MAX PLANCK INSTITUTE
FOR GRAVITATIONAL PHYSICS
(ALBERT EINSTEIN INSTITUTE)

LISA Data Analysis I: PSDs, TDI response, sensitivity curves

LISA School for Early-career Scientists
11th of October 2025, Les Houches

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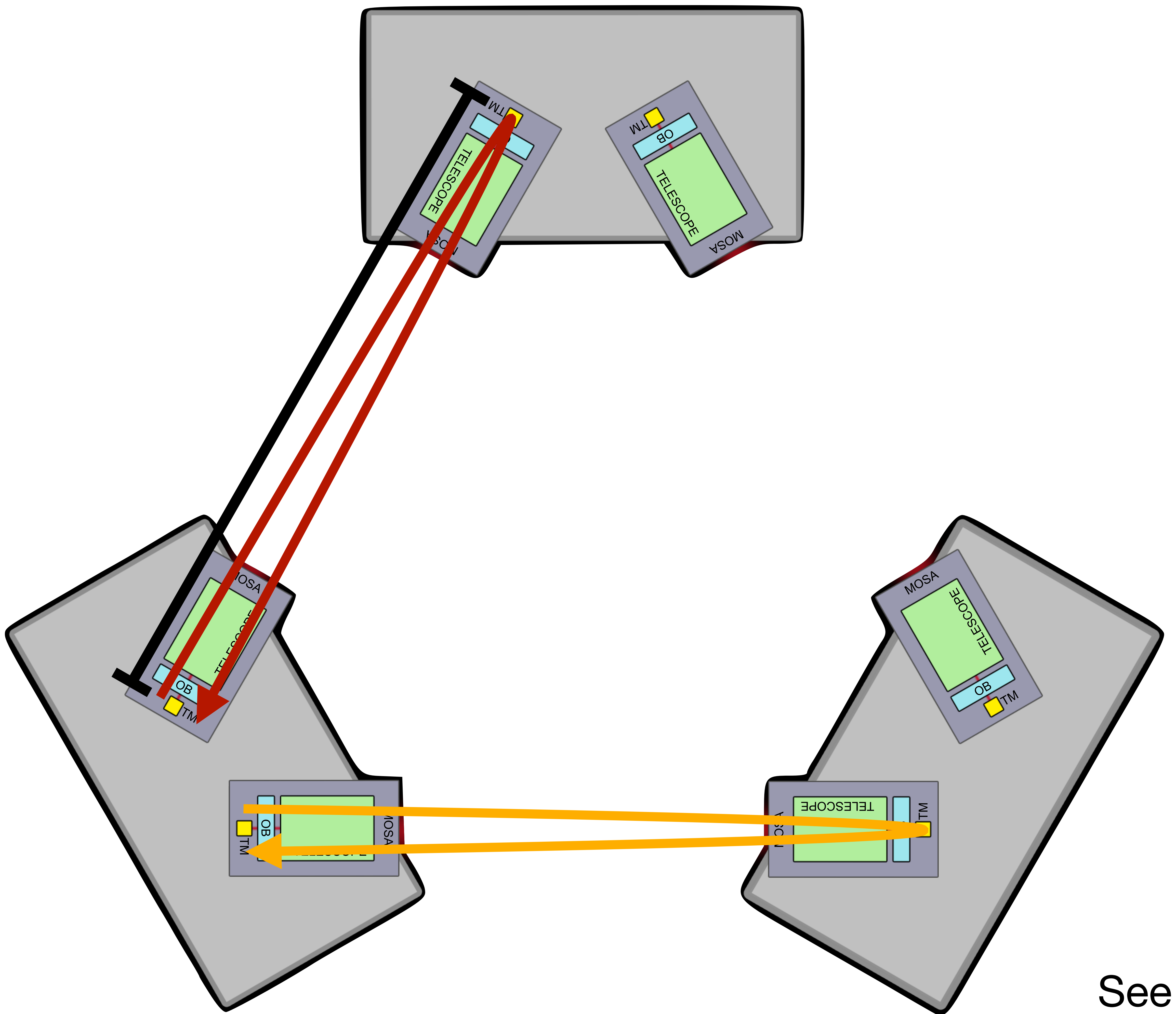
How to build a space based GW detector

An (over)-simplified LISA summary

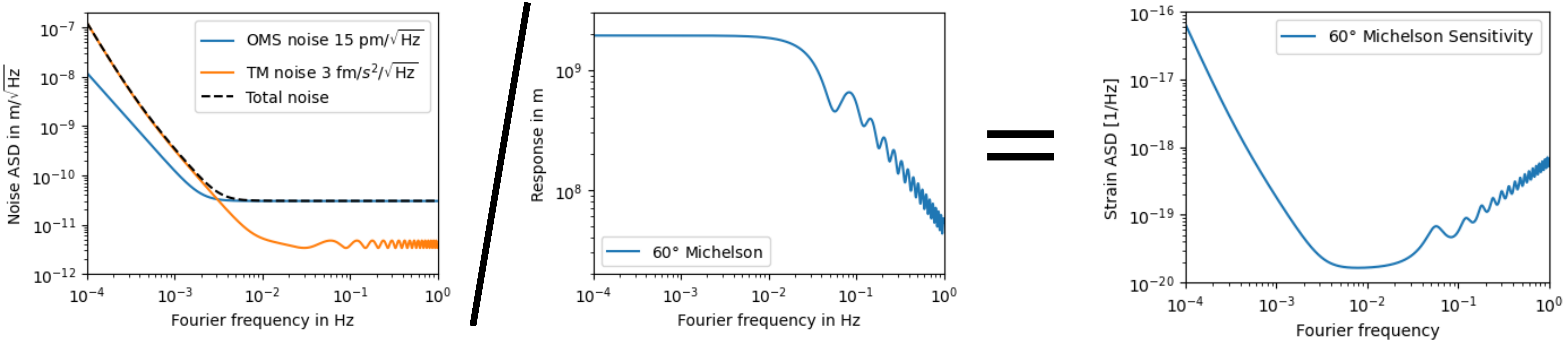


- Basic measurement principle:
  - Measure separation between free-falling test-masses
- Targeted performance:
  - Test-mass (TM) free-fall purity $3\,\mathrm{fm/s^2/\sqrt{Hz}}$
  - Optical metrology system (OMS) $15\,\mathrm{pm/\sqrt{Hz}}$
- Simplest possible interferometer: Michelson

See ‘LISA Redbook’, arXiv: 2402.07571, for comprehensive LISA details

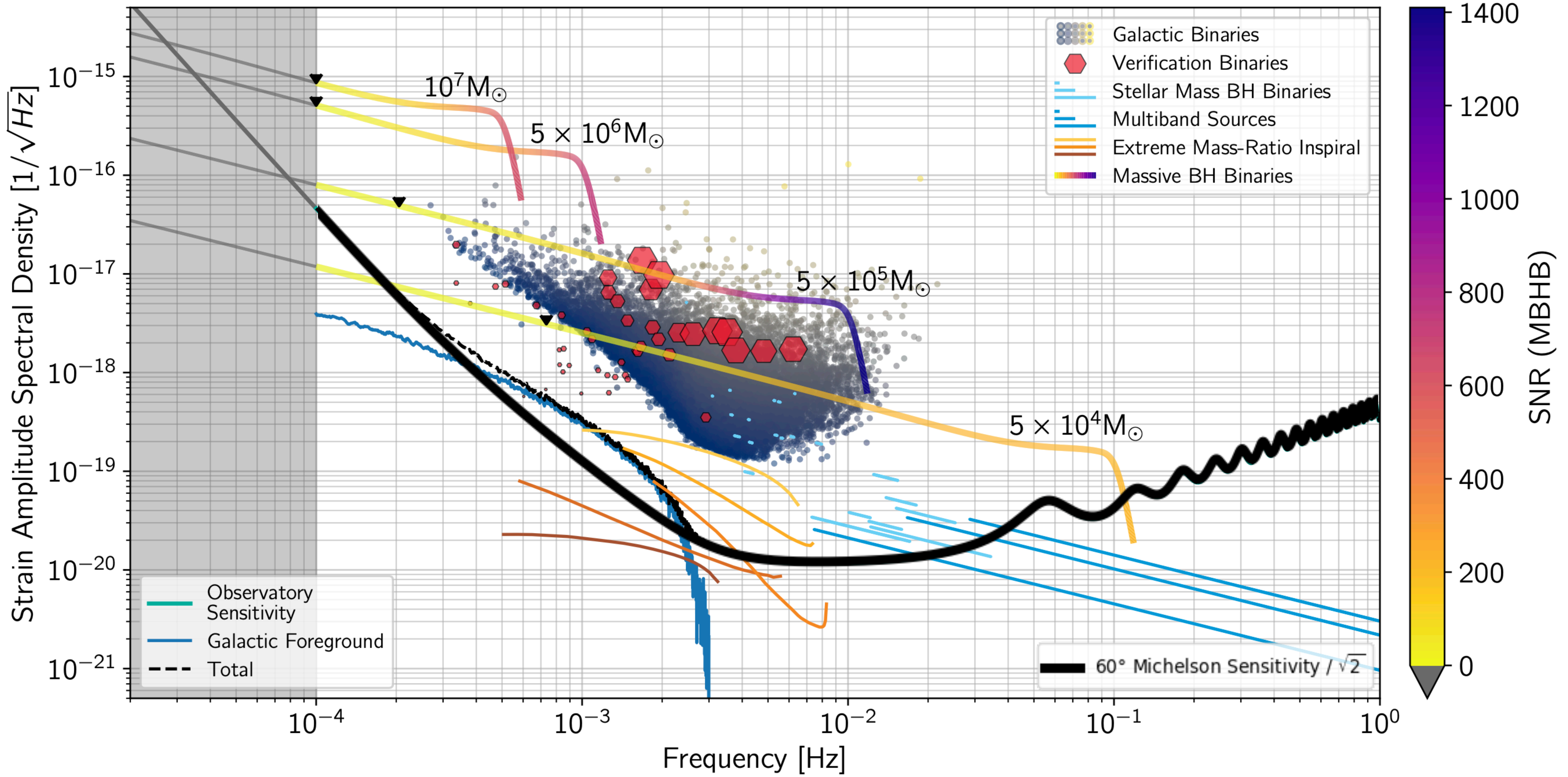
GW Sensitivity for one detector

- $S_h = S_{noise}/R$
- S_{noise}: Noise PSD in detector,
- R: sky and polarisation averaged GW response



Plot source: ‘LISA Redbook’, arXiv:2402.07571
Detailed derivation of this sensitivity curve: arXiv:2108.01167

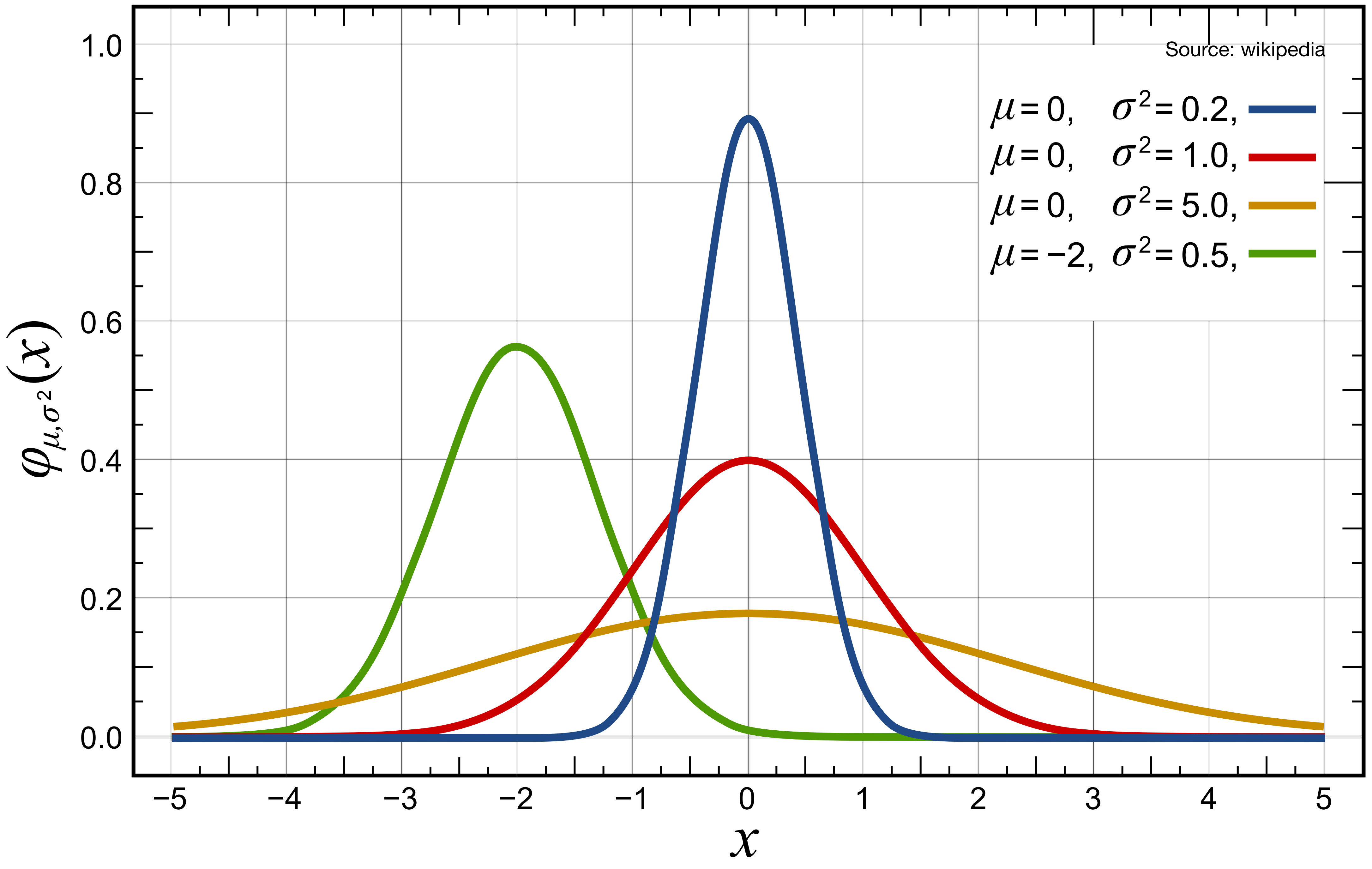
LISA Sensitivity



Some details on PSDs

Background

Random variable



- Random variable X: fully described by probability density $p(X)$
- Very important example: normal distribution

$$p(X) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{X-\mu}{2\sigma^2}}$$

- Normal distributed (Gaussian) variables fully characterised by
  - mean/expectation value $\mu = E[X] = \int_{-\infty}^{\infty} Xp(X)dX$
  - and variance $\sigma^2 = E[\,|X-\mu|^2]$

Background

Multivariate random variable

- Consider set of jointly distributed random variable, arranged as vector $\mathbf{X} = [X_1 \ldots X_n]$
- Mean becomes a vector, computed component-wise $\mu = E[\mathbf{X}]$
- Second-order structure becomes **covariance matrix**

$$\Sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)^*]$$

- Multivariate normal distribution again completely described by these two quantities:

$$p(\mathbf{X}) = \frac{1}{\sqrt{(2\pi)^N |\Sigma|}} e^{-\frac{1}{2}(\mathbf{X}-\mu)^\dagger \Sigma^{-1} (\mathbf{X}-\mu)}$$

- Intuition: Σ describes how strongly each X_i varies (diagonal) and how strongly they co-vary (off-diagonal)

Background

Stochastic processes

- Stochastic processes $X(t), Y(t)$: sets of random variables, labelled by an external parameter (time).

- Covariance becomes cross-correlation function: $R_{XY}(t, t') = E[X^*(t)Y(t')]$

- For ‘wide-sense stationary’ processes: this only depends on the lag $\tau = t - t'$

$$R_{XY}(\tau) = < X^*(t)Y(t + \tau) >$$

- In practice: often only one realization. Assumed property: ergodicity, ie.,

$$R_{XY}(\tau) = \lim_{T -> \infty} \int_{-T/2}^{T/2} X^*(t)Y(t + \tau)\mathrm{d}t$$

Background

Power spectral denstities, Cross-spectral densities

- The cross-spectral density of two WSS random processes is the Fourier transform of the cross-correlation function (Wiener-Kinchin theorem):

$$S_{XY}(f) = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-i2\pi f\tau} d\tau$$

- The power spectral density is simply S_{XX}, ie., the Fourier transform of the auto-covariance function.

- Alternative formulations (equivalent under some assumptions):

$$S_{XY}(f) = \lim_{T\to\infty} \frac{1}{T}[\tilde{X}^{*}_{T}(f)\tilde{Y}_{T}(f)], \qquad \text{with } \tilde{V}_{T}(f) \equiv \int_{-T/2}^{T/2} V(t) e^{-i2\pi f} dt$$

- or implicitly via $E[\tilde{X}^{*}(f)\tilde{Y}(f')] = S_{XY}(f)\delta(f-f')$

Background

Estimating PSDs/CSDs: periodograms

- In practice: data are time series, finite duration.
- Discretize previous expression:

$$\tilde{V}_T(f) = \int_{-T/2}^{T/2} V(t) e^{-i2\pi f t} dt \approx \sum_{m=-N/2}^{N/2-1} V(m\Delta t) e^{-i2\pi f (m\Delta t)} \Delta t$$

- Such that we can write:

$$\tilde{V}_T\left(k\frac{f_s}{N}\right) \approx \Delta t \underbrace{\sum_{m=-N/2}^{N/2} V(m\Delta t) e^{-i2\pi k \frac{m}{N}}}_{\mathrm{DFT}[V]_k}$$

Background

Estimating PSDs/CSDs: periodograms

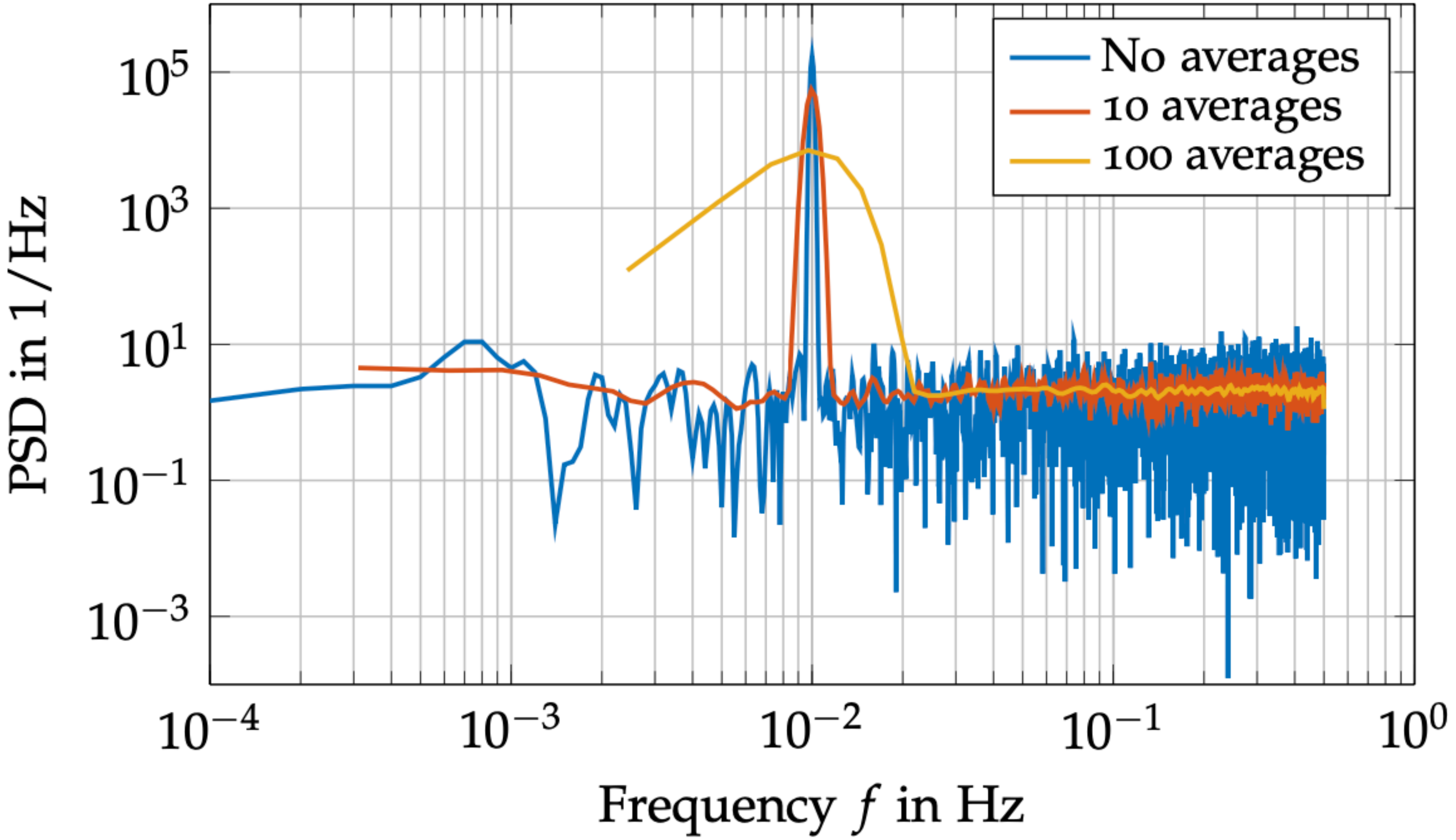
- PSDs are typically estimated by averaging over periodograms: :

$$
\begin{aligned}
S_{XY}\left(k\frac{f_s}{N}\right) &= \lim_{T\to\infty}\frac{1}{T}\left[\tilde{X}_T^*\left(k\frac{f_s}{N}\right)\tilde{Y}_T\left(k\frac{f_s}{N}\right)\right],\\
&\approx \lim_{N\to\infty}\frac{1}{N\Delta t}\left[\Delta t\,\mathrm{DFT}[X_T]_k^\dagger\,\Delta t\,\mathrm{DFT}[Y_T]_k^\dagger\right]\\
&\approx \frac{1}{Nf_s}\left[\mathrm{DFT}[X_T]_k^\dagger\,\mathrm{DFT}[Y_T]_k\right]
\end{aligned}
$$

- Note: a lot of ‘tricks’ and technical details to deal with complicated spectra and improve statistics!

Background

Example: harmonic signal in white noise



- Summary: the PSD tells us how noise power is distributed frequency-by-frequency

Noise PSDs and TDI

LISA model ingredients

Mostly delays

- For LISA, time domain model mostly linear (+ delays)
- Example: in a single link, uncorrelated OMS and TM noise appear as

$$\eta_{12} = N_{12}^{oms} + N_{12}^{tm} + D_{12}N_{21}^{tm}$$

- In the frequency domain, **constant** delays become exponentials:

$$FT[V(t-d)](f) = \int_{-\infty}^{\infty} V(\tau - d)e^{-i2\pi f\tau}d\tau = \int_{-\infty}^{\infty} V(\tau)e^{-i2\pi f(\tau - d)}d\tau = FT[V(t)](f)e^{-i2\pi fd}$$

- We get:

$$\tilde{\eta}_{12} = \tilde{N}_{12}^{oms} + \tilde{N}_{12}^{tm} + e^{-i2\pi fd_{12}}\tilde{N}_{21}^{tm}$$

TDI

… in the equal arm approximation

- We have, using short-hand $D_{ij}D_{ji} \equiv D_{iji}$,
  - For 0th generation TDI: $X_0 = \eta_{12} + D_{12}\eta_{21} - \eta_{13} - D_{13}\eta_{31}$
  - For 1st generation TDI: $X_1 = (1 - D_{131})(\eta_{12} + D_{12}\eta_{21}) - (1 - D_{121})(\eta_{13} + D_{13}\eta_{31})$
  - For 2nd generation TDI: $X_2 = (1 - D_{131} - D_{13121} + D_{1213131})(\eta_{12} + D_{12}\eta_{21}) - (1 - D_{121} - D_{12131} + D_{1312121})(\eta_{13} + D_{13}\eta_{31})$

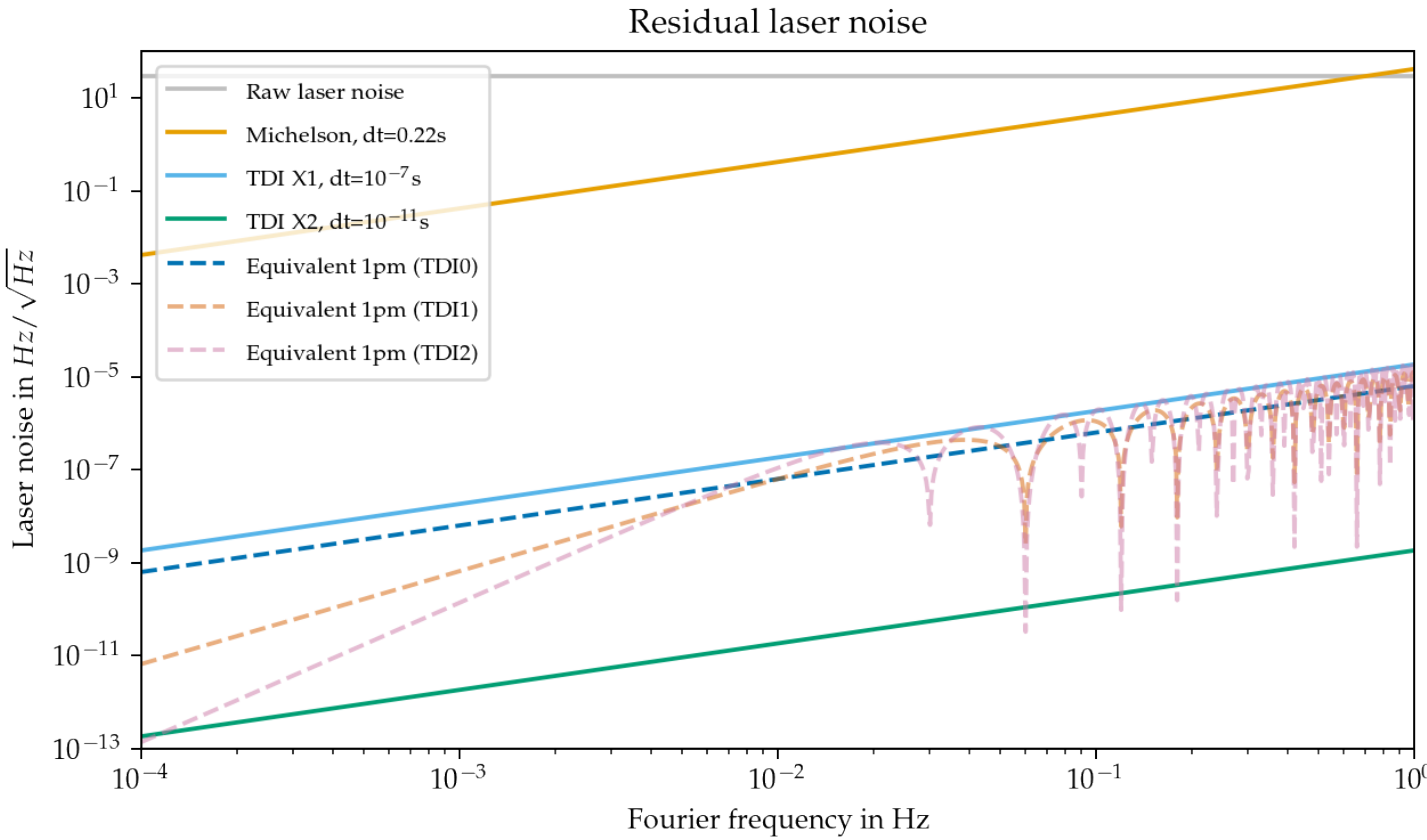
- If we work in the equal arm approximation, this simplifies drastically:

$$X_2 \approx (1 - D^4)X_1 \approx (1 - D^4)(1 - D^2)X_0$$

TDI

… in the equal arm approximation, in the frequency domain

- Assuming equal arms, $\tilde{X}_2 \approx \left(1 - e^{-i8\pi fd}\right)\left(1 - e^{-i4\pi fd}\right)\tilde{X}_0$
- Consequence: $PSD[X_2] \sim |\tilde{X}_2|^2 = 16\sin^2(4\pi fd)\sin^2(2\pi fd)PSD[X_0]$
- Under this assumption: **anything** entering the single links will receive the same global factor!
- Sensitivity of 2nd generation TDI **identical** to simple Michelson
- **Careful**: this does **not** hold with unequal arms, and not for noises not entering the single link directly!



TDI

From X to X,Y,Z

- Typically: we use not just X, but at least X, Y and Z

- We need to consider not just the PSD, but the cross-spectral density matrix:

$$\boldsymbol{\Sigma} = \begin{pmatrix} S_{XX} & S_{XY} & S_{XZ} \\ S_{YX} & S_{YY} & S_{YZ} \\ S_{ZX} & S_{ZY} & S_{ZZ} \end{pmatrix}$$

- If we not only consider equal arms, but also symmetric noise levels across the constellation, this simplifies:

$$\boldsymbol{\Sigma} \approx \begin{pmatrix} S_{XX} & S_{XY} & S_{XY} \\ S_{XY} & S_{XX} & S_{XY} \\ S_{XY} & S_{XY} & S_{XX} \end{pmatrix}$$

TDI

From X, Y, Z to A, E, T

$$\mathbf{\Sigma} \approx \begin{pmatrix} S_{XX} & S_{XY} & S_{XY} \\ S_{XY} & S_{XX} & S_{XY} \\ S_{XY} & S_{XY} & S_{XX} \end{pmatrix}$$

- The symmetric matrix has Eigenvalues

$$S_{XX} - S_{XY}, \quad S_{XX} - S_{XY}, \quad S_{XX} + 2S_{XY}$$

- By rotating XYZ into the Eigenbasis, we get uncorrelated channels:

$$A = \frac{-X + Z}{\sqrt{2}}, \quad E = \frac{X - 2Y + Z}{\sqrt{6}}, \quad T = \frac{X + Y + Z}{\sqrt{3}}$$

- Remarkably: independent of values of S_{XX}, S_{XY}!
- But: many approximations; in general, use full covariance matrix

From PSD to sensitivity curves

Sensitivity $\simeq$ reciprocal integrand of optimal SNR

- One can define the sensitivity via the optimal SNR of a matched filter:

$$\mathrm{SNR}^2 = 4\mathrm{Re}\int_{f_{min}}^{f_{max}} E[\tilde{X}_i \tilde{X}_j^*](\Sigma^{-1})_{ij}\,\mathrm{d}f$$

- For the signal, write $\tilde{X}_i = r_{ij}\tilde{h}_j$, where $h_i \in h_+, h_x$
- r_{ij} : encodes detector response for given sky direction:
  - Includes projection on single links (see Henri's lecture, also https://arxiv.org/abs/2302.12573 for a simplified frequency domain version)
  - Projection onto respective TDI channels (complex exponentials)

Sensitivity $\simeq$ reciprocal integrand of optimal SNR

- Assuming equal power in both polarisations, $E[\tilde{h}_k^* \tilde{h}_l] = \frac{1}{2} P_h \delta_{lk}$, we can simplify:

$$\mathrm{SNR}^2 = 2\mathrm{Re} \int_{f_{min}}^{f_{max}} P_h Tr[\Sigma^{-1} R]$$

- Here, R is the response matrix $rr^\dagger$
- We can now average over sky directions to get

$$< \mathrm{SNR}^2 > = 2\mathrm{Re} \int_{f_{min}}^{f_{max}} P_h Tr[\Sigma^{-1} < R >]$$

- Sensitivity:

$$S_h(f) \equiv \frac{1}{Tr[\Sigma^{-1} < R >]}$$

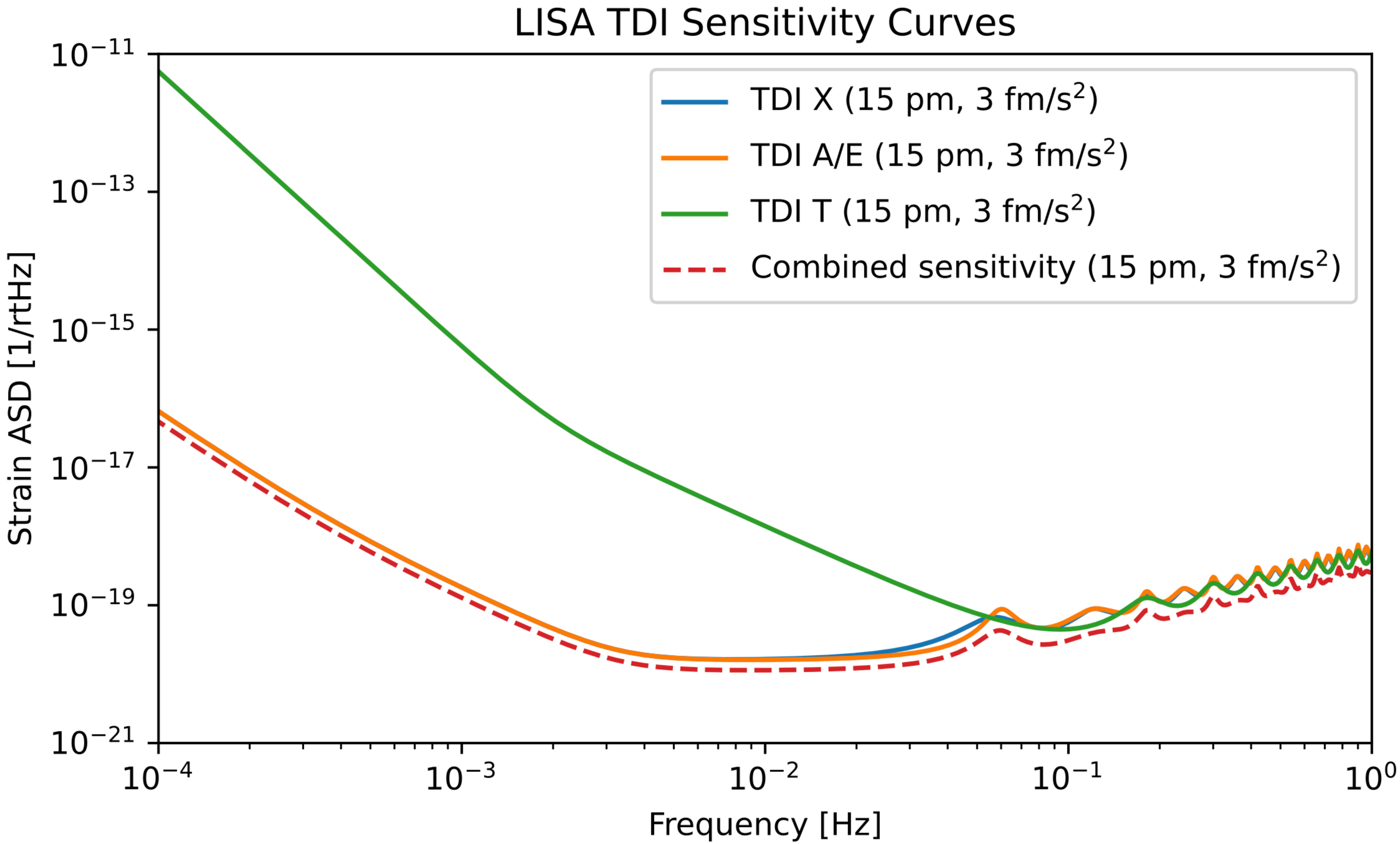
Back to AET

- For orthogonal channels: $S_h^{-1} = \mathrm{Tr}[\Sigma^{-1}\hat{R}] = (S_h^A)^{-1} + (S_h^E)^{-1} + (S_h^T)^{-1}$

- At low frequencies:

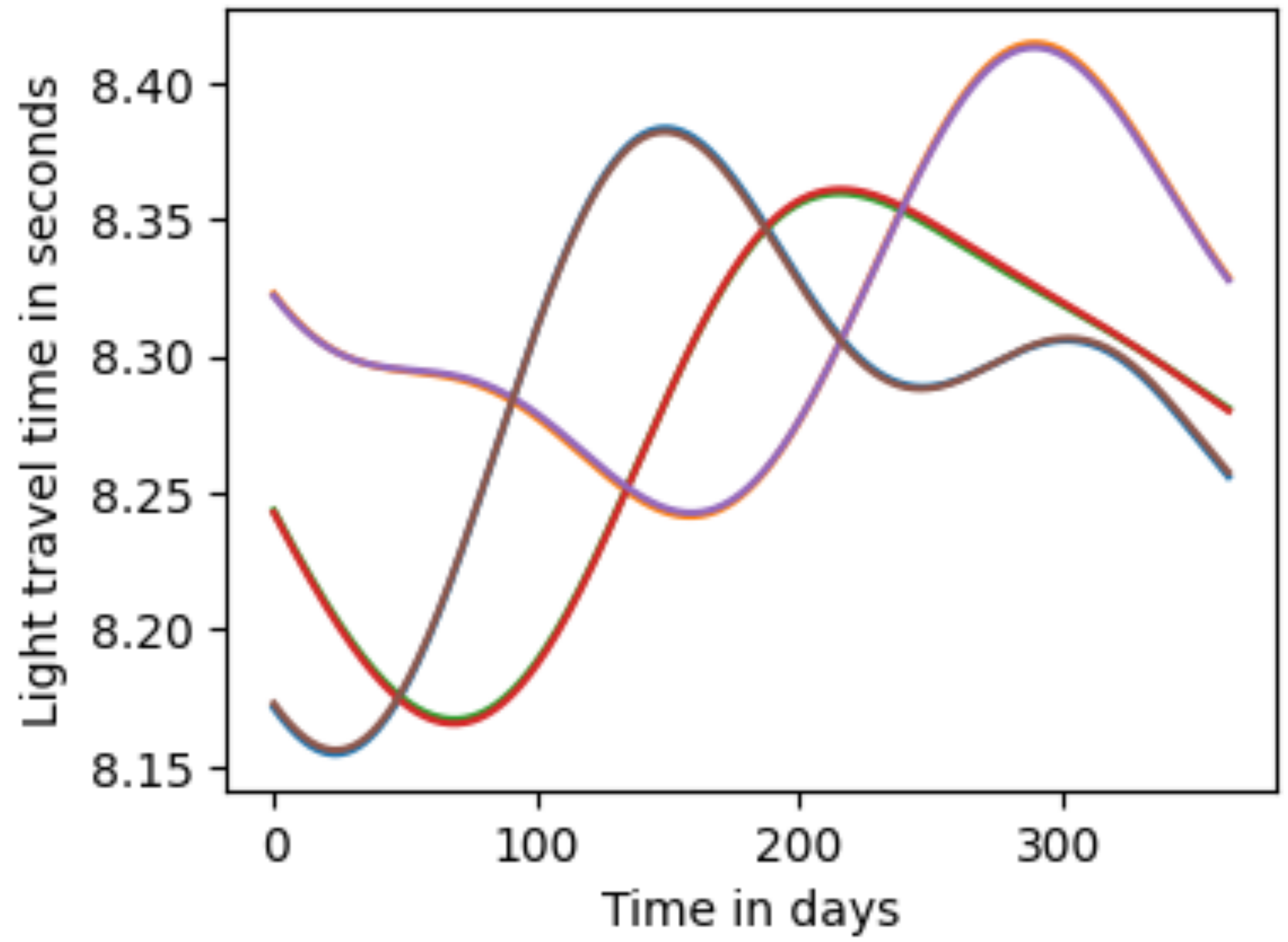
$$S_h^A = S_h^E = S_h^X, \quad S_h^T = 0$$

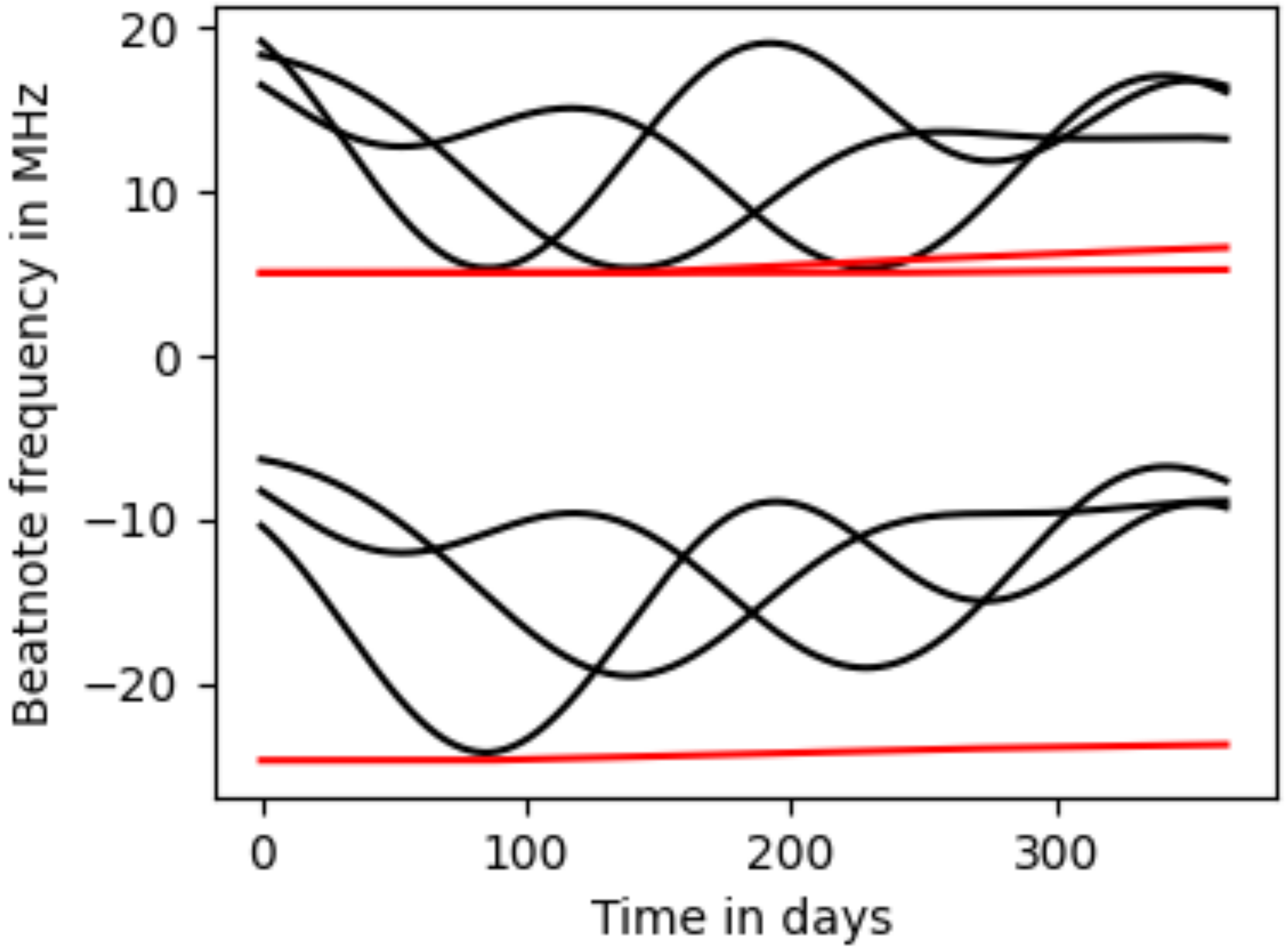
$$\implies S_h = \frac{S_h^X}{2}$$

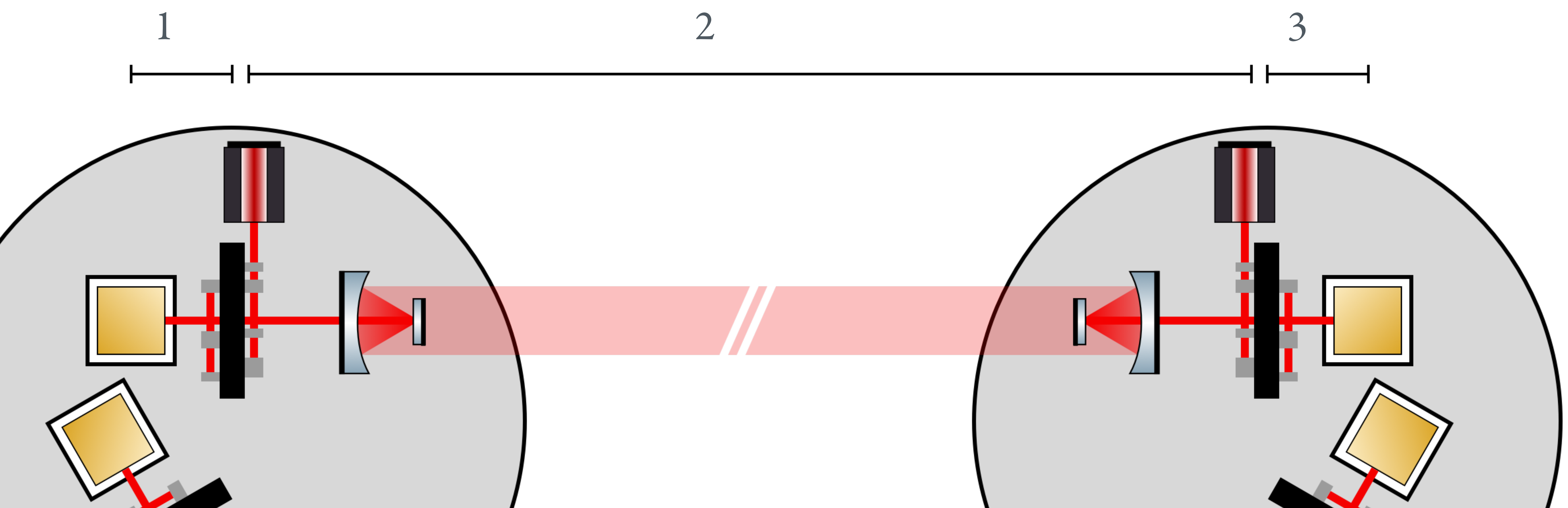


Increasing complexity

- Arm lengths are not equal, but time-varying
- Not just laser noise reduction! See previous talks for details
    - L0-L1 pipeline combines various measurements, reducing laser noise, spacecraft jitter, clock noises and TTL
    - Most corrections on single link level
    - Exception clock correction: another step of TDI, different transfer functions and non-stationary!







Full-LISA model example

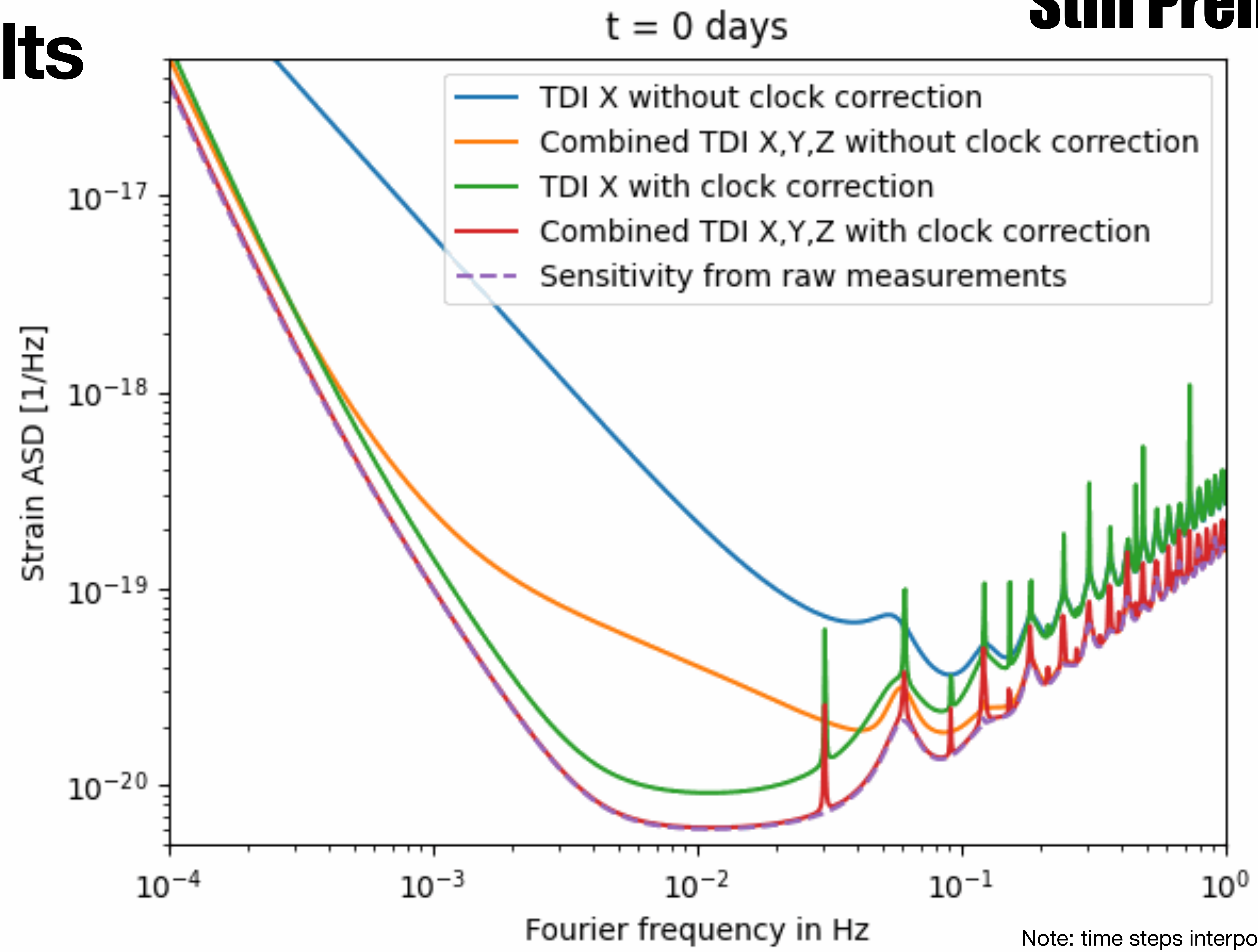
- Consider a covariance matrix for fundamental noises, including the ones suppressed by L0-L1
  - Model in current prototype: 63 noises, assumed uncorrelated (no TTL so far)
- Project these noises into the covariance matrix of the raw measurements produced by LISA.
  - Following current simulation model: 30 longitudinal interferometric measurements
- Further project this into the TDI variables with or without clock correction
- Do the same for signal response
- Compute sensitivity and compare optimal case vs. L0-L1 baseline, using

$$S_h = \frac{1}{\mathrm{Tr}[C^{-1}\hat{R}]}$$

- Note: current results assume **stationarity,** slightly **reduced laser noise level** and **simplified models** for individual components

Results

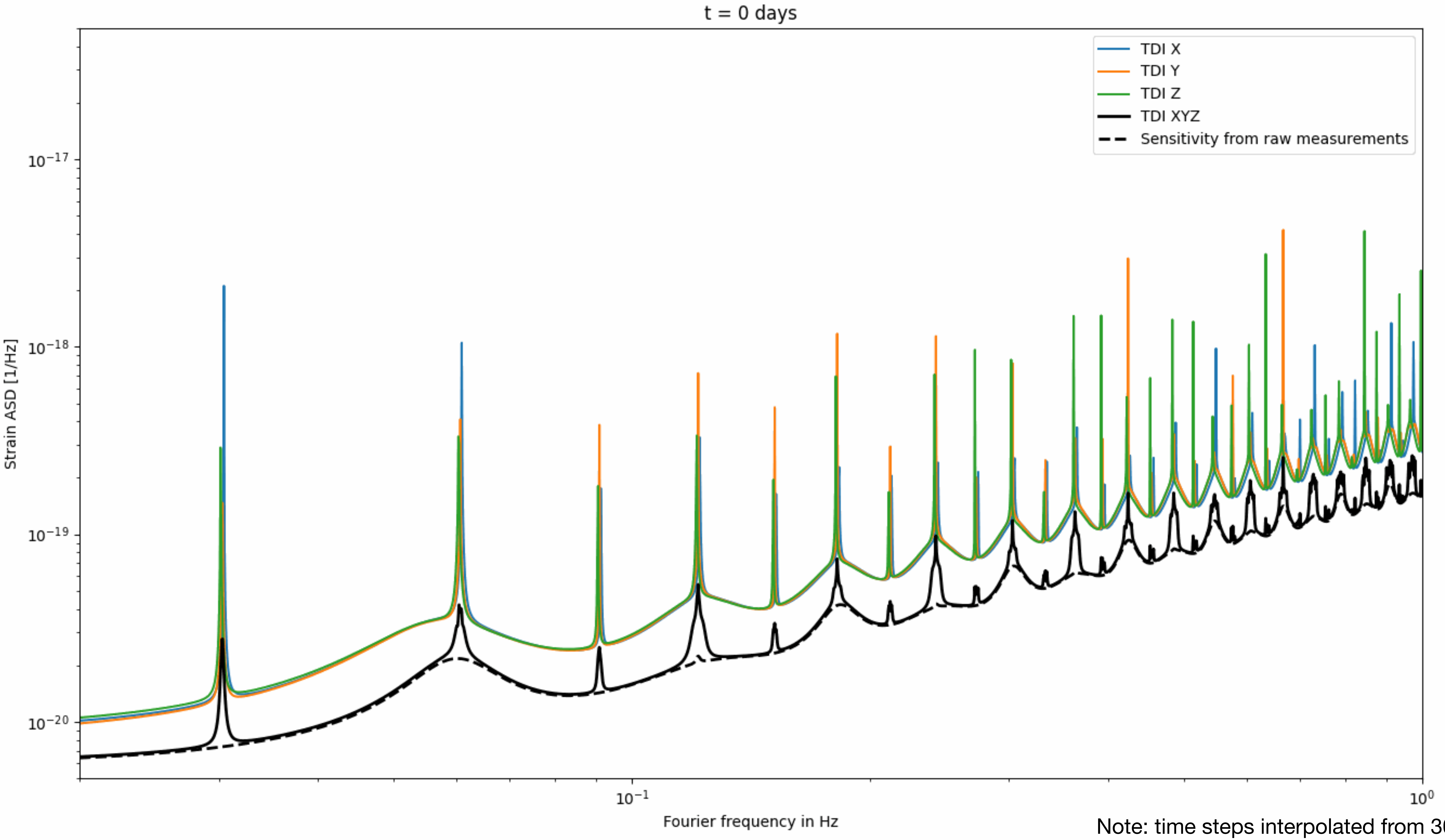
Still Preliminary!!!



Note: time steps interpolated from 30 day cadence

Zoom on high frequencies

Still Preliminary!!!

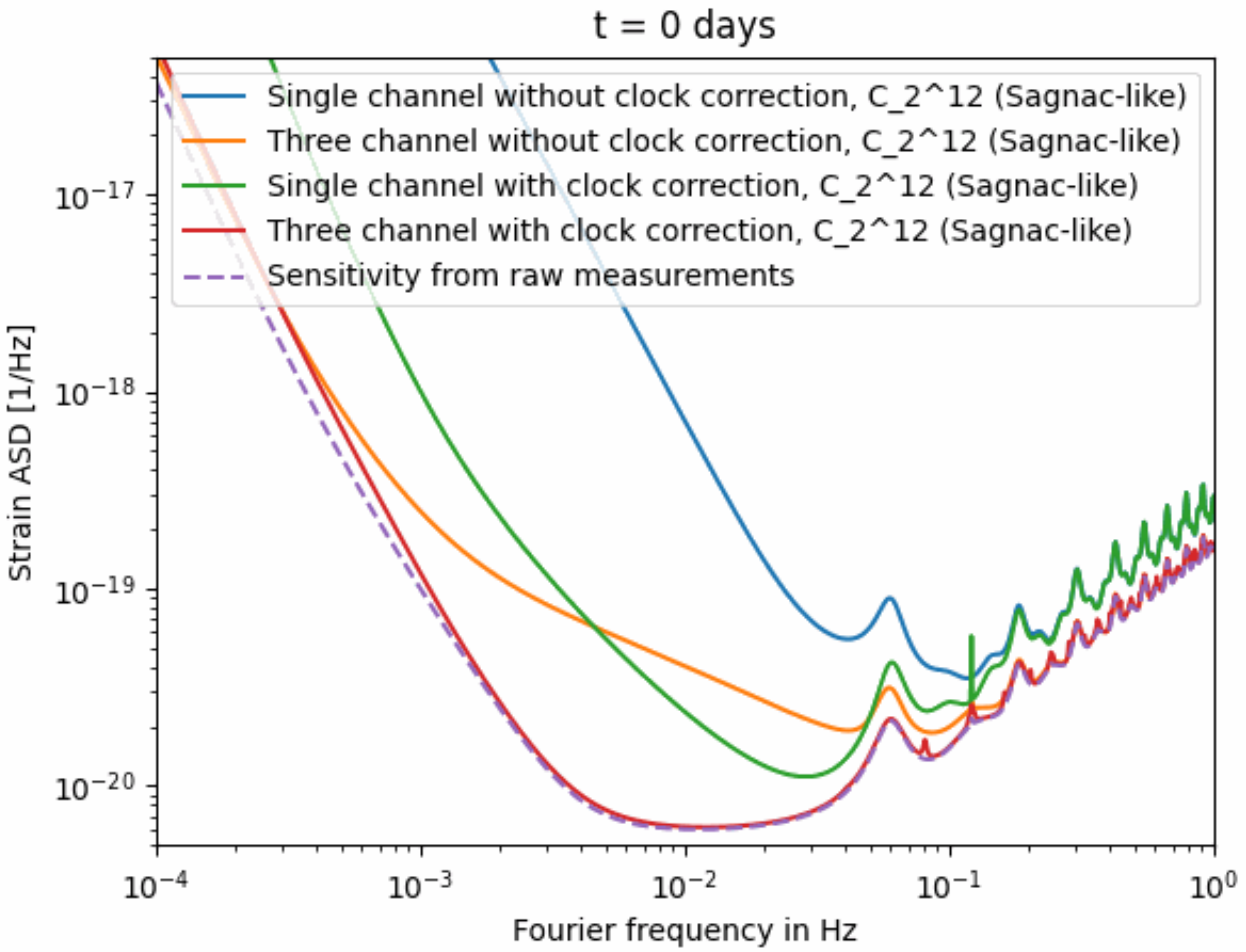


Note: time steps interpolated from 30 day cadence

Outlook

Preliminary!!!

- Precise impact on data analysis needs to be studied
  $\Longrightarrow$ Plan to include in DDPC data sets
- Height of peaks depend on the offset lock plan
  $\Longrightarrow$ could be optimised to reduce impact
- Specific to the choice of TDI combination
  $\Longrightarrow$ could be fixed by moving to different basis or different method (eg., TDI-∞)
- Model used should be updated to include TTL and more realistic models for individual noises



Ref. for C_2^12: 10.1103/PhysRevD.105.023009
Note: time steps interpolated from 30 day cadence