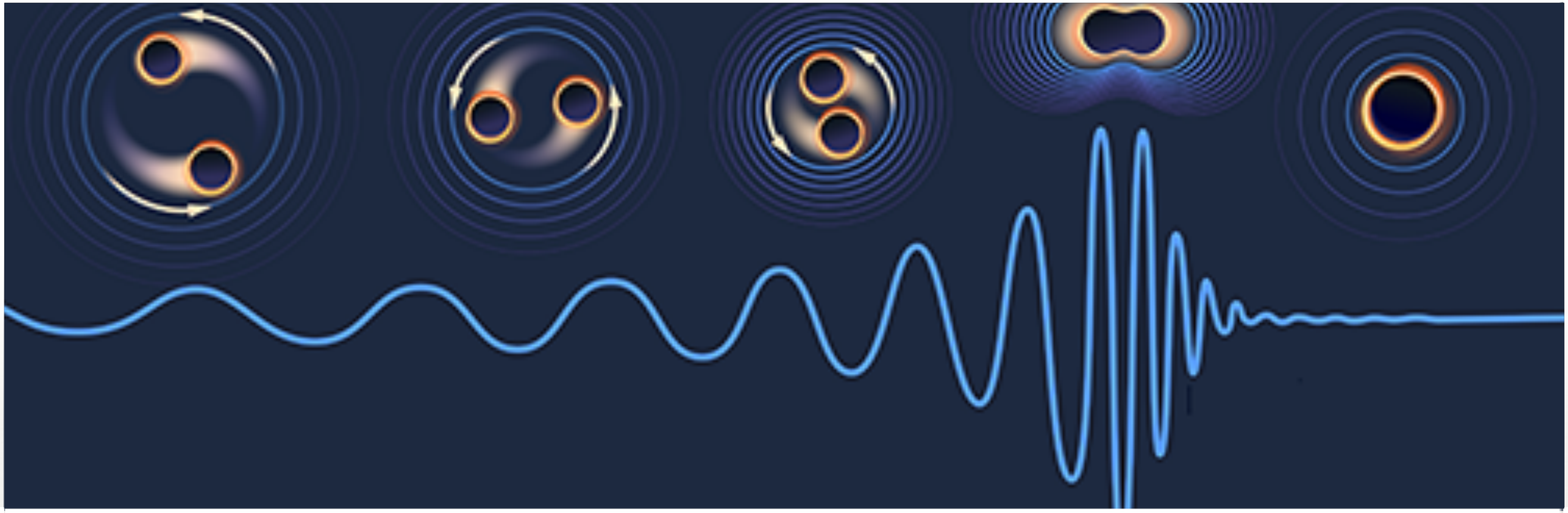


LISA School for Early Career Scientists

Black Hole Perturbation Theory for Inspiral and Ringdown

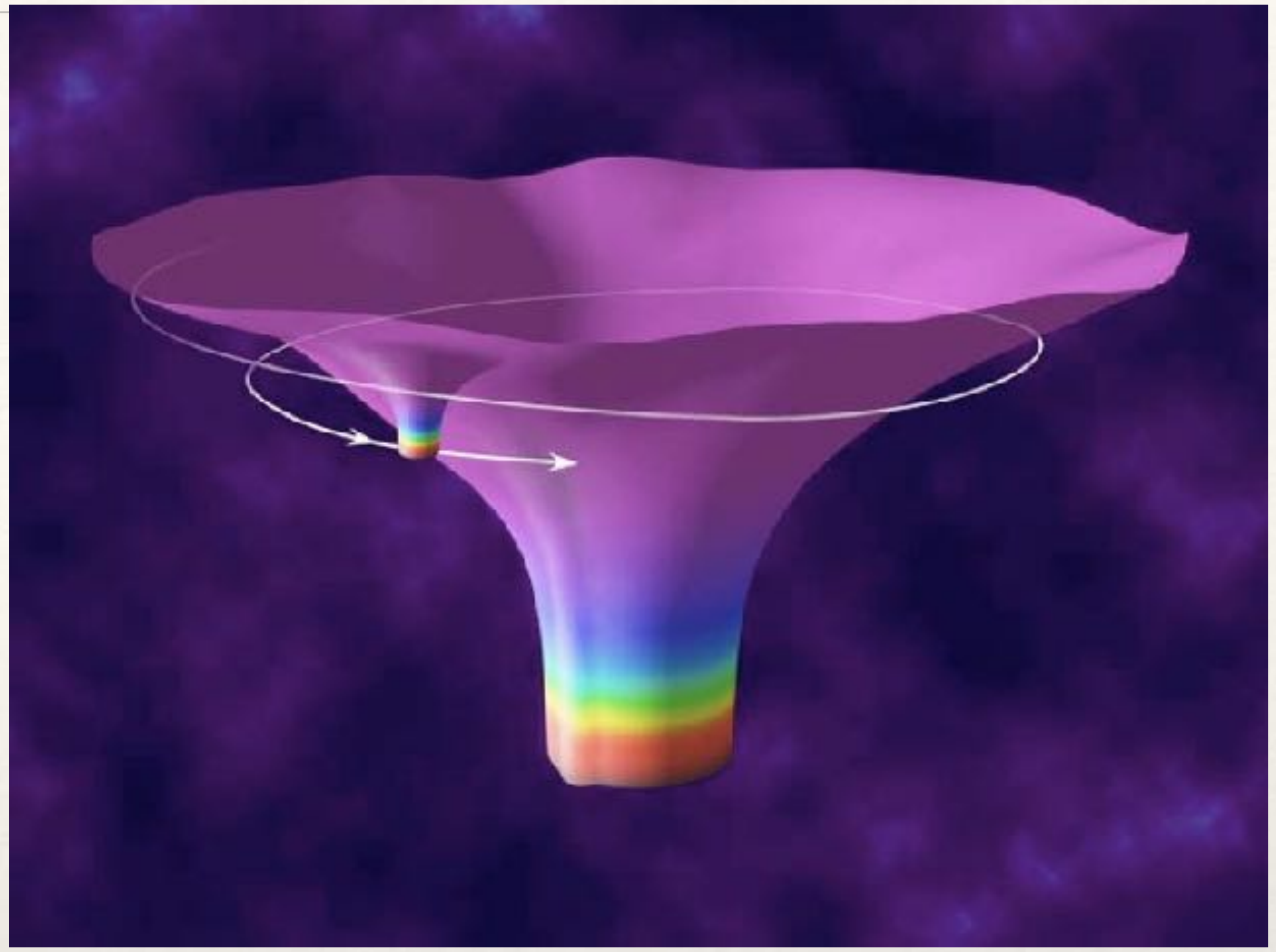
Universitat de les Illes Balears

IAC3 Institut d'Aplicacions Computacionals de Codi Comunitari

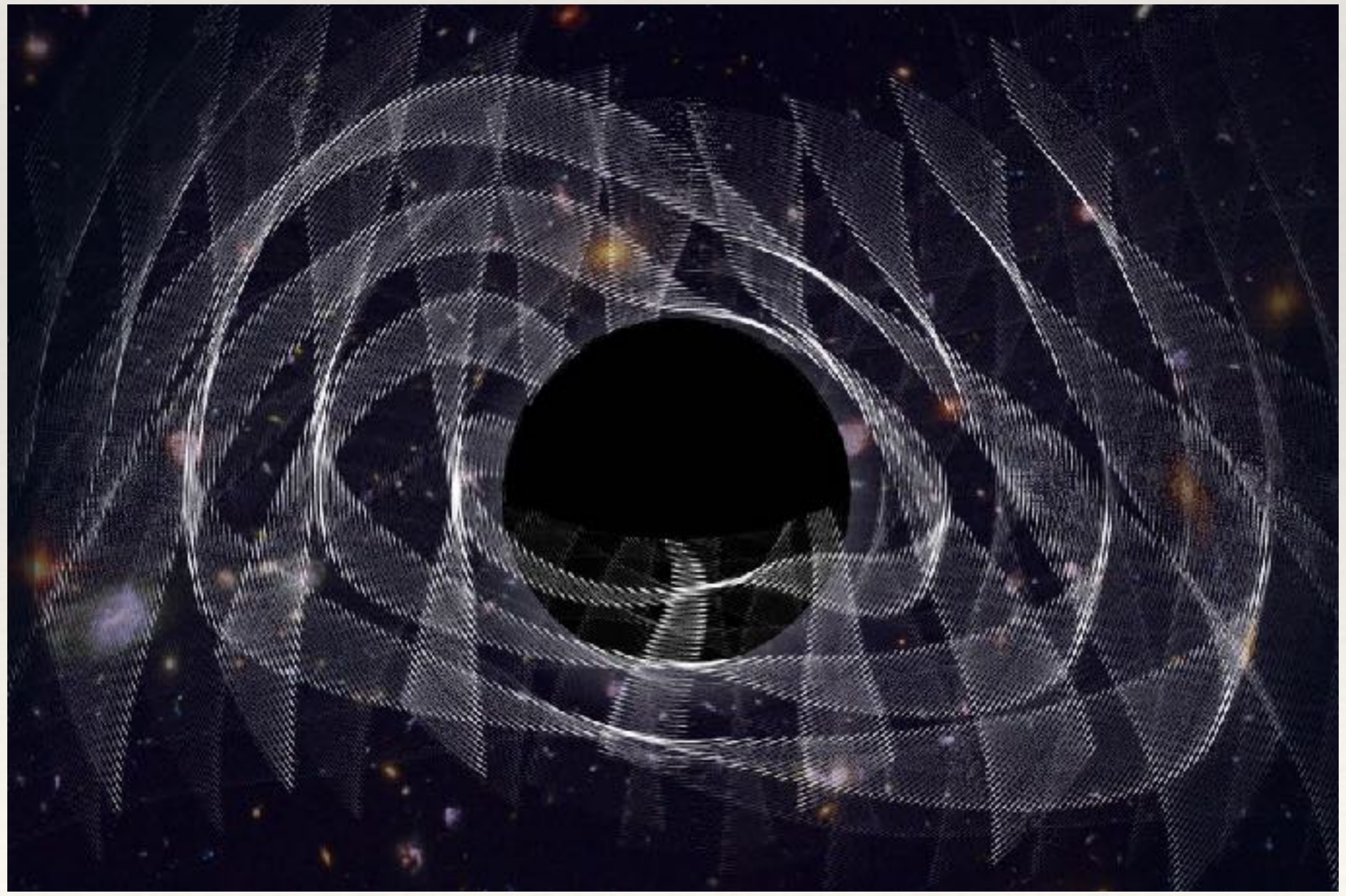
Marta Colleoni

- We will focus on two applications of perturbation theory (BHPT)
- **Evolution of EMRIs**
  - Perturbation parameter given by the mass ratio:

$$\epsilon = m_2/m_1 \ll 1$$



- **Ringdown**: quasi-normal ringing of the remnant produced by a compact binary merger
  - A merger of two BHs is a pretty violent event!
  - Still, can apply perturbation theory after the merger has relaxed to a nearly stationary Kerr or Schwarzschild BH.



The Teukolsky equation: intro

- Teukolsky equation is a powerful tool to study perturbations of BH spacetimes
- One can solve only 1 complex second-order differential equation, as opposed to ten coupled equations
- The Teukolsky equation is separable into radial ODEs by decomposing the angular dependence into **spin-weighted spheroidal harmonic** modes.
- In particular, it allows to:
    - Compute the quasi-normal-modes of perturbed BHs
    - Obtain the necessary input for adiabatic inspirals

Null tetrad

- A set of linearly independent four-vectors

$$\{\ell^{\mu}, n^{\mu}, m^{\mu}, \bar{m}^{\mu}\}$$

Choose ℓ^{μ}, n^{μ} so that they align with some “special” null geodesics that

- are related to ingoing and outgoing radiation
- satisfy certain relationships that greatly simplify the perturbation equations

Metric can be written as $g_{\alpha\beta} = -\ell_{\alpha}n_{\beta} - \ell_{\beta}n_{\alpha} + m_{\alpha}\bar{m}_{\beta} + m_{\beta}\bar{m}_{\alpha}$

- We can project arbitrary tensors into the this tetrad in the usual way e.g. $R_{ab} = e^{\mu}_{a} e^{\nu}_{b} R_{\mu\nu}$ where $e^{\mu}_{a} = \partial x^{\mu}/\partial y^{a}$ (which give us tetrad vector components in coordinate basis)

Newman-Penrose scalars

- In th tetrad formalism, one can study perturbations in terms of some complex scalar quantities $\Psi_n, n = 0...4$ obtained by contracting the Weyl tensor with the tetrad four-vectors:

$$\Psi_0 := C_{\alpha\beta\gamma\delta}\,\ell^\alpha m^\beta \ell^\gamma m^\delta$$
$$\Psi_1 := C_{\alpha\beta\gamma\delta}\,\ell^\alpha n^\beta \ell^\gamma m^\delta$$
$$\Psi_2 := C_{\alpha\beta\gamma\delta}\,\ell^\alpha m^\beta \bar{m}^\gamma n^\delta$$
$$\Psi_3 := C_{\alpha\beta\gamma\delta}\,\ell^\alpha n^\beta \bar{m}^\gamma n^\delta$$
$$\Psi_4 := C_{\alpha\beta\gamma\delta}\,n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

$$C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{2}{n-2}\left(g_{\alpha[\gamma}R_{\delta]\beta} - g_{\beta[\gamma}R_{\delta]\alpha}\right) + \frac{2}{(n-1)(n-2)}R\, g_{\alpha[\gamma}g_{\delta]\beta}.$$

- These complex scalars encode the 10 degrees of freedom of the Weyl tensor.
- The Riemann tensor can be decomposed into a tracefree part (Weyl tensor) and a trace part (Ricci tensor)

The physical interpretation of Ψ_0 and Ψ_4

- The Einstein field equations imply that Riemann tensor is related to the distribution of matter in spacetime
- The trace-free part, on the other hand, represents the true gravitational degrees of freedom of the Riemann tensor, or, in other words the true radiative degrees of freedom of the gravitational field
- E.g. in vacuum flat spacetime, knowing that
- $R_{0i0j} = C_{0i0j} = -\frac{1}{2}\ddot{h}_{ij}$ (all other components are zero)
- One can use the previous definitions to show that, asymptotically $\Psi_4 \propto (\ddot{h}_+ - i\ddot{h}_\times)$
- The numerical prefactor depends on the tetrad normalisation

The Teukolsky equation

- $\Psi_0 = \Psi_4 = 0$ in the background and $\delta\psi \to \delta\psi + \xi^\alpha \nabla_\alpha \psi$, so perturbations of these two scalars are gauge invariant in linear perturbation theory and represent the physical radiative degrees of freedom of the gravitational field
- In regions where they satisfy the homogeneous Teukolsky equations, the two scalars above are not independent (they are related by some identities called the Teukolsky-Starobinsky identities)
- The scalars satisfy some master equations that are fully separable into a radial and angular parts

$$\Psi(t,r,\theta,\phi) = \sum_{l=|s|}^{\infty} \sum_{m=-l}^{l} e^{im\phi} \int_{-\infty}^{+\infty} d\omega \; e^{-i\omega t} \; {}_s\psi_{lm\omega}(r) \; S_{lm}(\theta; a\omega)$$

$$\left[\frac{d}{d\chi}\left((1-\chi^2)\frac{d}{d\chi}\right) + a^2\omega^2\chi^2 - \frac{(m+s\chi)^2}{1-\chi^2} - 2as\omega\chi + s + A\right] {}_sS_{\ell m} = 0,$$

$$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) + \frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - {}_s\lambda_{\ell m}\right] {}_s\psi_{\ell m\omega} = {}_sT_{\ell m\omega}$$

The Teukolsky equation

- The angular part contains some functions ${}_sS_{\ell m}$ called *spin-weighted spheroidal harmonics* (not to be confused with spin-weighted **spherical** harmonics!)
- *Spin-weighted spheroidal harmonics* come up as natural basis functions to work with in perturbation theory (and in particular ringdown GW signals)

How do these functions relate to the more familiar ${}_sY_{\ell m}$?

- In Schwarzschild, spin-weighted spheroidal harmonics reduce to spin-weighted spherical harmonics
- In Kerr, we can relate the two basis functions via spherical-spheroidal mixing coefficients

$$\int {}_sS^*_{\ell' m' m'}\ {}_sY_{\ell m} d\Omega = \mu_{m\ell\ell' m'}(a)\delta_{m,m'}$$

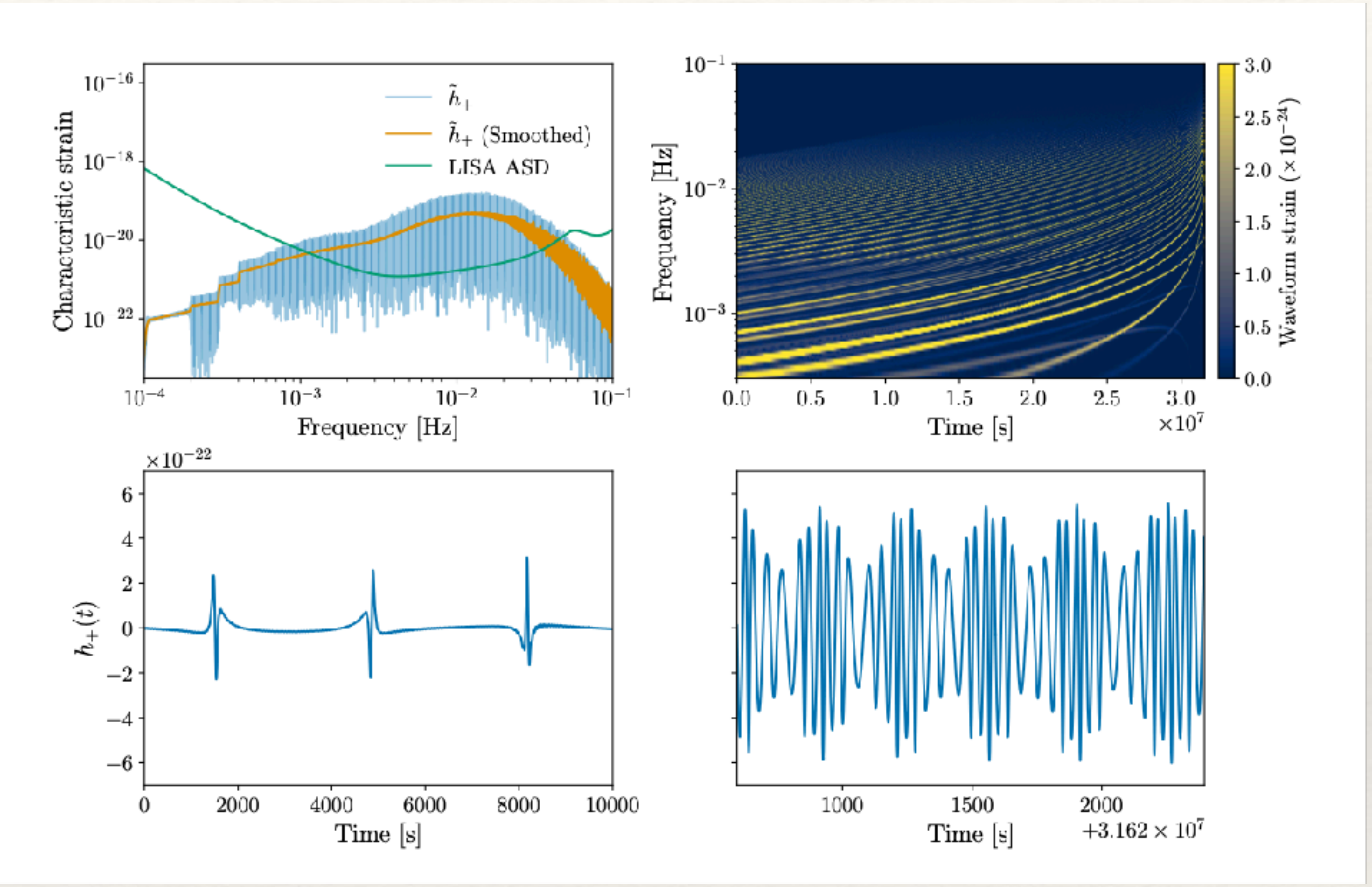
- Mode mixing between spherical harmonics in Kerr happens between modes with the same m because of the axial symmetry of the spacetime

Adiabatic inspirals

Adiabatic inspiral: general facts

Multiple voices due to the interplay of radial, azimuthal and polar motion

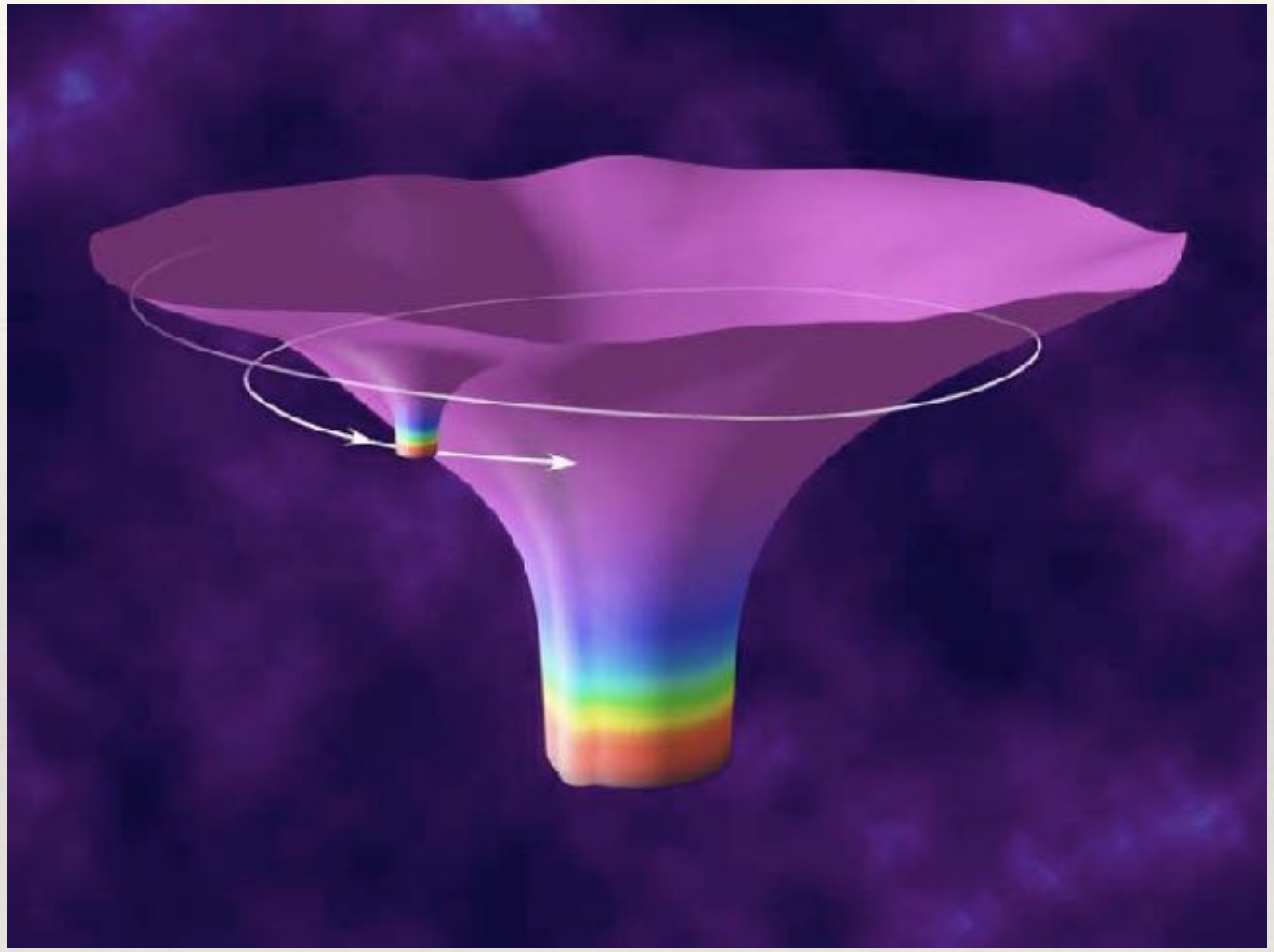
- Intermediate and extreme mass ratio binaries, with mass ratios roughly in the range $[10^2, 10^4]$ and $[10^4, 10^6]$
- Orbits are expected to be inclined and eccentric with respect to the primary's spin axis, giving rise to a complex mode structure
- During inspiral there's a natural separation of timescales: orbital timescale is much faster than radiation reaction timescale
- When the orbit approaches the last stable orbit, adiabaticity hypothesis breaks down



Chapman-Bird et al., https://arxiv.org/pdf/2506.09470

Computational recipe

- Excluding *resonances:*
  - Assumes that the inspiral evolves slowly due to gravitational radiation reaction
  - Compute the evolution of the fundamental phases describing polar, radial and azimuthal motion $\Phi_r, \Phi_\theta, \Phi_r$, together with evolution of orbital elements.
  - Compute forcing functions and amplitudes associated for snapshot via Teukolsky equation
  - Expensive! Precompute data and interpolate them so that waveform generation is fast
  - Evolve the orbital parameters via flux-balance laws



Sourced Teukolsky equation

- For adiabatic inspirals, we need to solve the sourced Teukolsky equation

$$O[\Psi_s] = \mathcal{T}$$

- The source term is rather involved (see, e.g. Hughes, Phys. Rev. D 61, 084004 (2000))
- Let's just say it depends on the projections on the null tetrad of the stress-energy tensor of a geodesic:

$$\begin{aligned} T^{\alpha\beta}(x) &= \mu \int d\tau \, u^\alpha u^\beta \delta^{(4)} \left[x - z(\tau)\right] \\ &= \mu \int d\tau \, u^\alpha u^\beta \left(-g\right)^{1/2} \delta\left[t - t(\tau)\right] \delta\left[r - r(\tau)\right] \delta\left[\theta - \theta(\tau)\right] \delta\left[\phi - \phi(\tau)\right] \end{aligned}$$

Integrability of Kerr Geodesics

- Geodesic orbits in Kerr are characterised by 3 constants of motion, related to the presence of two Killing vectors and a Killing tensor
  - E (conserved energy per unit rest mass)
  - L_z (conserved z-component of angular momentum per unit rest mass)
  - $C = K^{\mu\nu}u_\mu u_\nu$ (Carter constant normalised per unit squared mass)
- The conservation of E and L can be intuitively understood in terms of the symmetry of the Kerr metric under time-translations and rotations around the BH spin axis
- Carter constant has more elusive meaning: for a non-spinning BH or in the weak field ($r \to \infty$), it can be identified with $L^2 - L_z^2$

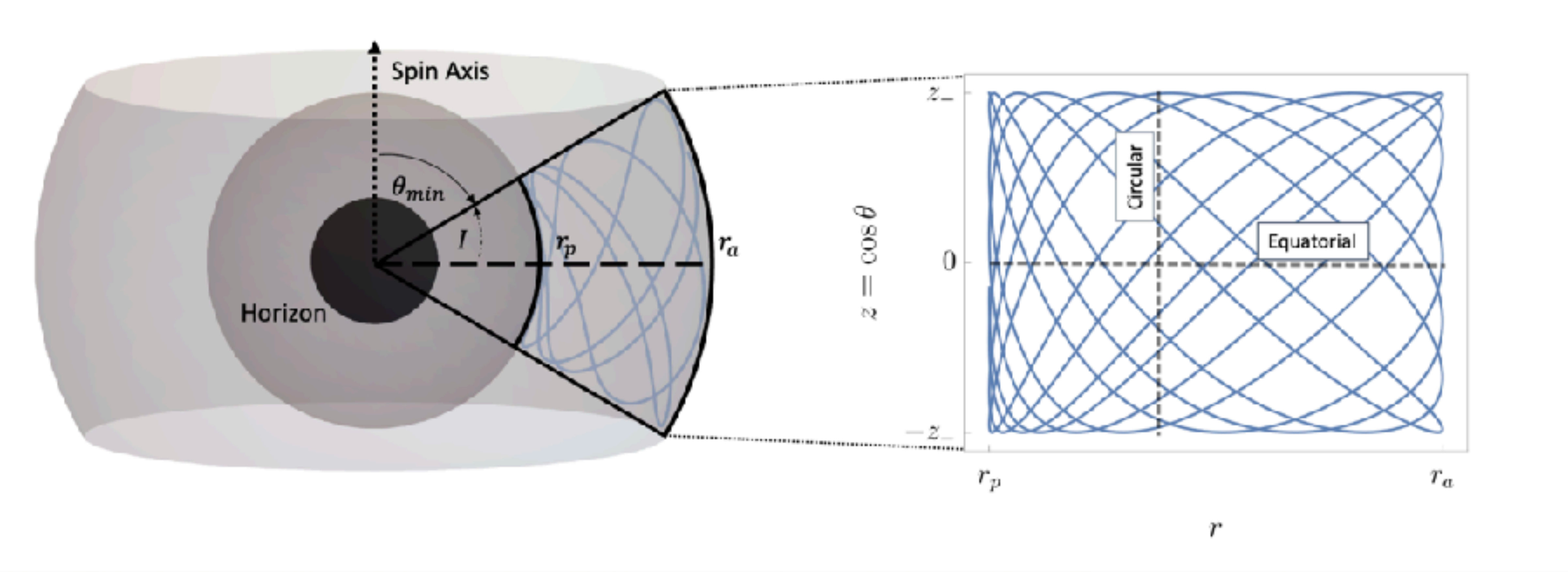
Orbital parametrisation

Motion parametrised by quasi-Keplerian orbital elements:

$$\text{(semi-latus rectum)}: p = \frac{2r_a r_p}{r_a + r_p}$$

$$\text{(eccentricity)}: \quad e = \frac{r_a - r_p}{r_a + r_p}$$

$x_I = \cos I$ (cosine of the orbital inclination)



Lynch&Burke arXiv:2411.04955

Input from Teukolsky equation (1)

- The radial part of the solution to Teukolsky eq. $\psi_{\ell m\omega}(r)$ has asymptotic behaviour

$$\psi_{\ell m\omega}(r,\omega) \to Z^{\infty}_{\ell m\omega} r^3 e^{i\omega r_*}, \quad r \to \infty \quad \text{Orbit-averaged GSF contribution regular at spatial infinity}$$

$$\to Z^{H}_{\ell m\omega} \Delta e^{i(\omega - m\Omega_H) r_*}, \quad r \to r_+ \quad \text{Orbit-averaged GSF contribution regular at the horizon}$$

where r_* is the tortoise coordinate, r_+ is the radius of the event horizon and $\Omega_H = \dfrac{a}{2Mr_+}$ (horizon rotation frequency)

- The $Z^{H/\infty}_{\ell m\omega}$ coefficients are key quantities entering the calculation of energy, angular momentum flux, etc, e.g. for fluxes to infinity

$$\left(dE/dt\right)^{\infty} = \sum_{\ell mkn} \frac{|Z^{\infty}_{\ell mkn}|^2}{4\pi\omega^2_{mkn}}$$

Forcing terms

- Time-averaged GW fluxes $\frac{dC}{dt} = -\left(\frac{dC}{dt}\right)^{\infty} - \left(\frac{dC}{dt}\right)^{\mathscr{H}}$ with $C = \{E, L_z, Q\}$
- Enter forcing term for orbital elements α

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \frac{\nu}{M}\left[\hat{f}_{\alpha}^{(0)}(a, p, e, x_I) + \mathcal{O}(\nu)\right]$$

$$\hat{f}_{\alpha}^{(0)} = -\frac{m_1^2}{m_2}\left[\frac{\partial \alpha}{\partial E}\left\langle\frac{dE}{dt}\right\rangle_{\mathrm{GW}} + \frac{\partial \alpha}{\partial L}\left\langle\frac{dL}{dt}\right\rangle_{\mathrm{GW}}\right],$$

Chapman-Bird et al.arXiv:2506.09470

Constants of motion slowly evolve under the dissipative effect of radiation emitted out to infinity and into the horizon

Putting everything together

- One also needs for each snapshot the mode amplitudes

$$H_{\ell mn}(t) = A_{\ell mn}(t)\,{}_{-2}S_{\ell mn}(\theta;\hat{\omega}_{mn})e^{im\phi},$$

associated with each combination of (m,n,k) indices associated with the triperiodic motion of the orbit

$$\Phi_{mkn}(t) = m\Phi_{\phi}(t) + k\Phi_{\theta}(t) + n\Phi_{r}(t).$$

- These are given by

$$A_{\ell mn}(t) = -2\frac{\hat{Z}^{\infty}_{\ell mn}(a,p,e,x_I)}{\hat{\omega}^2_{mn}(a,p,e,x_I)}.$$

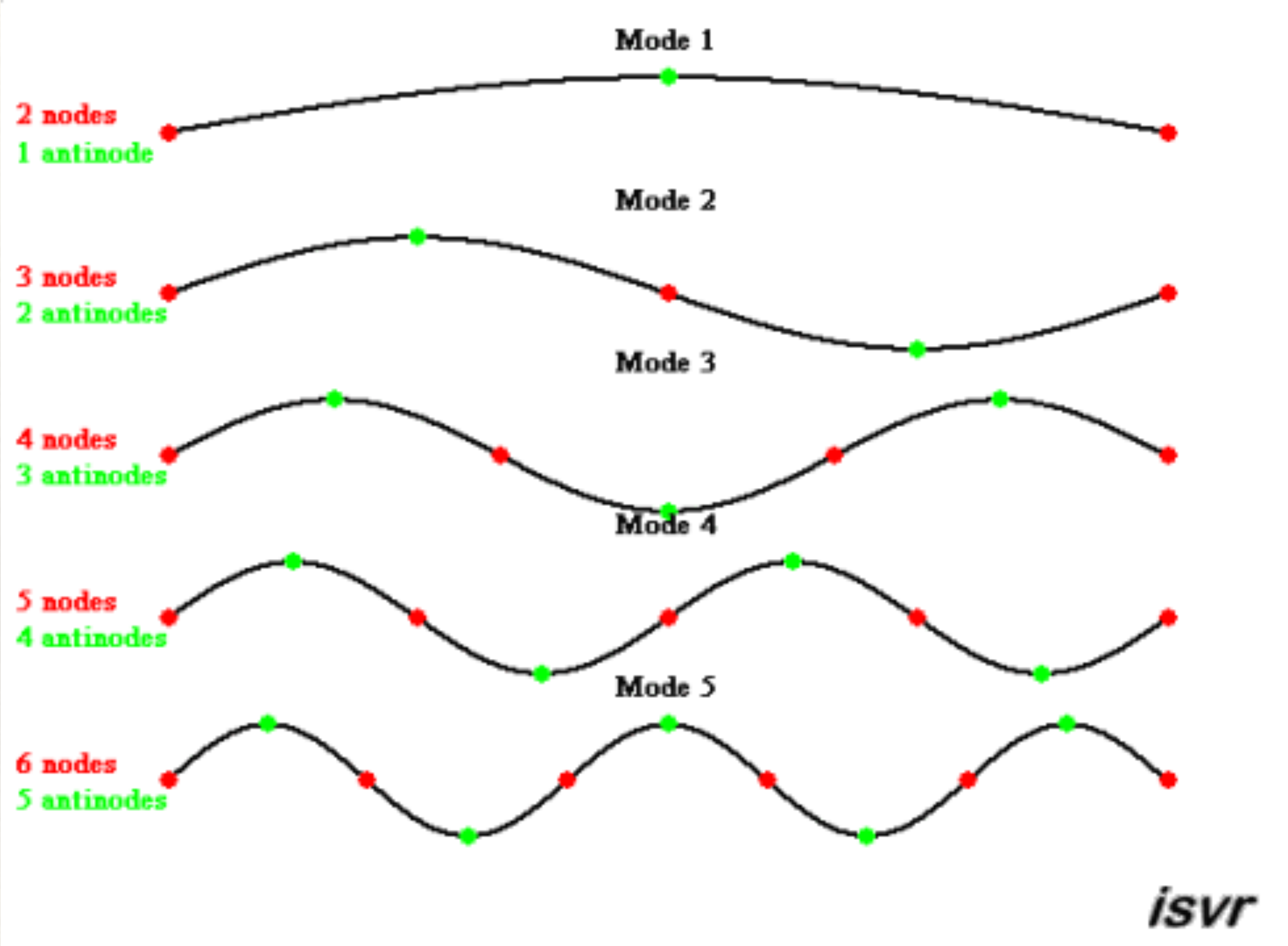
- Putting everything together

$$h(t) = \frac{\mu}{d_{\mathrm{L}}}\sum_{\ell mn} A_{\ell mn}(t)\,{}_{-2}Y_{\ell m}(\theta,\phi)e^{-\Phi_{mn}(t)}.$$

Ringdown

Normal modes

- $$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} = 0$$
- Imposing some boundary conditions, $\psi(t,0) = \psi(t,L) = 0$
- Can write solution as a linear superposition of normal modes $\psi(t,x) = \sum_n A_n e^{-i\omega_n t}\psi_n(x)$
- Characteristic vibration frequencies that are related to the tension and density of the string



Quasi-normal modes

- In BH spacetimes, energy is dissipated through the horizon and there's a potential term, whose precise form depends on the type of master equation employed
- The corresponding "vibration modes" of a perturbed BH become complex valued quantities, called quasi-normal-modes (QNMs)
- QNMs correspond to perturbations satisfying physically meaningful boundary conditions:
  - Nothing should come out of the horizon: "ingoing boundary condition"
  - Nothing should come in from infinity: "outgoing boundary condition"

$$\Psi(r_\star \to +\infty) \to \mathrm{e}^{-i\omega(t-r_\star)},$$
$$\Psi(r_\star \to -\infty) \to \mathrm{e}^{-i\omega(t+r_\star)}.$$

Quasi-normal modes

- For each (ℓ, m) index, there's a discrete set of complex frequencies $\omega_{\ell,m,n}$

$$\omega_{\ell,m,n} = \omega^R_{\ell,m,n} + i\omega^I_{\ell,m,n}$$

where the real part gives the oscillation frequency and the imaginary part determines the damping time. $\tau = 1/|\omega_I|$

- The fundamental mode n = 0 is the least damped mode, i.e. the longest-lived
- QNMs frequencies can be computed via numerical or approximate methods, e.g. Leaver's method Proc. Roy. Soc. Lond. A 402 (1985) 285–298.
- Waveform models directly incorporate some of these results

Ringdown description in IMR models

- If no mode-mixing:

$$h_{\ell m}^{\text{merger-RD}}(t) = \nu \tilde{A}_{\ell m}(t) e^{i\tilde{\phi}_{\ell m}(t)} e^{-i\sigma_{\ell m0}t}$$

where $\sigma_{\ell m0}$ is the complex frequency of the least-damped quasi-normal-mode

$$\sigma_{\ell m0} = 2\pi f_{\ell m0} - i/\tau_{\ell m}$$

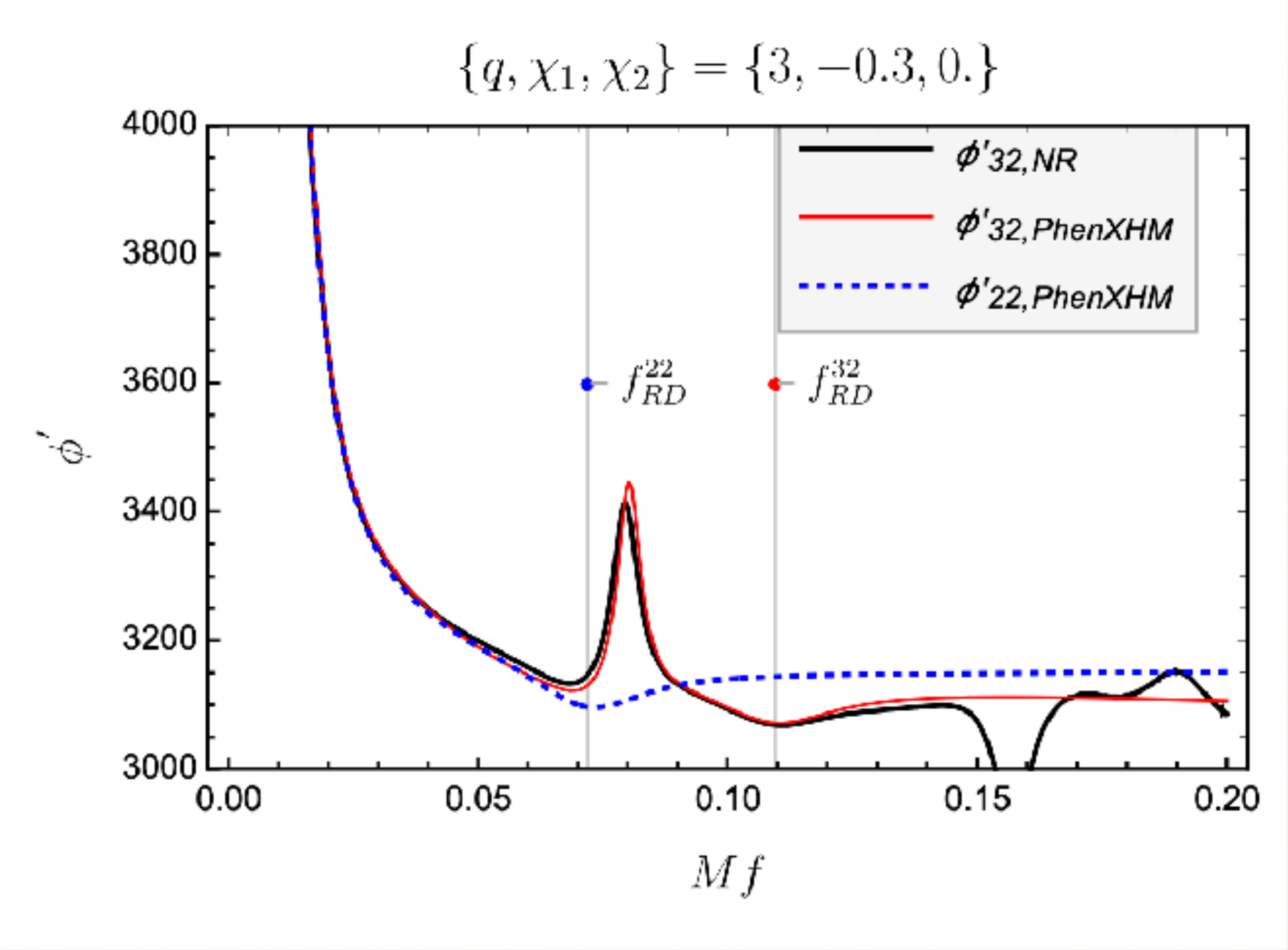
  - $\tilde{A}_{\ell m}(t), \tilde{\phi}_{\ell m}(t)$ are some amplitudes and phases with coefficients fitted to NR simulations and test-particle waveforms.
  - Ansatz above is inspired by *linear perturbation theory*
  - However, effectively these amplitudes and phases capture other non-linear contributions present in the numerical data

Mode-mixing

- Let's focus for simplicity on fundamental modes, $n = 0$
- Start from the ringdown strain expressed in terms of ${}_sS_{\ell m}$:
  $h = \sum_{\ell' m} {}_sS_{\ell' m} h^S_{\ell' m}$
- Now we can use the spherical-spheroidal mixing coefficients to convert this into an expansion in terms of the ${}_sY_{\ell m}$ basis:

$$h = \sum_{\ell' \ell m} {}_sY_{\ell m} \mu^*_{m \ell \ell' m} h^S_{\ell' m}$$

- Depending on the magnitude of $\mu_{m\ell\ell' m}$ and $h^S_{\ell' m'}$, some harmonics might show evidence of ***mode-mixing*** with some of their neighbours, e.g. (3,2) will mix with the (2,2)



García-Quirós PRD 102, 064002 (2020)

QNMs at second order in PT

GR is a nonlinear theory, second-order effects in BH perturbation theory

$$\delta G_{ab}\left[h_{ab}^{(1)}\right] = 0,$$

$$\delta G_{ab}\left[h_{ab}^{(2)}\right] = -\,\delta^2 G_{ab}\left[h_{ab}^{(1)}, h_{ab}^{(1)}\right],$$

will also manifest in the ringdown signal

The frequency spectrum of quadratic QNMs is distinct from the linear QNM spectrum; there will be additional frequencies, e.g.

$$\omega_{\ell_1 m_1 n_1 \times \ell_2 m_2 n_2} = \omega_{\ell_1 m_1 n_1} + \omega_{\ell_2 m_2 n_2}$$

Detection prospects with LISA investigated by one of the participants! Yi+, PRD 109, 124029 (2024)

Numerical Relativity

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What is Numerical Relativity?

- NR attempts to solve the Einstein Field Equations (EFEs)

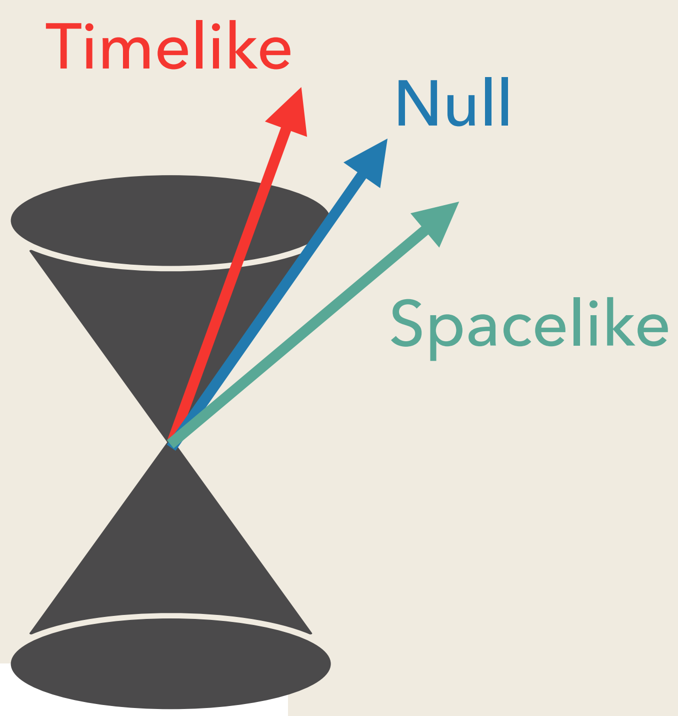
$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$$

  on a computer

- These equations have two physical degrees of freedom

- NR requires us to solve the Cauchy problem– i.e. the process of constructing a solution to a partial differential equation given data on some boundary or initial hypersurface

Foliations of spacetime



- A Cauchy surface is a space like hypersurface Σ in a Lorentzian manifold $\mathscr{M}$ such that each timeline or null curve without end points intersects Σ exactly once

- A family of such surfaces is called a **foliation** of the spacetime

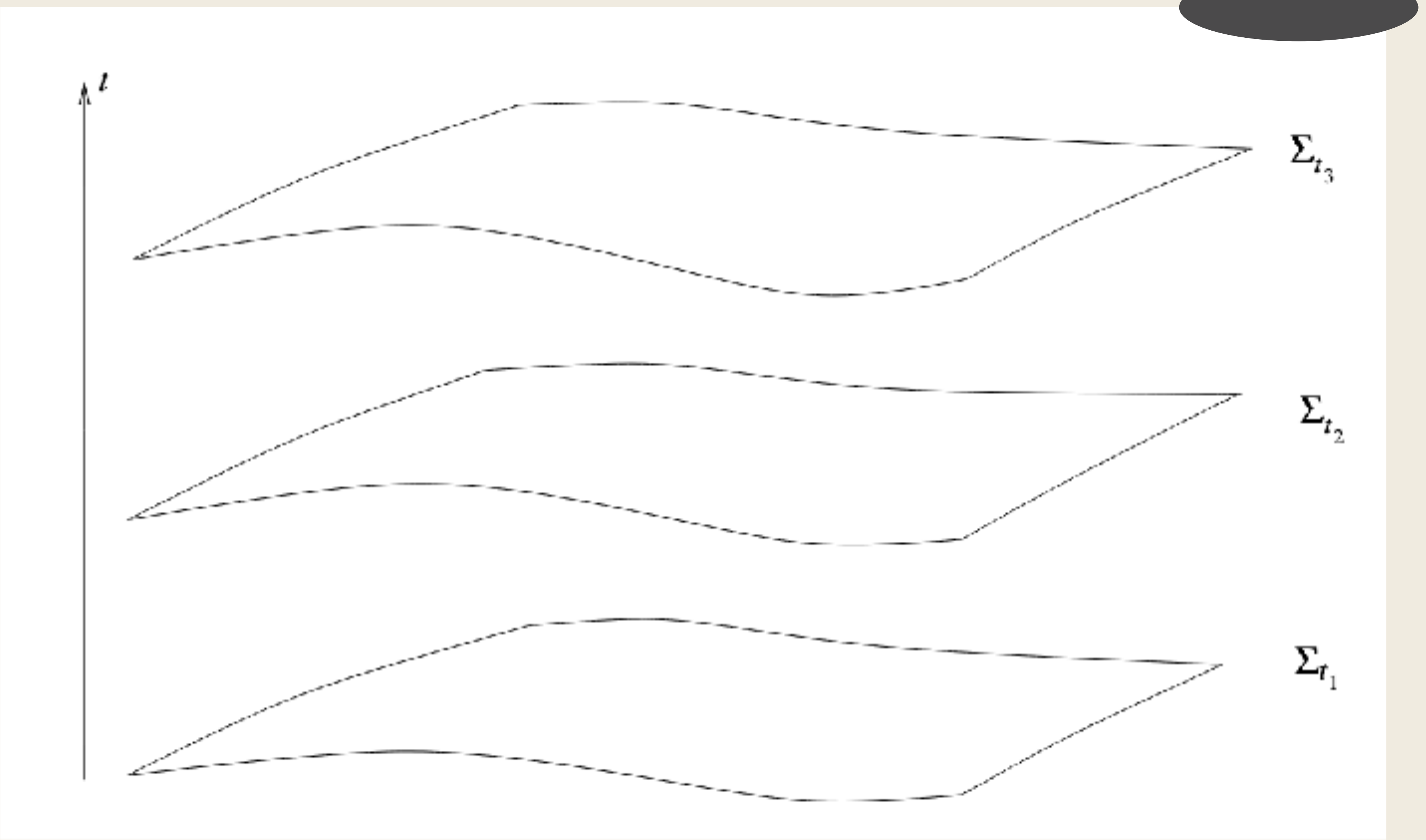


Illustration of a foliation of spacetime. Taken from https://www.damtp.cam.ac.uk/user/us248/Lectures/Notes/gwnr.pdf

Foliations of spacetime

- Label these hyper surfaces by a constant t
- Normal to hypersurface therefore $\nabla_\mu t$
- Unit normal:
  $n_\mu = -\alpha \nabla_\mu t$
  where α is the **lapse function**

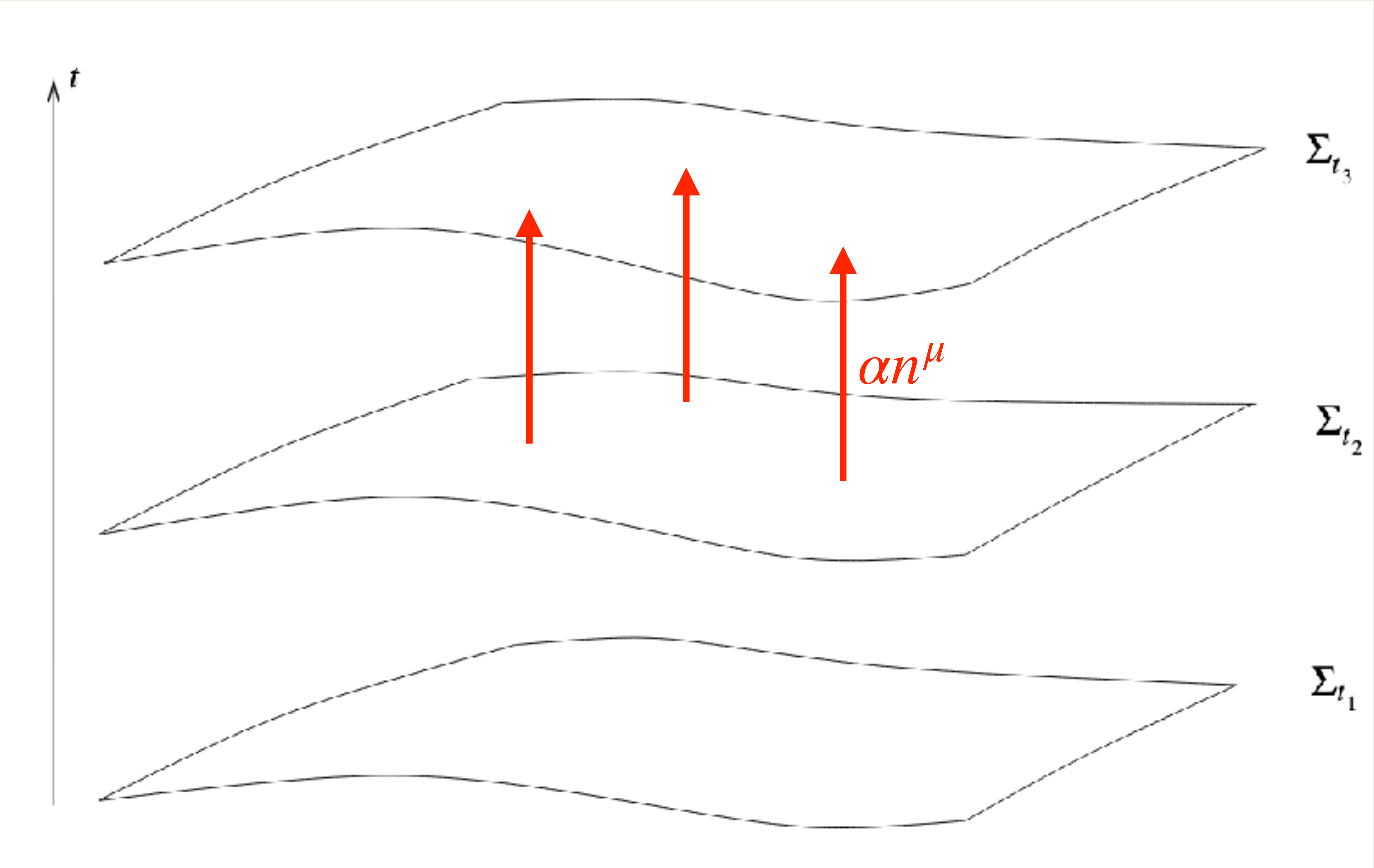


Illustration of a foliation of spacetime. Taken from https://www.damtp.cam.ac.uk/user/us248/Lectures/Notes/gwnr.pdf

Foliations of spacetime

- We require a **proper** time derivative, given by
  $(\partial_t)^m u = \alpha n^\mu + \beta^\mu$
  where β^μ is the **shift**

- The induced metric on Σ is given by
  $\gamma_{ab} = g_{ab} + n_a n_b$

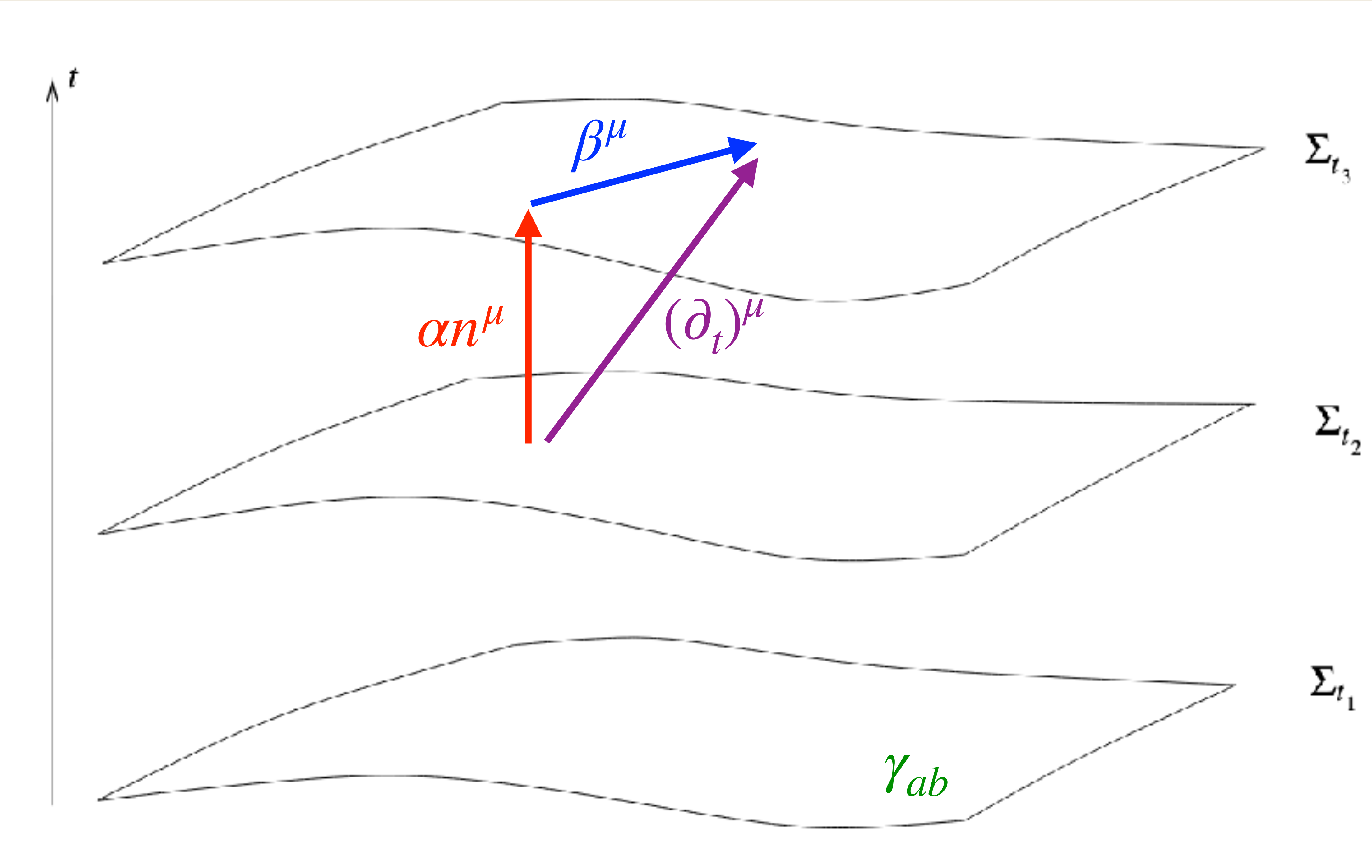


Illustration of a foliation of spacetime. Taken from
https://www.damtp.cam.ac.uk/user/us248/Lectures/Notes/gwnr.pdf

Foliations of spacetime

- α and β^μ determine how co-ordinates evolve from one slice Σ_t to $\Sigma_{t+\mathrm{dt}}$ along the time direction $(\partial_t)^\mu$
- α determines how much proper time elapses between time slices along the normal vector n^μ
- β^μ determines how much spatial co-ordinates are shifted wrt the normal vector n^μ

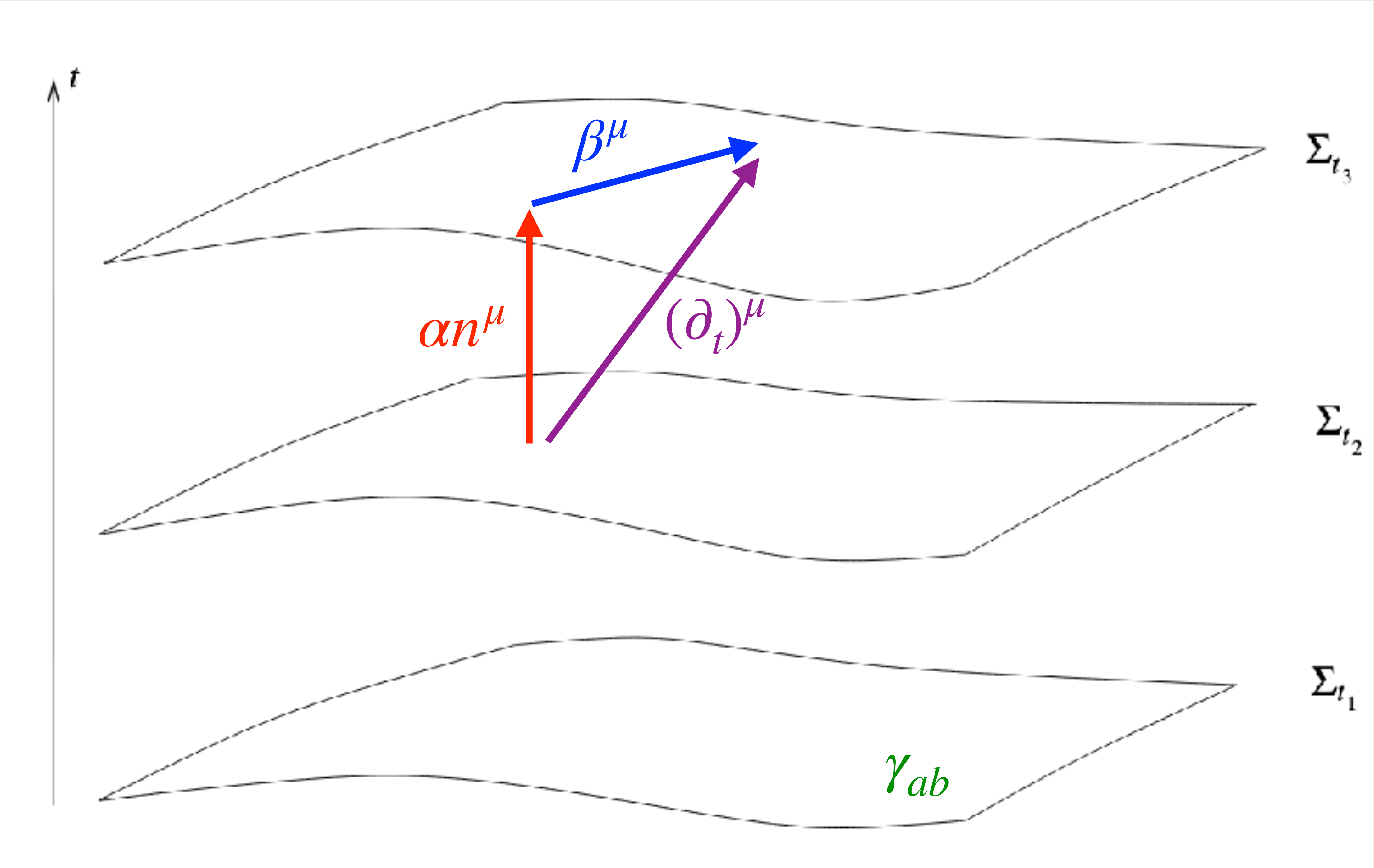


Illustration of a foliation of spacetime. Taken from https://www.damtp.cam.ac.uk/user/us248/Lectures/Notes/gwnr.pdf

Extrinsic curvature

- n^μ does not remain normal to Σ as we parallel transport it from point P to point Q
- The extent to which n^μ deviates from the normal direction defines the **extrinsic curvature**

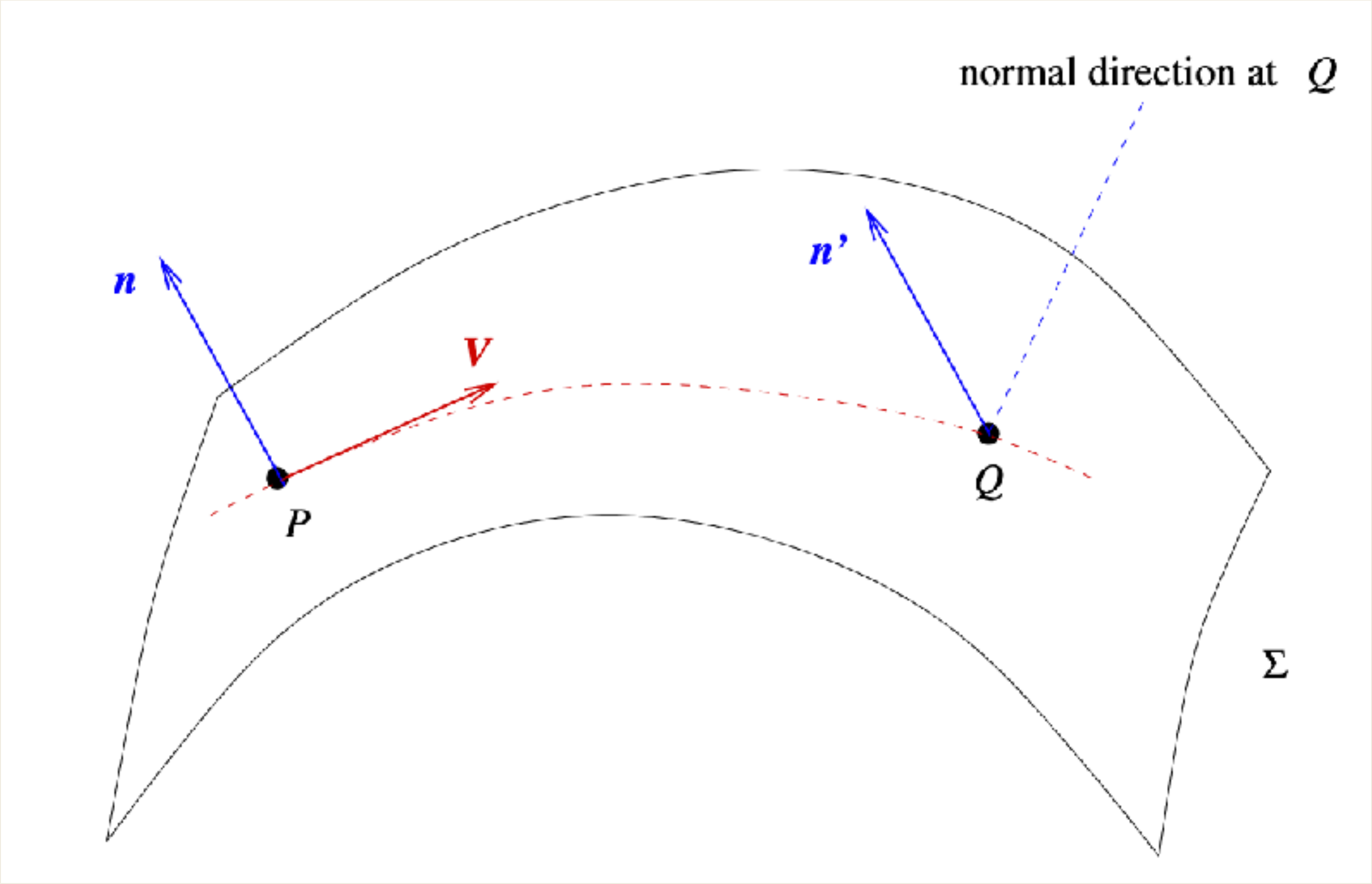


Illustration of curve embedding. Taken from https://www.damtp.cam.ac.uk/user/us248/Lectures/Notes/gwnr.pdf

Extrinsic curvature

- The extrinsic curvature $K_{\mu\nu}$ can be written in terms of the acceleration of normal observers $a_\beta = n^\mu \nabla_\mu n_\beta$:

$$K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu$$

- $K_{\mu\nu}$ can also be written as the Lie derivative of the 3-metric

$$K_{\mu\nu} = -\frac{1}{2}\mathscr{L}_{\mathbf{n}}\gamma_{\mu\nu}$$

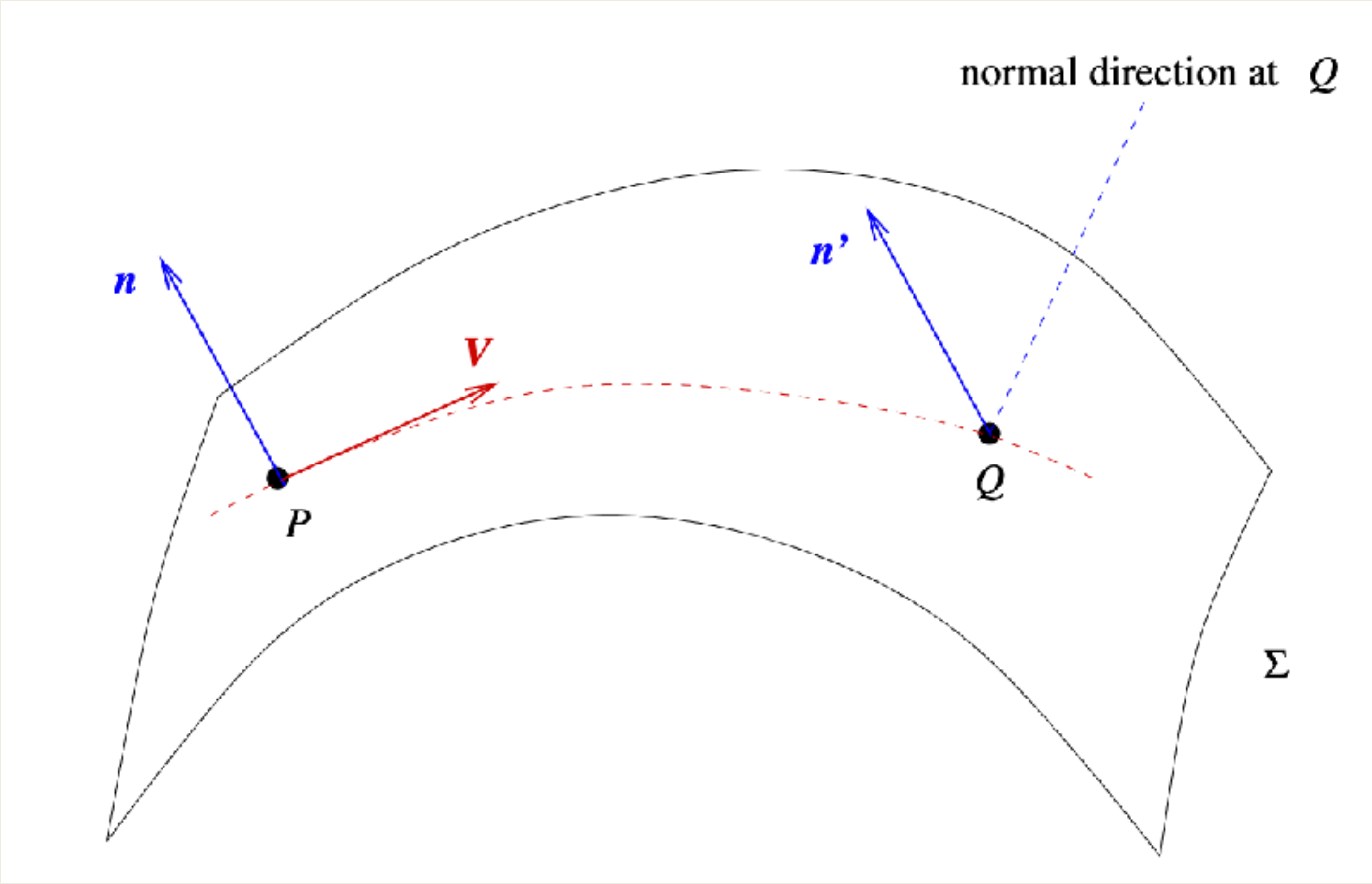


Illustration of curve embedding. Taken from
https://www.damtp.cam.ac.uk/user/us248/Lectures/Notes/gwnr.pdf

ADM formalism

- Need to evolve the spatial metric γ_{ab} and extrinsic curvature $K_{\mu\nu}$

- Return to the EFEs:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} \qquad \circledast$$

- Steps to obtain constraint and evolution equations:
  1. Project $\circledast$ twice onto time
  2. Project $\circledast$ once onto space and once onto time
  3. Take the Lie derivative along n^μ of the spatial metric
  4. Project the trace-revered version of $\circledast$ twice onto space

ADM formalism

- Resulting expressions:
  1. Hamiltonian contraint: $\mathscr{R} + K^2 - K^{\mu\nu}K_{\mu\nu} = 16\pi\rho$
  2. Momentum constraint: $D_\alpha K - D_\mu K^\mu_\alpha = -8\pi j_\alpha$
  3. Induced metric evolution equation: $\mathscr{L}_\mathbf{n}\gamma_{\alpha\beta} = -2K_{\alpha\beta}$
  4. Extrinsic curvature evolution equation:
     $$\mathscr{L}_\mathbf{n}K_{\alpha\beta} = -\frac{1}{\alpha}D_\alpha D_\beta\alpha - 2K_{\alpha\mu}K^\mu_\beta + \mathscr{R}_{\alpha\beta} + KK_{\alpha\beta} - 8\pi[S_{\alpha\beta} - \frac{1}{2}\gamma_{\alpha\beta}(S-\rho)]$$
- Analogous to Maxwell's field equations in electromagnetism

ADM formalism

- Inspect the form of the 4 equations which make up the **ADM** or **3+1 formulation** of the EFEs:
  - 4 equations; no time derivatives
  - Constraints preserved under evolution equations
  - 2 degrees of freedom (as expected)
- Obtain intial data from simulations by solving constraint equations

Alternative formalisms

- The ADM formalism is not well-posed
- Current NR codes employ alternative formulations of the EFEs, such as
  - Generalised harmonic gauge (GHG)
  - BSSN, Z4 gauges

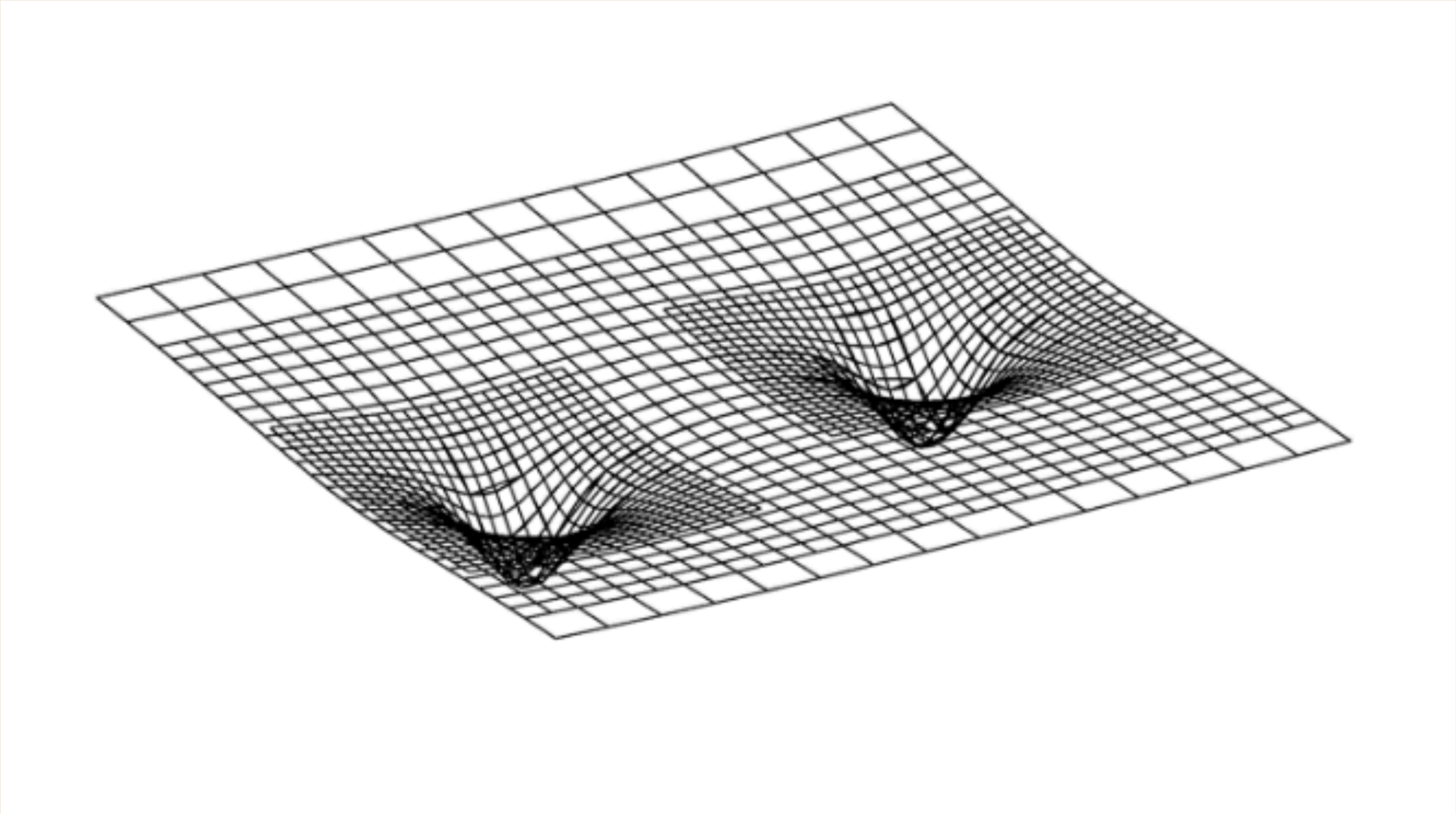
Types of NR codes

- Different NR codes use
  - different formulations of the EFEs
  - different treatments of the singularities
  - different numerical methods

- Widely used BBH NR codes:
  - SpEC/SpECTRE
  - BAM, Einstein Toolkit, MAYA, LazEv…

Numerical methods

- Finite differencing



Demonstration of Adaptive Mesh Refinement
From https://www.phys.ufl.edu/courses/phz6607/fall20/Reports/Daniel_George_Numerical_Relativity.pdf

- Spectral decomposition
  - Formulate series representation:

$$f(x) = \sum_{n=1}^{N} f^{(n)} \phi^{(n)}(x)$$

  - Error falls off exponentially

Treatment of singularities

- Moving punctures

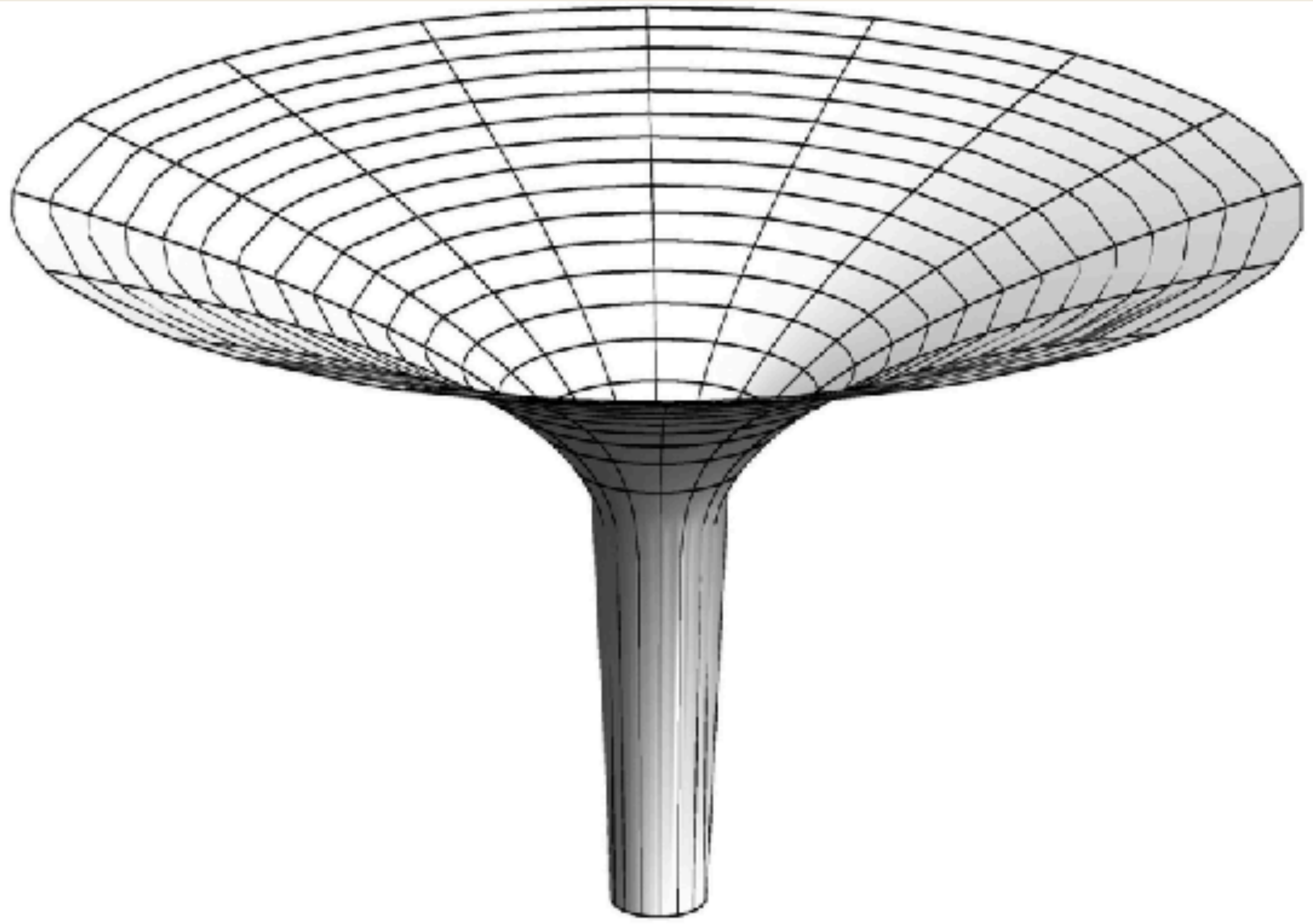


FIG. 2. Embedding diagram of a two-dimensional slice (t = const, $\theta = \pi/2$) of the maximal solution (17)–(20). The distance to the rotation axis is R. In contrast to Fig. 1 there is only one asymptotically flat end. The other end is an infinitely long cylinder with radius $R_0 = 3M/2$.

Hannam, M., et al. "Wormholes and trumpets: Schwarzschild spacetime for the moving-puncture generation." *Physical Review D* 78.6 (2008): 064020.

- Excision

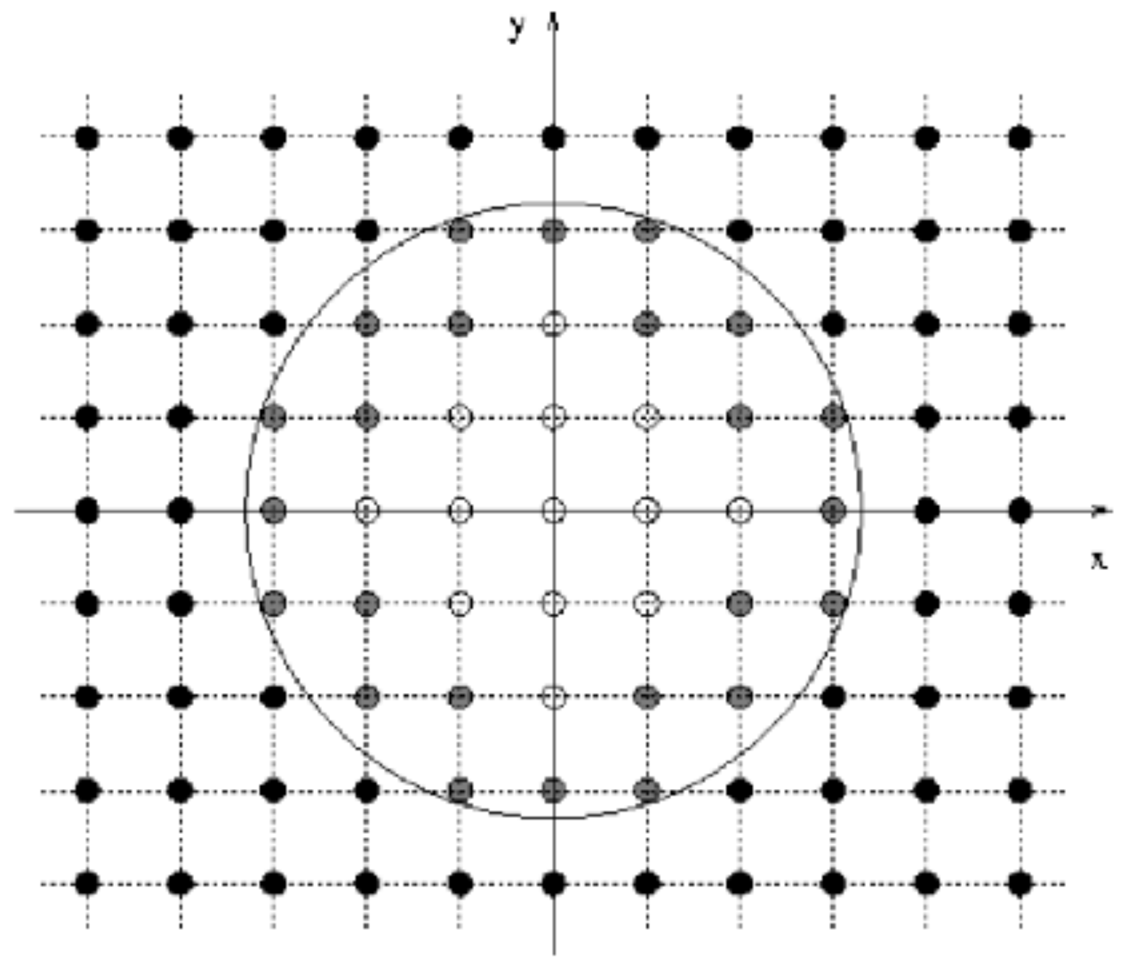
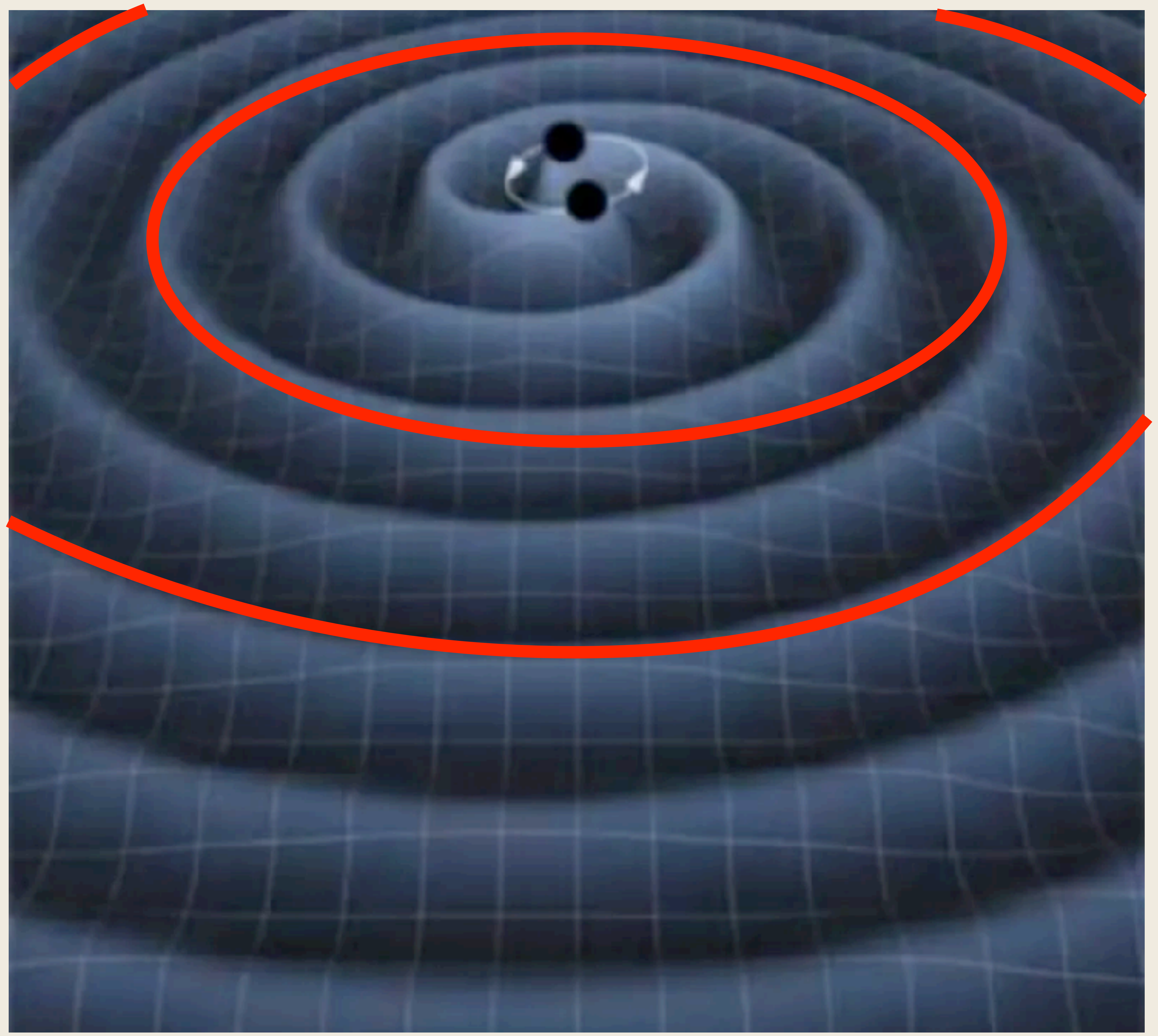


Figure 3. Illustration of BH excision with one spatial dimension suppressed. Black grid points are updated regularly in time, white points inside the AH (large circle) are excluded from the time evolution and gray points mark the excision boundary and need to be updated in time using sideways differencing operators [35], extrapolation [58] or are filled in through regular update with spectral methods [59].

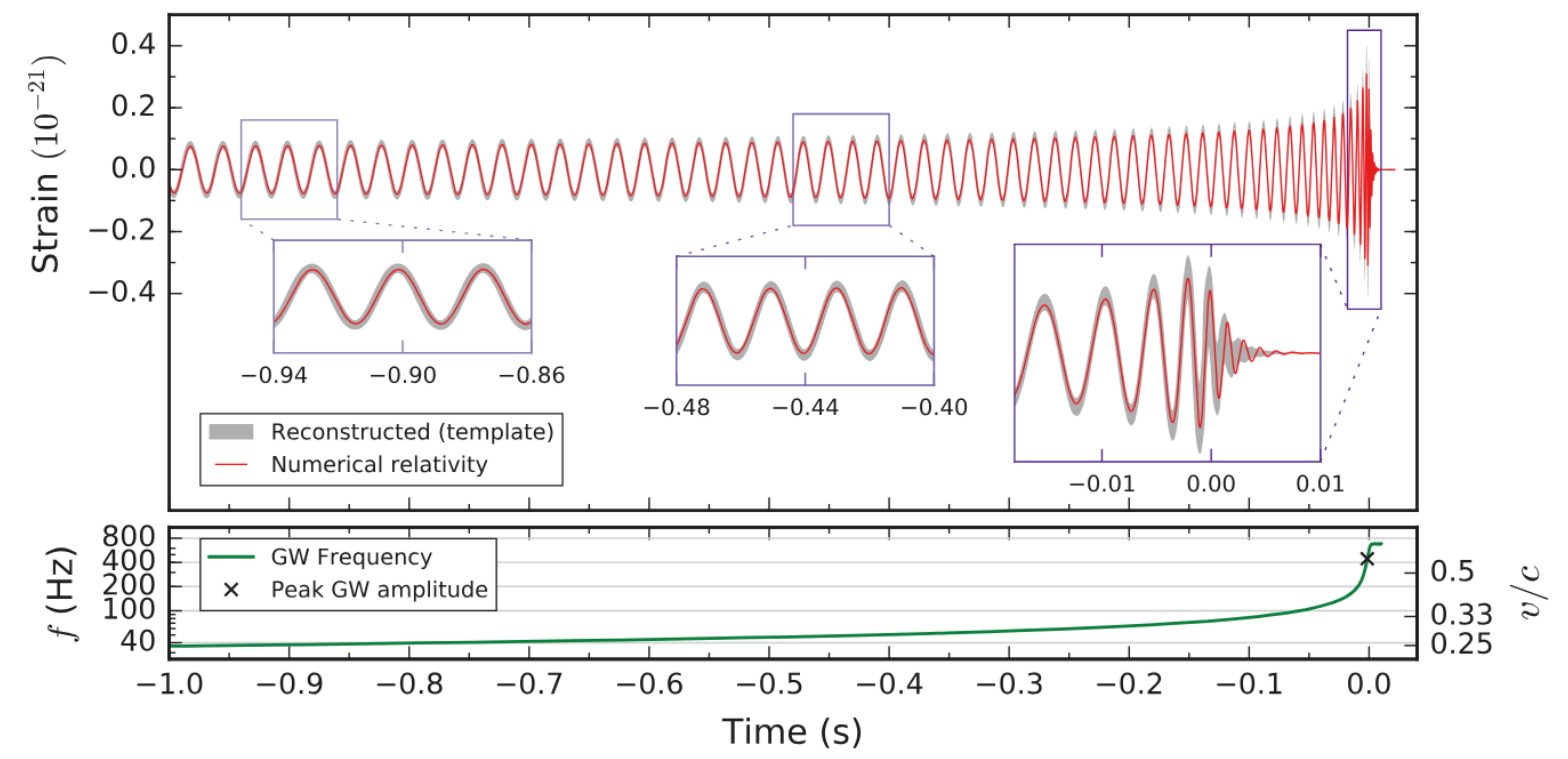
Extraction of Gravitational Waves

- The spacetime around the black holes is evolved at the centre of the simulation
- Gravitational waves propagate outwards and are then extracted
- We measure the metric perturbation $\Psi_4 = \ddot{h}_+ - \ddot{h}_\times$
- Obtain the strain via Fixed Frequency Integration



https://medium.com/@AndreKosmos/the-numerical-relativity-breakthrough-for-binary-black-holes-502e6f2b4b9e

Gravitational waveforms



LIGO-Virgo Collaboration "GW151226: observation of gravitational waves from a 22-solar-mass binary black hole coalescence." Physical review letters 116.24 (2016): 241103.

Limitations of numerical waveforms

- NR waveforms are the closest we can get to an exact solution of the EFEs, but they are limited by **numerical error**
- 2 main sources of numerical error:
  - Finite extraction radii
  - Finite resolution
- Accuracy is dominated by the phase error

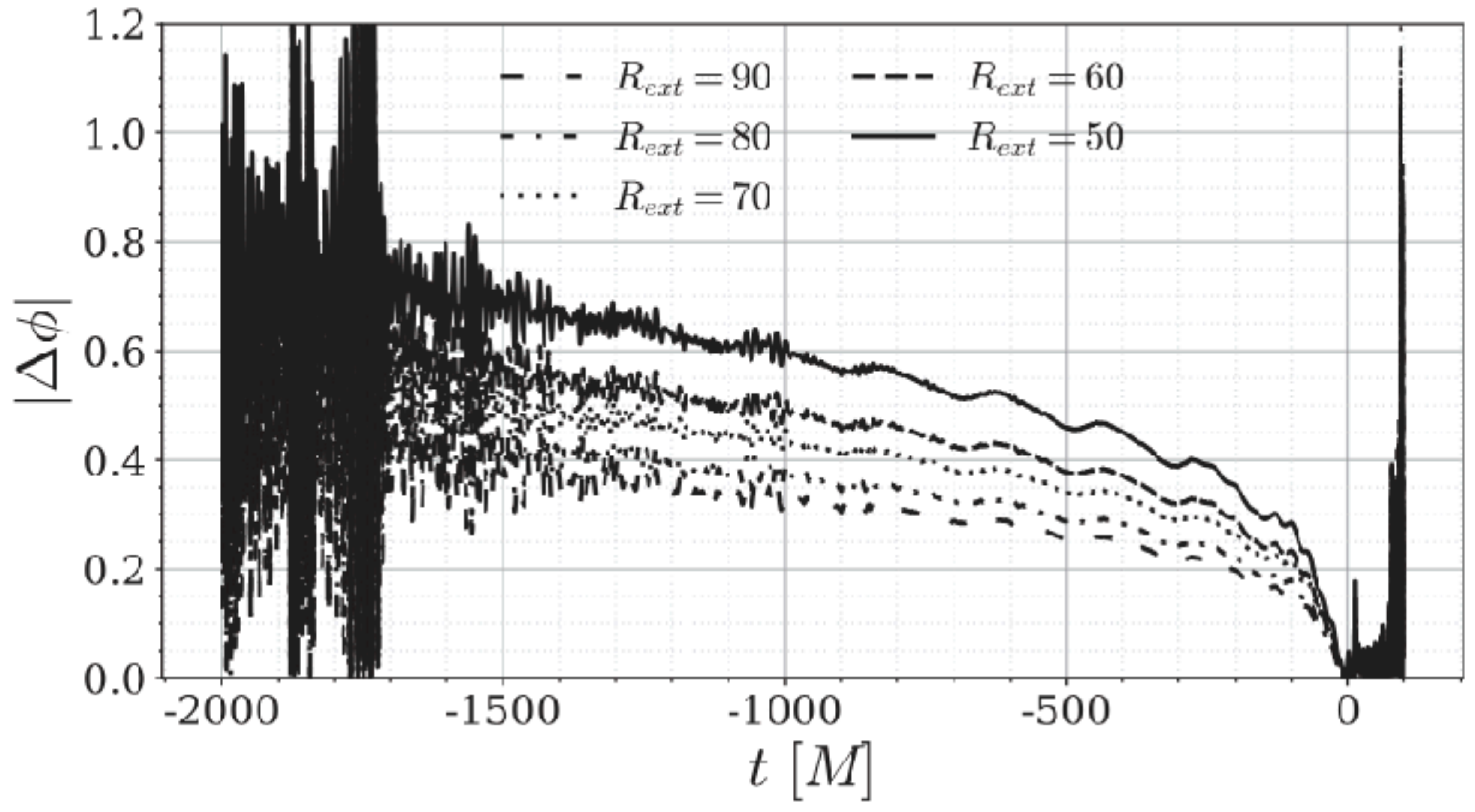


FIG. 7. Extraction radius dependence of the absolute error in the time domain co-precessing phase, relative to the Richardson-extrapolated phase. The phases have been aligned at merger.

Hamilton, E., et al. "Catalog of precessing black-hole-binary numerical-relativity simulations." Physical Review D 109.4 (2024): 044032.

Limitations of numerical waveforms

- Richardson extrapolation:
  $q^* = q + e_i \Delta^i$
- Measuring convergence requires 3 waveforms:
  $C = \dfrac{q(\Delta_1) - q(\Delta_2)}{q(\Delta_2) - q(\Delta_3)} = \dfrac{\Delta_1^n - \Delta_2^n}{\Delta_2^n - \Delta_3^n}$
- Find convergence order n which minimises
  $\delta = |q(\Delta_1) - q(\Delta_2)| - C|q(\Delta_2) - q(\Delta_3)|$

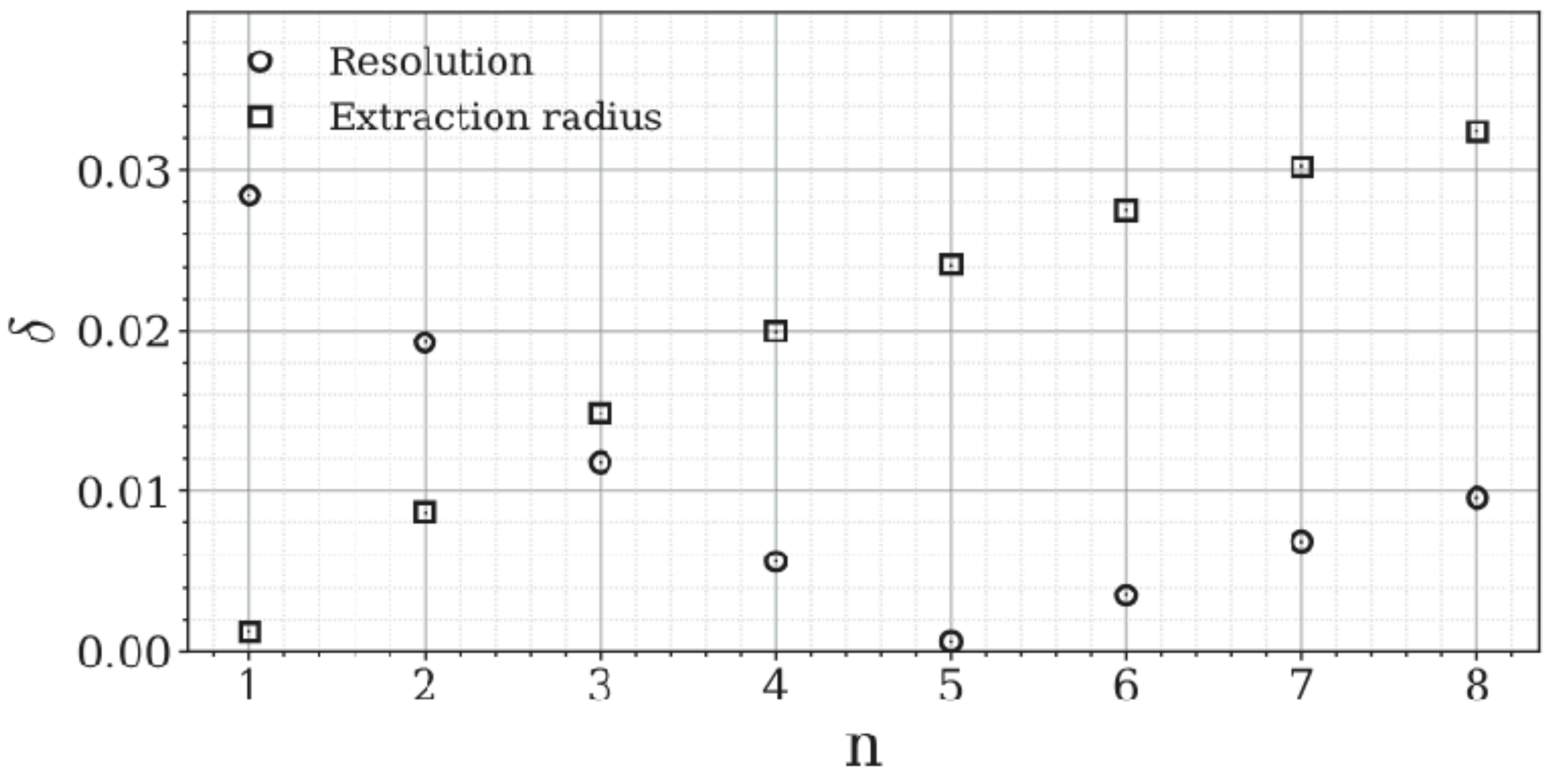
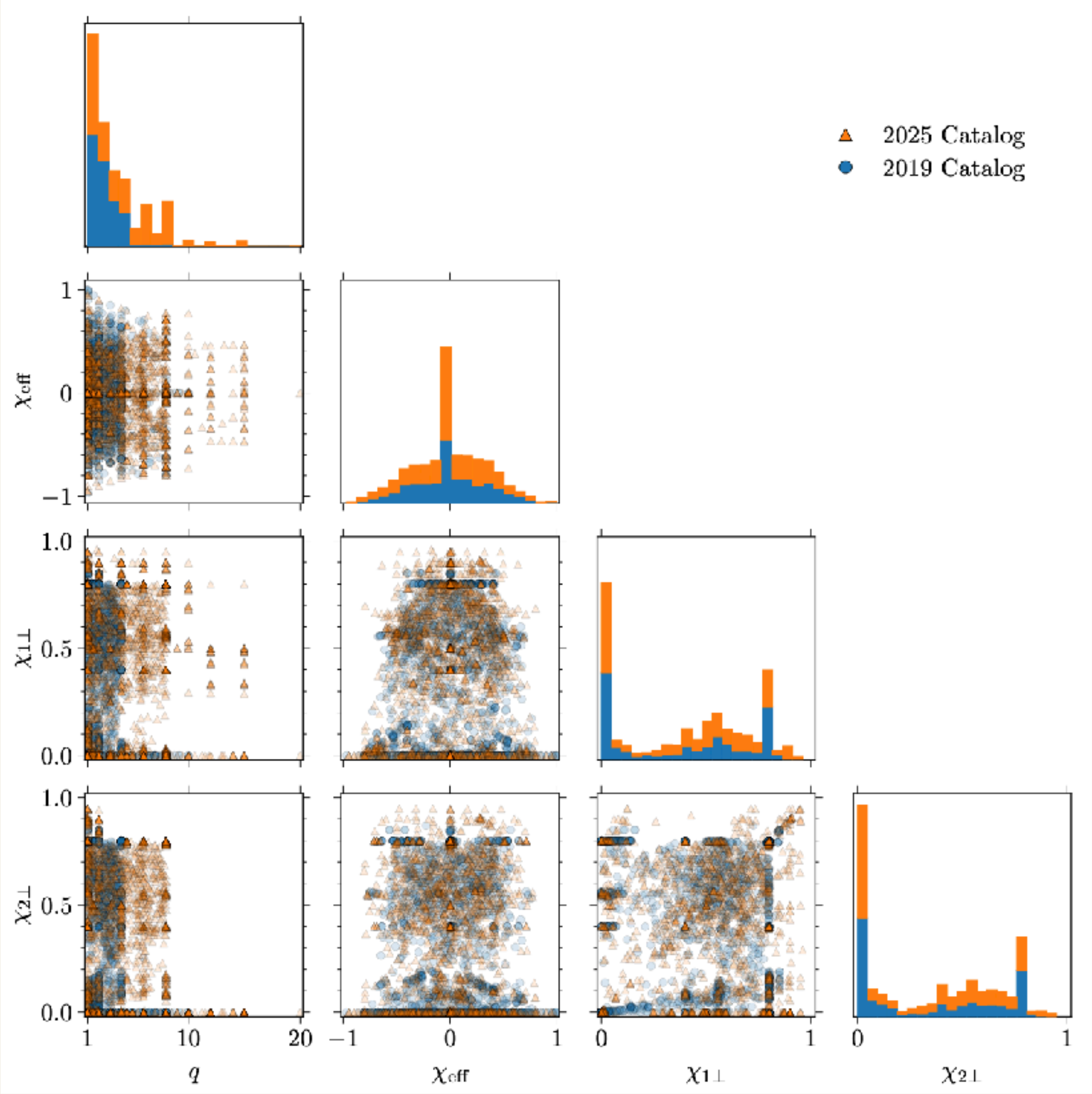


FIG. 5. The value δ as given by Eq. (9) as a function of convergence order. The circle markers represent waveforms of differing resolution. The square markers represent waveforms at differing extraction radii. The configuration $q = 8$, $\chi = 0.8$, $\theta_{\rm LS} = 150°$ was used in this analysis.

Hamilton, E., et al. "Catalog of precessing black-hole-binary numerical-relativity simulations." Physical Review D 109.4 (2024): 044032.

Current NR catalogues

- There is a range of publicly available NR catalogues covering different parameter spaces and suitable for different applications:
  - BBH: BAM, MAYA, RIT, SXS
  - BNS: CoRe, SACRA
- Largest available catalogue is the SXS catalogue



Scheel, Mark A., et al. "The SXS Collaboration's third catalog of binary black hole simulations." Classical and Quantum Gravity (2025).

Acknowledgments

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