $D_{12} \to \dot D_{12}$



[Bayle,Hartwig,Staab 2021]

Clock noise: technical details

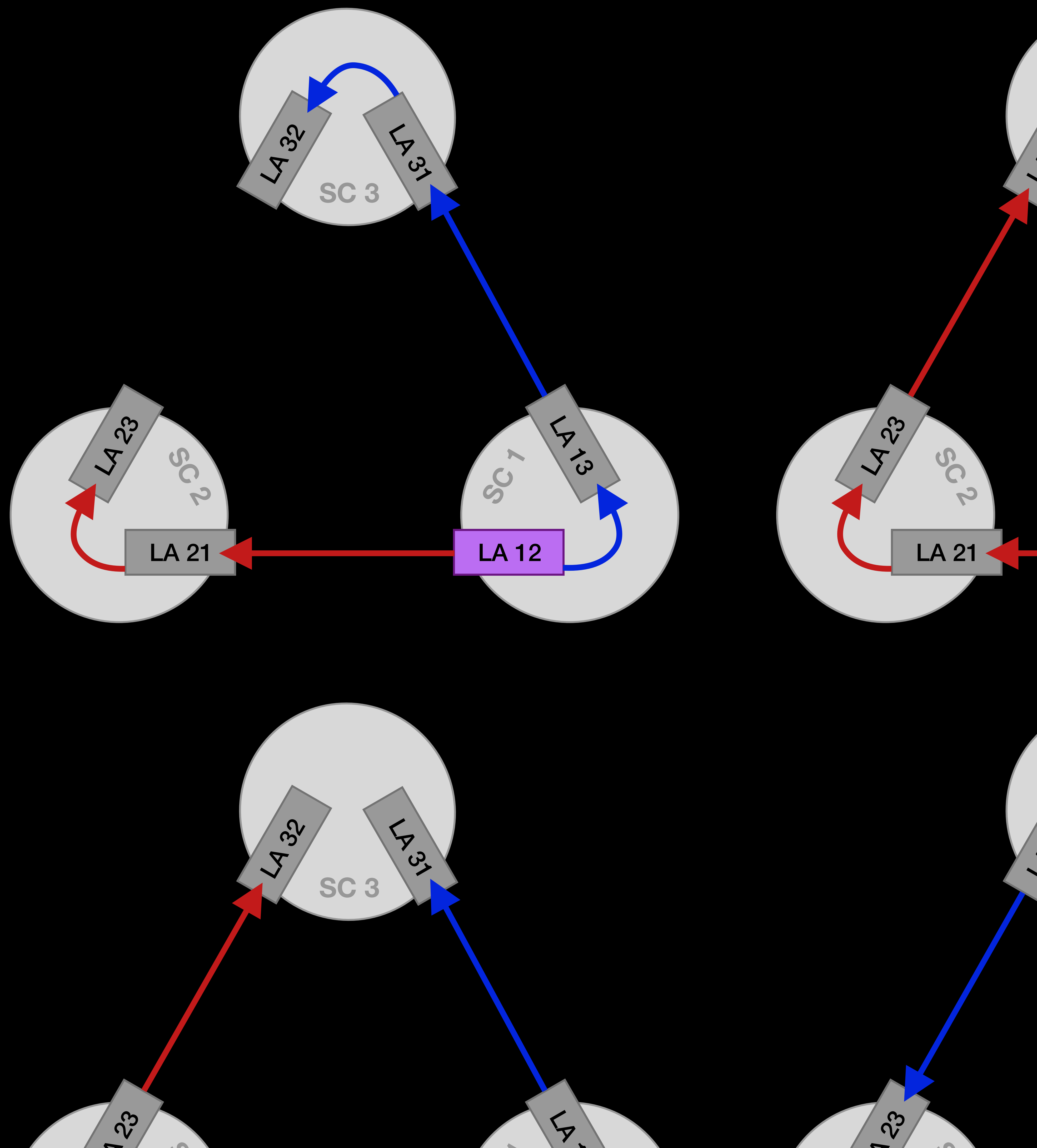
Further complications: beyond laser noise suppression



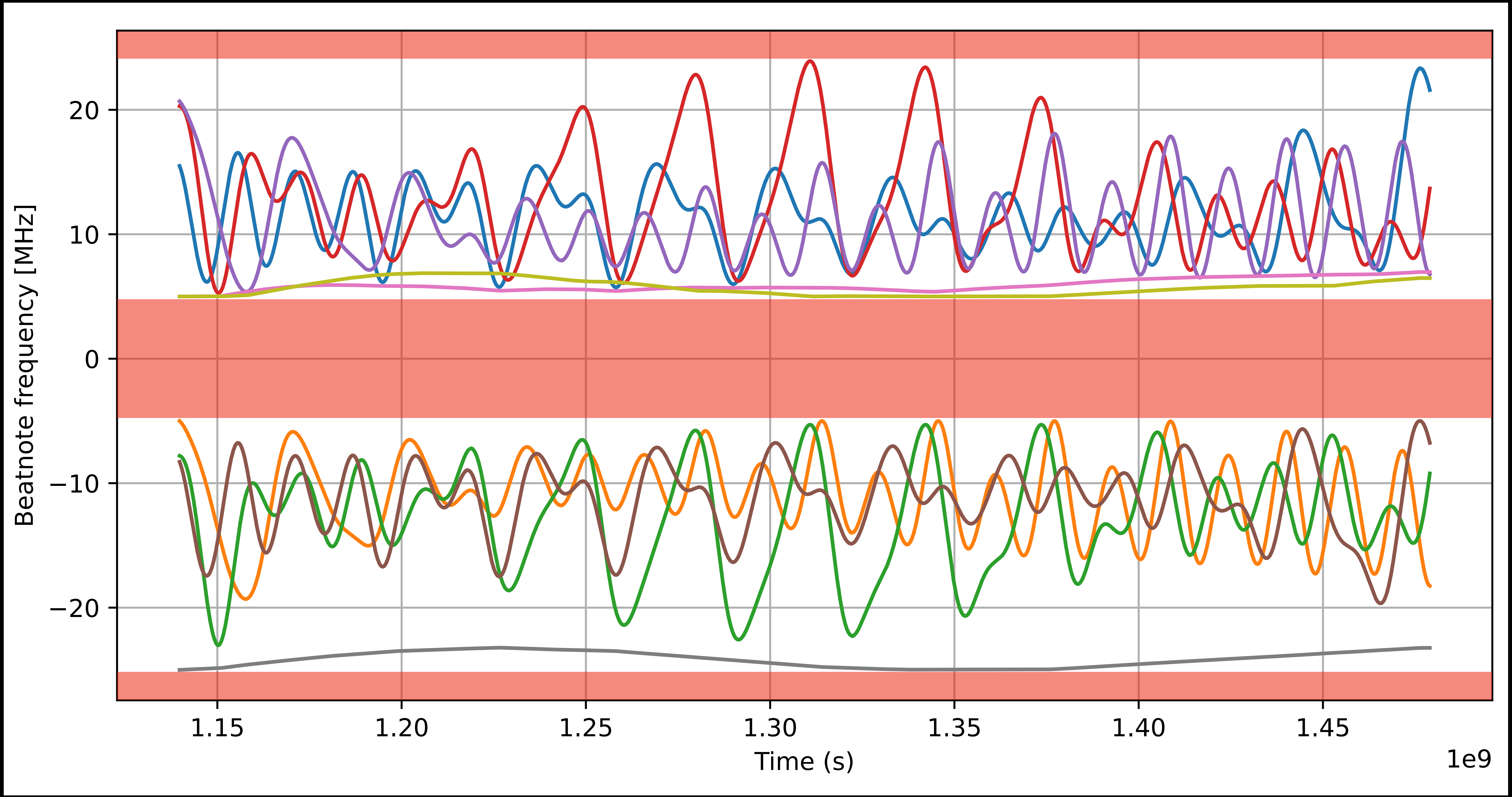
- Spacecraft are moving!
- Heterodyne interferometry:
  - Signals are up to 25 MHz beatnotes
  - pm distance fluctuations with $\lambda = 1064\,\mathrm{nm}$ $\implies$ μcycle phase shifts

Laser Locking & Frequency Plan

- Due to time-varying Dopplers, beatnotes are not guaranteed to fall in phasemeter validity frequency range (5 to 25 MHz)
  - Doppler shifts: 10s of MHz
- Solution: lock lasers with precomputed frequency plan
- Interesting problem in computational geometry (many configurations possible), see [Heinzel et al.]

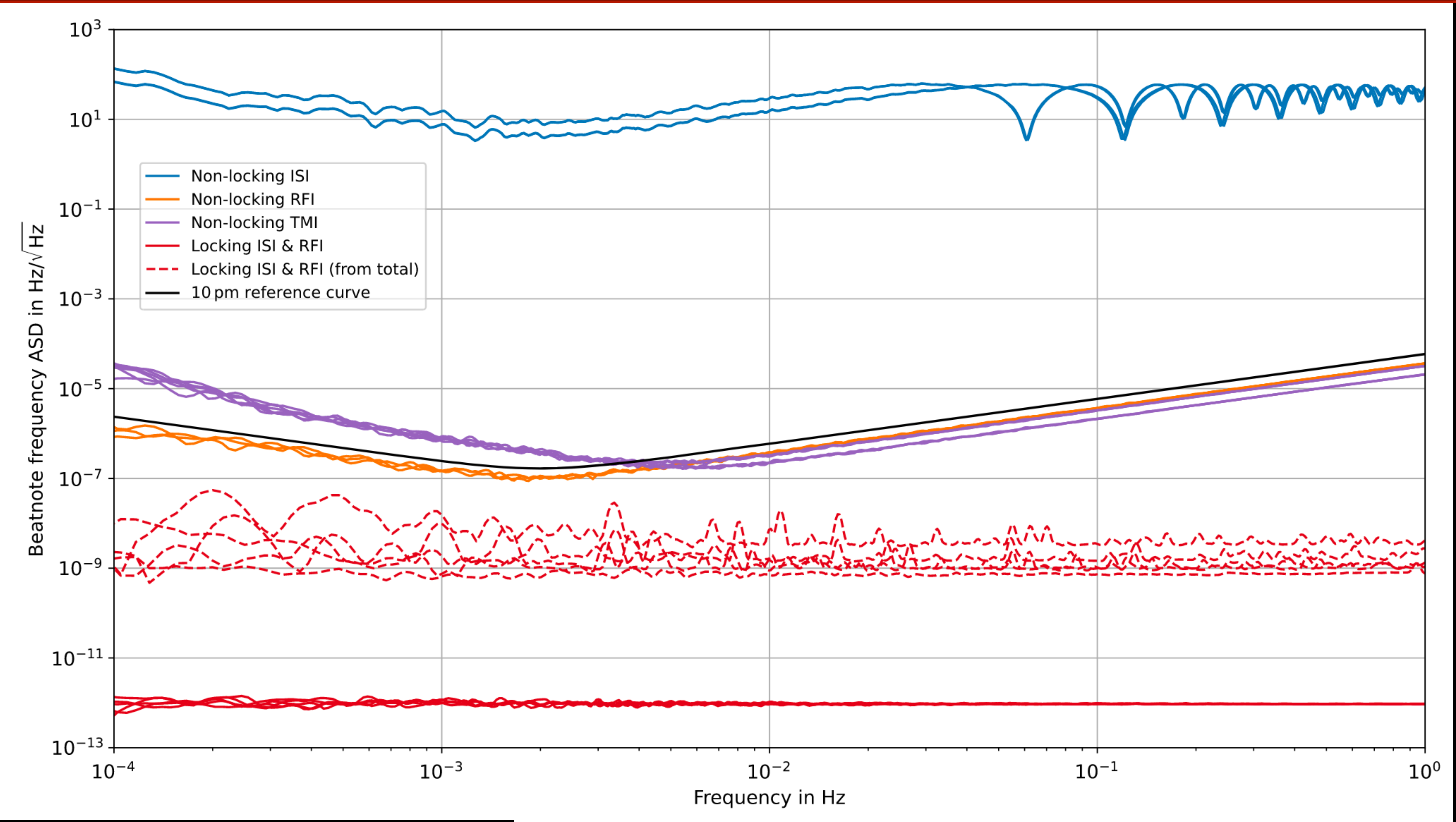


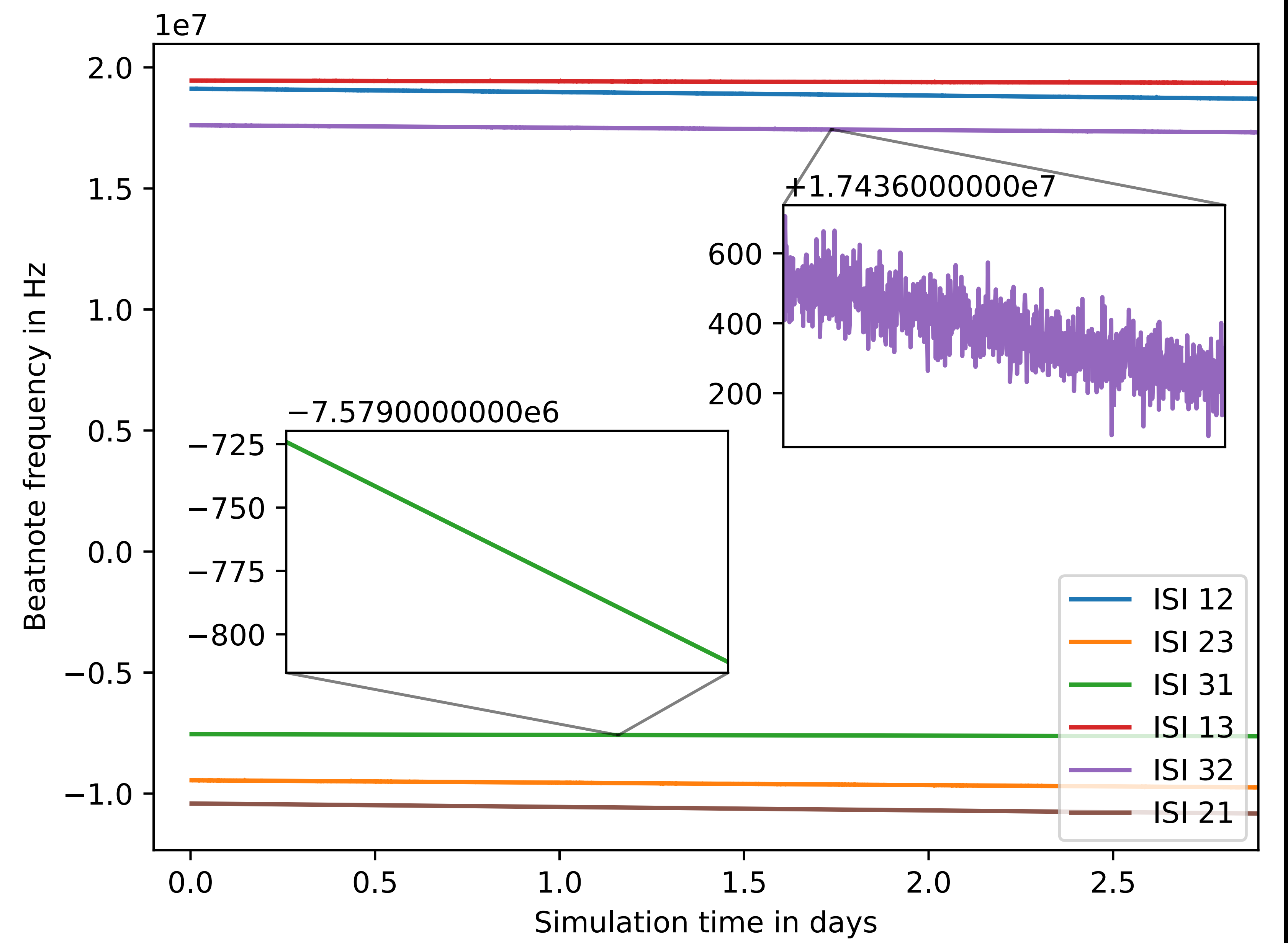
Laser Locking & Frequency Plan



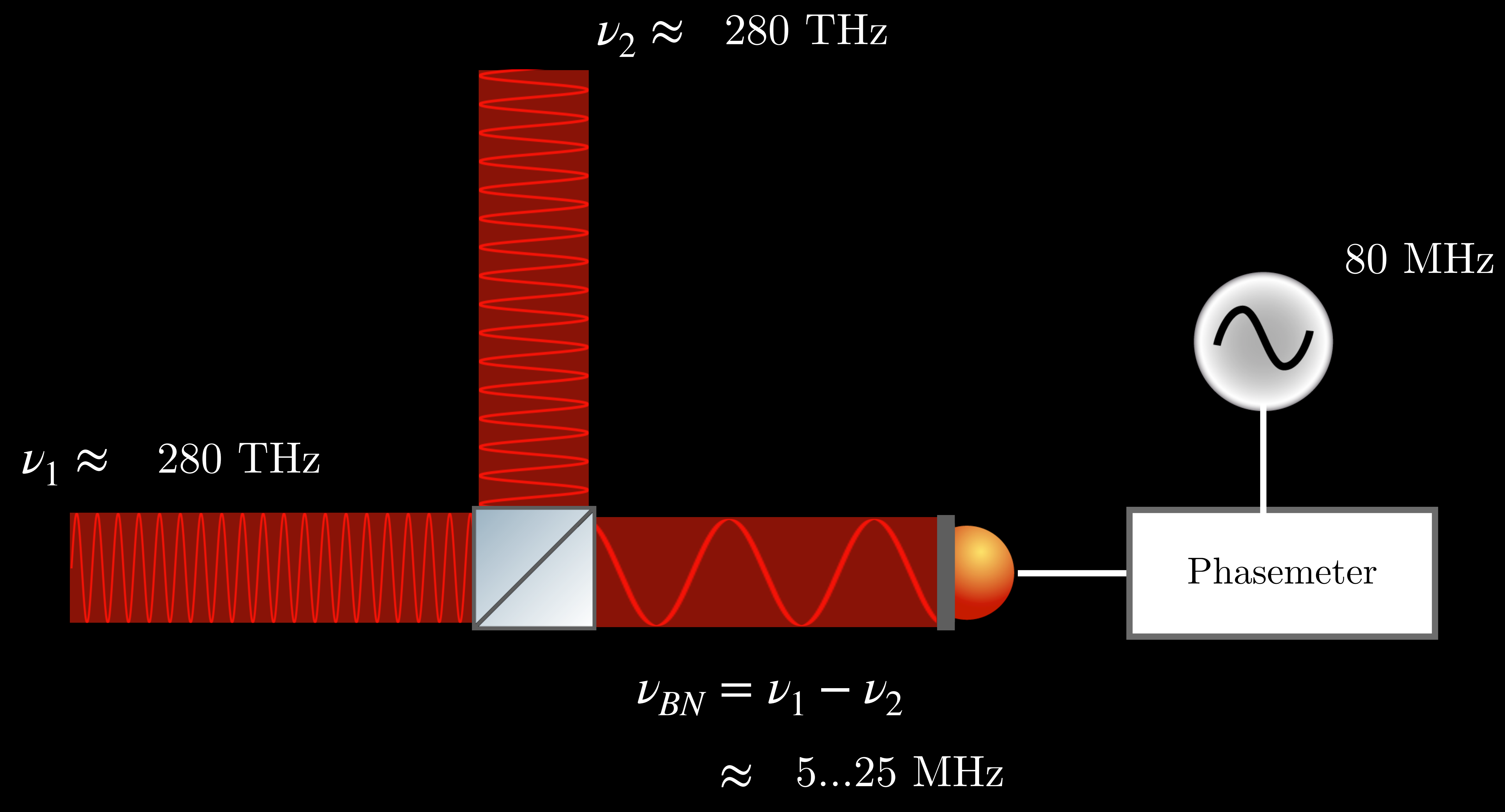
Laser locking side effects

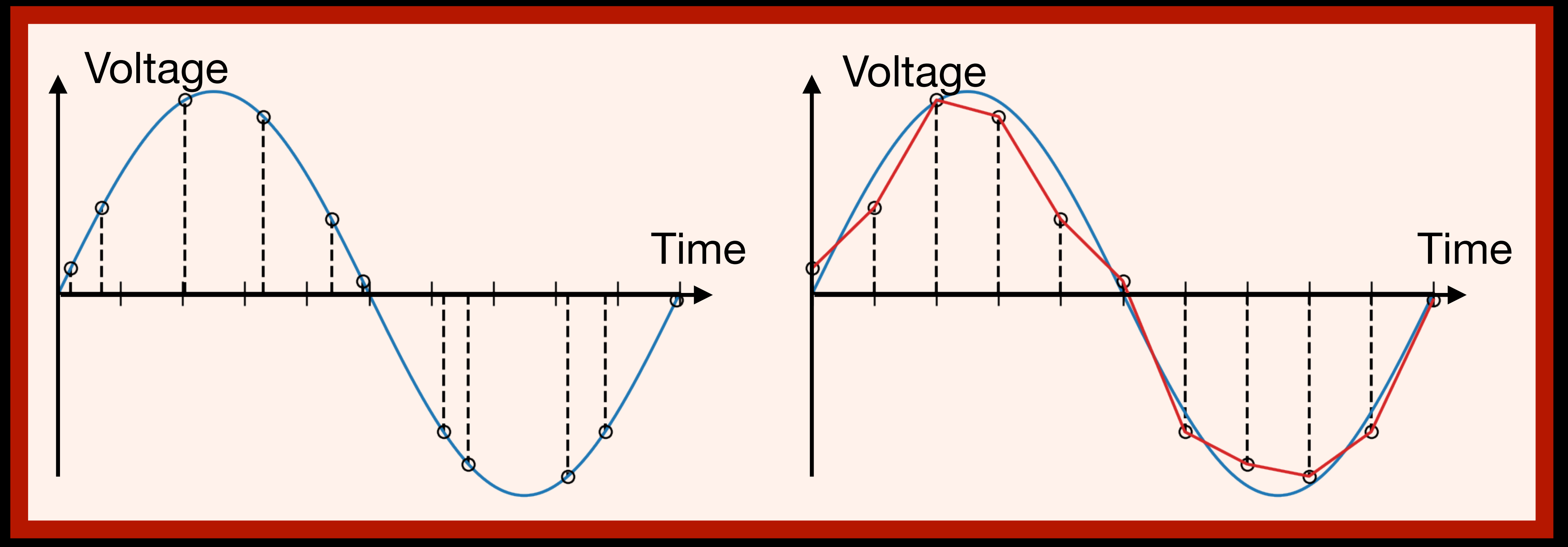
- Locking control loop: adjust local laser frequency based on measured beatnote to drive it to desired value
- Side effect: information is ‘moved around’ between beatnotes
- Drastically changes raw measurements: GW (and secondary noises) only visible in non-locking beatnotes, locking ones follow nominal values
- Interestingly: this (almost) completely disappears on TDI level!
- Simple argument: TDI suppresses whatever comes out of the laser - laser locking simply makes that more complex



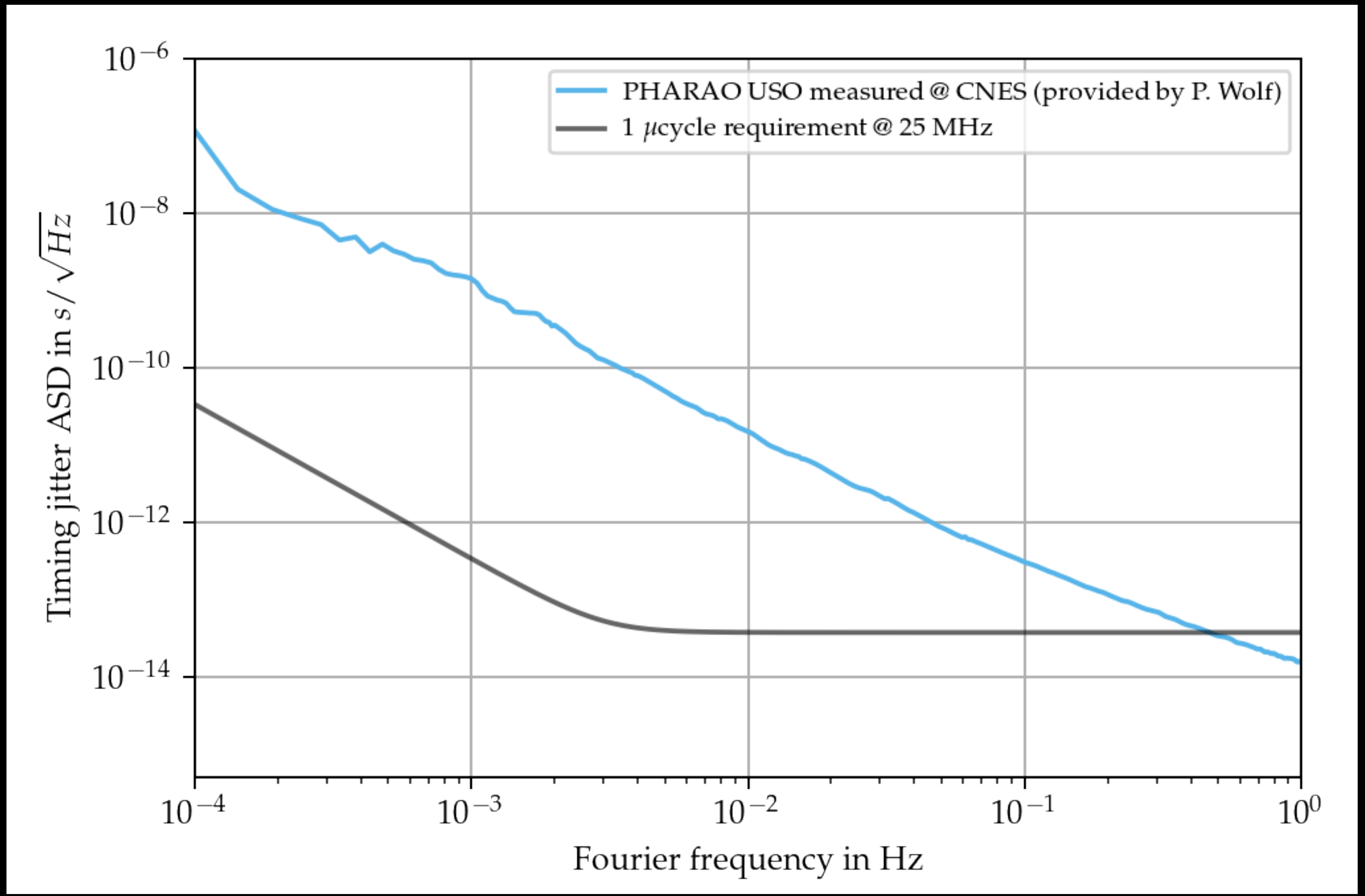


Clock and time-related issues





- Phase tracking requires comparison to local reference clock
- 25 MHz beatnotes require 40 fs/$\sqrt{\text{Hz}}$ timing precision for µcycle phase readout
- Existing space-qualified clocks fall short by a few orders of magnitude!



Note: Illustrative, numbers to be seen as placeholders

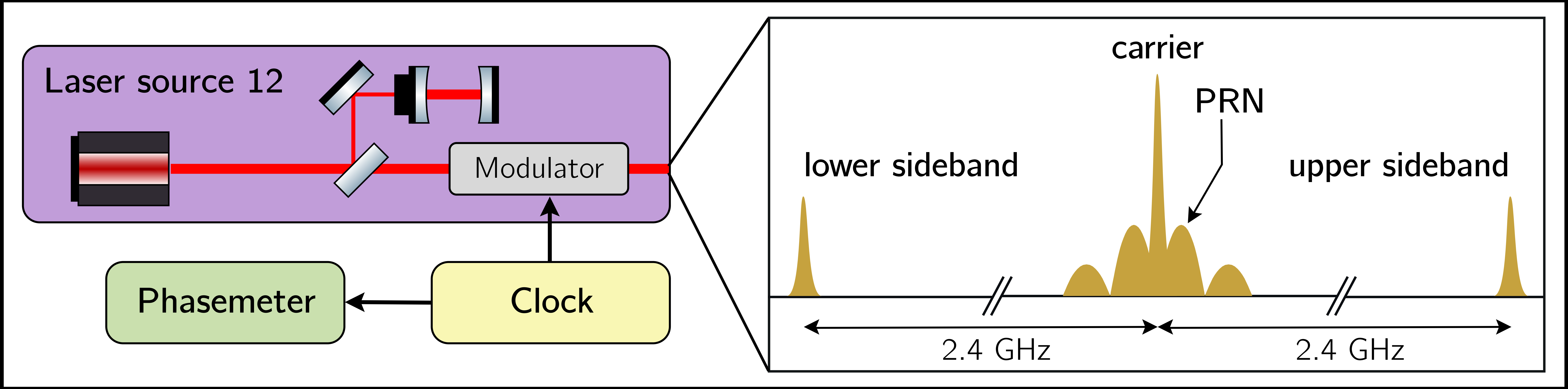
Beam Modulation: GHz Sidebands

- Phase modulation used to measure the in-band part of this clock noise

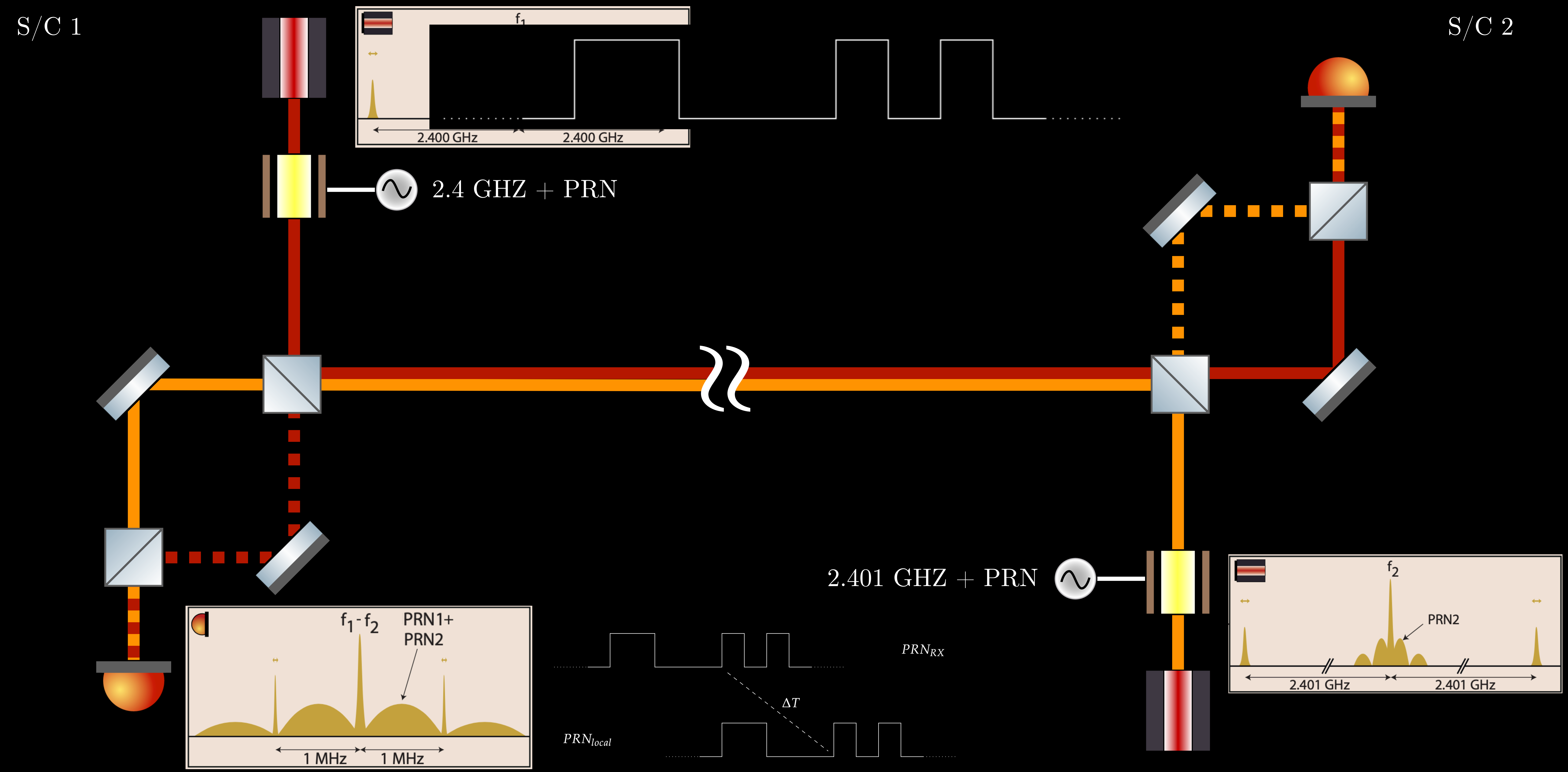
$$E(\tau) \approx E_0(\tau) e^{i\Phi_c(\tau)} e^{im\cos(\Phi_m(\tau))}$$

- Modeled as "independent" sideband beams (expansion with Bessel functions)

$$e^{im\cos(\Phi_m(\tau))} \approx 1 + \frac{im}{2} e^{i\Phi_m(\tau)} + \frac{im}{2} e^{-i\Phi_m(\tau)}$$



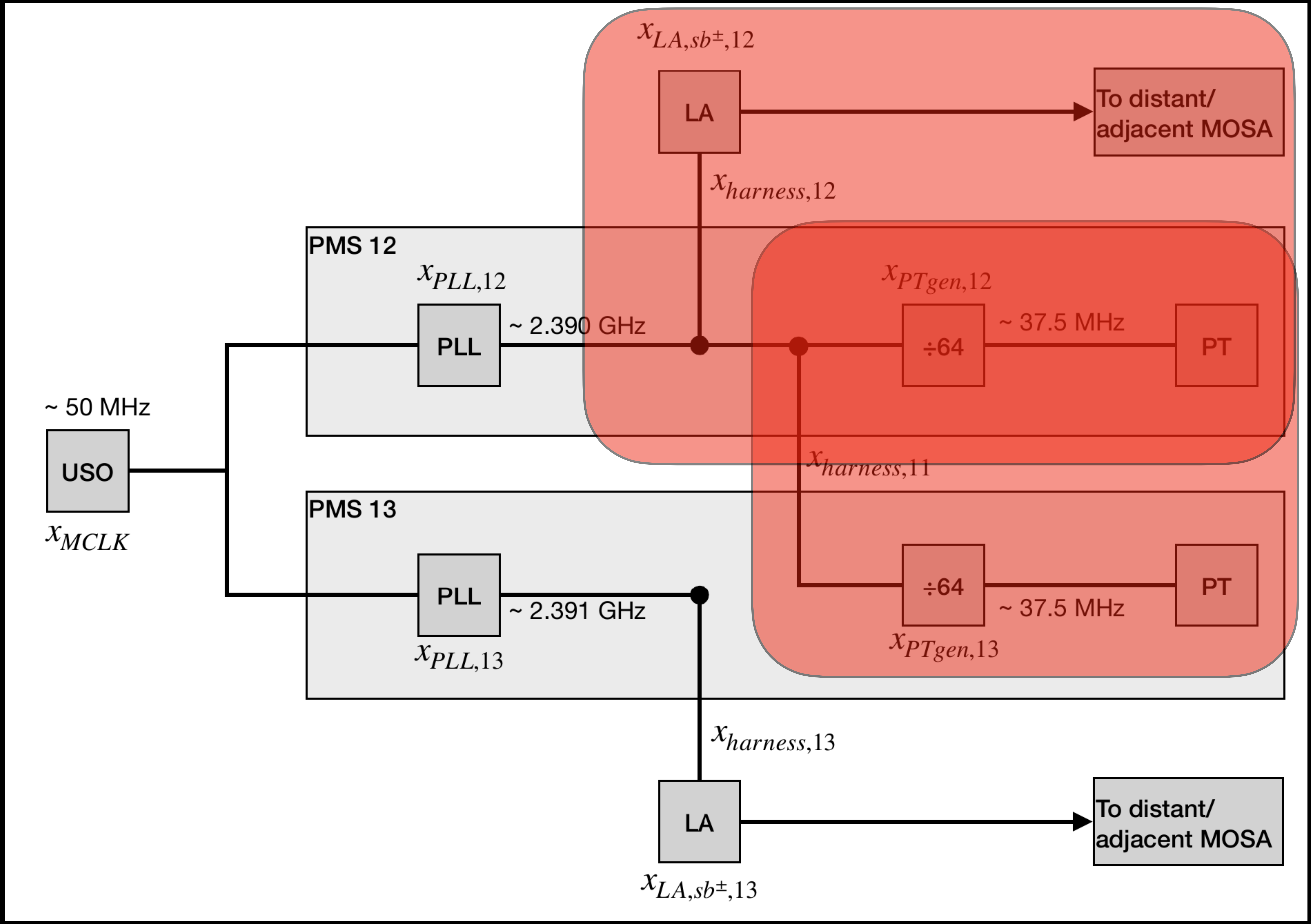
Clock and time-related issues



Note: Illustrative, numbers to be seen as placeholders

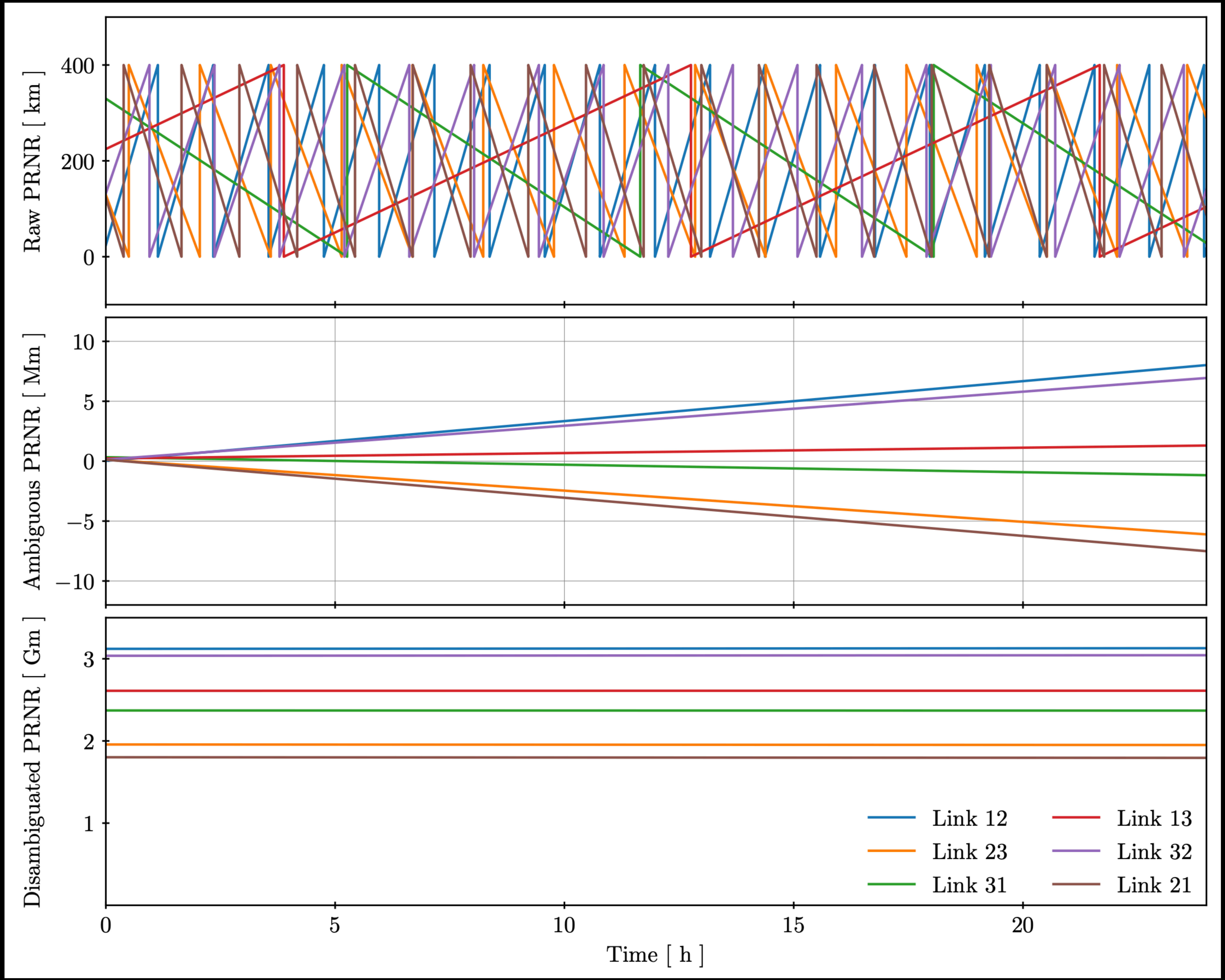
Frequency distribution system details

- Critical paths (electric):
  - Pilot tone vs. pilot tone
  - Pilot tone vs. **left** laser assembly
- Critical paths must perform (around) the 40 fs/$\sqrt{\text{Hz}}$ timing precision mark
- Right handed modulation: Can be corrected using sideband beatnote in RFI



PRN: processing details

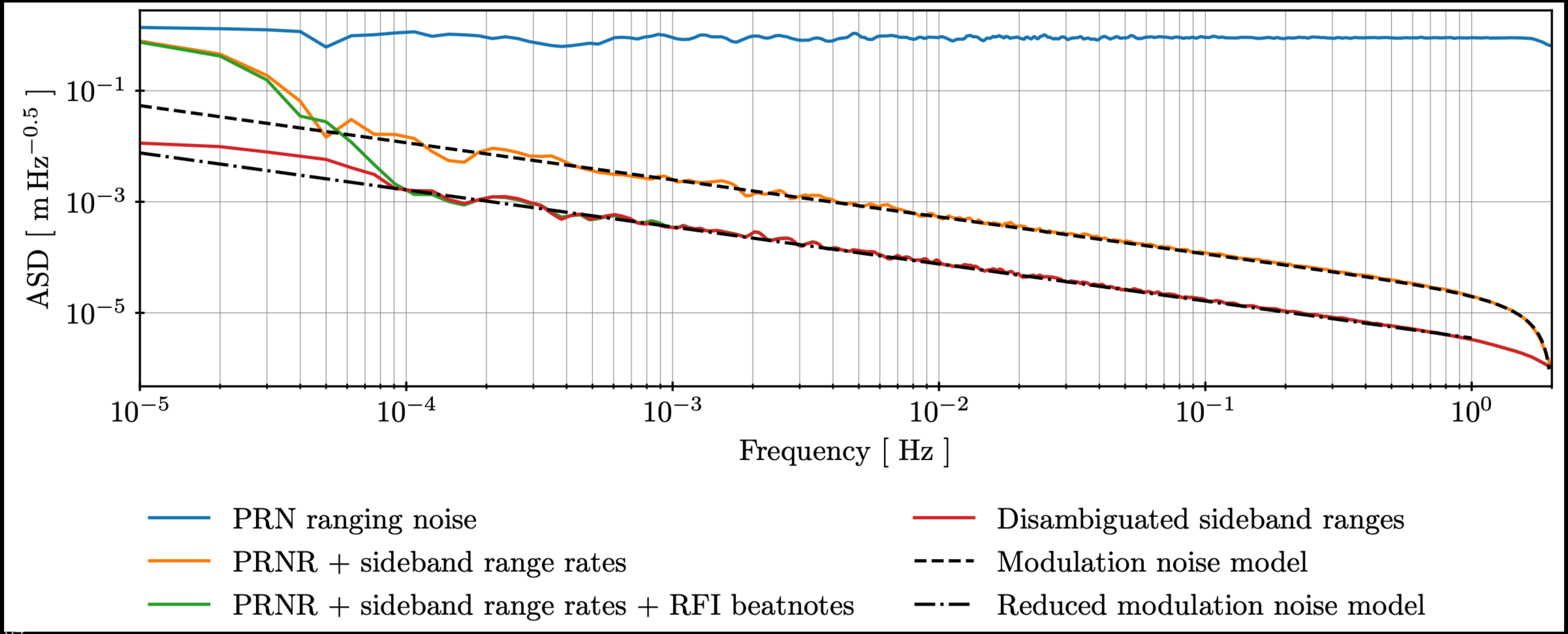
- PRN codes have finite length (≈400 km)
- Absolute value and dynamic range of pseudo-range much larger -> signal 'wraps' to code length
- First step: un-wrap raw codes, jumps can be detected automatically
- Second step: use ground-based observations (or TDI-R) to find ambiguity/offsets



[Reinhard, 2025]

PRN: processing details

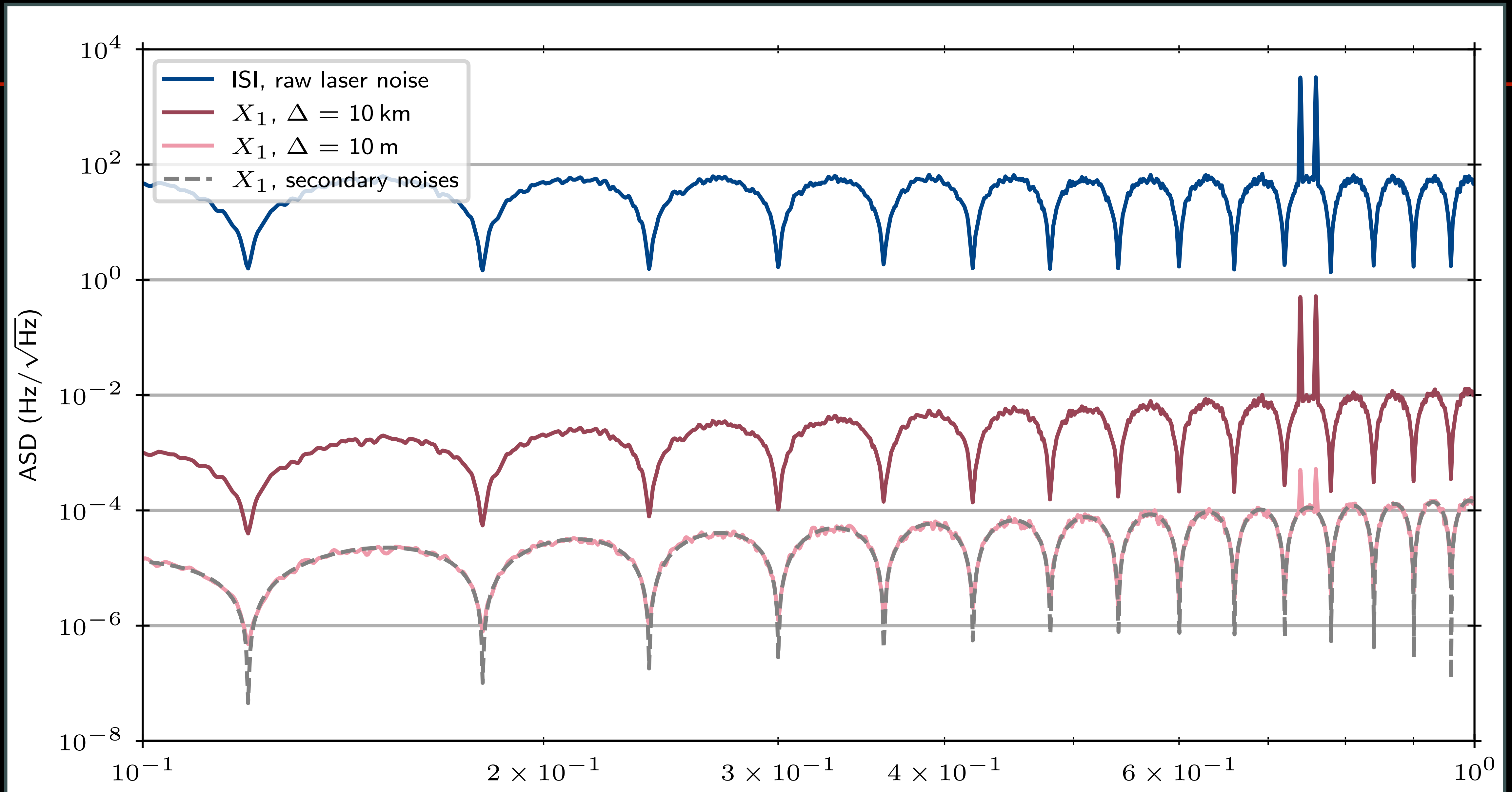
- PRN and sideband fundamentally measure the same quantity: pseudo range
- PRN is absolute, but noisy
- Sideband is precise, but ambiguity even more challenging: 12 cm instead of 400km
- Combined: low-noise, high accuracy pseudo-range measurement



[Reinhard, 2025]

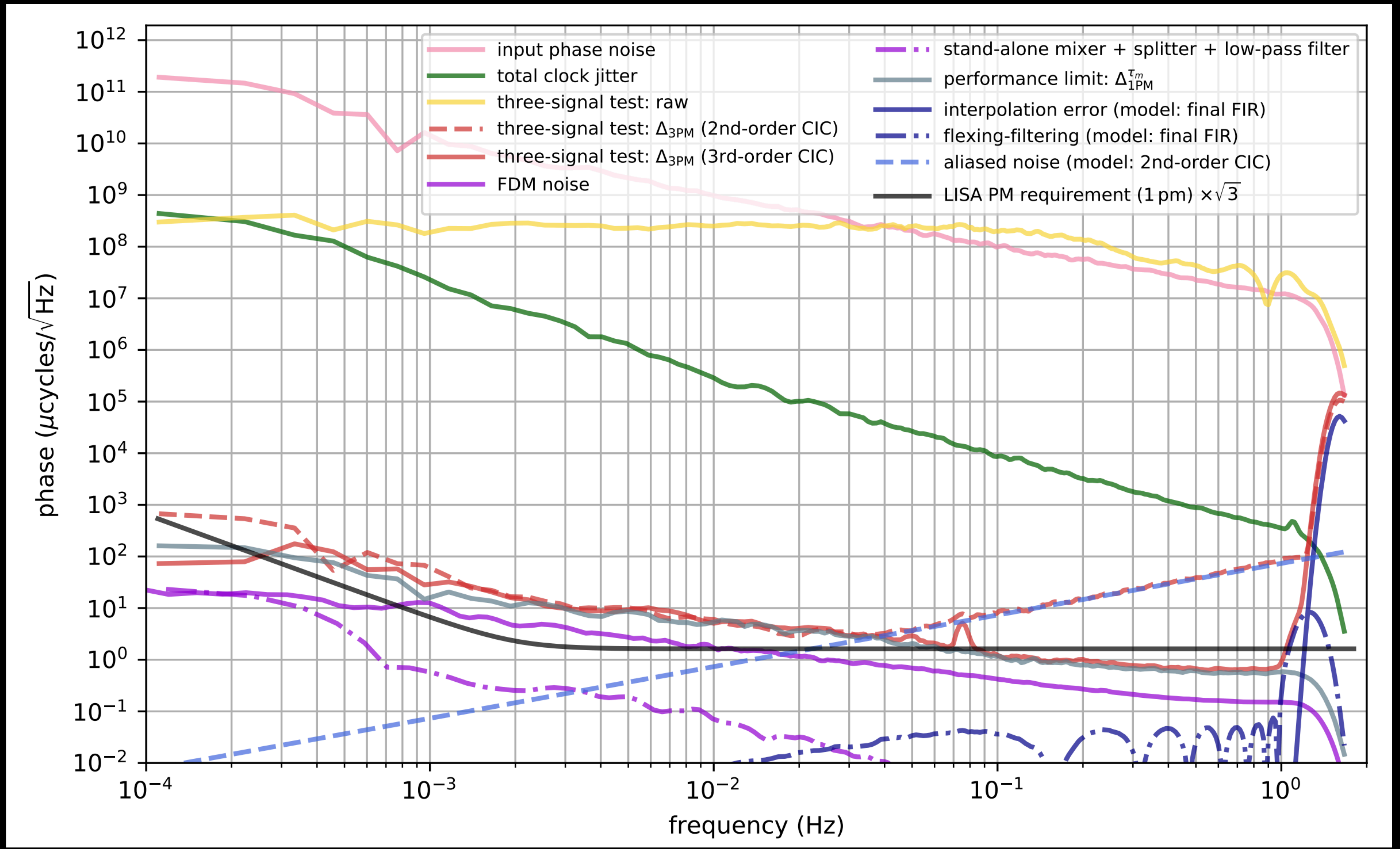
Alternative/backup to PRN: TDIR

- TDI-ranging (TDI-R): Fit ranging bias by minimizing noise in TDI combination [Tinto et al., 2004]



[Staab et al., in prep.]

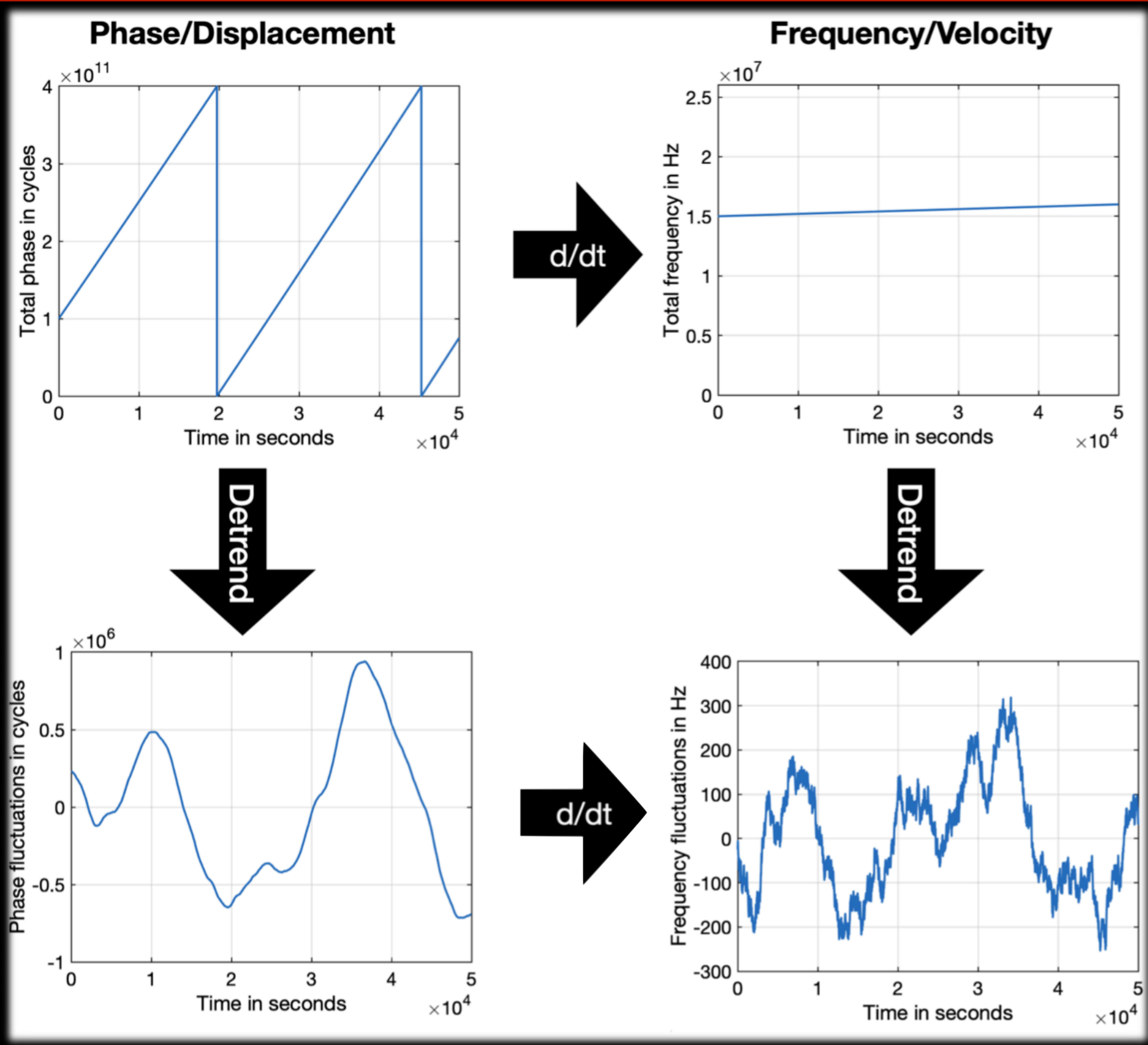
- Relative clock noise measurements, absolute ranging via PRN and TDI ranging have been demonstrated in hardware demonstrators, e.g., Hexagon experiment [Yamamoto et al., 2022, 2024]



Clock noise algorithm

TDI with or without detrending

- For signals expressed in total phase (or total frequency):
  - Each time shift (Doppler shift) applied in TDI couples to MHz beatnote
  - Previous logic on TDI with unsynchronised clocks directly applies
  - Clock noise correction included in main laser noise reduction step!



- If data is detrended: time shifts applied to residuals $\phi(t) \approx -\omega\delta t + \varphi(t - \delta t)$
  - Time shifts applied to ϕ don't couple to ω $\implies$ relaxed requirements for laser noise reduction, but time shifts cannot correct for $\omega\delta t$

Clock noise with detrending

- With detrending:

$$\eta_{12} = D_{12}\phi_2 - \phi_1 + D_{12}b_{23}q_2 - a_{12}q_1$$

$$\eta_{13} = D_{13}\phi_3 - \phi_1 - (b_{12} + a_{13})q_1$$

- After TDI:

$$\text{TDI} = \sum_{i,j \in I_2} P_{ij}\eta_{ij} \approx \sum_{i,j,k \in I_3^+} [P_{ki}D_{ki} - P_{ik}]b_{ij}q_i - \sum_{i,j \in I_2} P_{ij}a_{ij}q_i$$

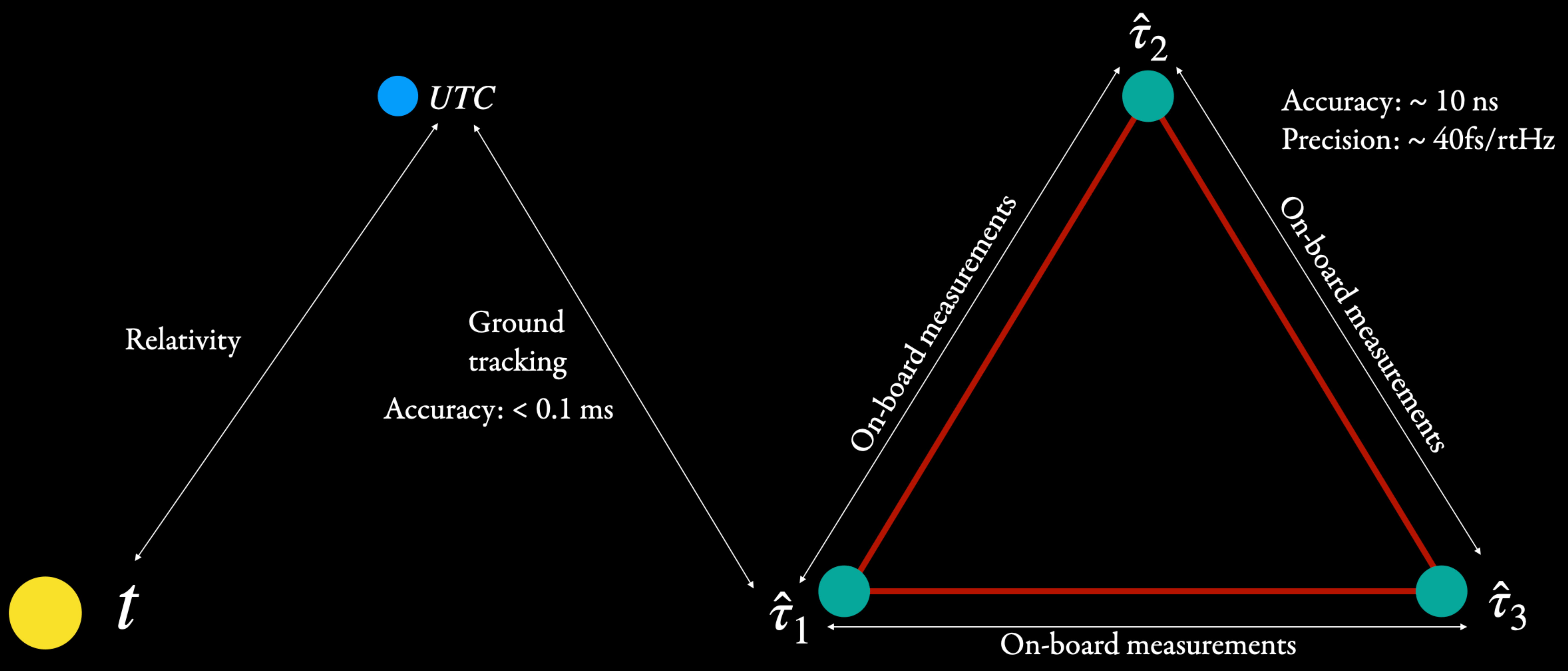
- Clock noise terms remain, and need to be removed in an extra processing step, using sideband measurements

$$r_{ij} = D_{12}q_2 - q_1$$

- Correction non-trivial data combination of r_{ij}, but can be constructed for any geometric TDI combination

Clock and time-related issues: summary

Note: Illustrative, numbers to be seen as placeholders

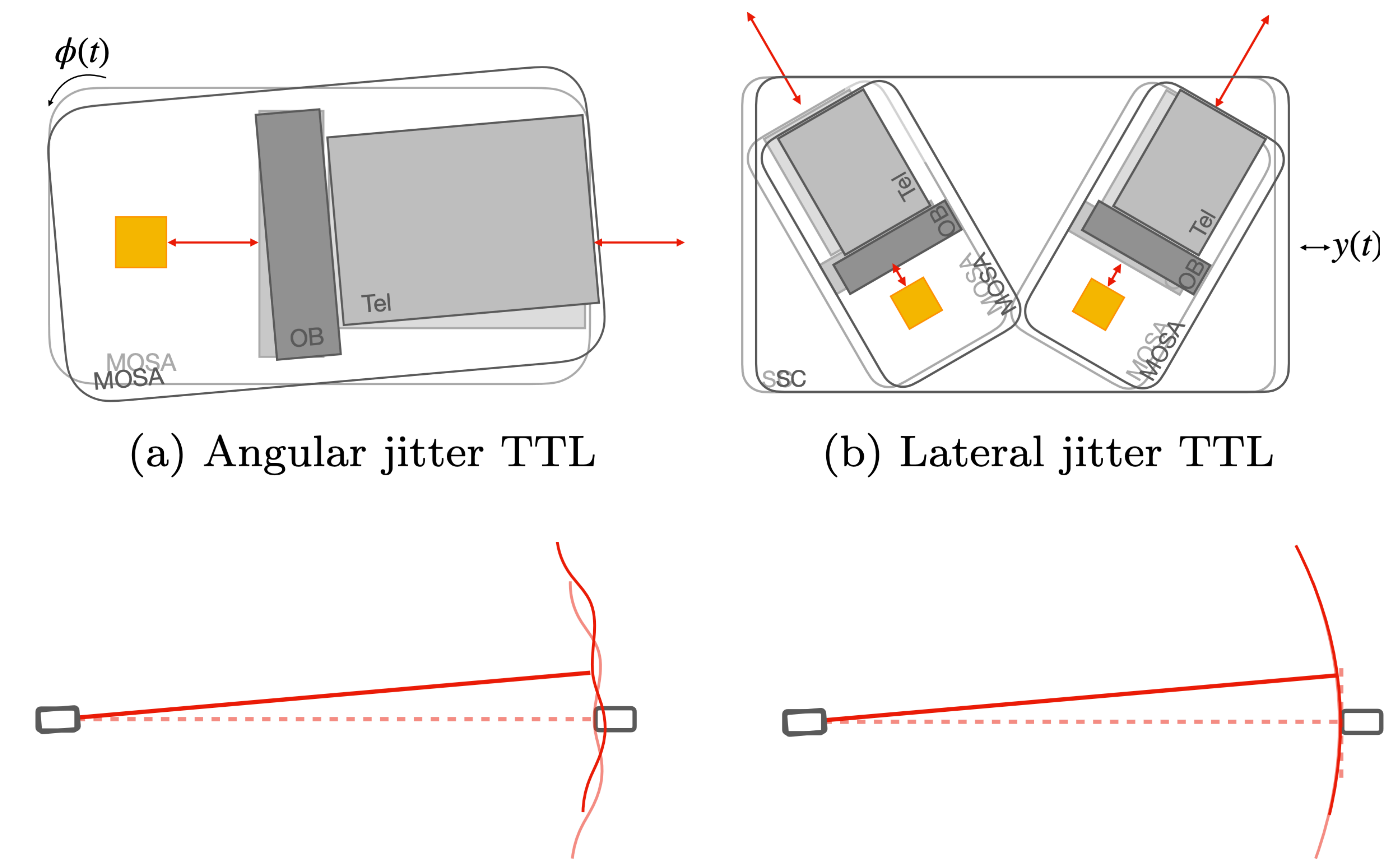


- Inter-SC accuracy: needed for laser noise reduction
- Inter-SC precision: needed for clock noise reduction
- Absolute accuracy wrt. TCB: mostly needed for astrophysical DA

Angular jitters: Tilt-to-length (TTL) couplings

TTL coupling

- Tilt-to-length:
  - Coupling of other degrees of freedom into desired longitudinal TM-TM measurement
- Typical origins:
  - Optical element misalignment
  - Wavefront errors
- Some compensation in hardware, but expected impact of TTL exceeds requirements



(a) Angular jitter TTL (b) Lateral jitter TTL

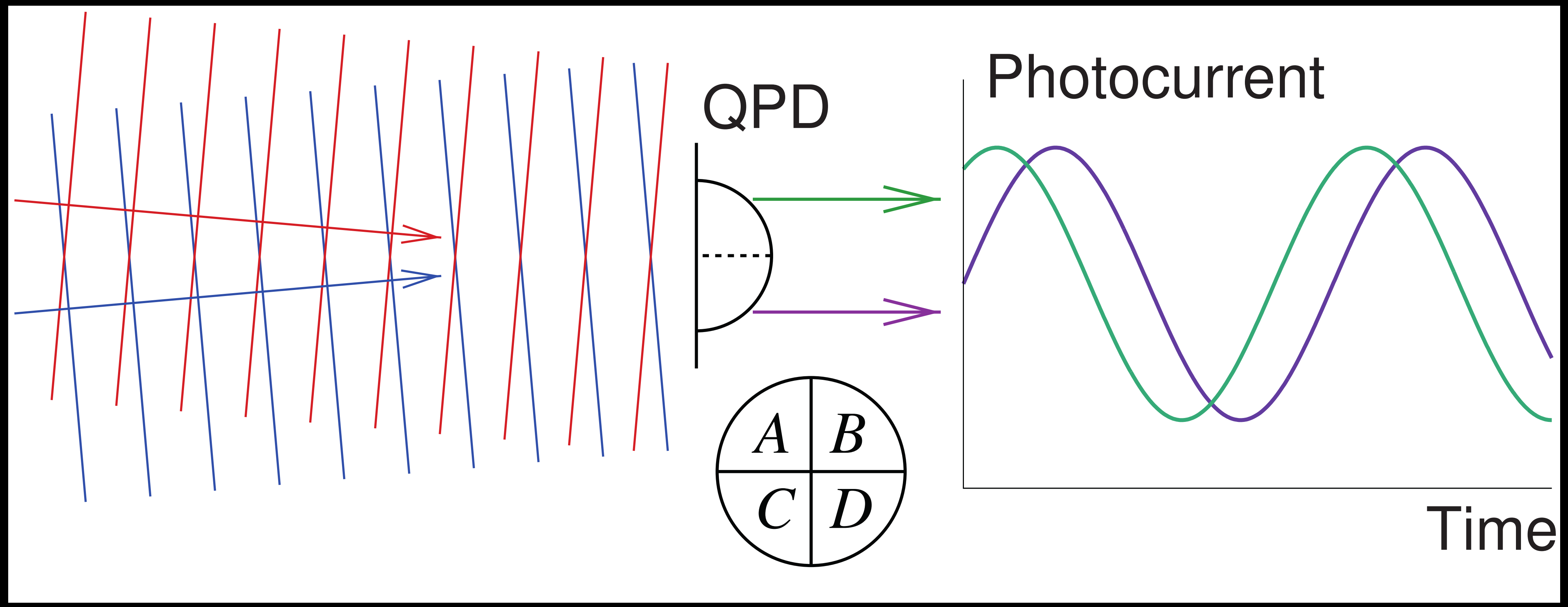
(c) Angular transmitter jitter TTL

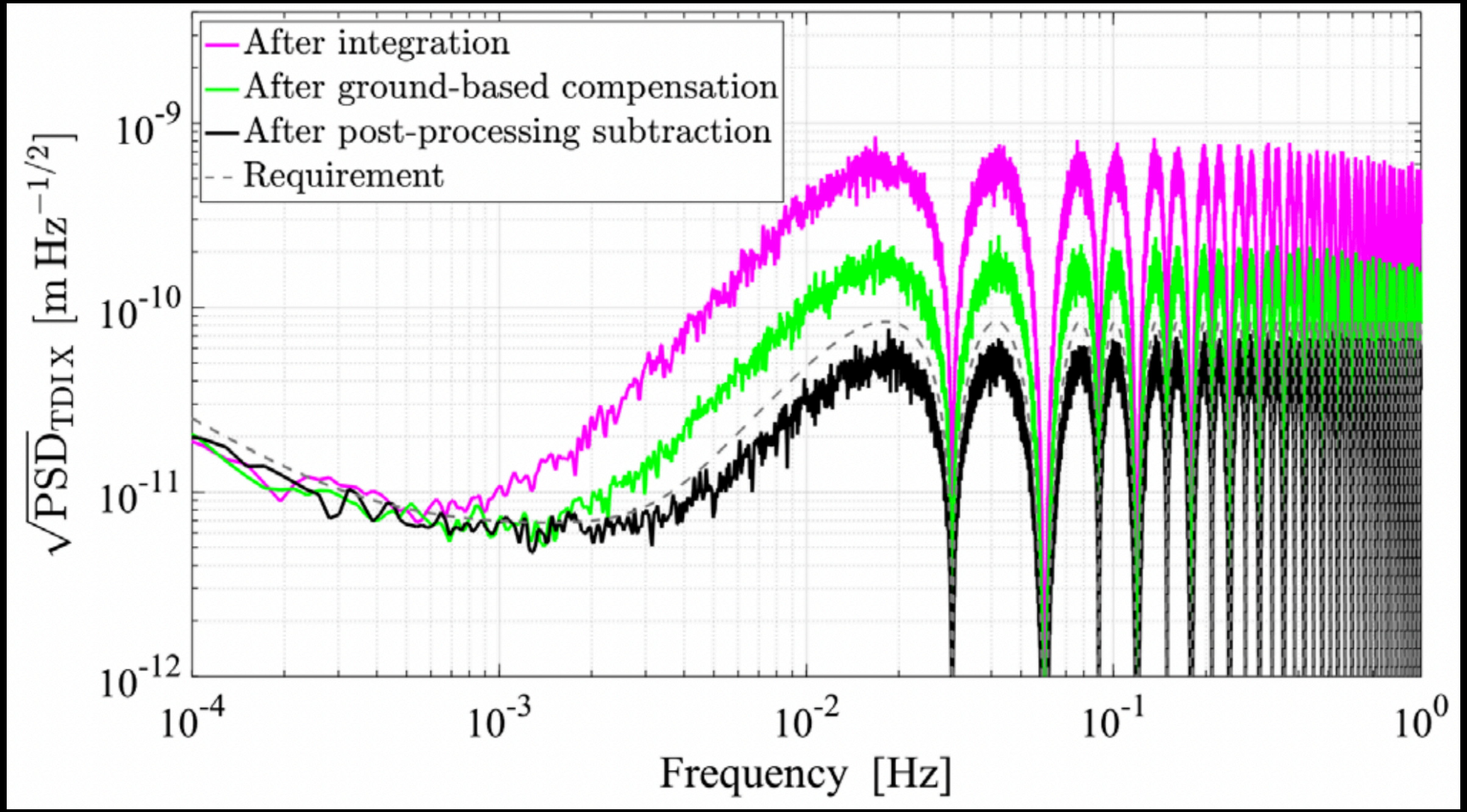
Details: Wanner et al. 10.1103/PhysRevD.110.022003

TTL compensation

Details: Paczkowski et al. 10.1103/PhysRevD.106.042005

- Differential wavefront sensing (DWS) allows to measure angular tilts of 2 beams by combining outputs of a quadrant photodiode
- TTL subtraction:
  - Assume linear model with a set of 24 coefficients relating tilt angles to pathlength changes
  - TTL coefficients are not known sufficiently well a-priori to subtract jitters
  - Fit DWS measurement coupling factors by minimizing the noise

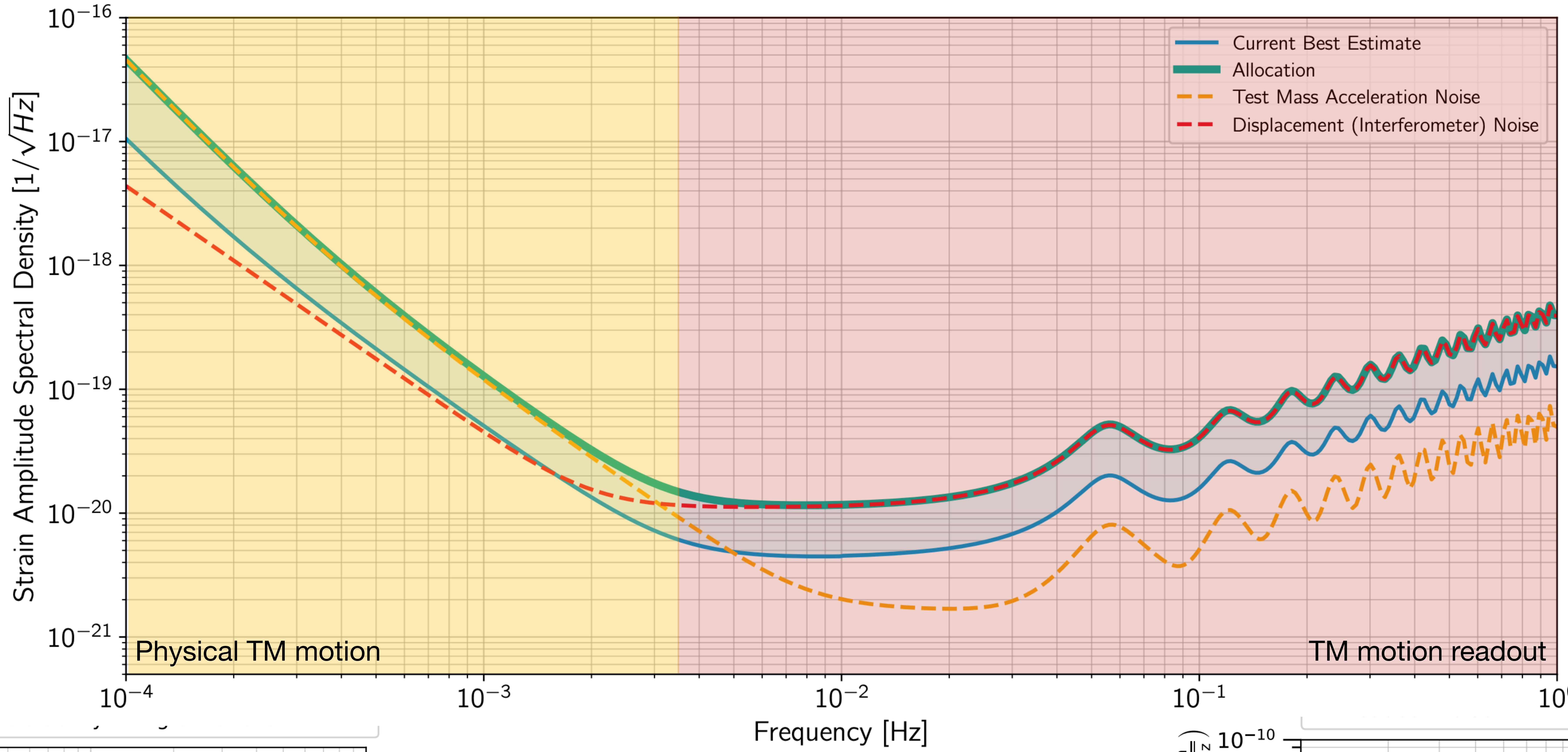


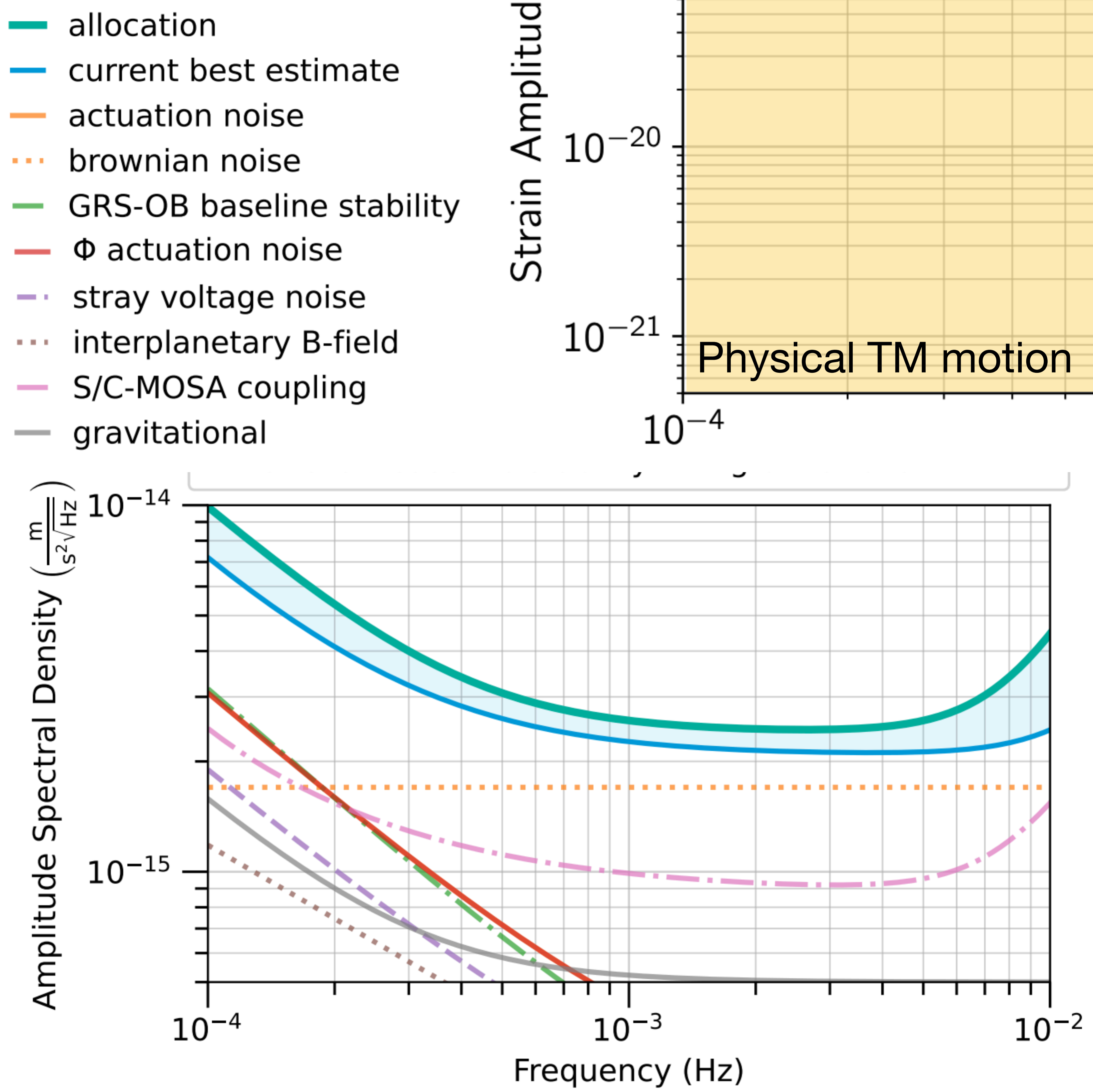


- Assumes requirement noise level (flat at high frequency)

LISA Performance and Sensitivity

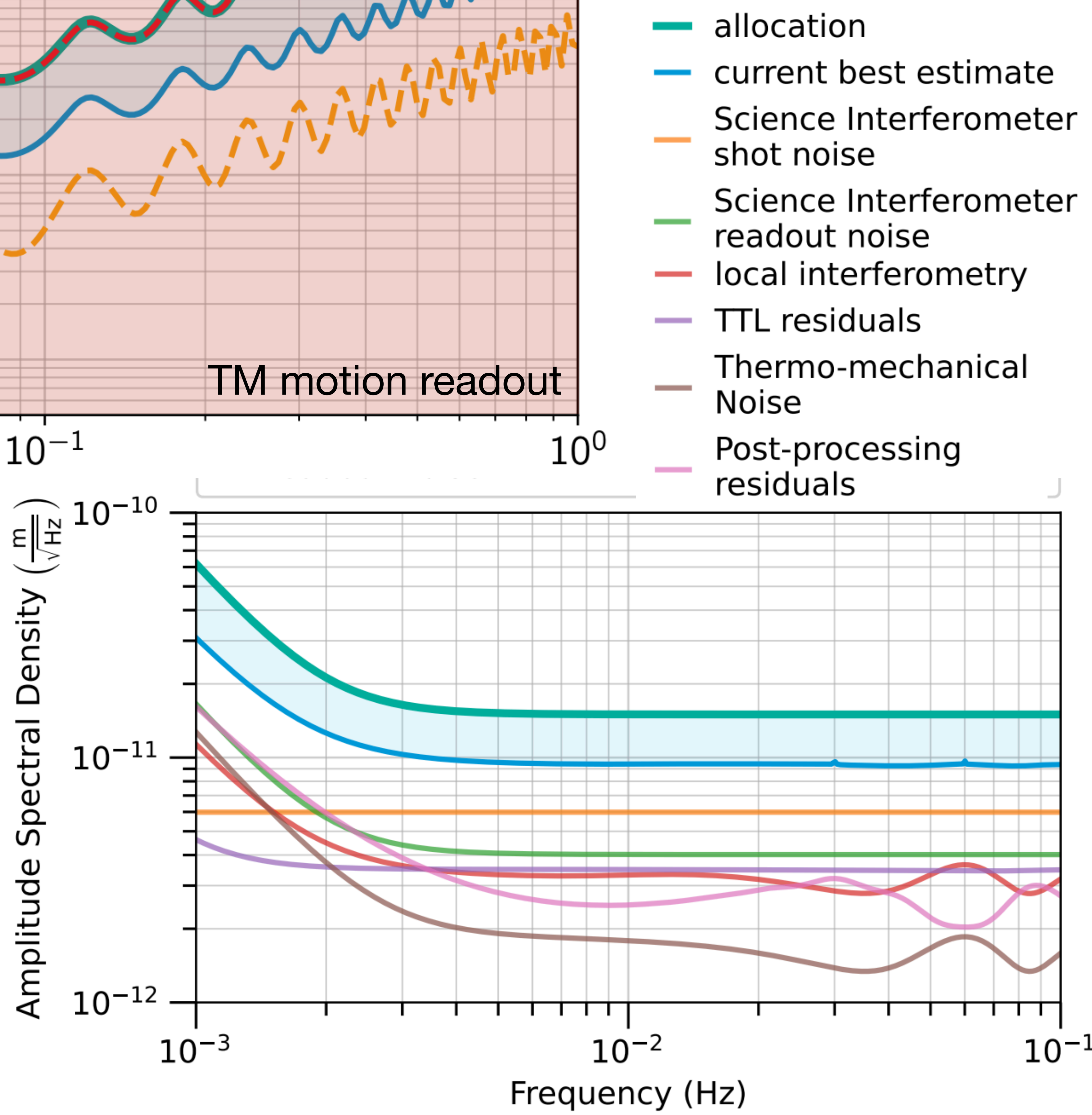
Main limiting noise sources left after TDI





$$S_h(f) \approx \frac{T^X_{acc}S^X_{acc} + T^X_{oms}S^X_{oms}}{2\,R^X_{GW}}$$

!!!



Thank you for your attention!