Waveforms I: Perturbation Models

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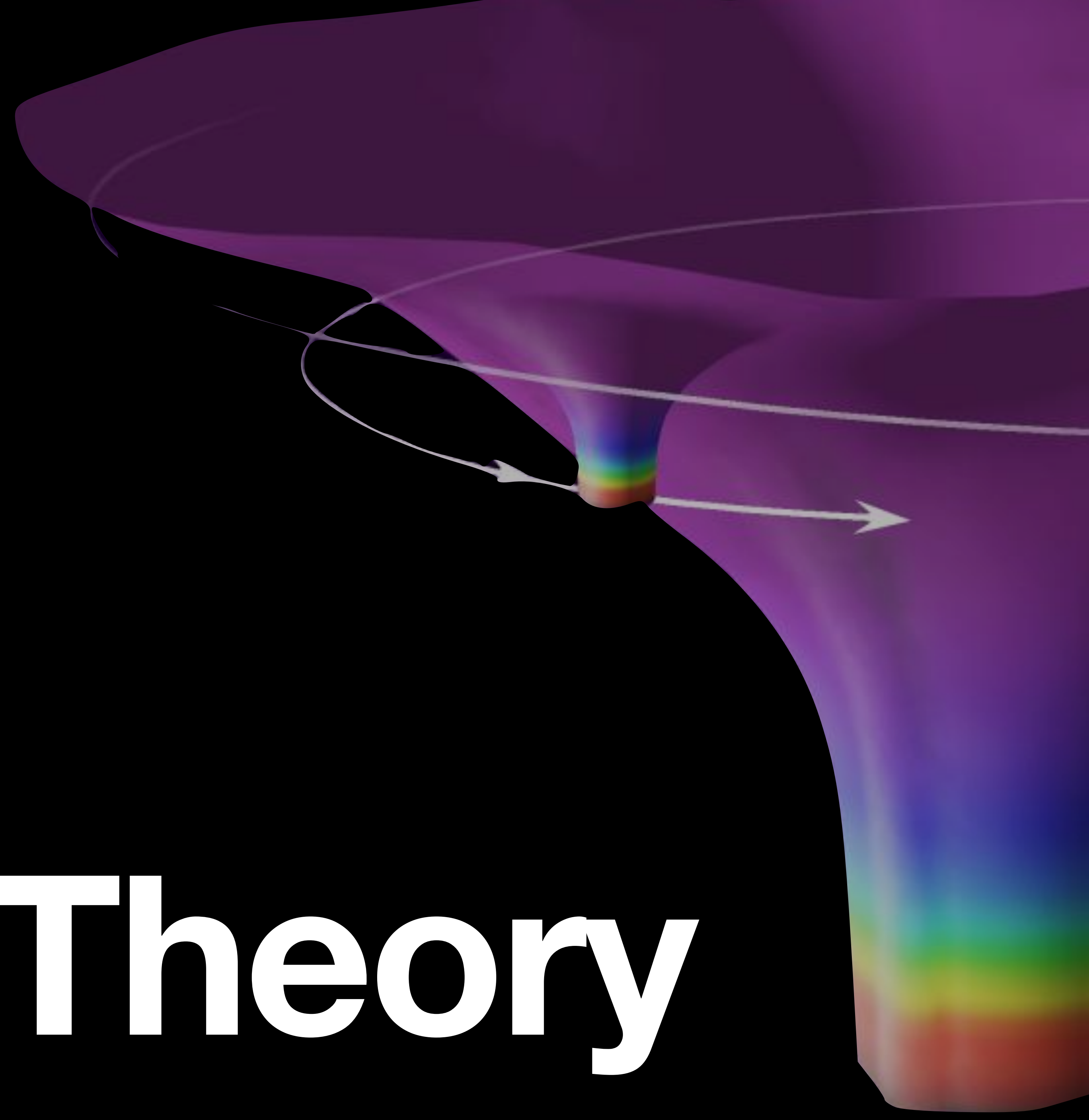
White Paper: arXiv:2311.01300
Redbook: arXiv:2402.07571

Les Houches LISA School for Early Career Scientists, October, 2025



Part A Post-Newtonian Theory

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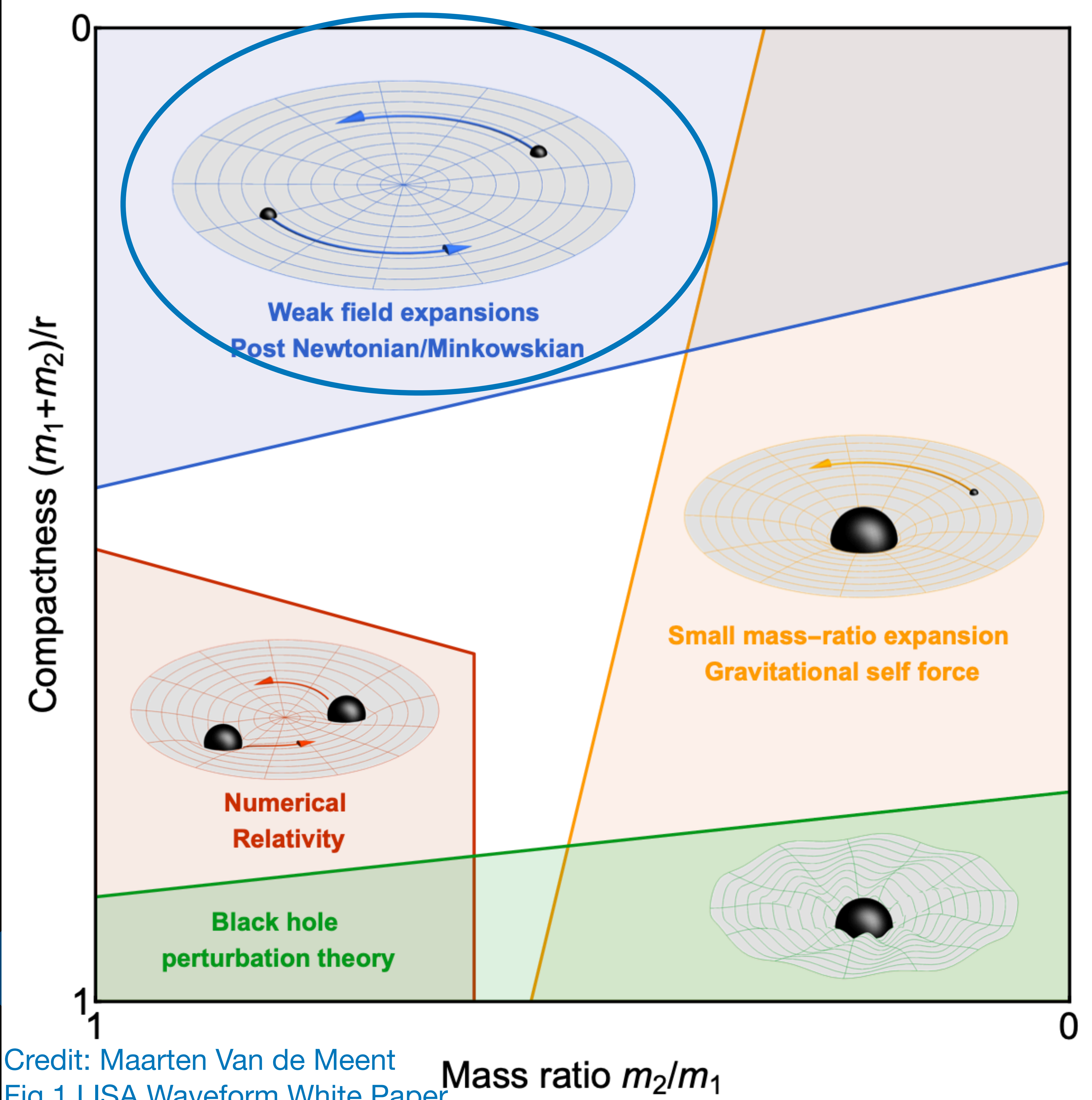
Post-Newtonian / Minkowskian Approximations

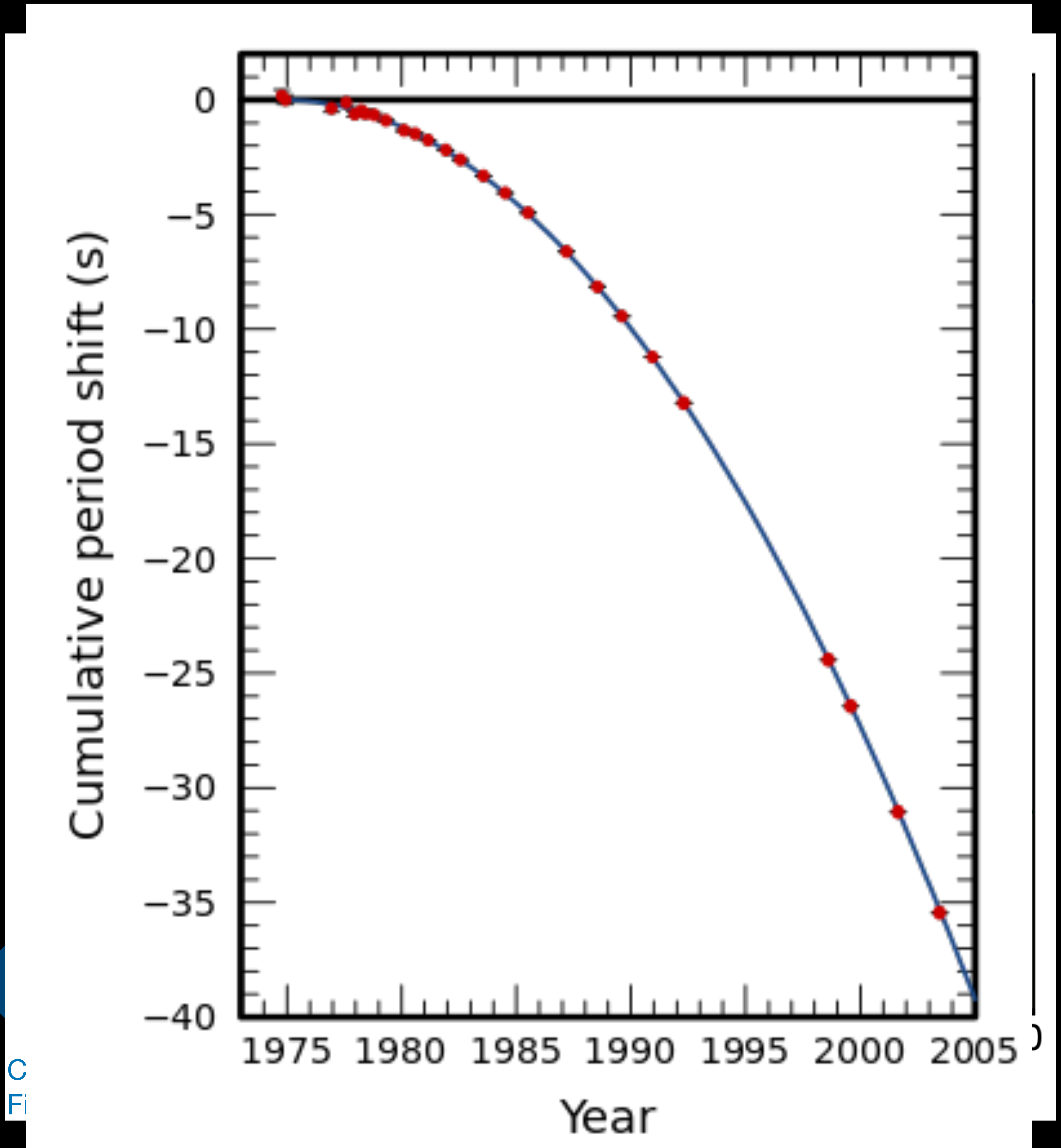


Credit: Maarten Van de Meent
Fig.1 LISA Waveform White Paper

Post-Newtonian / Minkowskian

- Perturbation from flat spacetime
$$g_{ab} = \eta_{ab} + \epsilon h_{ab} + \mathcal{O}(\epsilon^2),$$
- Post-Newtonian (PN): system velocity
$$\Rightarrow \epsilon \sim v^2/c^2,$$
- Post-Minkowskian: Gravitational Potential $\Rightarrow \epsilon \sim G,$
- Application: Inspiral, all masses

Post-Newtonian / Minkowskian Approximations



Post-Newtonian / Minkowskian

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$$\Rightarrow \epsilon \sim v^2/c^2,$$

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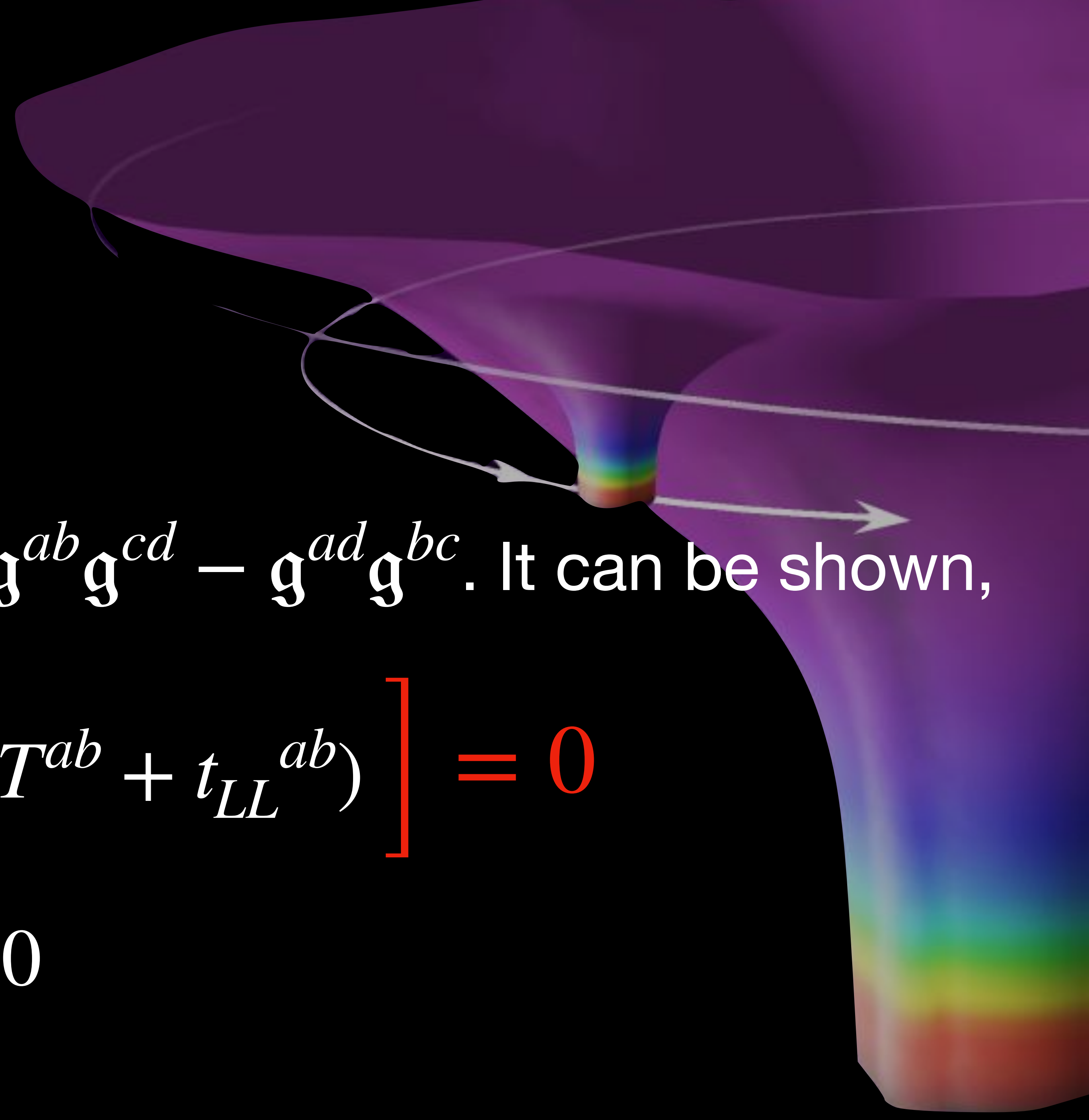
- Application: Inspiral, all masses

- Initially implemented by Einstein
- Used to confirm Hulse-Taylor binary pulsar

The Landau-Lifshitz Formalism

Harmonic Coordinates and a Wave Equation



Introduce 'gothic inverse metric', $\mathfrak{g}^{ab} = \sqrt{-g}\, g^{ab}$, and the tensor density, $H^{acbd} = \mathfrak{g}^{ab}\mathfrak{g}^{cd} - \mathfrak{g}^{ad}\mathfrak{g}^{bc}$. It can be shown,

$$\partial_b \partial_{cd} H^{acbd} = 2(-g)G^{ab} + \frac{16\pi G}{c^4}(-g){t_{LL}}^{ab} = \partial_b \left[\frac{16\pi G}{c^4}(-g)(T^{ab} + {t_{LL}}^{ab}) \right] = 0$$

- H^{acbd} has same symmetries of Riemann tensor (you can check) $\Rightarrow \partial_{bcd} H^{acbd} = 0$
- Conservation for the total energy-momentum pseudotensor.

Introduce harmonic coordinates, $\partial_b \mathfrak{g}^{ab} = 0$ and the potentials, $h^{ab} = \eta^{ab} - \mathfrak{g}^{ab} \Rightarrow \partial_b h^{ab} = 0$,

$$\Rightarrow \partial_{cd} H^{acbd} = -\Box h^{ab} + h^{cd}\partial_{cd} h^{ab} - \partial_c h^{ad} \partial_d h^{bc} = \frac{16\pi G}{c^4}(-g)(T^{ab} + {t_{LL}}^{ab}),$$

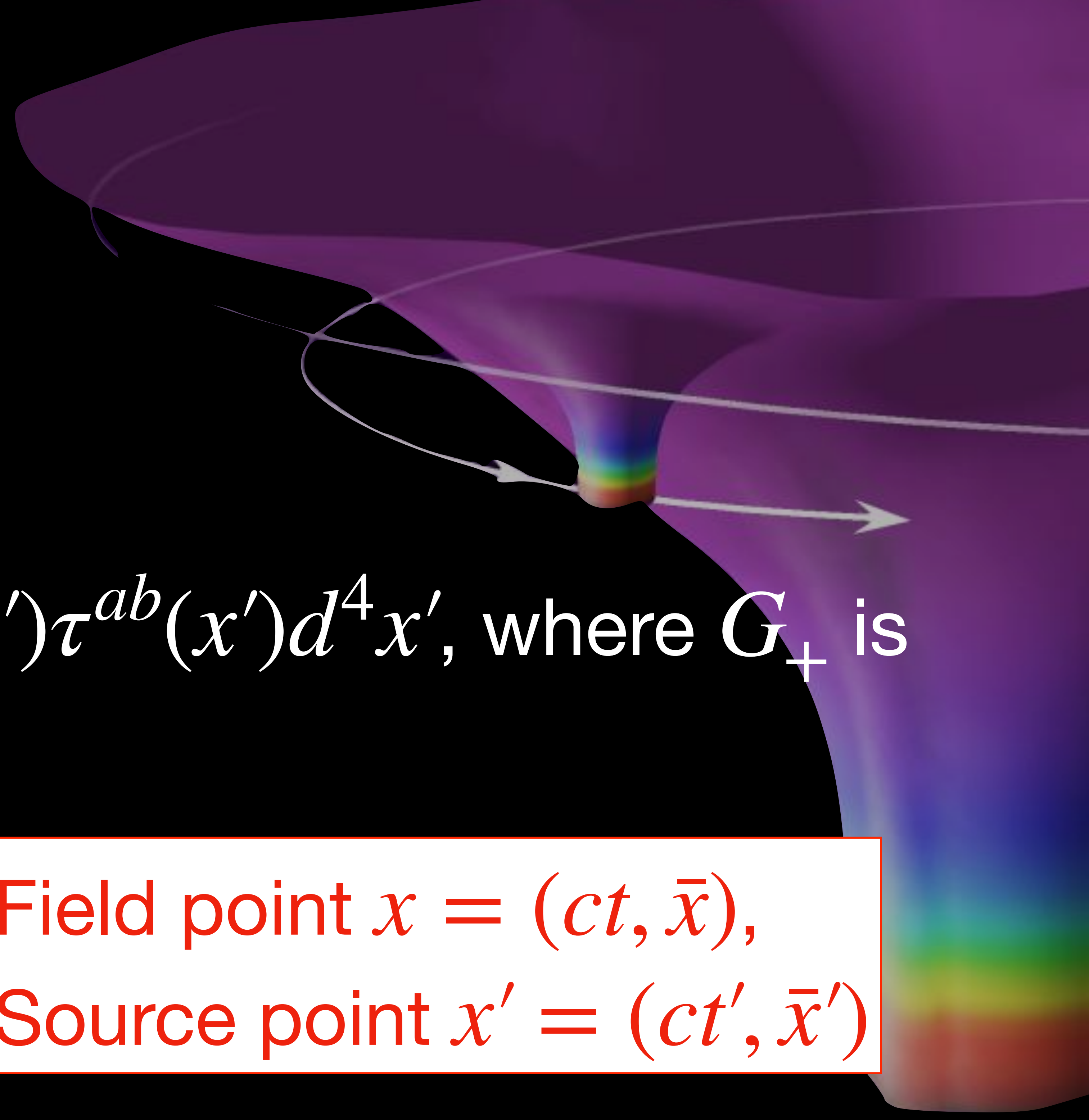
$$-\frac{16\pi G}{c^4}(-g){t_H}^{ab}$$

$$\Box h^{ab} = -\frac{16\pi G}{c^4}(-g)(T^{ab} + {t_{LL}}^{ab}) + \boxed{h^{cd}\partial_{cd} h^{ab} - \partial_c h^{ad}\partial_d h^{bc}} = -\frac{16\pi G}{c^4}\tau^{ab},$$

Marta yesterday: the stress-energy pseudotensor, $\tau^{\mu\nu} = (-g)\left(T^{\mu\nu} + t^{\mu\nu}_{LL} + t^{\mu\nu}_{H}\right)$ $\Rightarrow \partial_b \tau^{ab} = 0$

Solving the Wave Equation

Green's Functions



Wave Equation: $\Box h^{ab} = -\dfrac{16\pi G}{c^4}\tau^{ab}, \Rightarrow h^{ab}(x) = \dfrac{4G}{c^4}\displaystyle\int G(x,x')\tau^{ab}(x')d^4x'$, where G_+ is the retarded Green's function satisfying,

$$\Box G_+(x,x') = -4\pi\delta(x-x').$$

Field point $x = (ct, \bar{x})$,
Source point $x' = (ct', \bar{x}')$

For flat spacetime,

$$G_+(x,x') = \frac{\delta(ct - ct' - |\bar{x} - \bar{x}'|)}{|\bar{x} - \bar{x}'|} = 2\Theta(ct - ct')\delta[(ct - ct')^2 - |\bar{x} - \bar{x}'|^2],$$

$$\Rightarrow h^{ab} = \frac{4G}{c^4}\int \frac{\delta(ct - ct' - |\bar{x} - \bar{x}'|)}{|\bar{x} - \bar{x}'|}\tau^{ab}(x')d^4x' = \frac{4G}{c^4}\int \frac{\tau^{ab}(t - |\bar{x} - \bar{x}'|/c, \bar{x}')}{|\bar{x} - \bar{x}'|}d^34x'$$

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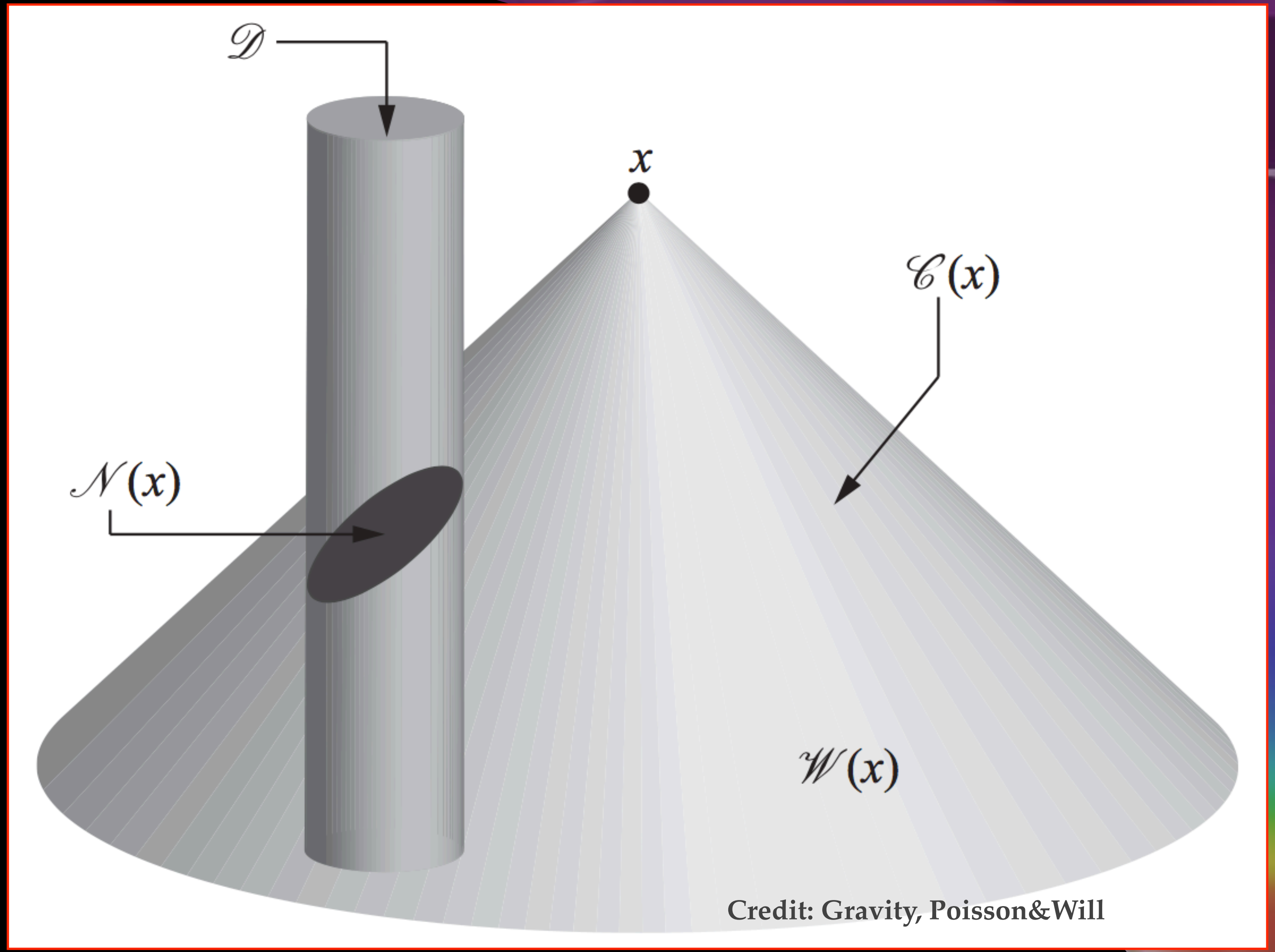
Solving the Wave Equation

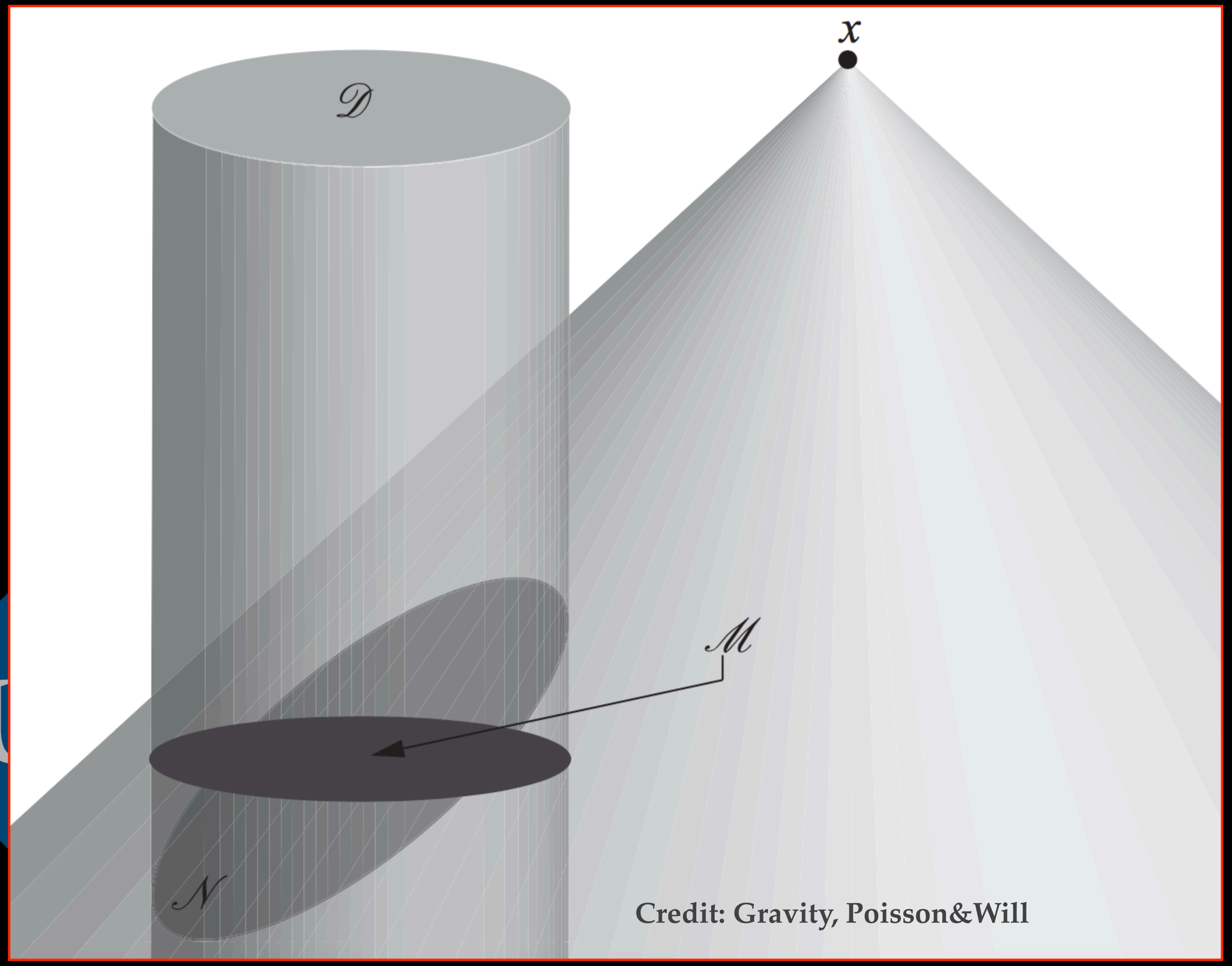
Integration Domains

Domain: past light cone of the field point, x; divide into near zone and wave zone, $h^{ab} = h^{ab}_{\mathcal{N}} + h^{ab}_{\mathcal{W}}$,

$$\Rightarrow h^{ab}_{\mathcal{N}/\mathcal{W}} = \frac{4G}{c^4}\int_{\mathcal{N}/\mathcal{W}} \frac{\tau^{ab}(t - |\bar{x} - \bar{x}'|/c, \bar{x}')}{|\bar{x} - \bar{x}'|} d^3x'$$



Near zone integration with far zone field point



$$\Rightarrow h^{ab}_{\mathcal{N}} = \frac{4G}{c^4}\int_{\mathcal{N}}\int \frac{\tau^{ab}(t - |\bar{x} - \bar{x}'|/c, \bar{y})}{|\bar{x} - \bar{x}'|}\delta(\bar{y} - \bar{x}') d^3y d^3x',$$

$\bar{x}'$ small

$$= \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}} \tau^{ab}(t - |x|/c, \bar{x}')x'^L d^3x'\right],$$

Far-away zone ($r \to \infty$),

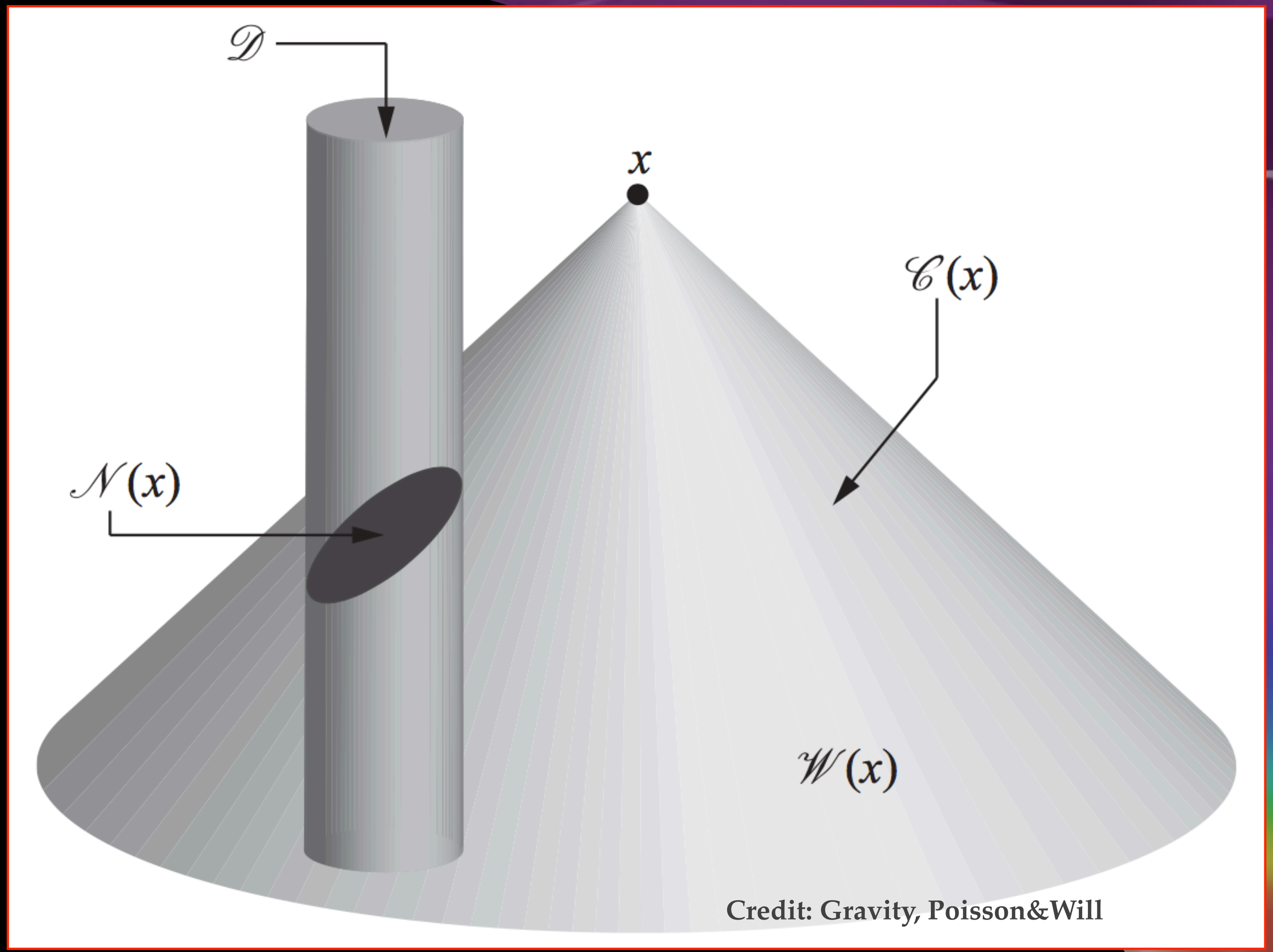
$$\Rightarrow h^{ab}_{\mathcal{N}}(x) = \frac{4G}{rc^4}\sum_{\ell=0}^{\infty}\frac{1}{\ell!} n_L\left(\frac{\partial}{\partial\tau}\right)^\ell \int_{\mathcal{M}} \tau^{ab}(\tau, \bar{x}')x'^L d^3x' + \mathcal{O}(r^{-2}).$$

Solving the Wave Equation

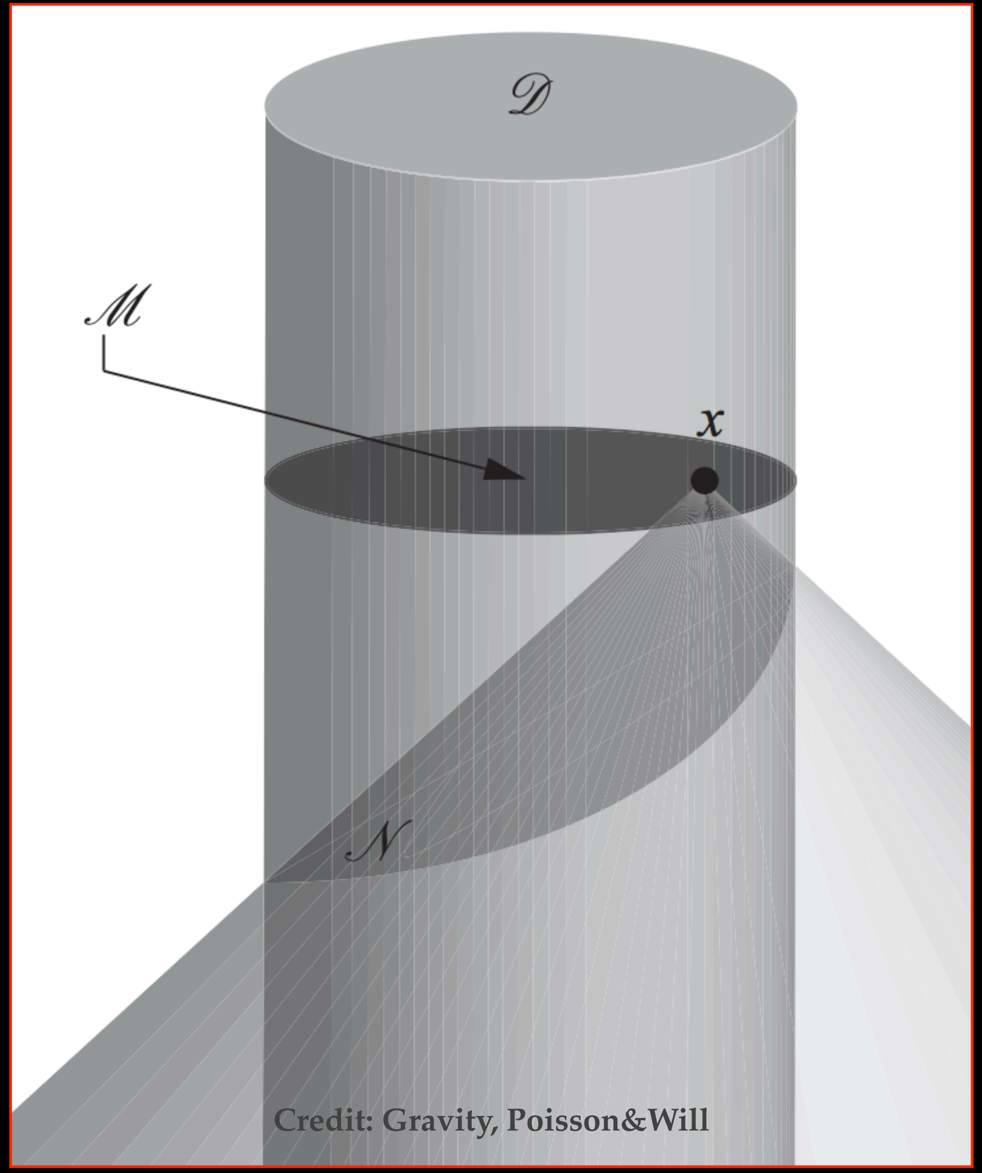
Integration Domains

Domain: past light cone of the field point, x; divide into near zone and wave zone, $h^{ab} = h^{ab}_{\mathcal{N}} + h^{ab}_{\mathcal{W}}$,

$$\Rightarrow h^{ab}_{\mathcal{N}/\mathcal{W}} = \frac{4G}{c^4}\int_{\mathcal{N}/\mathcal{W}} \frac{\tau^{ab}(t - |\bar{x}-\bar{x}'|/c, \bar{x}')}{|\bar{x}-\bar{x}'|} d^3x'$$



Near zone integration with near zone field point



$|\bar{x}-\bar{x}'|$ is small => Taylor expand τ^{ab},

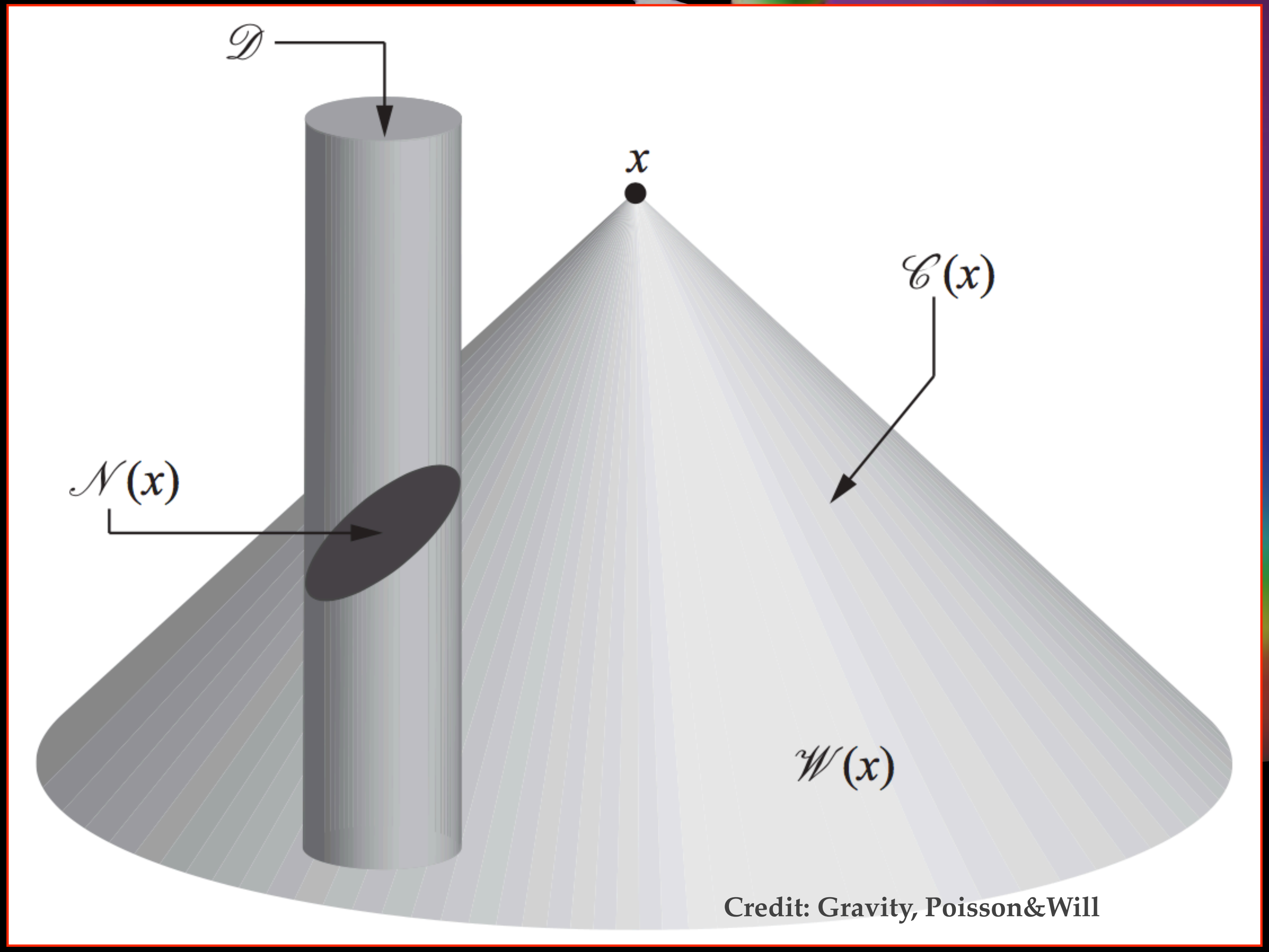
$$\Rightarrow h^{ab}_{\mathcal{N}} = \frac{4G}{c^4}\int_{\mathcal{N}} \frac{\tau^{ab}(t,\bar{x}')}{|\bar{x}-\bar{x}'|} - \frac{\partial}{c\partial t}\tau^{ab}(t,\bar{x}') + \ldots d^3x',$$

$$= \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!c^{\ell}}\int_{\mathcal{M}} \tau^{ab}(t,\bar{x}')|\bar{x}-\bar{x}'|^{\ell-1} d^3x',$$

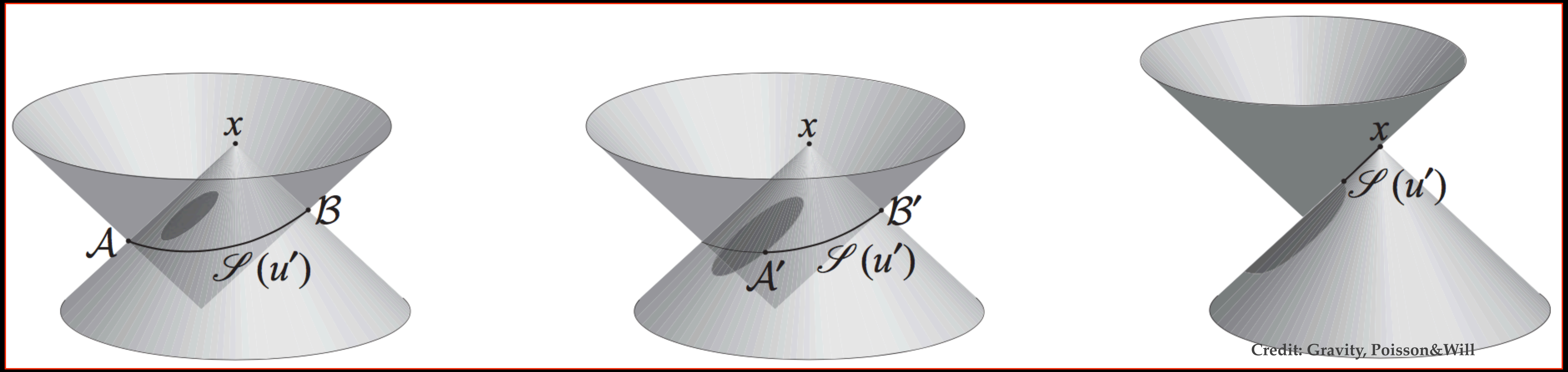
Solving the Wave Equation

Integration Wave Zone

$$\Rightarrow h^{ab}_{\mathscr{W}} = \frac{4G}{c^4}\int_{\mathscr{W}} \frac{\tau^{ab}(t-|\bar{x}-\bar{x}'|/c,\bar{x}')}{|\bar{x}-\bar{x}'|}d^3x'$$

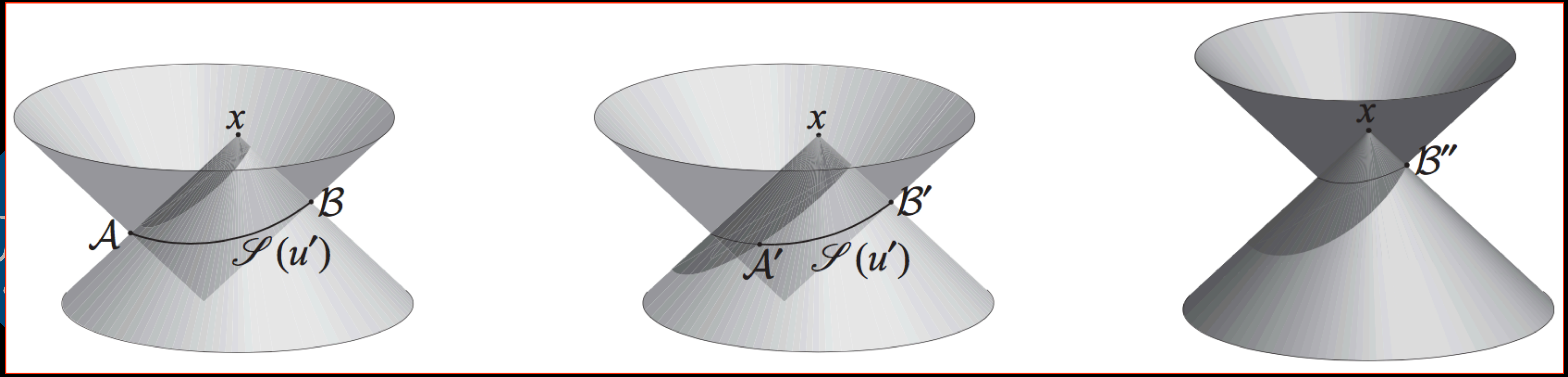


Wave zone integration with wave zone field point



$$h^{\alpha\beta}_{\mathcal{W}} = \frac{n^{<L>}}{r}\left[\int_{\cancel{0}\;\mathcal{R}-r}^{\mathcal{R}} ds f(\tau-2s)A(s,r) + \int_{\mathcal{R}}^{\infty} ds f(\tau-2s)B(s,r)\right]$$

Wave zone integration with near zone field point



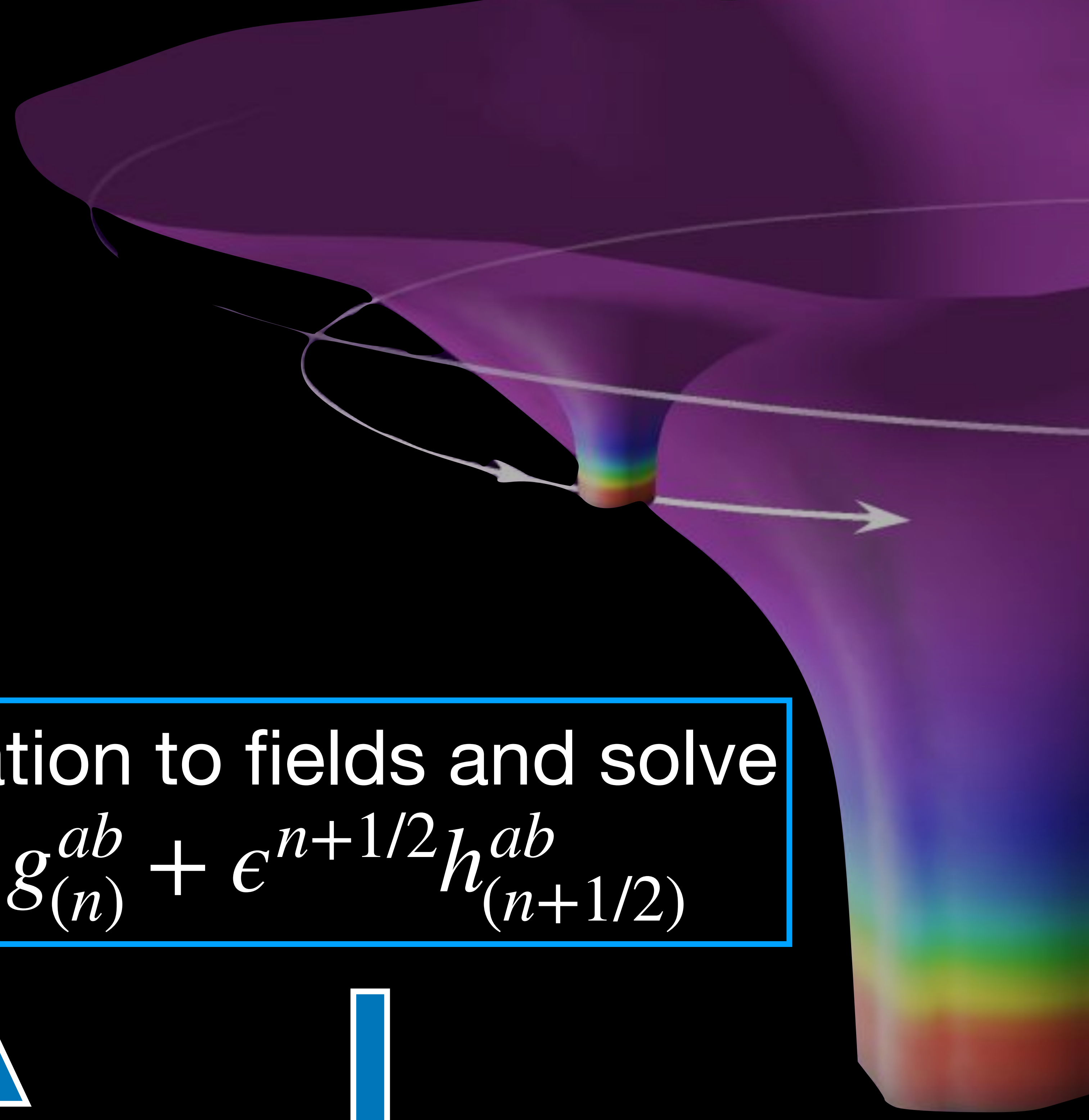
$$A(s,r) := \int_{\mathcal{R}}^{r+s}\frac{P_\ell(\xi)}{r'^{(n-1)}}dr',$$

$$B(s,r) := \int_{s}^{r+s}\frac{P_\ell(\xi)}{r'^{(n-1)}}dr',$$

Solving PN

Reiteratively

Solve simultaneously for source term τ^{ab} and metric g^{ab}



Flat metric
$g^{ab} = \eta^{ab}$

Calculate 0PN source term
τ^{ab}

Add perturbation to fields and solve
$g^{ab}_{(n+1/2)} = g^{ab}_{(n)} + \epsilon^{n+1/2} h^{ab}_{(n+1/2)}$

Calculate next +0.5PN source term
$\tau^{ab}_{(n+1/2)}$

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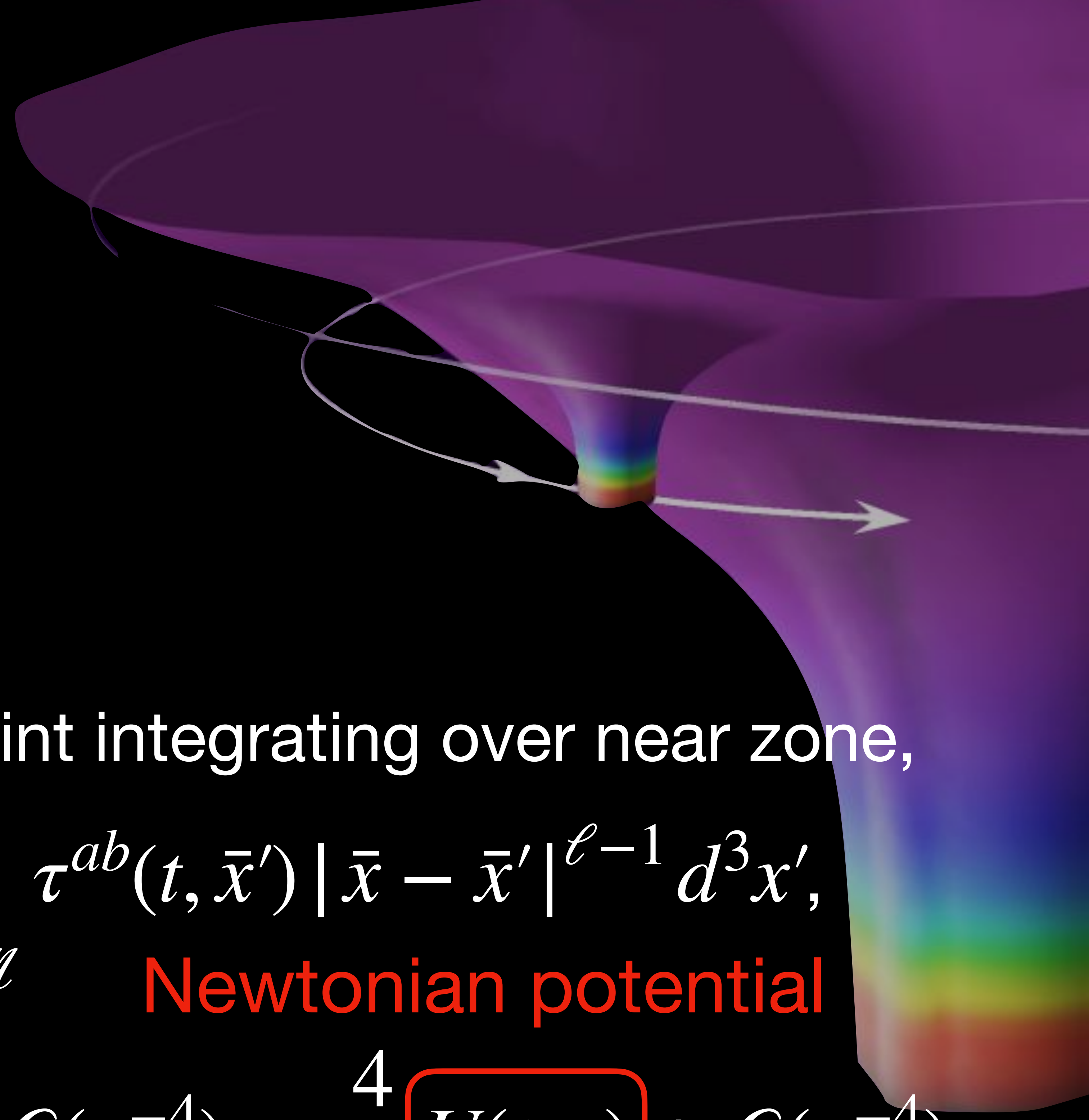
Example: Perfect Fluid



Energy-momentum tensor,

$$T^{ab} = (\rho + \mathscr{E}/c^2 + p/c^2)u^a u^b + p g^{ab} \ ,$$

and rescaled mass density, $\rho^* = \sqrt{-g}\gamma\rho$,

$$\Rightarrow \rho = [1 - 1/2(v/c)^2 + \mathcal{O}(c^{-4})]\rho^*,$$

Assuming η^{ab},

$$c^{-2}T_0^{00} = \rho^* + \mathcal{O}(c^{-2}),$$

$$c^{-1}T_0^{0j} = \rho^* v^j + \mathcal{O}(c^{-2}),$$

$$T_0^{ij} = \mathcal{O}(1),$$

Recall for near zone field point integrating over near zone,

$$h^{ab} = \frac{4G}{c^4}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell\,!c^\ell}\int_{\mathscr{M}} \tau^{ab}(t,\bar{x}')\,|\bar{x}-\bar{x}'|^{\ell-1}\,d^3x',$$

Newtonian potential

$$h_1^{00} = \frac{4G}{c^2}\int\frac{\rho^*}{|x-x'|}d^3x' + \mathcal{O}(c^{-4}) \equiv \frac{4}{c^2}\boxed{U(t,x)} + \mathcal{O}(c^{-4})$$

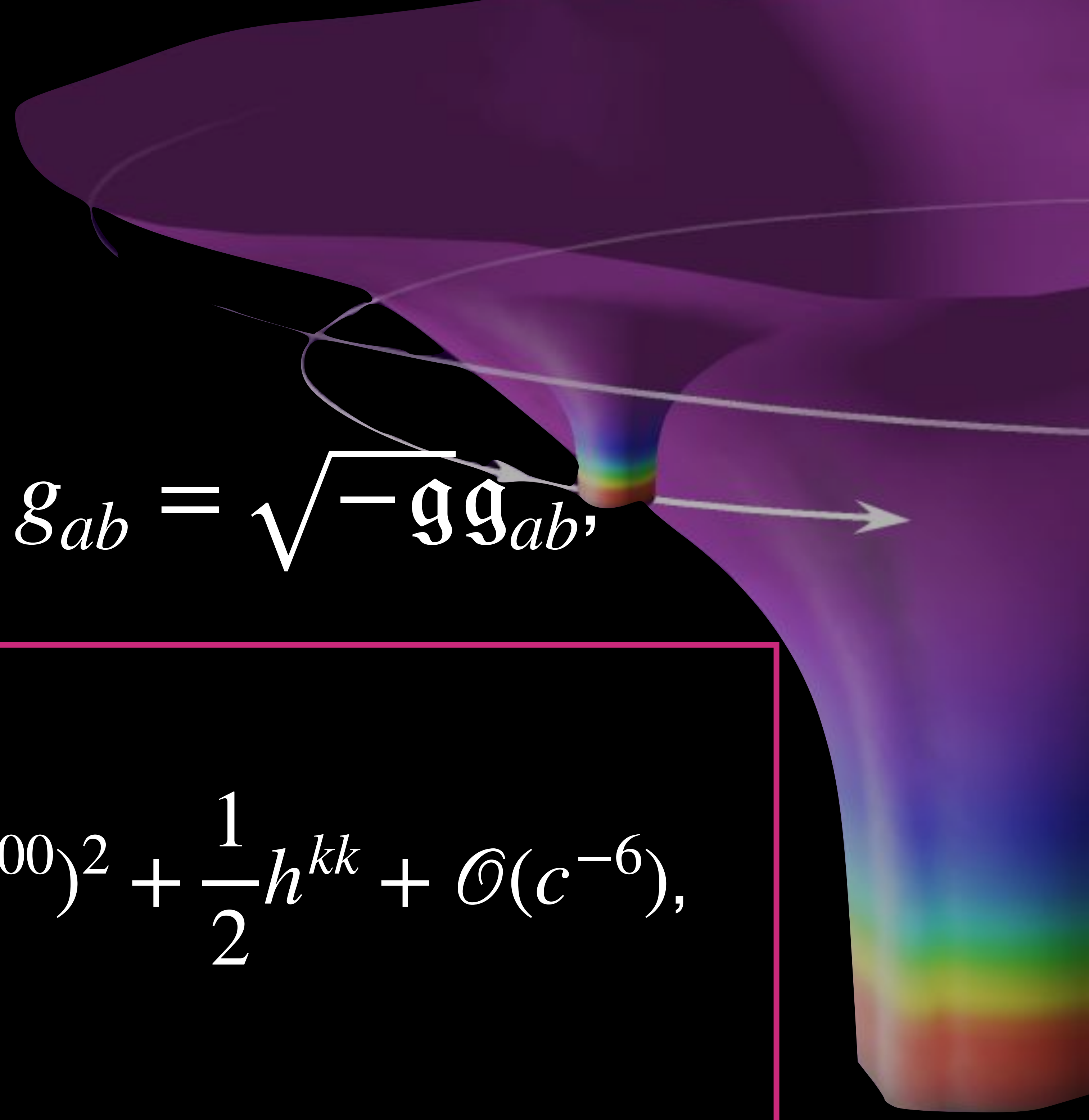
$$h_1^{0j} = \frac{4G}{c^3}\int\frac{\boxed{\rho^* v^j(t,x')}}{|x-x'|}d^3x' + \mathcal{O}(c^{-4}) \equiv \frac{4}{c^3}\boxed{U^j(t,x)} + \mathcal{O}(c^{-4})$$

$$h_1^{jk} = \mathcal{O}(\boxed{c^{-4}}),$$

Mass current density

Vector potential

Example: Perfect Fluid



Recall gothic inverse metric, $\mathfrak{g}^{ab} = \sqrt{-g}\,g^{ab} = \eta^{ab} - h^{ab} \Rightarrow g_{ab} = \sqrt{-\mathfrak{g}}\,\mathfrak{g}_{ab},$

This gives,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \frac{1}{2}h\eta_{\alpha\beta} + h_{\alpha\mu}h^{\mu}{}_{\beta} - \frac{1}{2}hh_{\alpha\beta} + \left(\frac{1}{8}h^2 - \frac{1}{4}h^{\mu\nu}h_{\mu\nu}\right)\eta_{\alpha\beta} + O(G^3),$$

So we have,

$$g_{00} = -1 + \frac{1}{2}h^{00} - \frac{3}{8}(h^{00})^2 + \frac{1}{2}h^{kk} + \mathcal{O}(c^{-6}),$$

$$g_{0j} = -h^{0j} + \mathcal{O}(c^{-5}),$$

$$g_{jk} = \delta_{jk}(1+\tfrac{1}{2}h^{00}) + \mathcal{O}(c^{-4}).$$

Finally

$$g_{00} = -1 + \frac{2}{c^2}U + \mathcal{O}(c^{-4}),$$

$$g_{0j} = -\frac{4}{c^3}U^j + \mathcal{O}(c^{-4}),$$

$$g_{jk} = \delta_{jk}(1 + \frac{2}{c^2}U) + \mathcal{O}(c^{-4}).$$

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Post-Newtonian / Minkowskian

Current Status (from LISA Waveform White Paper)

PN order	Dynamics: non-spinning	Dynamics: spinning SO	Dynamics: spinning SS	Dynamics: spinning higher spins	Dissipative flux: non-spinning	Dissipative flux: spinning SO	Dissipative flux: spinning SS	Dissipative flux: spinning higher spins
0	$\surd$	-	-	-	-	-	-	-
1	$\surd$	-	-	-	-	-	-	-
1.5	-	$\surd$	-	-	-	-	-	-
2	$\surd$	-	$\surd$	-	-	-	-	-
2.5	$\surd$	$\surd$	-	-	$\surd$	-	-	-
3	$\surd$	-	$\surd$	-	-	-	-	-
3.5	$\surd$	$\surd$	-	$\surd$ (S^3)	$\surd$	-	-	-
4	$\surd$	-	$\surd$	$\surd$ (S^4)	$\surd$	$\surd$	-	-
4.5	*	$\surd$	-	$\surd$ (S^3)	$\surd$	-	$\surd$	-
5	*	-	$\surd$	$\surd$ (S^4)	$\surd$	$\surd$	-	-
5.5	*			$\surd$ (S^5)	$\surd$	$\surd$	$\surd$	-
6				$\surd$ (S^6)	$\surd$	$\surd$	$\surd$	$\surd$ (S^3)
6.5				$\star$	$\surd$	$\surd$	$\surd$	
7				$\star$	$\surd$			

Table 9. Waveform White Paper

- Recent progress in PN: new 4.5PN dissipative results.
- PM research active, with interactions with the amplitudes community.
- Challenges at 5PN order: finite size and tidal effects enter.
- MBHBs will need longer NR or higher PN.

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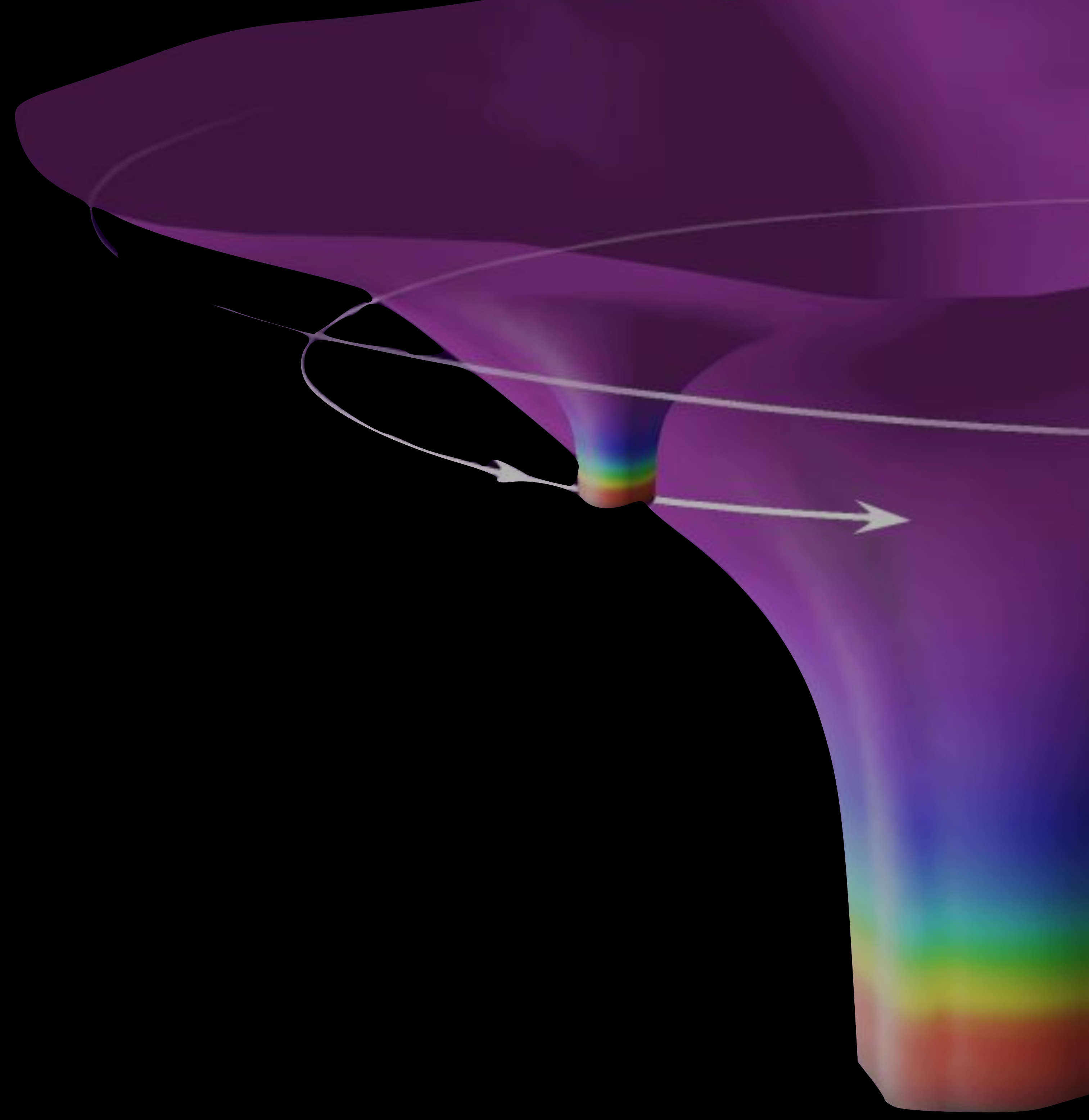
Part B The Self-Force



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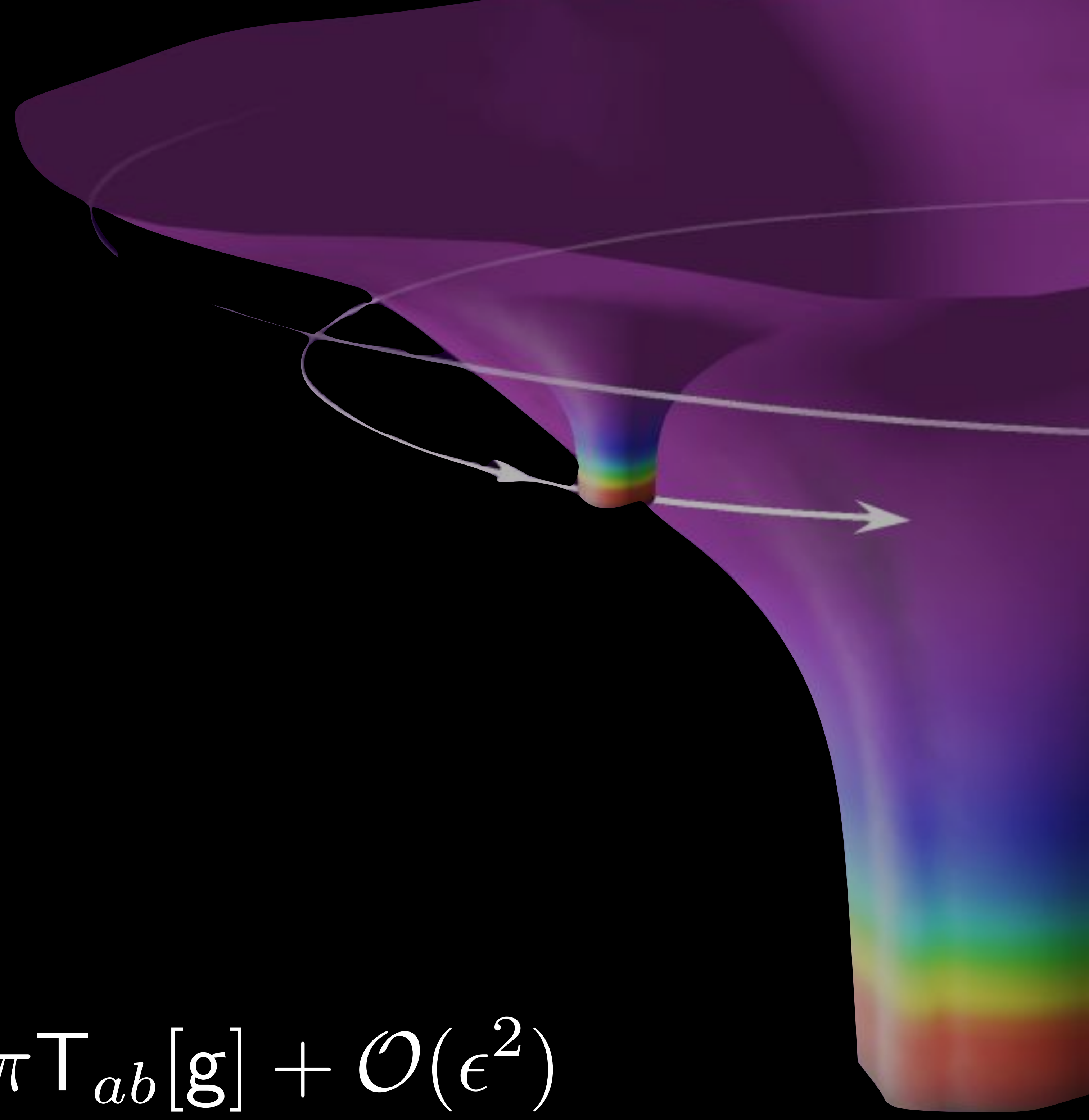

What is the Self-Force?

Metric Perturbation

Perturb the background space time:

Far zone: $$\mathrm{g}_{\mu\nu} = g^{(M)}_{\mu\nu} + \epsilon h_{\mu\nu} + \mathcal{O}(\epsilon^2)$$

Einstein Field Equations:

$$\mathsf{G}_{ab}[\mathrm{g}] = 8\pi \mathsf{T}_{ab}[\mathrm{g}], \quad \rightarrow \quad \cancel{G}_{ab}[g] + \epsilon\delta G_{ab}[h] = \epsilon 8\pi \mathsf{T}_{ab}[\mathrm{g}] + \mathcal{O}(\epsilon^2)$$

$$\Box \bar{h}_{ab} + 2R_{acbd}\bar{h}^{cd} = -16\pi \mathsf{T}_{ab}[\mathrm{g}] + \mathcal{O}(\epsilon)$$

Stress-Energy Tensor:

$$S_p = -m\int_\gamma \sqrt{-\mathrm{g}_{ab}\dot{z}^a\dot{z}^b}\,d\lambda$$

$$\rightarrow \quad \mathsf{T}^{ab}[\mathrm{g}](x) = m\int_\gamma \frac{\mathrm{g}^a{}_{a'}\mathrm{g}^b{}_{b'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}{\sqrt{-\mathrm{g}_{a'b'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}}\delta_4(x, \mathsf{z}(\lambda'))d\lambda',$$

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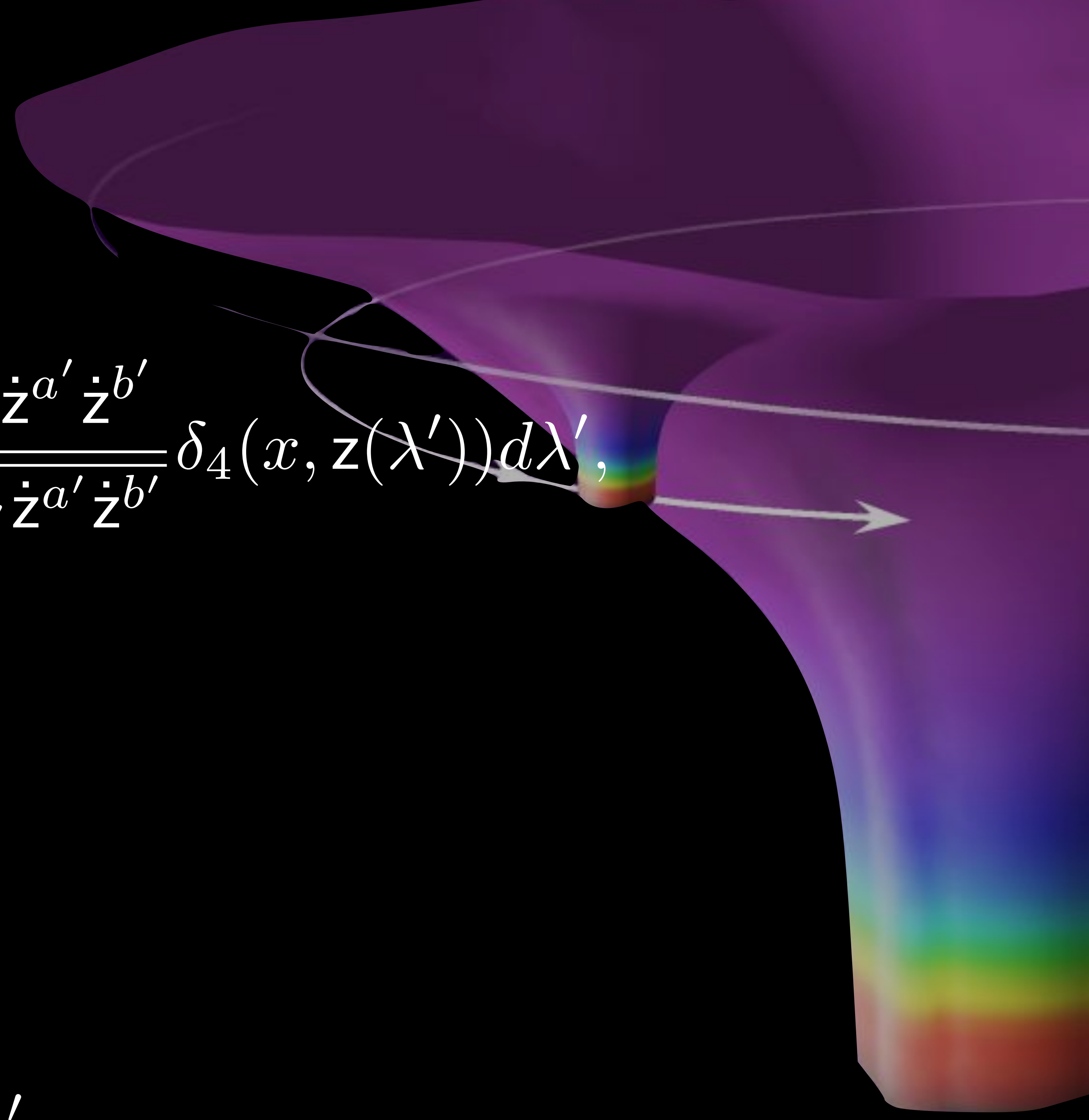
What is the Self-Force?

Resulting Equation of Motion



$$\mathsf{T}^{ab}[\mathsf{g}](x) = m\int_\gamma \frac{\mathsf{g}^a{}_{a'}\mathsf{g}^b{}_{b'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}{\sqrt{-\mathsf{g}_{a'b'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}}\delta_4(x,\mathsf{z}(\lambda'))d\lambda',$$

Conservation:

$$\begin{aligned}
\mathsf{T}^{ab}{}_{;b}[\mathsf{g}](x) &= -m\int_\gamma \frac{\mathsf{g}^a{}_{a'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}{\sqrt{-\mathsf{g}_{a'b'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}}\partial_{b'}\delta_4(x,\mathsf{z}(\lambda'))d\lambda',\\
&= m\int_\gamma \frac{d}{d\lambda'}\left[\frac{\mathsf{g}^a{}_{a'}\dot{\mathsf{z}}^{a'}}{\sqrt{-\mathsf{g}_{a'b'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}}\right]\delta_4(x,\mathsf{z}(\lambda'))d\lambda',\\
&= m\int_\gamma \frac{\mathsf{g}^a{}_{a'}}{\sqrt{-\mathsf{g}_{a'b'}\dot{\mathsf{z}}^{a'}\dot{\mathsf{z}}^{b'}}}\left[\frac{\mathsf{D}\dot{\mathsf{z}}^{a'}}{d\lambda'} - k'\dot{\mathsf{z}}^{a'}\right]\delta_4(x,\mathsf{z}(\lambda'))d\lambda',
\end{aligned}$$

$$k = \frac{1}{\sqrt{-\mathsf{g}_{ab}\dot{\mathsf{z}}^a\dot{\mathsf{z}}^b}}\frac{d}{d\lambda}\left(\sqrt{-\mathsf{g}_{ab}\dot{\mathsf{z}}^a\dot{\mathsf{z}}^b}\right)$$

$$\rightarrow \quad \frac{\mathsf{D}\dot{\mathsf{z}}^a}{d\lambda} = k\dot{\mathsf{z}}^a \quad \rightarrow \quad \frac{D\dot{\mathsf{z}}^a}{d\lambda} = \left(\Gamma^a{}_{bc} - \mathsf{\Gamma}^a{}_{bc}\right)\dot{\mathsf{z}}^b\dot{\mathsf{z}}^c + k\dot{\mathsf{z}}^a$$

In perturbed spacetime:

$$\begin{aligned}
k &= \frac{1}{\sqrt{1-h_{ab}u^{ab}}}\frac{d}{d\tau}\left(\sqrt{1-h_{ab}u^{ab}}\right),\\
&= -\frac{1}{2}h_{ab;c}u^{abc} + \mathcal{O}(\epsilon)
\end{aligned}$$

$$\rightarrow \quad \begin{aligned}
a^{a[0]} &= 0,\\
a^{a[1]} &= -\frac{1}{2}\left(h^a{}_{b;c} + h^a{}_{c;b} - h_{bc}{}^{;a} - u^{ad}h_{bc;d}\right)u^{bc},\\
&= -\frac{1}{2}\left(g^{ab} + u^{ab}\right)\left(2h_{bc;d} - h_{cd;b}\right)u^{cd}.
\end{aligned}$$

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Parameter Space

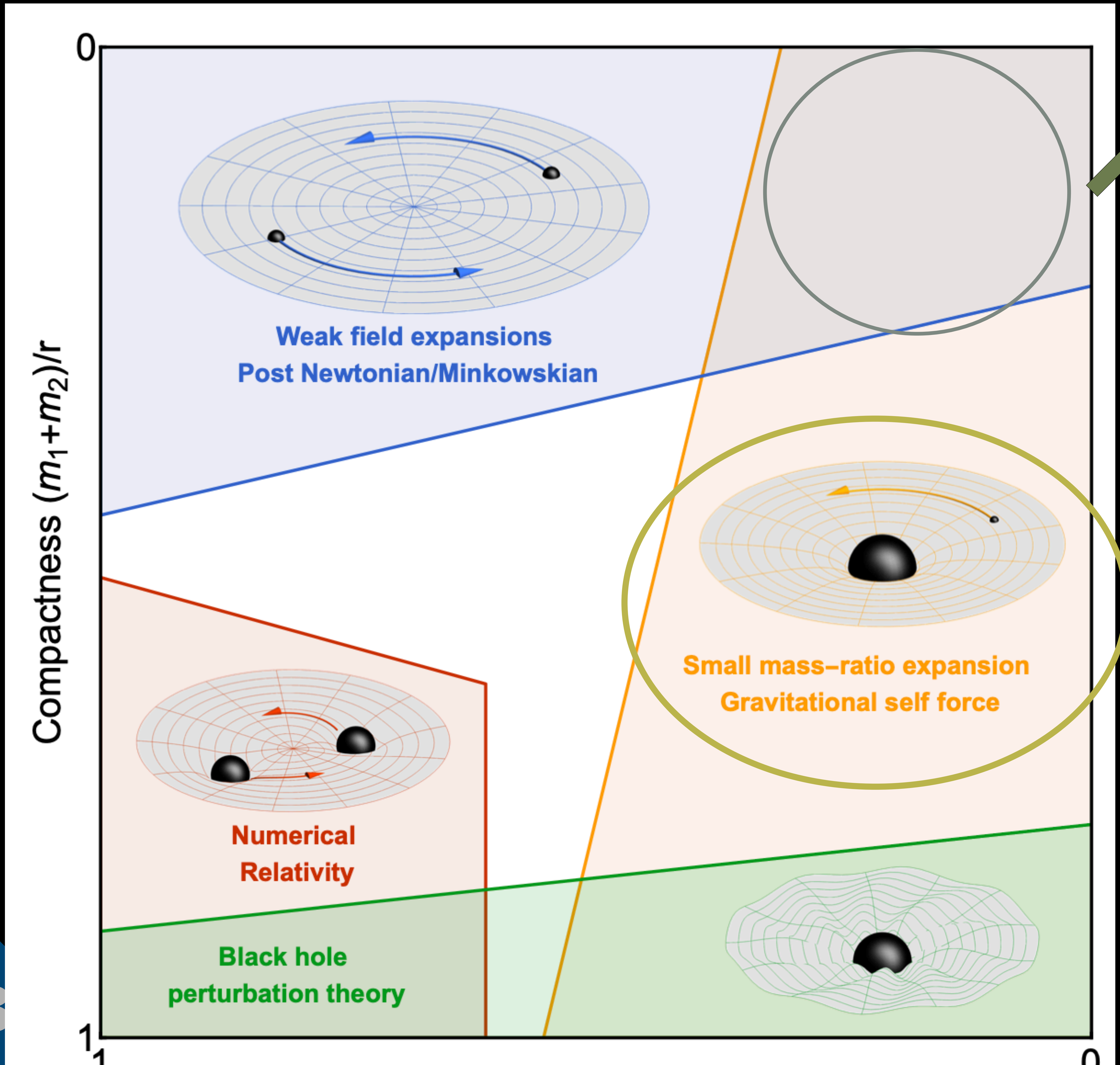


Figure credit: Maarten Van de Meent
Fig.1 Waveform White Paper

Comparison with PN

- Gauge invariant quantities
- Read off unknown PN coefficients
- Calibrate EOB

Waveforms

- **Highly accurate 1st order Kerr eccentric inclined**

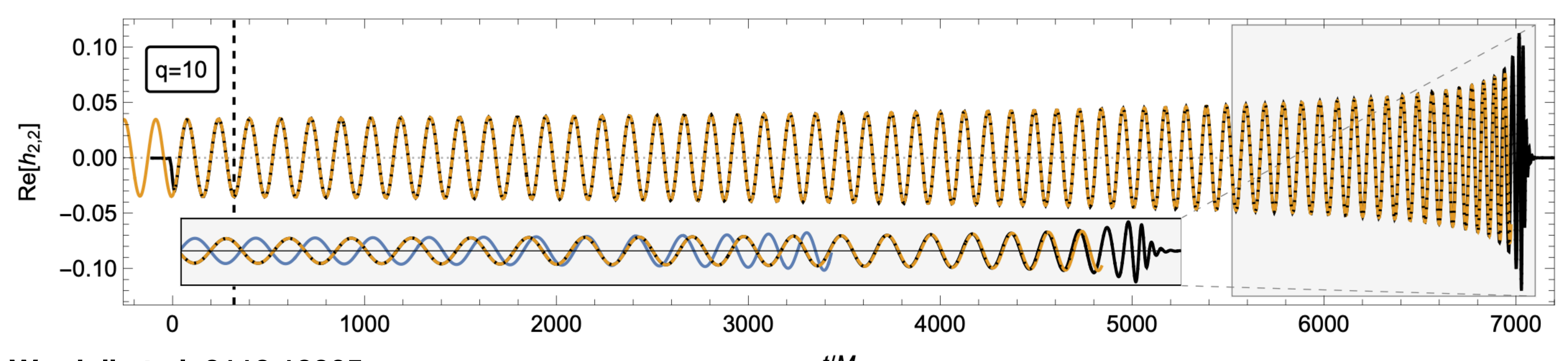
$$\mathsf{g}_{ab} = g_{ab} + \epsilon h^{(1)}_{ab} + \epsilon^2 h^{(2)}_{ab} + \ldots$$

- **2nd order**
- **Orbital Evolution / Waveform generation**

$$\frac{d\Omega}{dt} = F[h^{(1)}, h^{(2)}], \qquad \frac{d\varphi}{dt} = \Omega\,.$$

$h^{(1)}_{ab}$: Linear perturbation (Teukolsky)
- good handle

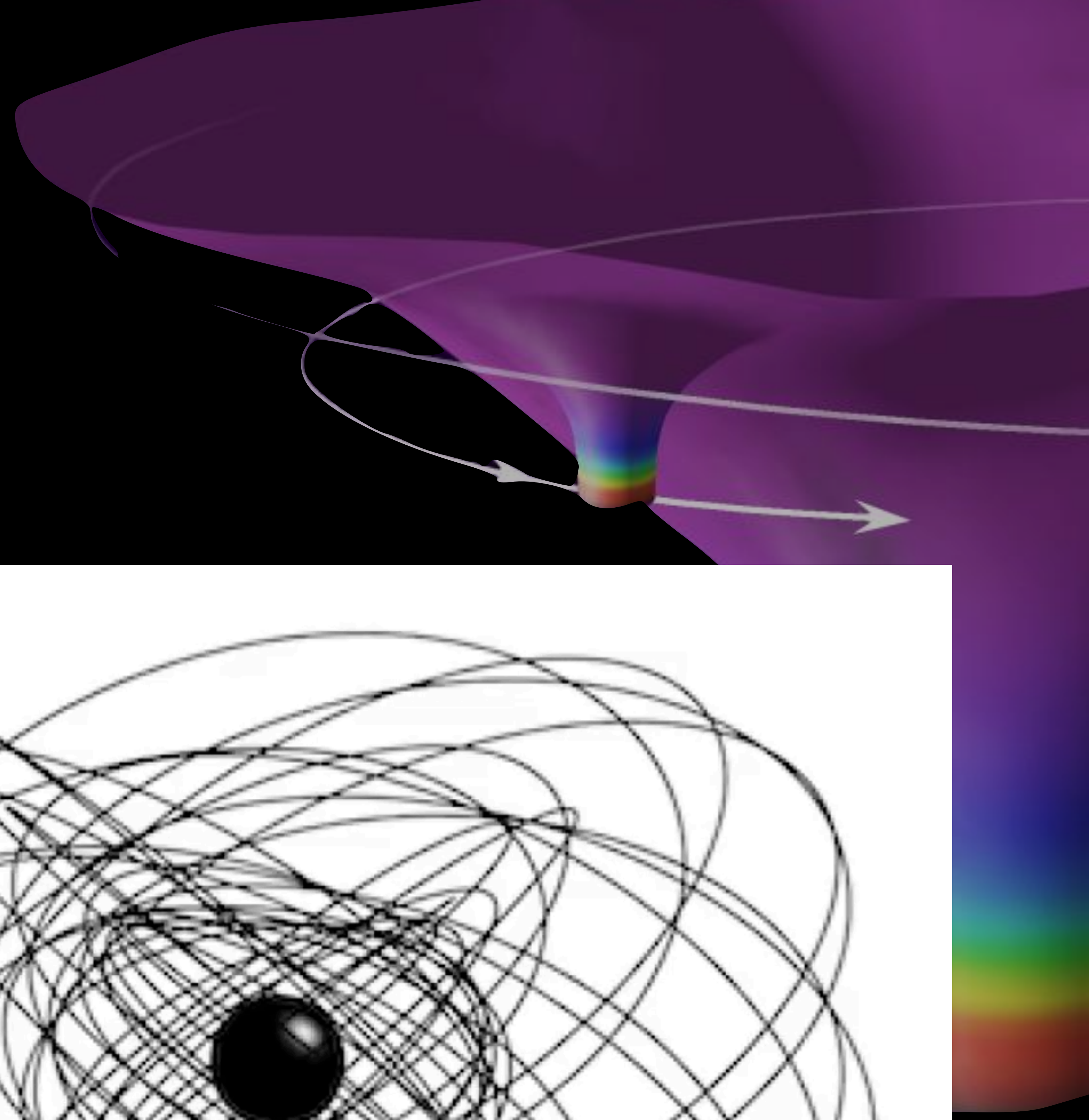
$h^{(2)}_{ab}$: Difficult
- only calculated for Schwarzschild circular



Wardell et al. 2112.12265

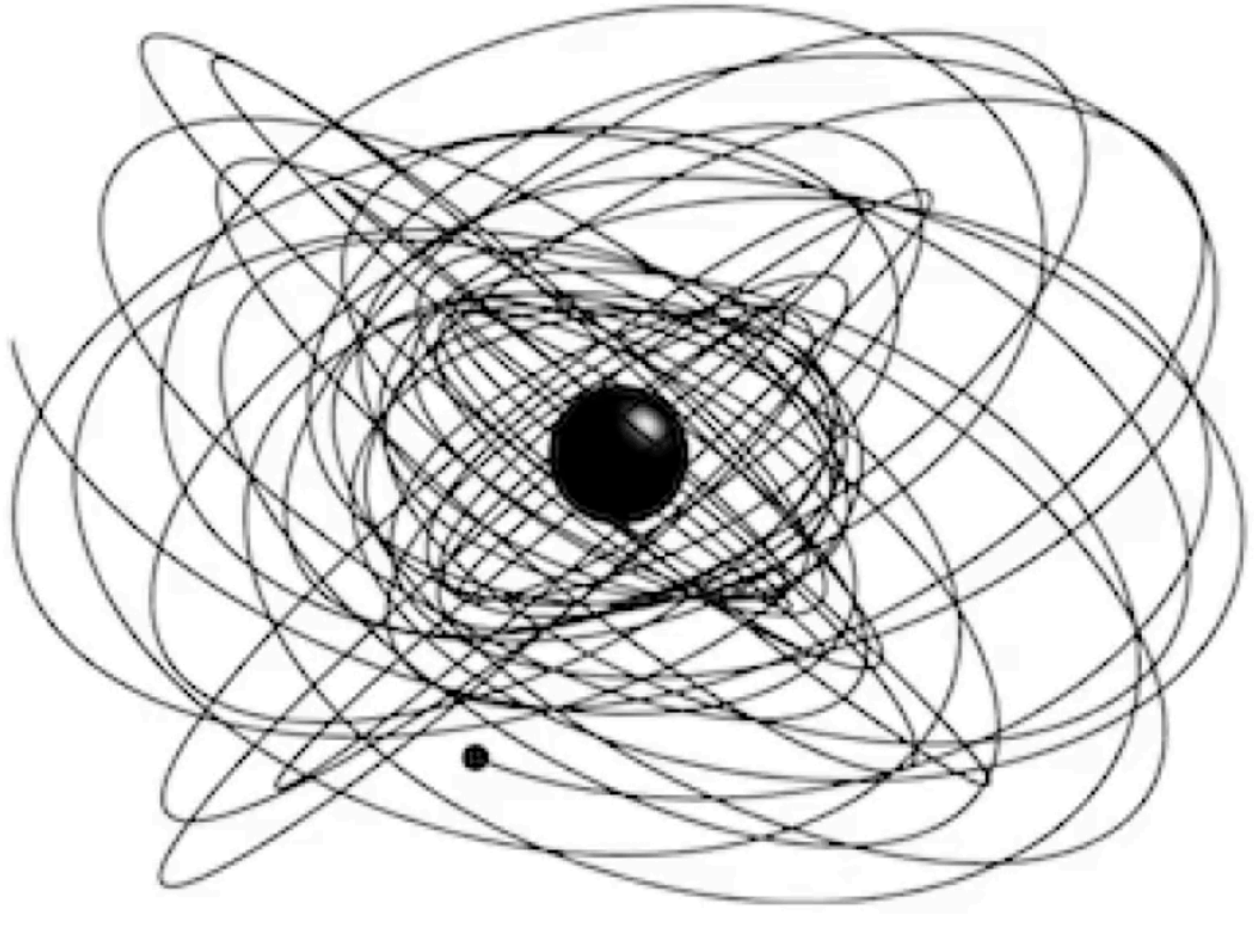
Zero Order

Test mass on geodesic in Kerr spacetime



- Geodesic characterised by 3 constants:
  $J_A = (E, L_z, Q)$
  - E: Energy,
  - L_z: Azimuthal Angular Momentum
  - Q: Carter constant related to orbital inclination
- Phases $\varphi_A = (\varphi_r, \varphi_\theta, \varphi_\phi)$ with frequencies

$$\frac{d\varphi_A}{dt} = \Omega_A(J_B)$$



[image credit: Steve Drasco]

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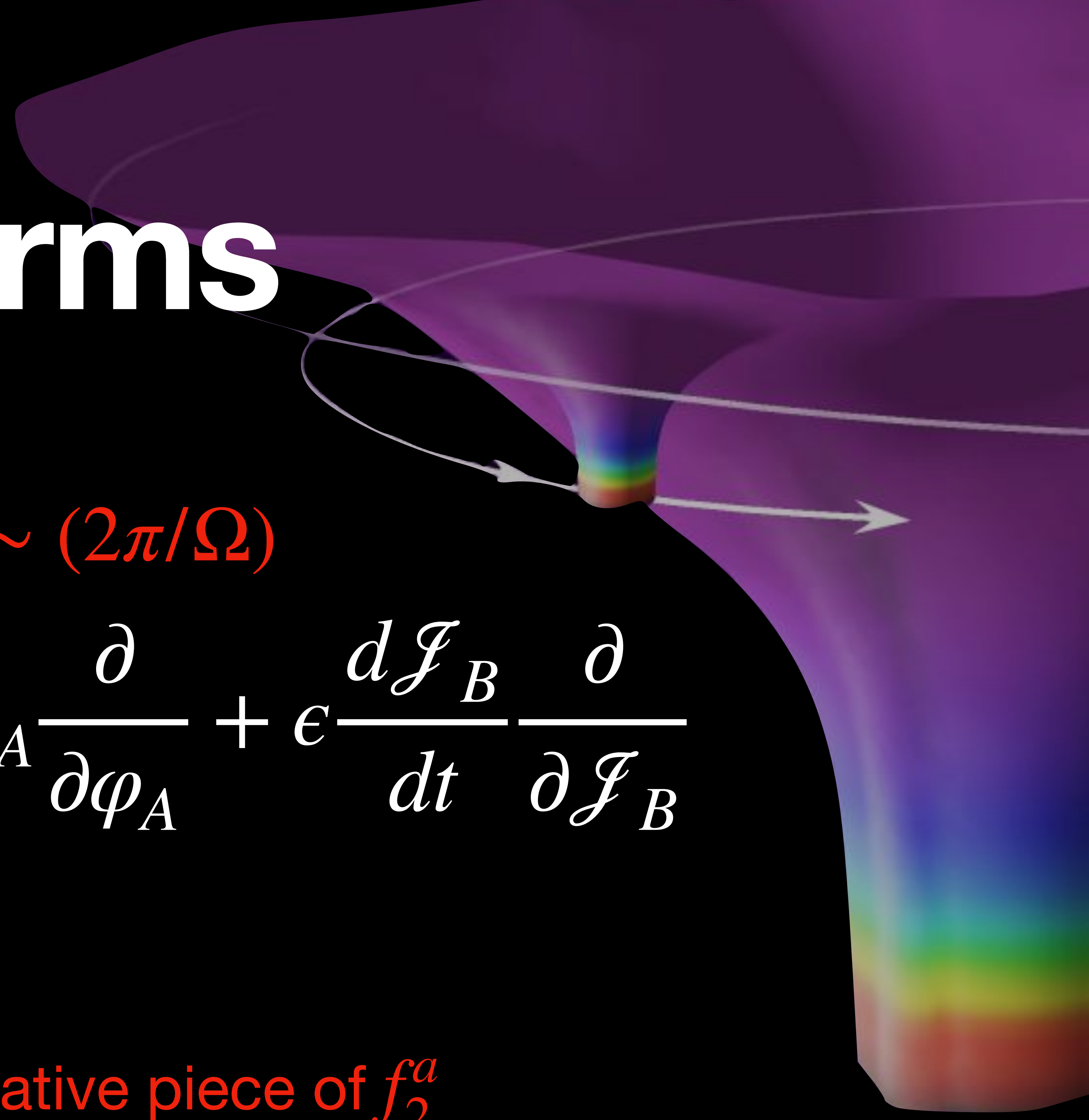
Adding the self-force for waveforms
(Perturbations)



$t_{rad} \sim (2\pi/\epsilon\Omega) \gg t_{orb} \sim (2\pi/\Omega)$

Two-timescale: $\frac{d}{dt} \rightarrow \Omega_A \frac{\partial}{\partial \varphi_A} + \epsilon \frac{d\mathcal{J}_B}{dt} \frac{\partial}{\partial \mathcal{J}_B}$

- Orbit description:
  - Perturbed variables $(\tilde{\varphi}_A, \tilde{J}_A)$ for orbit.
  - Full set of system parameters: $\mathcal{J}_A \sim (\tilde{J}_A, M_{BH}, J_{BH})$.
  - Phase frequencies: $\frac{d\tilde{\varphi}_A}{dt} = \Omega_A^{(0)}(\mathcal{J}_B) + \epsilon\Omega_A^{(1)}(\mathcal{J}_B) + \ldots,$

Dissipative piece of f_2^a

Conservative piece of f_1^a

Dissipative piece of f_1^a

  - Evolution (slow) of system parameters: $\frac{d\mathcal{J}_A}{dt} = \epsilon \left[F_A^{(0)}(\mathcal{J}_B) + \epsilon F_A^{(1)}(\mathcal{J}_B) + \mathcal{O}(\epsilon^2) \right]$
- Multiscale Waveform: $h_{lm} = \left[\epsilon h_{lm}^{(1)}(\mathcal{J}_A) + \epsilon^2 h_{lm}^{(2)}(\mathcal{J}_A) + \mathcal{O}(\epsilon^3) \right] e^{-im\varphi^\phi}$

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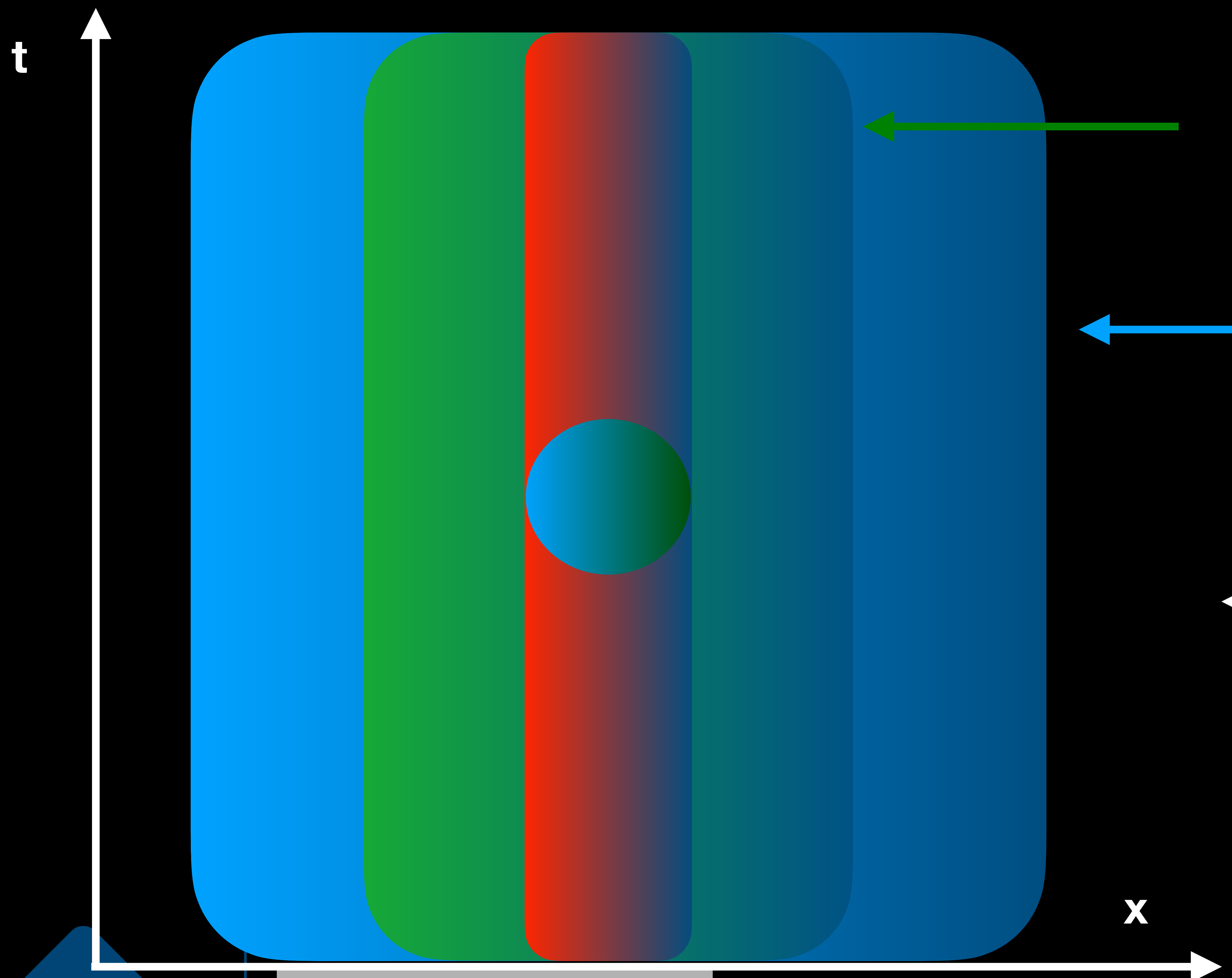
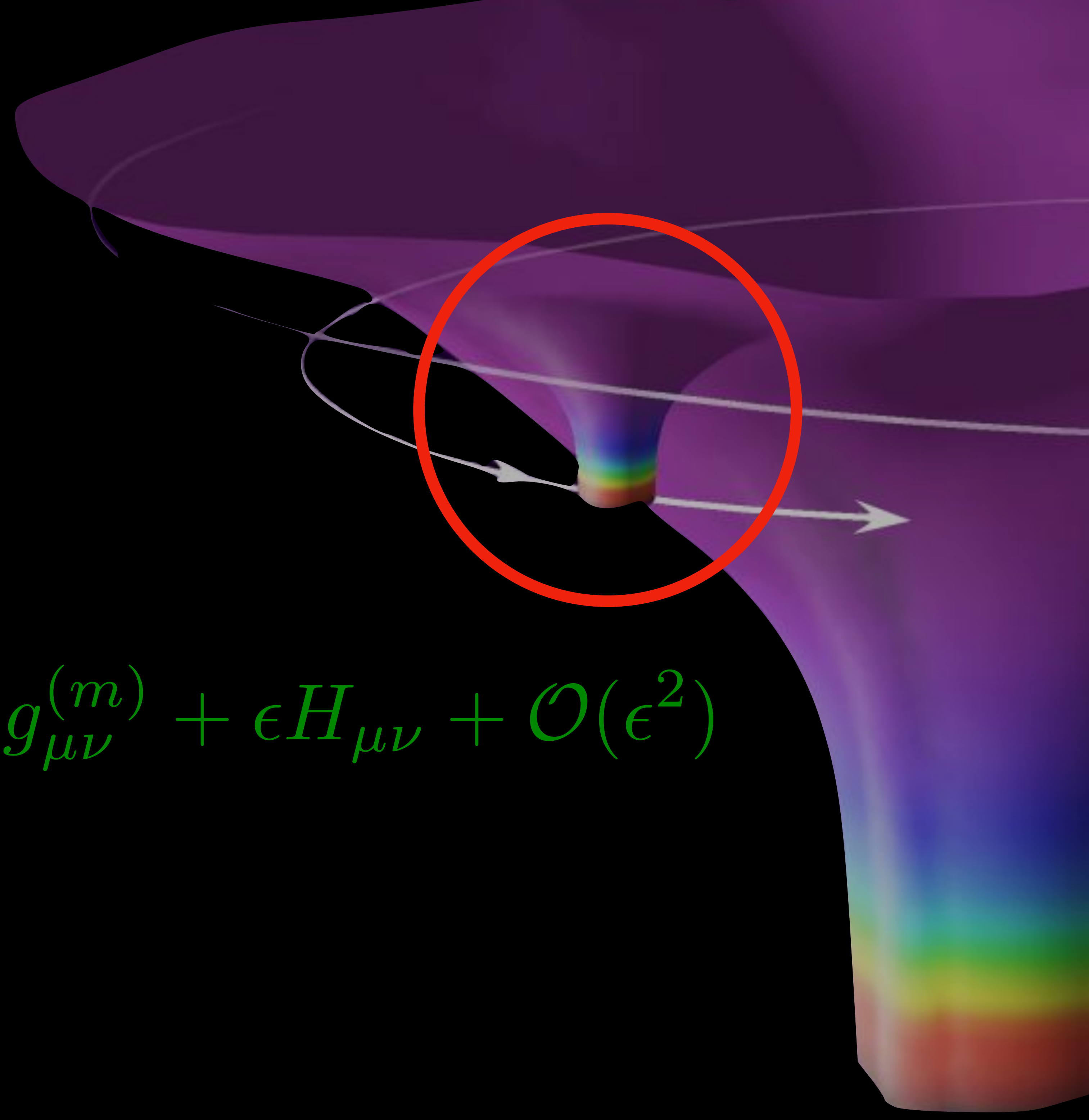
- Solve field equations for amplitudes h_{lmk^i} and forcing functions $F_A^{(n)}$ for given $\mathcal{J}_A$ (offline grid)
- Solve ODE's for time evolution (online)
- See FastEMRIWaveforms (FEW) (Chapman-Bird et el.) for waveforms in 10-100 ms

Flanagan et al., Pound et al., Miller et al., Van de Meent et al.

Regularising

Matched Expansions





Near zone:

$R \ll 1$

$$\mathbf{g}_{\mu\nu} = g^{(m)}_{\mu\nu} + \epsilon H_{\mu\nu} + \mathcal{O}(\epsilon^2)$$

Buffer zone:

$\epsilon \ll R \ll 1$

Far zone:

$R \gg \epsilon$
$(\epsilon \ll 1)$

$$\mathbf{g}_{\mu\nu} = g^{(M)}_{\mu\nu} + \epsilon h_{\mu\nu} + \mathcal{O}(\epsilon^2)$$

Solve in the buffer zone!

$\Rightarrow$ MiSaTaQuWa equations of motion

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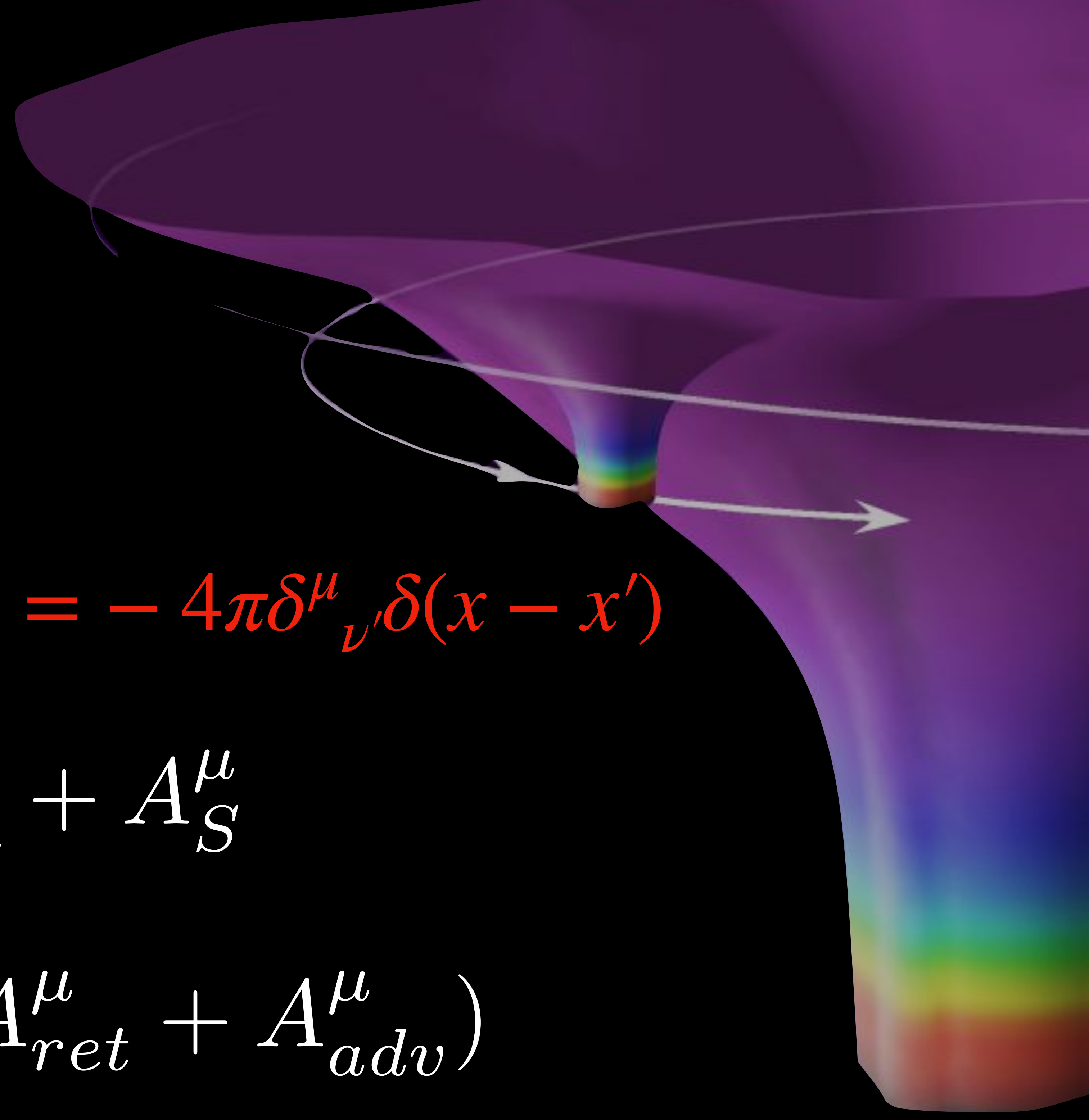
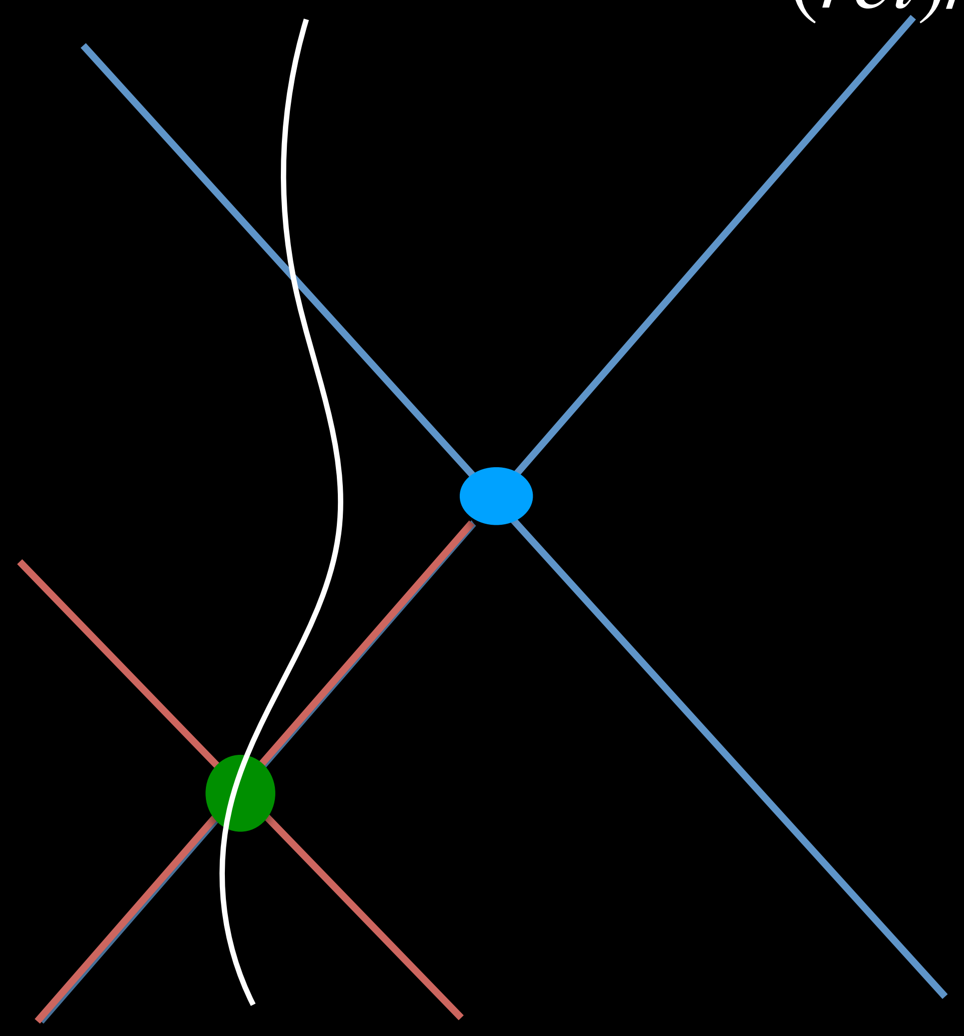
Regularising

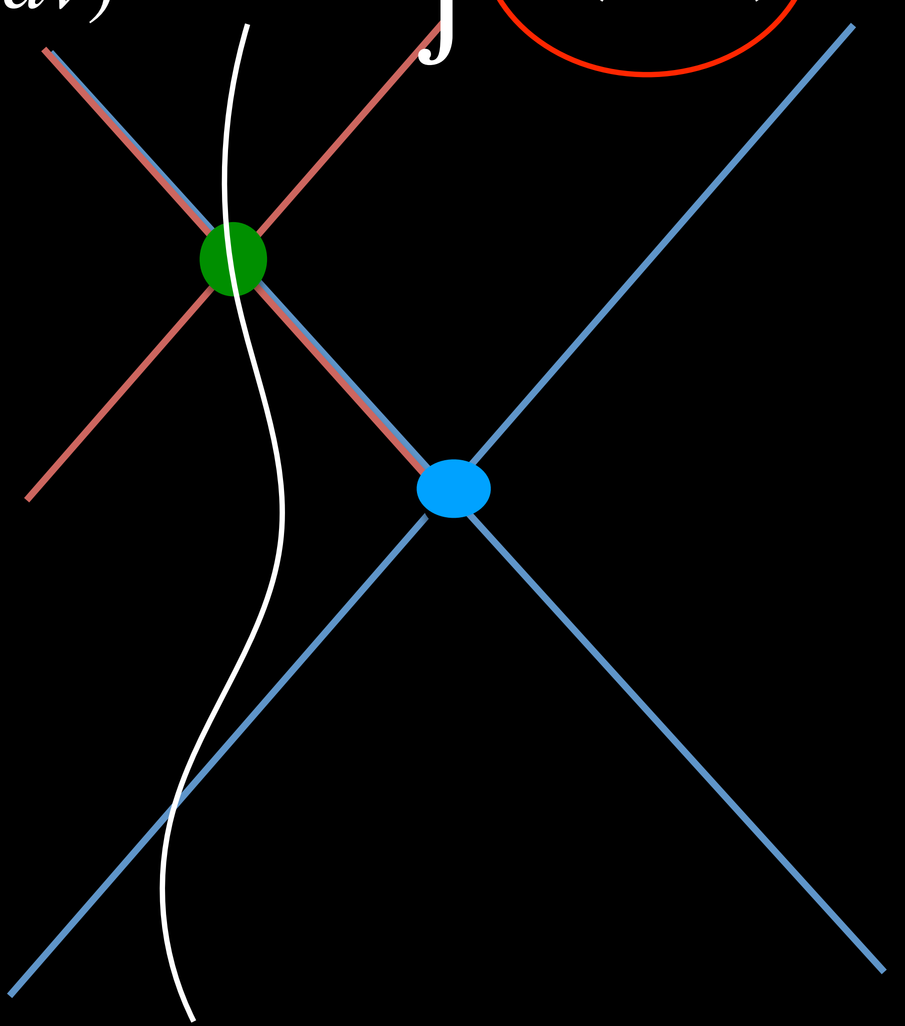
Flat Spacetime



- Electromagnetism $\Box A^{\mu} = -4\pi j^{\mu}$ singular!
- 2 Solutions: $A^{\mu}_{(ret)/(adv)}(x) = \int G_{(+/-)}{}^{\mu}{}_{\nu'} j^{\nu'}(x') d^4x'$



Retarded solution



Advanced solution

$$\Box G_{\pm}{}^{\mu}{}_{\nu'}(x,x') = -4\pi \delta^{\mu}{}_{\nu'}\delta(x-x')$$

$$A^{\mu}_{ret} = A^{\mu}_{R} + A^{\mu}_{S}$$

$$A^{\mu}_{S} = \frac{1}{2}\left(A^{\mu}_{ret} + A^{\mu}_{adv}\right)$$

$$\Box A^{\mu}_{S} = -4\pi j^{\mu}$$

$$A^{\mu}_{R} = A^{\mu}_{ret} - A^{\mu}_{S} = \frac{1}{2}\left(A^{\mu}_{ret} - A^{\mu}_{adv}\right)$$

$$\Box A^{\mu}_{R} = 0$$

$$F^{R}_{\mu\nu} = \partial_{\mu} A^{R}_{\nu} - \partial_{\nu} A^{R}_{\mu} \rightarrow m a_{\mu} = f^{ext}_{\mu} + e F^{R}_{\mu\nu} u^{\nu}$$

Dirac, 1938

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Regularising

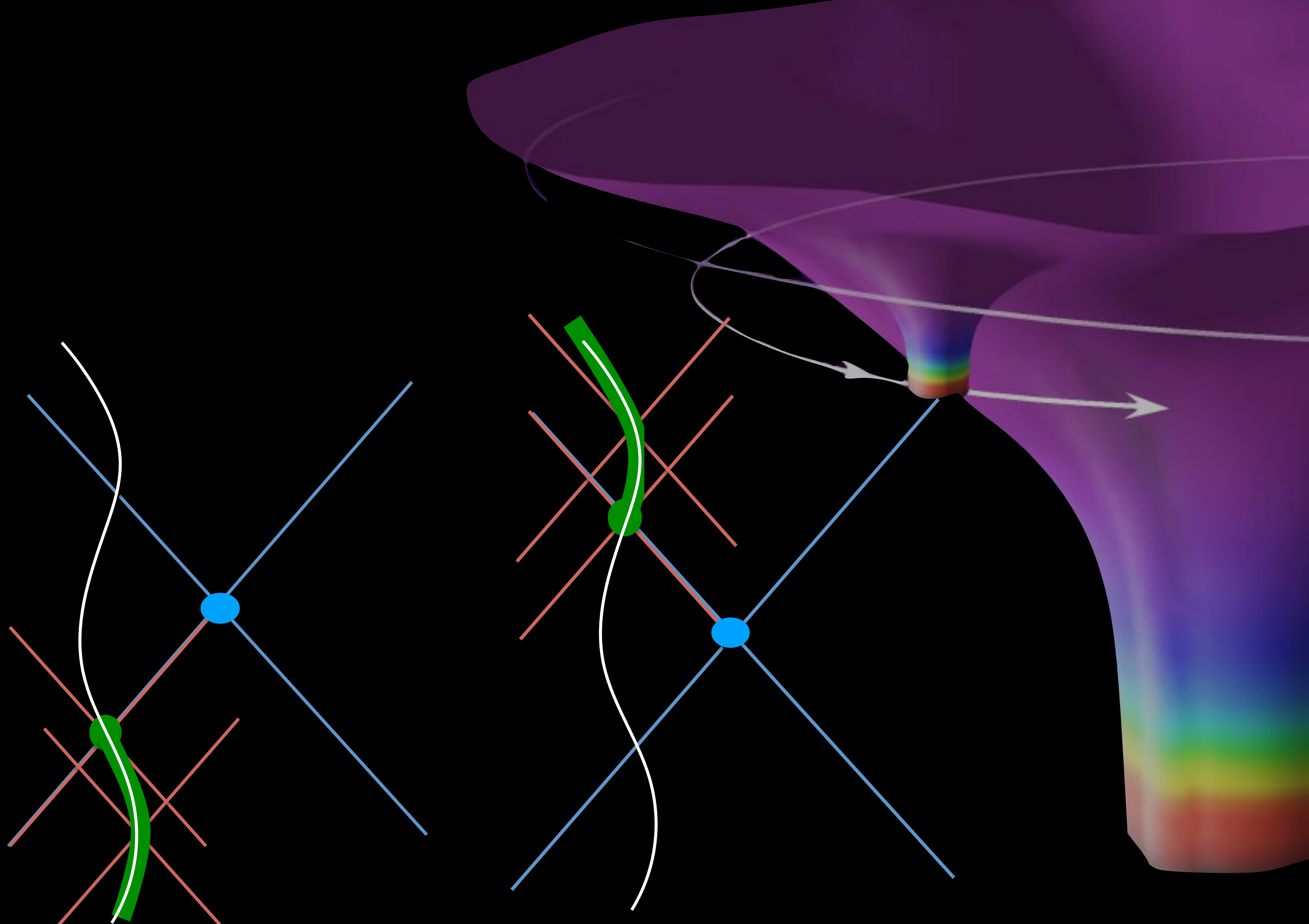
Curved Spacetime

- Electromagnetism: $\Box A^{\mu} - R^{\mu}{}_{\nu} A^{\nu} = -4\pi j^{\mu}$
- 2 Solutions: $A^{\mu}_{(ret)/(adv)}(x) = \int G_{(+/-)}{}^{\mu}{}_{\nu'} j^{\nu'}(x') \sqrt{-g(x')}\, d^4x'$
- Hadmard constructed Green's functions:

$$G_{\pm\mu\nu'}(x,x') = U_{\mu\nu'}(x,x')\delta_{\pm}(\sigma) - V_{\mu\nu'}(x,x')\Theta_{\pm}(-\sigma)$$

- Governing equation:

$$\Box G_{\pm\mu\nu'}(x,x') - R_{\mu}{}^{\gamma} G_{\pm\gamma\nu'}(x,x') = -4\pi g_{\mu\nu'}\delta_4(x,x')$$



Retarded

Advanced

Recall Flat Spacetime:

$$\Box G_{\pm}{}^{\mu}{}_{\nu'}(x,x') = -4\pi\delta^{\mu}{}_{\nu'}\delta(x-x')$$

$$\rightarrow \quad G_{\pm}{}^{\mu}{}_{\nu'} = \delta^{\mu}_{\nu'}\frac{\delta(ct - ct' \mp |x - x'|)}{|x-x'|},$$
$$= 2\Theta_{\pm}\delta\left[(ct-ct')^2 - |x-x'|^2\right],$$
$$= \delta_{\pm}(\sigma)$$

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Regularising

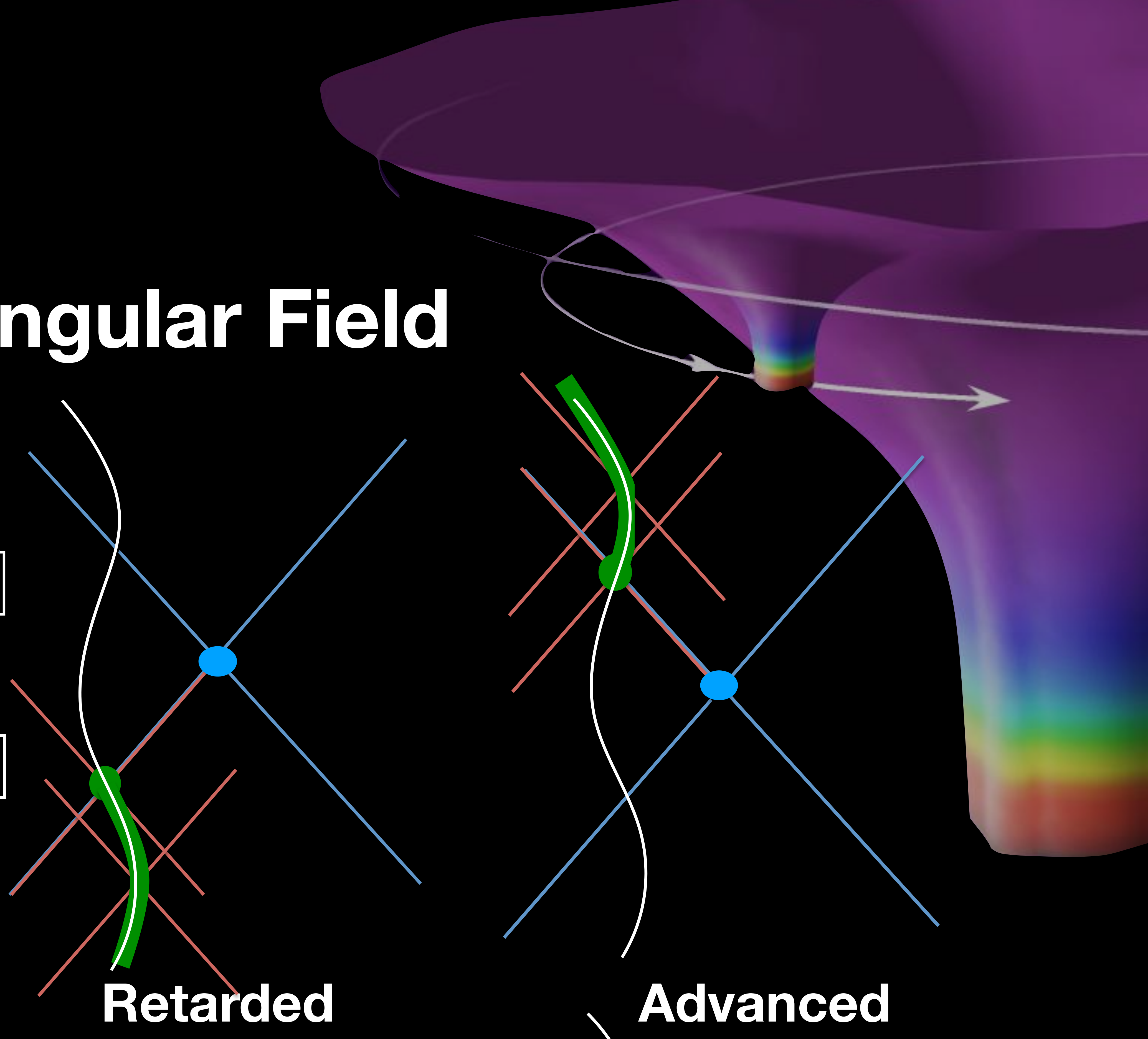
Curved Spacetime: Detweiler-Waiting Singular Field

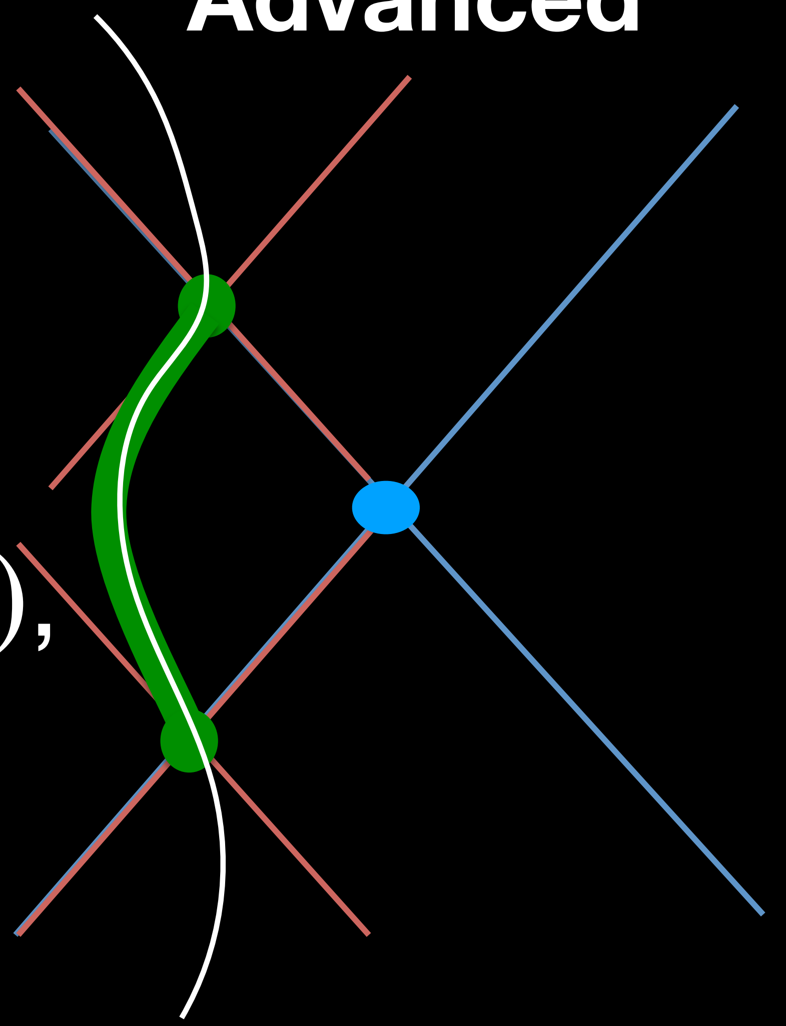
- Look for solutions of the type:

$$G_{(S)}(x,x') = \frac{1}{2}[G_{ret}(x,x') + G_{adv}(x,x') - H(x,x')]$$

$$G_{(R)}(x,x') = \frac{1}{2}[G_{ret}(x,x') - G_{adv}(x,x') + H(x,x')]$$

- Where
  - $H(x,x')$ is solution to homogeneous wave equation
  - Symmetric in its arguments eg. $H(x,x') = H(x',x)$
  - $G_{(S)}(x,x') = 0$ when $x \in I^{\pm}(x')$
  - $G_{(R)}(x,x') = 0$ when $x \in I^{-}(x')$, $G_{(R)}(x,x') = G_{ret}(x,x')$ when $x \in I^{+}(x')$





Recall: $G_{\pm}(x,x') = U(x,x')\delta_{\pm}(\sigma) - V(x,x')\Theta_{\pm}(-\sigma),$

$$\Rightarrow G_{(S)}(x,x') = \frac{1}{2}[U(x,x')\delta(\sigma) - V(x,x')(\Theta(\sigma)]$$

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Regularising

Curved Spacetime: Detweiler-Waiting Singular Field

- Scalar

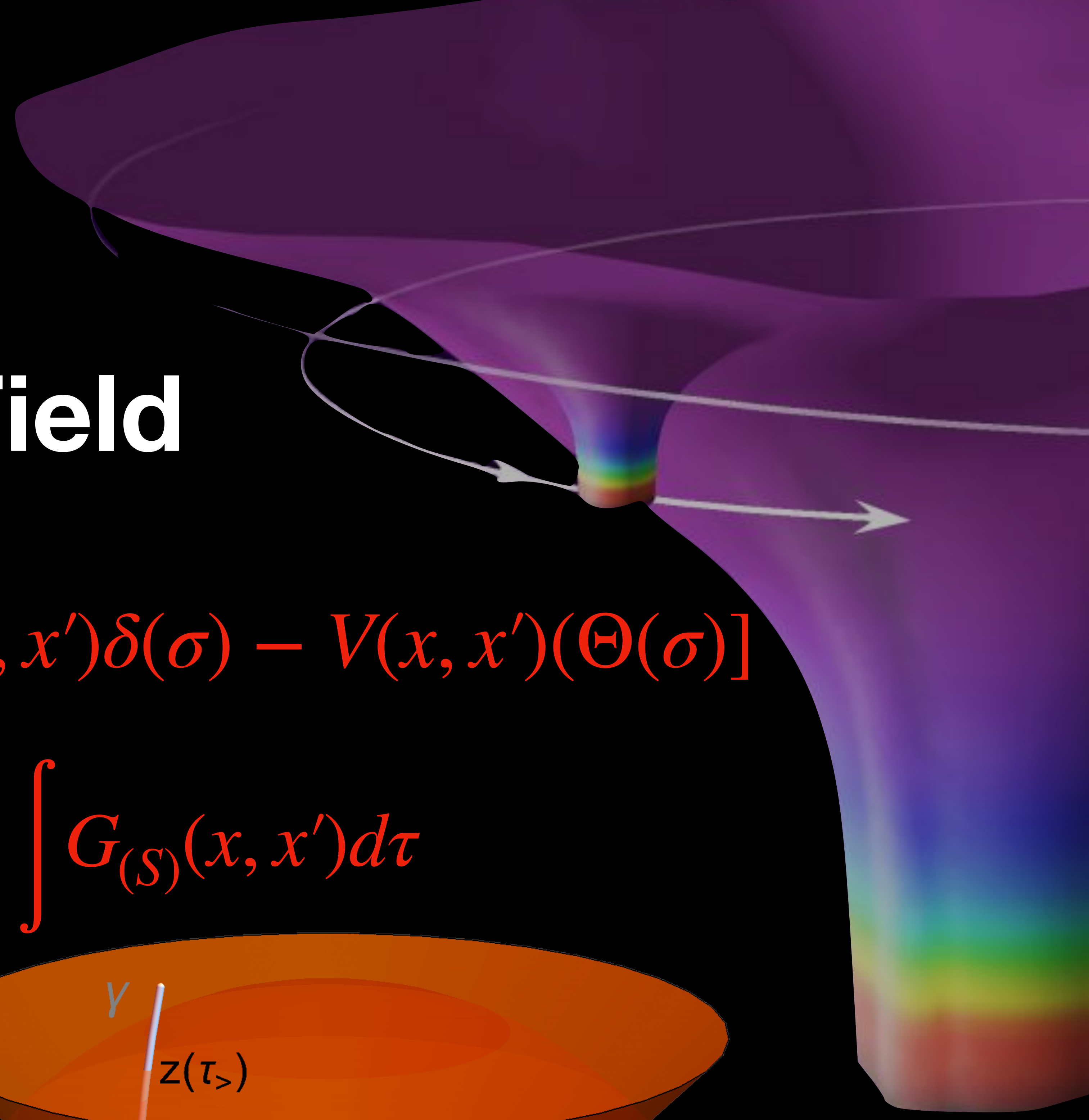
$$\Phi_{(S)} = \left[\frac{U(x,x)}{2\sigma_{a'}u^{a'}}\right]_{x=x_{ret}}^{x=x_{adv}} - \int_{\tau=\tau_{ret}}^{\tau=\tau_{adv}} V(x,x'(\tau))d\tau$$

- Electromagnetism

$$A^a_{(S)} = \left[\frac{u^{a'}U^a{}_{a'}(x,x)}{2\sigma_{a'}u^{a'}}\right]_{x=x_{ret}}^{x=x_{adv}} - \int_{\tau=\tau_{ret}}^{\tau=\tau_{adv}} u^{a'}V^a{}_{a'}(x,x'(\tau))d\tau$$

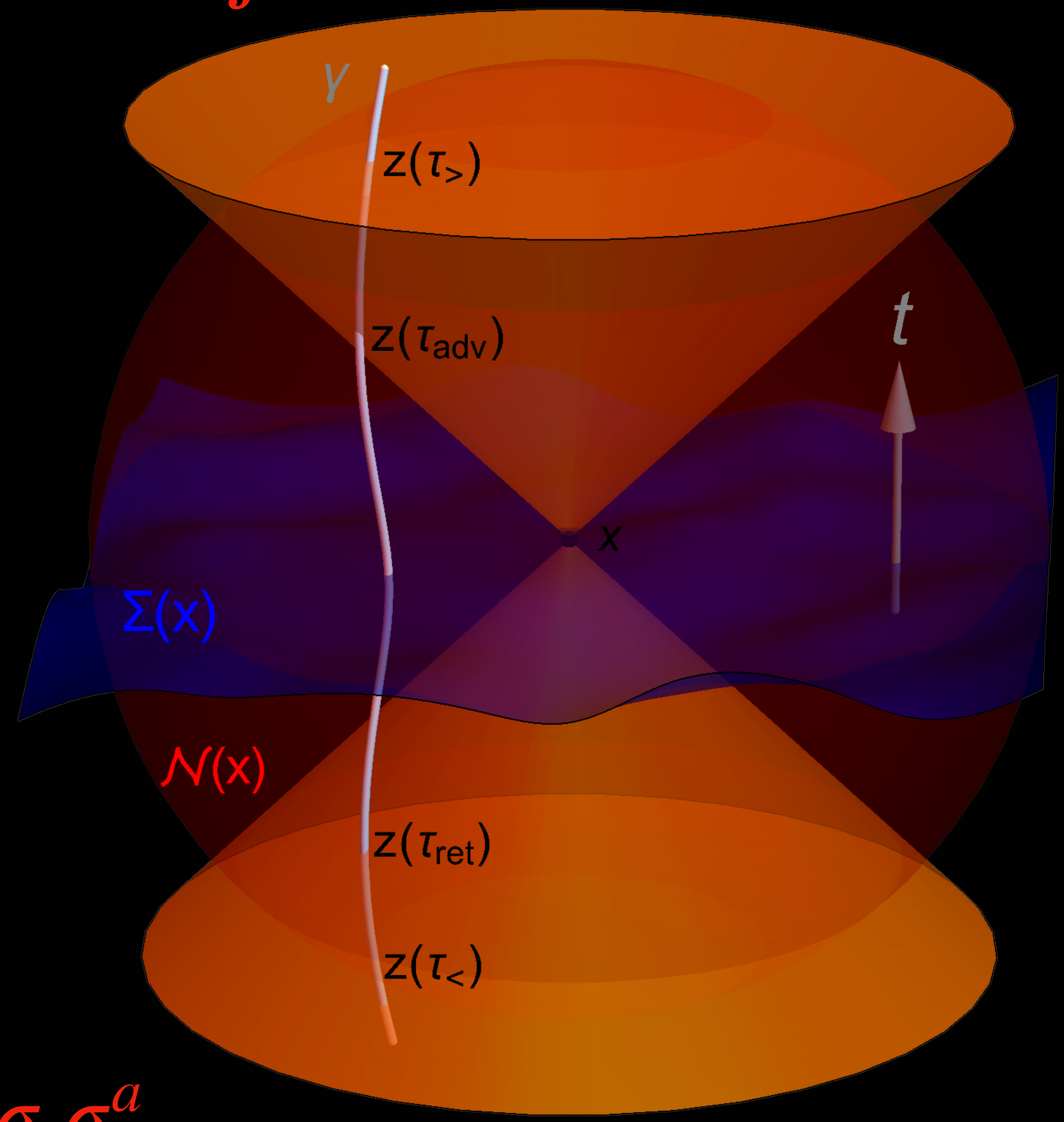
- Gravity

$$h^{ab}_{(S)} = \left[\frac{u^{a'}u^{b'}U^{ab}{}_{a'b'}(x,x)}{2\sigma_{a'}u^{a'}}\right]_{x=x_{ret}}^{x=x_{adv}} - \int_{\tau=\tau_{ret}}^{\tau=\tau_{adv}} u^{a'}u^{b'}V^{ab}{}_{a'b'}(x,x'(\tau))d\tau$$



$$G_{(S)}(x,x') = \frac{1}{2}[U(x,x')\delta(\sigma) - V(x,x')(\Theta(\sigma)]$$

$$\Phi_{(S)}(x) = \int G_{(S)}(x,x')d\tau$$



$$\sigma(x,x') = \frac{1}{2}g_{ab}(x)\delta x^{a'}\delta x^{b'} + A_{abc}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'} + B_{abcd}(x)\delta x^{a'}\delta x^{b'}\delta x^{c'}\delta x^{d'}\ldots$$

$$2\sigma = \sigma_a\sigma^a$$

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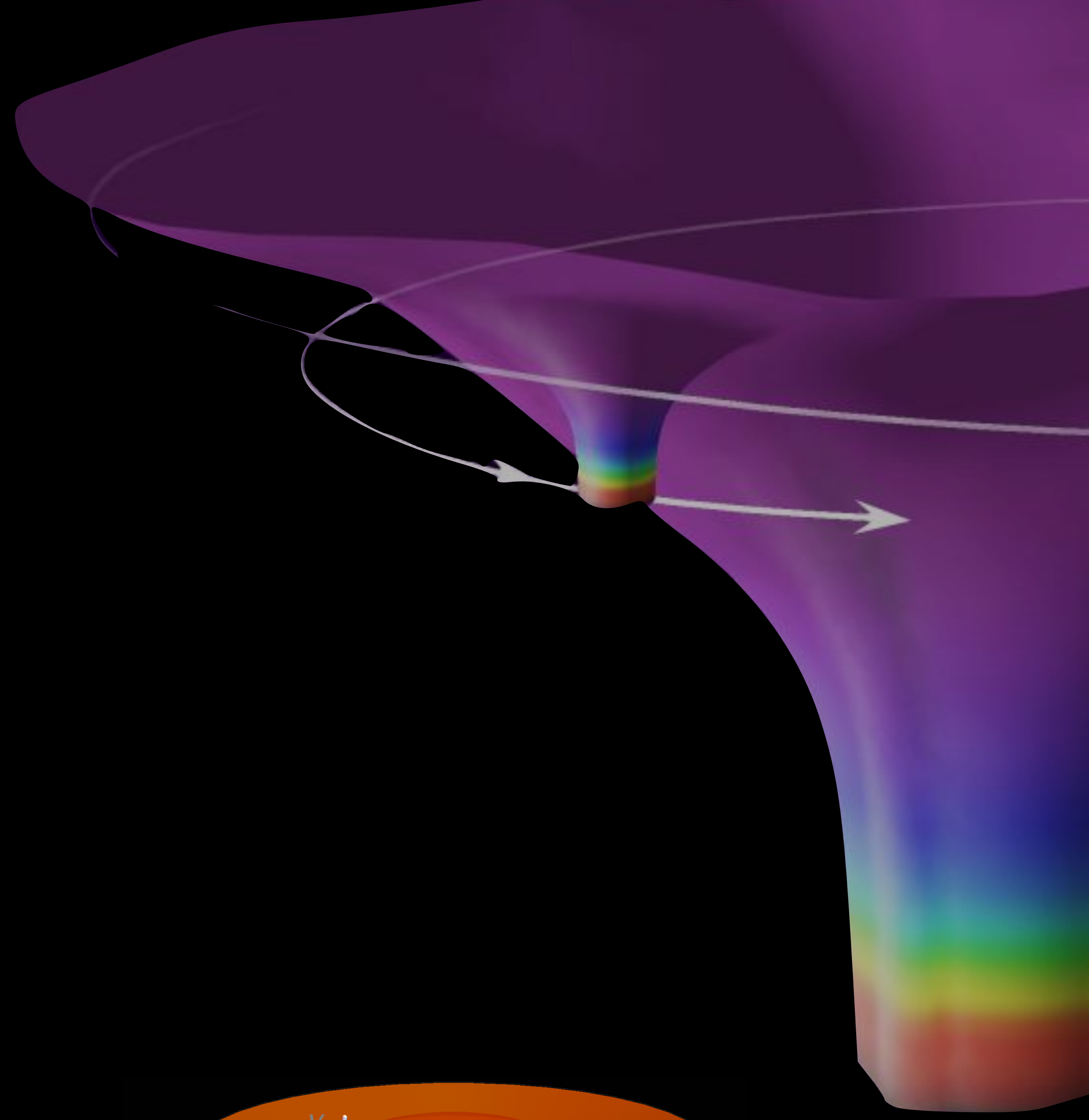
Self-Force Methods

Solving the wave equation



$$\Box \bar{h}_{ab} + 2R_{acbd}\bar{h}^{cd} = -16\pi \mathrm{T}_{ab}[\mathrm{g}] + \mathcal{O}(\epsilon)$$

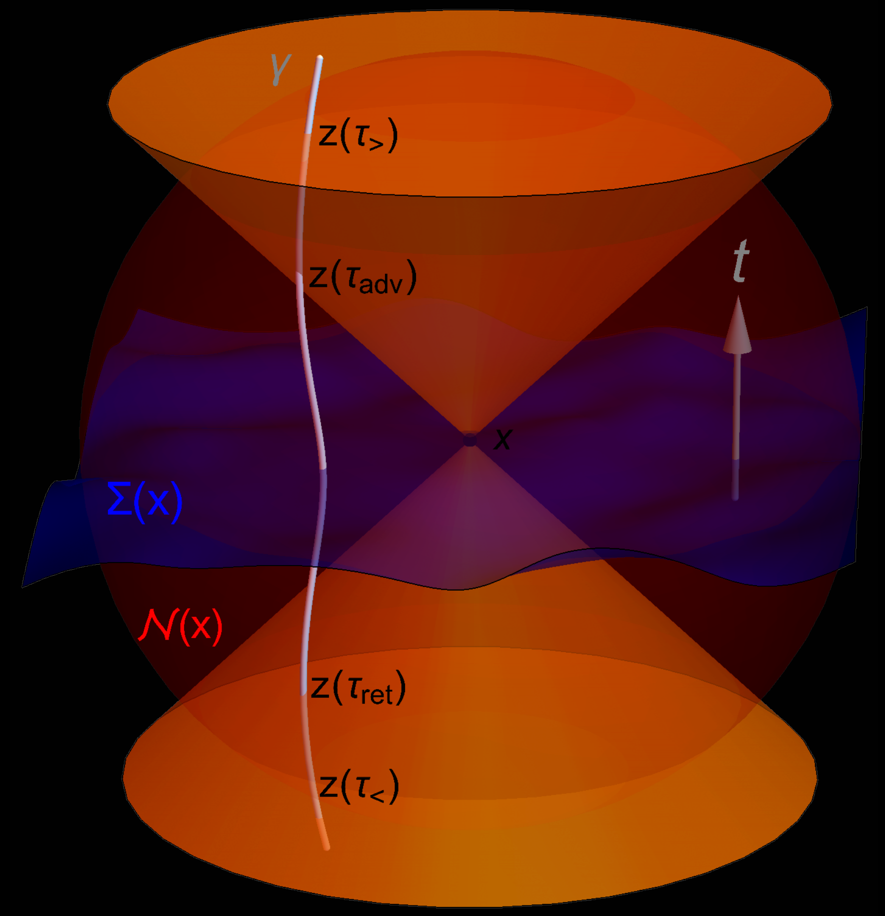
- Mode-Sum: Barack & Ori (2001)
  - Spheroidal harmonic decomposition $\Rightarrow \bar{h}^{(ret)}_{ab}$

analytic

$$F_a(\bar{x}) = \sum_{\ell}^{\infty} \left(F_a^{\ell(ret)}(\bar{x}) - F_a^{\ell(S)}(\bar{x}) \right),$$

numeric

$$F_a^{\ell(ret)/(S)}(\bar{x}) = \lim_{\Delta r \to 0} \frac{2\ell+1}{4\pi} \int F_a^{(ret)/(S)}(\bar{r}+\Delta r, \bar{t}, \alpha, \beta) P_\ell(\cos\alpha) d\Omega$$



- Effective Source: Golbourn & Barack, Vega & Detweiler (2007)

Numerically solve for regularised field

$$S_{eff} = \Box\, \Phi_{(R)} = -4\pi q \int \delta_4(x, x(\tau'))d\tau' - \Box \Phi_{(S)}$$

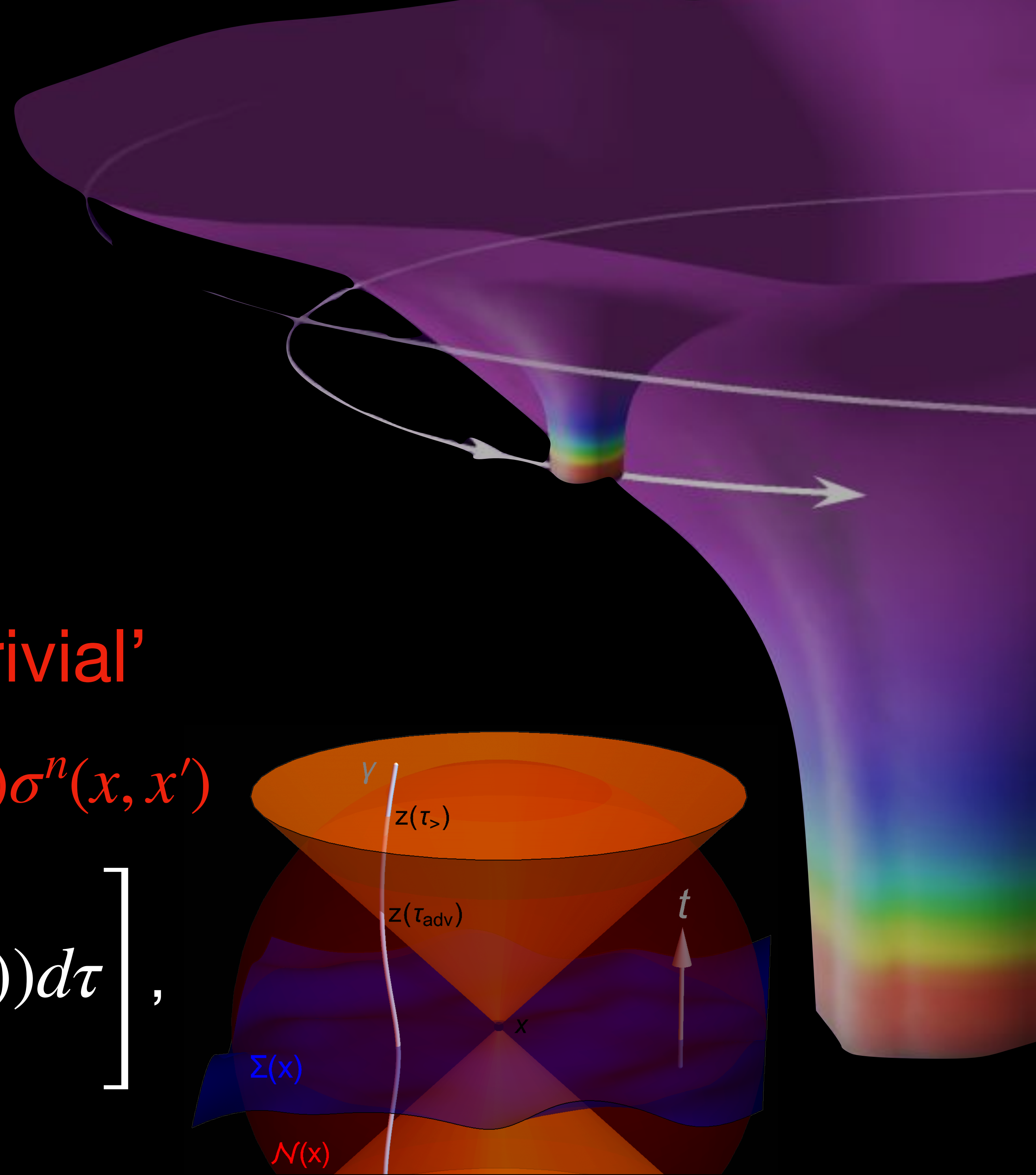
Self-Force Methods

Solving the wave equation

Matched Expansions: Schwarzschild Scalar, Casals et al. (2013)

Regularisation ‘trivial’

$$V(x,x') = \sum_{n=0}^{\infty} V_n(x,x')\sigma^n(x,x')$$



Locally: $G_+(x,x') = \frac{1}{2}[\cancel{U(x,x')}\delta_+(\sigma) - V(x,x')(\Theta_+(-\sigma)]$

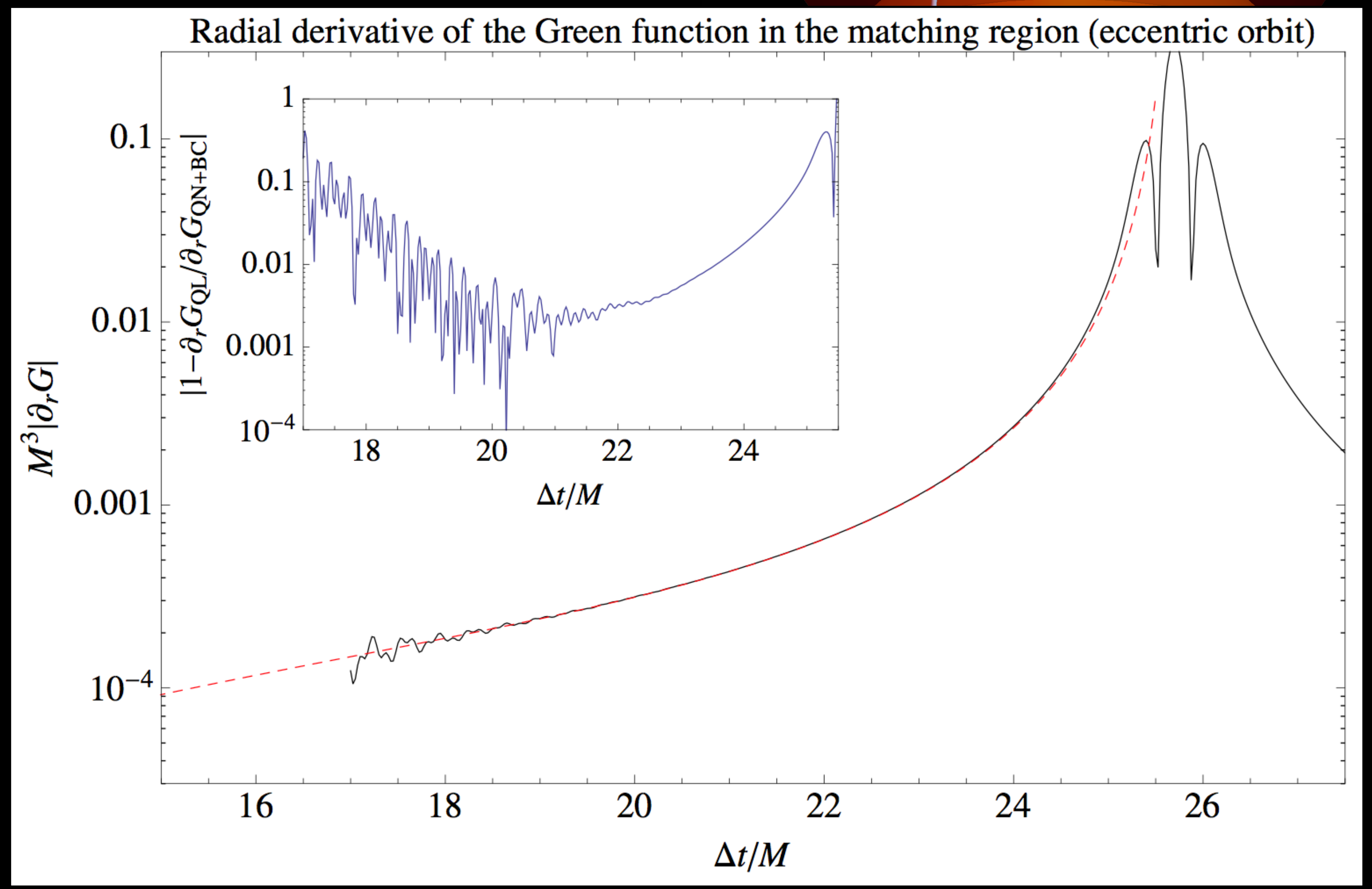
$$\Rightarrow f_a(x) = q^2\nabla_a \int_{\tau_<}^{\tau_+} G_{(R)}(x, x(\tau'))d\tau' = -q^2\nabla_a \left[\int_{\tau_<}^{\tau_+} V(x,x(\tau'))d\tau + \int_{-\infty}^{\tau_<} G^+(x,x(\tau'))d\tau\right],$$

Decompose Green Function into Fourier and spheroidal harmonic modes,

$$G^+ = \sum_{\ell,m}\int_{-\infty}^{\infty} d\omega e^{im(\varphi-\varphi')-i\omega(t-t')} S_{\ell m\omega}(\theta)S_{\ell m\omega}(\theta')G_{\ell m\omega}(r,r'),$$

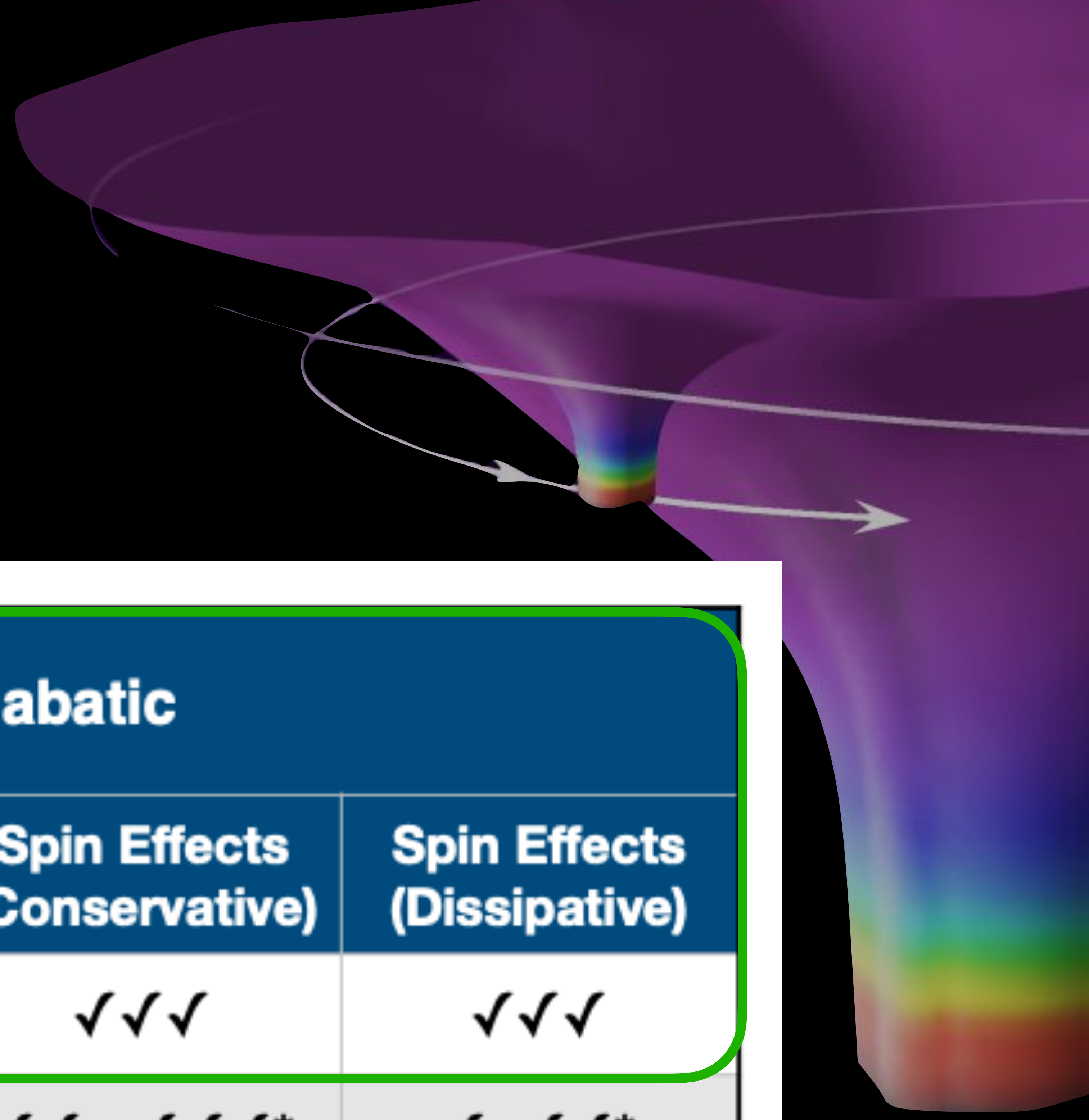
and use Residue Theorem,

$$G_{\ell m\omega} = \cancel{G^{HF}_{\ell m\omega}} + G^{QNM}_{\ell m\omega} + G^{BC}_{\ell m\omega}$$



Current Status

From LISA Waveform White Paper



Produced by Josh Mathews for LISA Waveform White Paper

Background Spacetime	Orbital Configuration	Adiabatic	Post-1-adiabatic			
		1SF (Dissipative)	1SF (Conservative)	2SF (Dissipative)	Spin Effects (Conservative)	Spin Effects (Dissipative)
Schwarzschild	Circular	✓✓✓	✓✓✓	✓✓✓	✓✓✓	✓✓✓
	Eccentric	✓✓✓	✓✓✓	✗	✓✓ , ✓✓✓*	✓, ✓✓*
Kerr	Circular	✓✓✓	✓✓	✗	✓,✓✓*	✓✓✓*
	Eccentric Equatorial	✓✓✓	✓✓	✗	✓,✓✓*	✓✓*
	Generic	✓✓✓	✓	✗	✓	✓*
	Resonances	✓✓✓	✓	✗	✗	✗

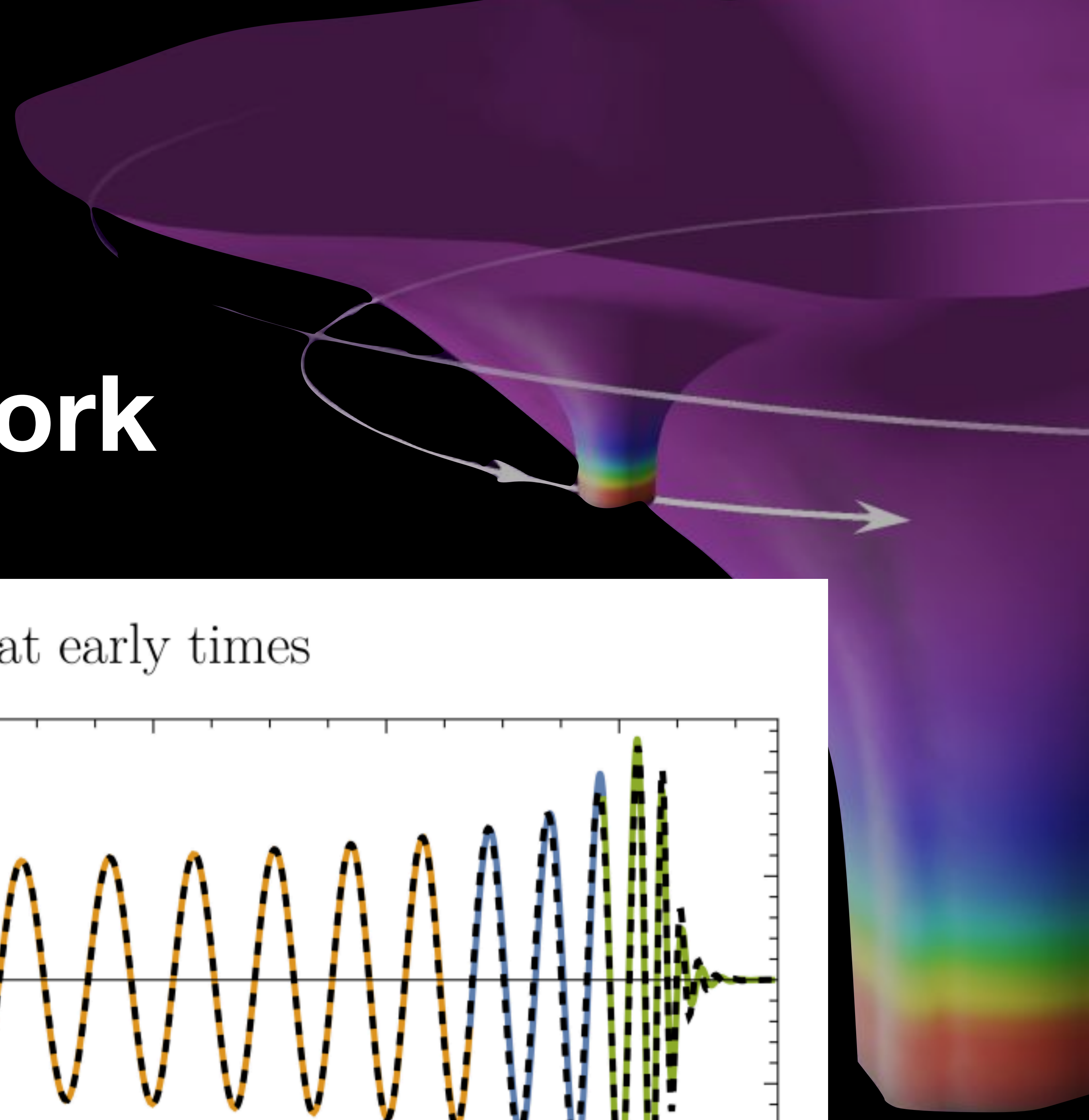
✓✓✓ Evolving Waveform ✓✓ Driven Inspiral ✓ Snapshot Calculation *(Anti-)Aligned Spin Only

Chapman-Bird et al., FEW waveforms (100ms):
Equatorial eccentric, $|a| < 0.999, \quad e < 0.9, \quad p < 200m$

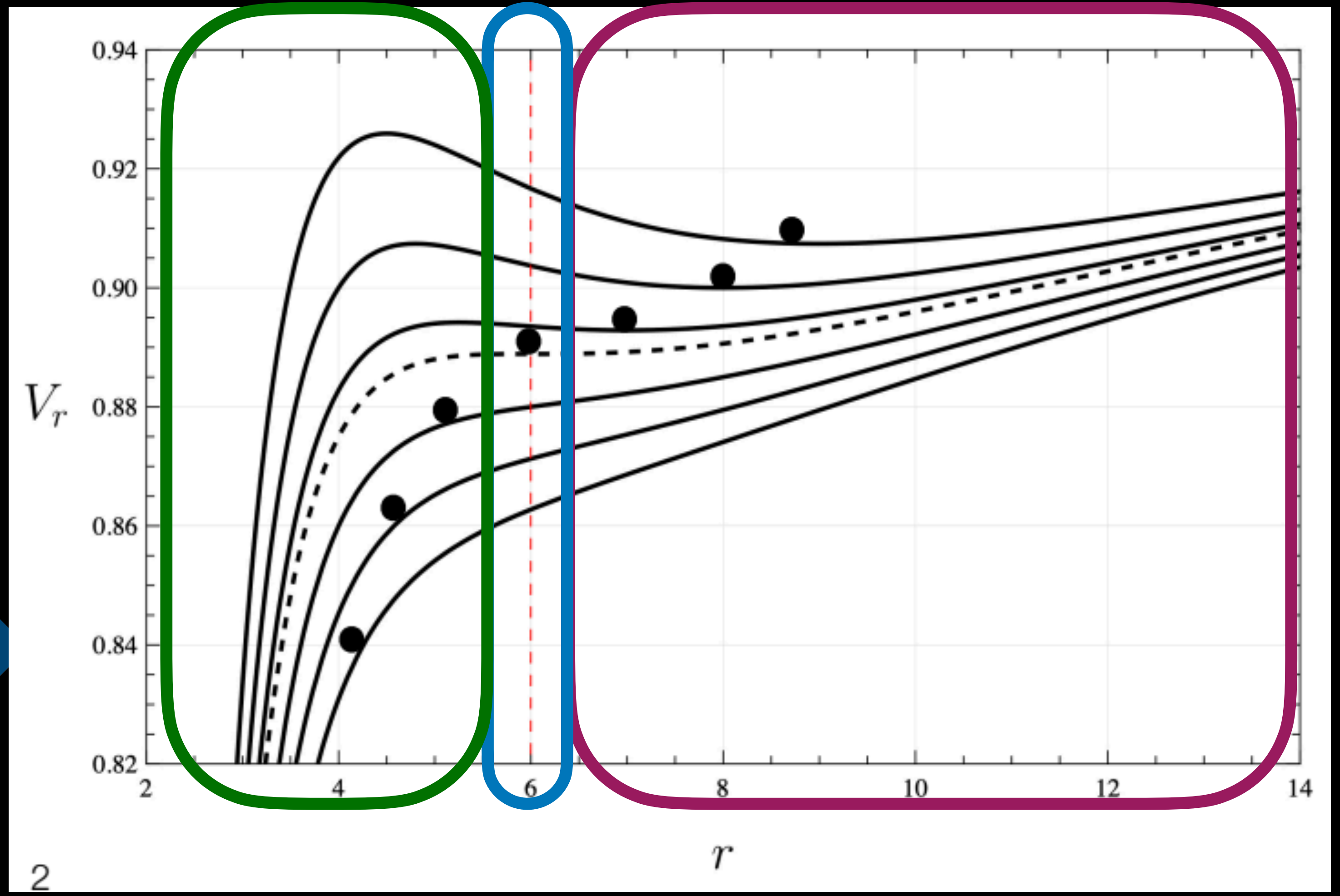
Wardell et al., https://bhptoolkit.org/WaSABI/
- Currently without secondary spin,
- Secondary spin waveforms on request

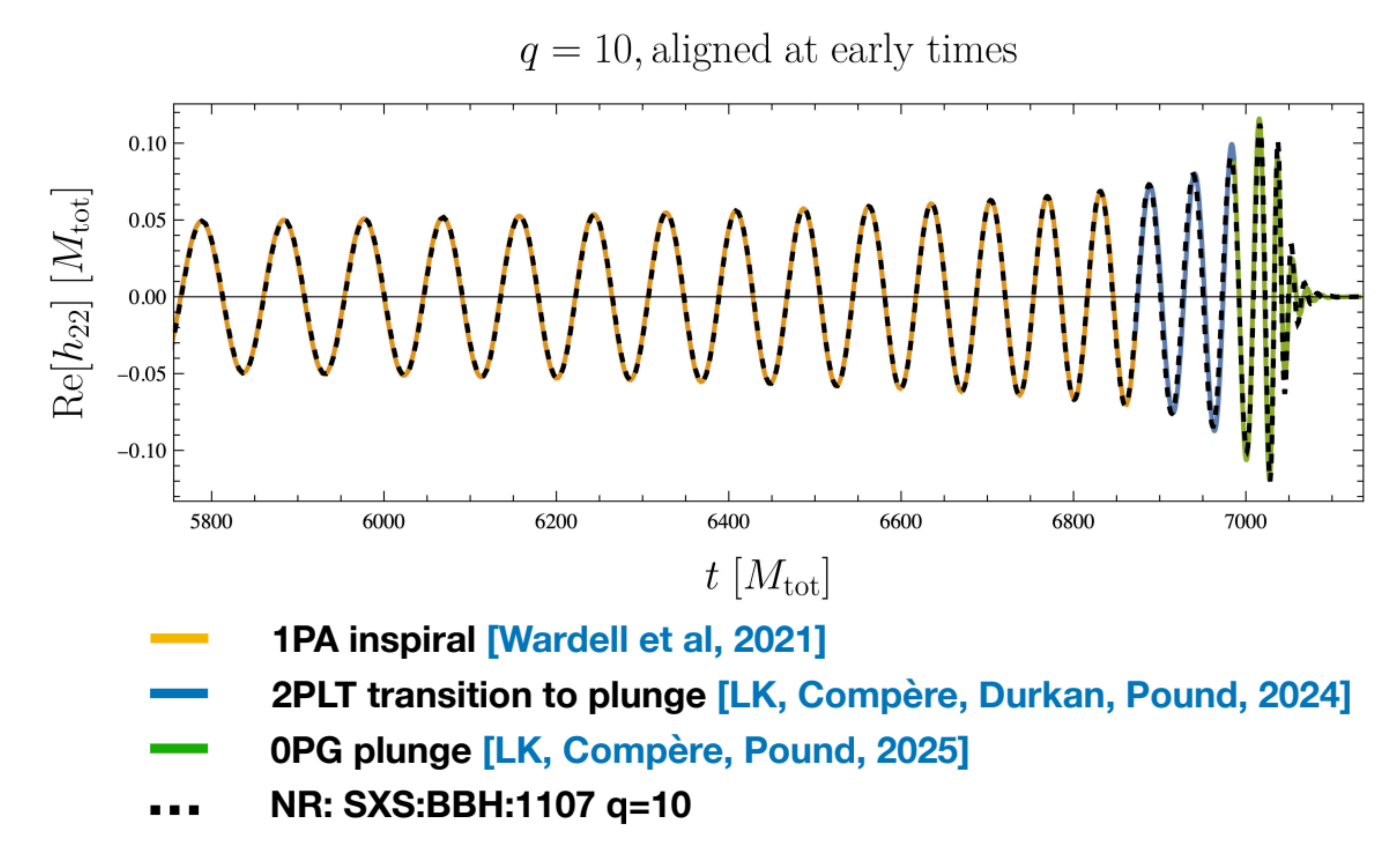
On the horizon …

Plunge and Merger within the self-force framework



- **1PA** timescale ϵ^{-1}
- **2PLT** (Post-Leading Transition) timescale $\epsilon^{-1/5}$ (scaled frequency adopted): $\Delta\Omega = (\Omega - \Omega_*)\epsilon^{-2/5}$
- **0PG** (Post-Geodesic) timescale ϵ^0



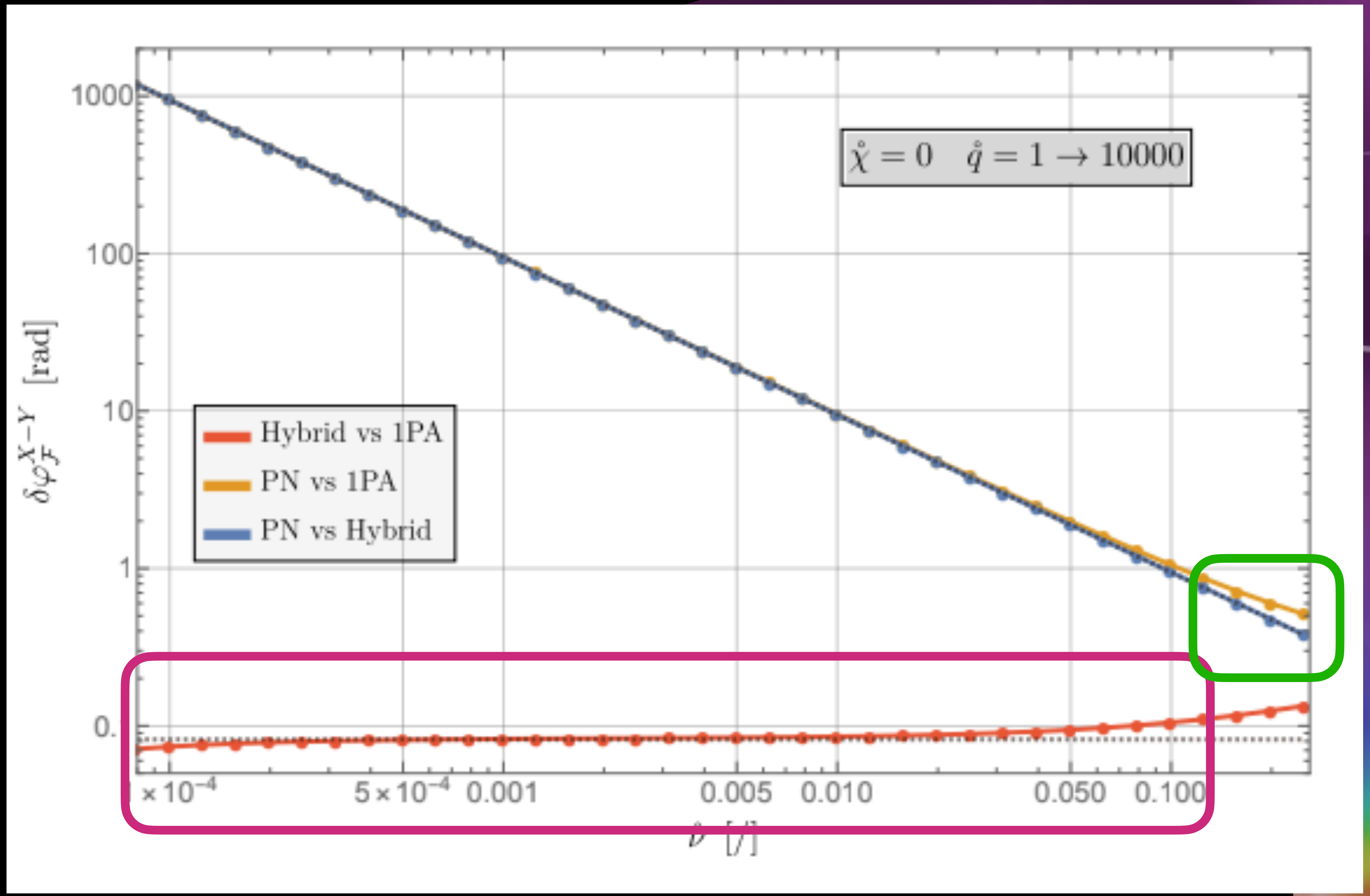


Taken from Lorenzo Küchler's Capra talk, 2025, see L.Küchler et al, arXiv:2506.02189

On the horizon …

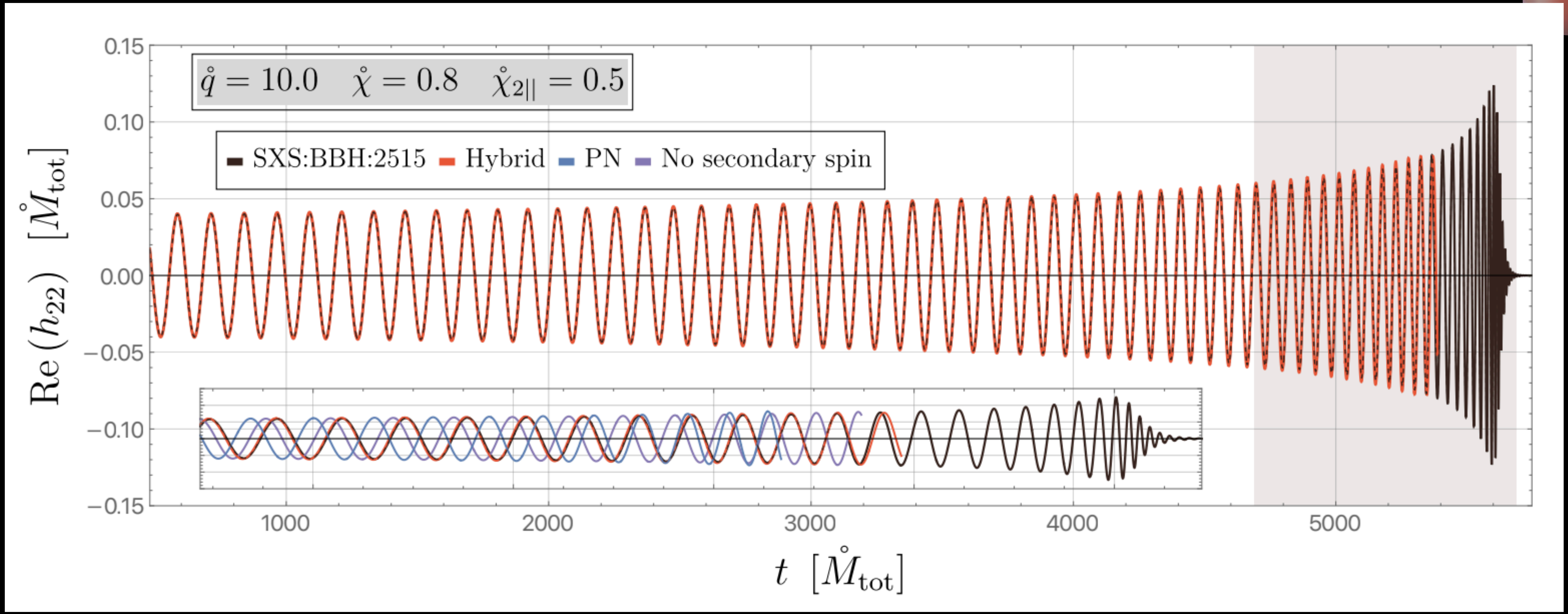
SF-PN Hybrid Inspirals

- Agreement with 1PA at high-mass ratios
- Agreement with PN at comparable masses (where 1PA deviates)



Taken from Loic Honet's Capra talk, 2025

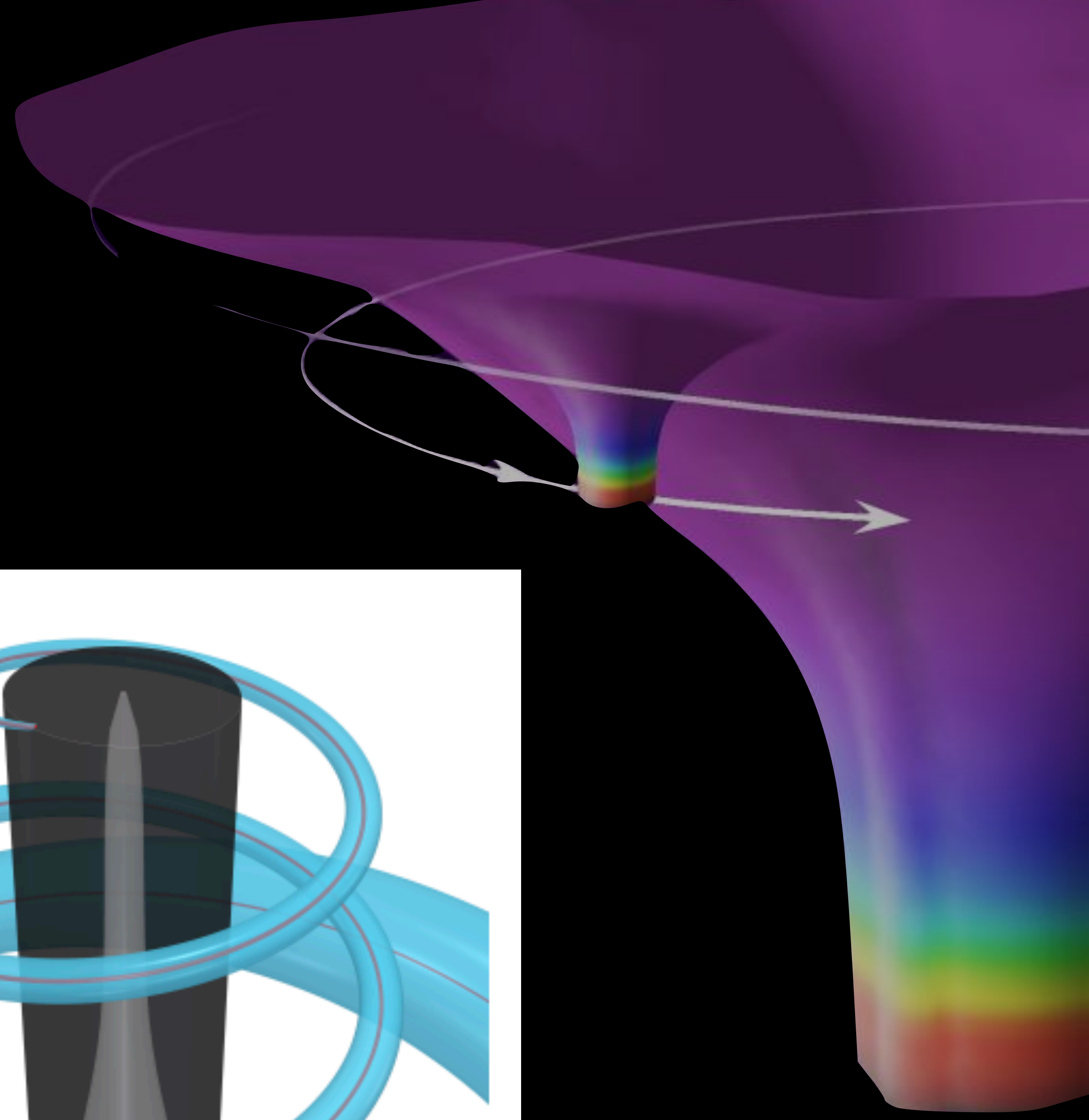
- 4.5PN Energy flux
- 1GSF Energy flux
- Evolution equations from flux-balance equation

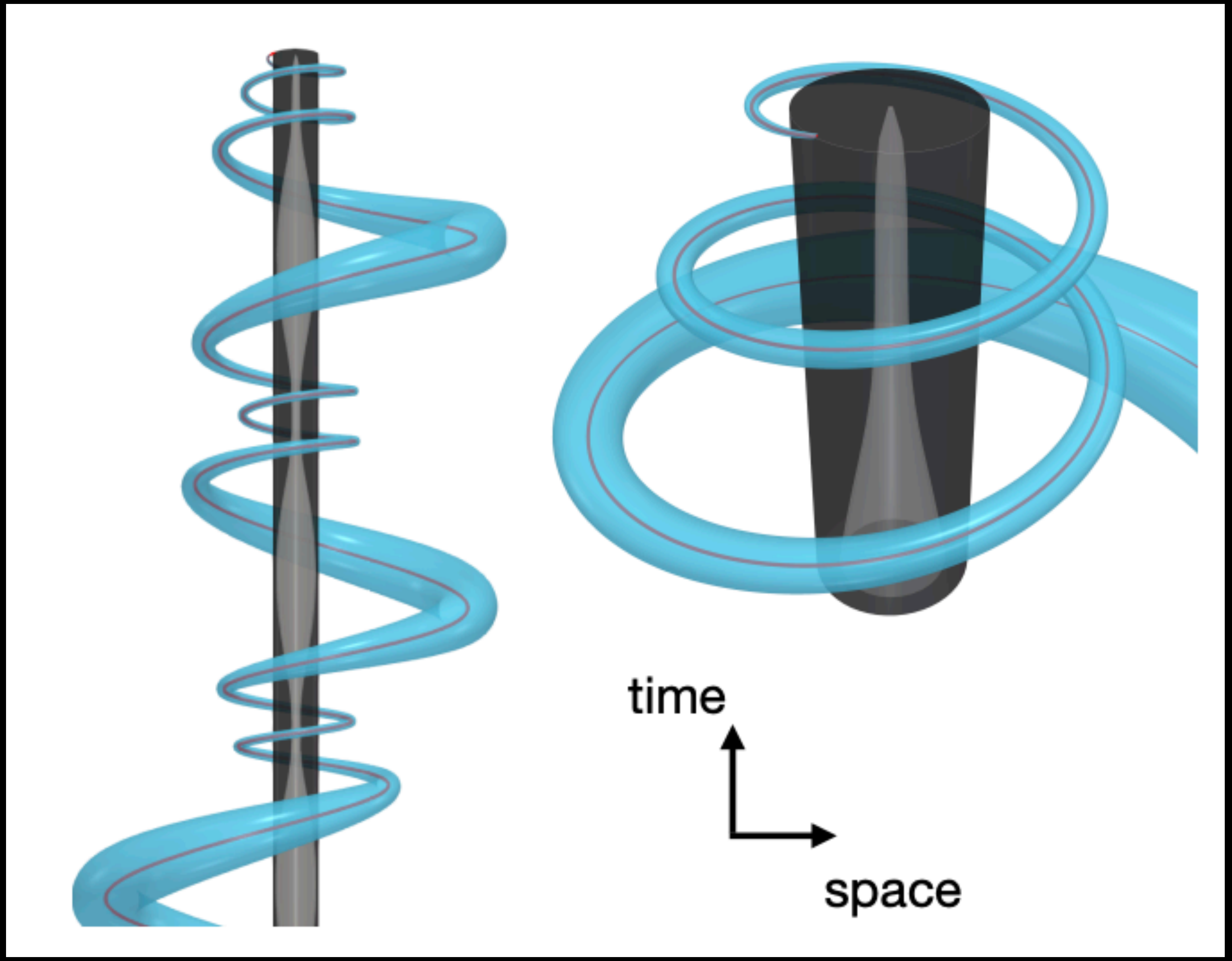


On the horizon …

Worldtube Excission

- Redline represents trajectory of smaller object.
- Cyan is the world tube excised around this object, within which a perturbative analytical solution is used.
- Field equations outside this world tube are solved with full 3+1 NR simulations
- Only in Schwarzschild scalar eccentric case (for now)





Taken from N.A.Wittek et al., arXiv:2410.22290

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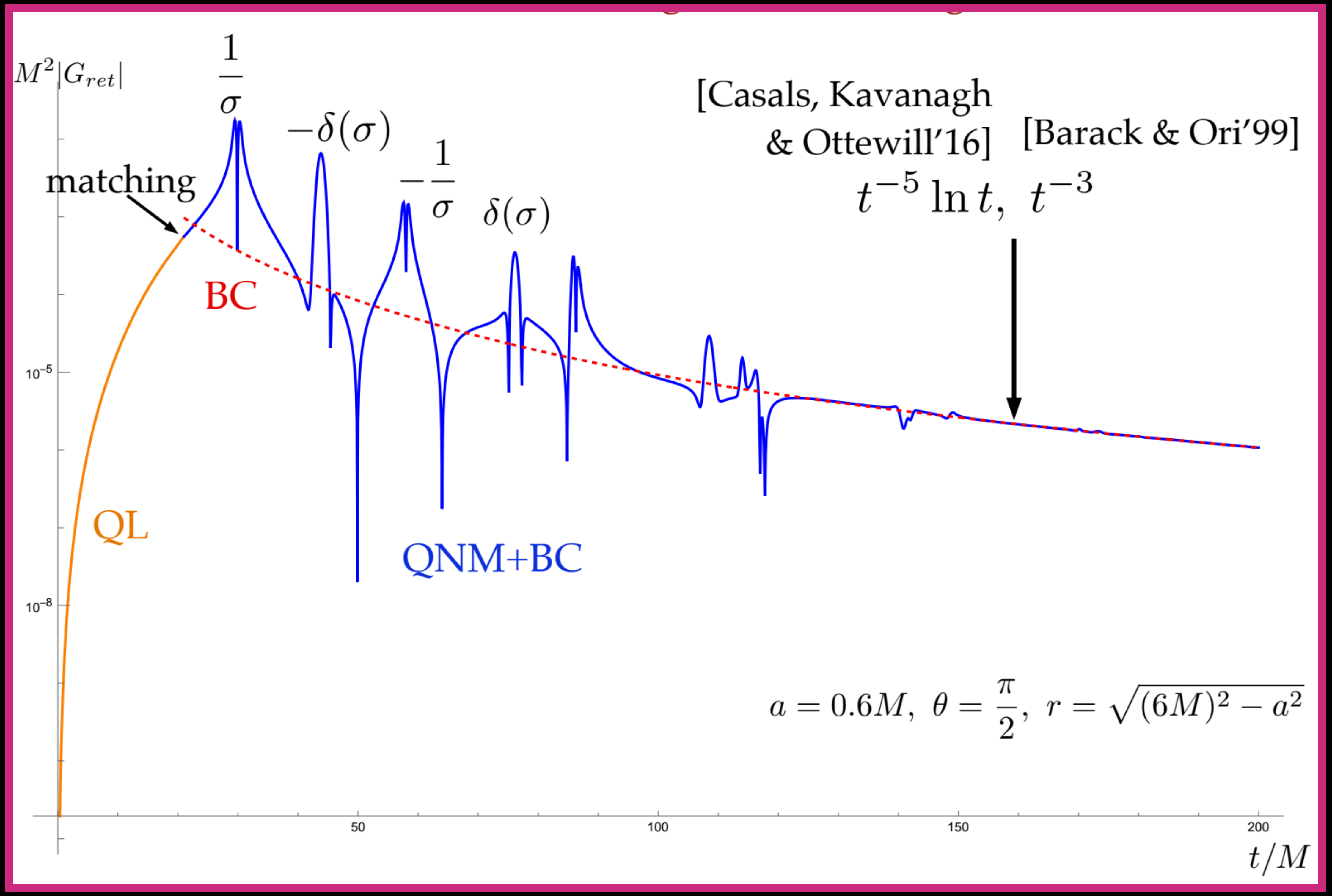
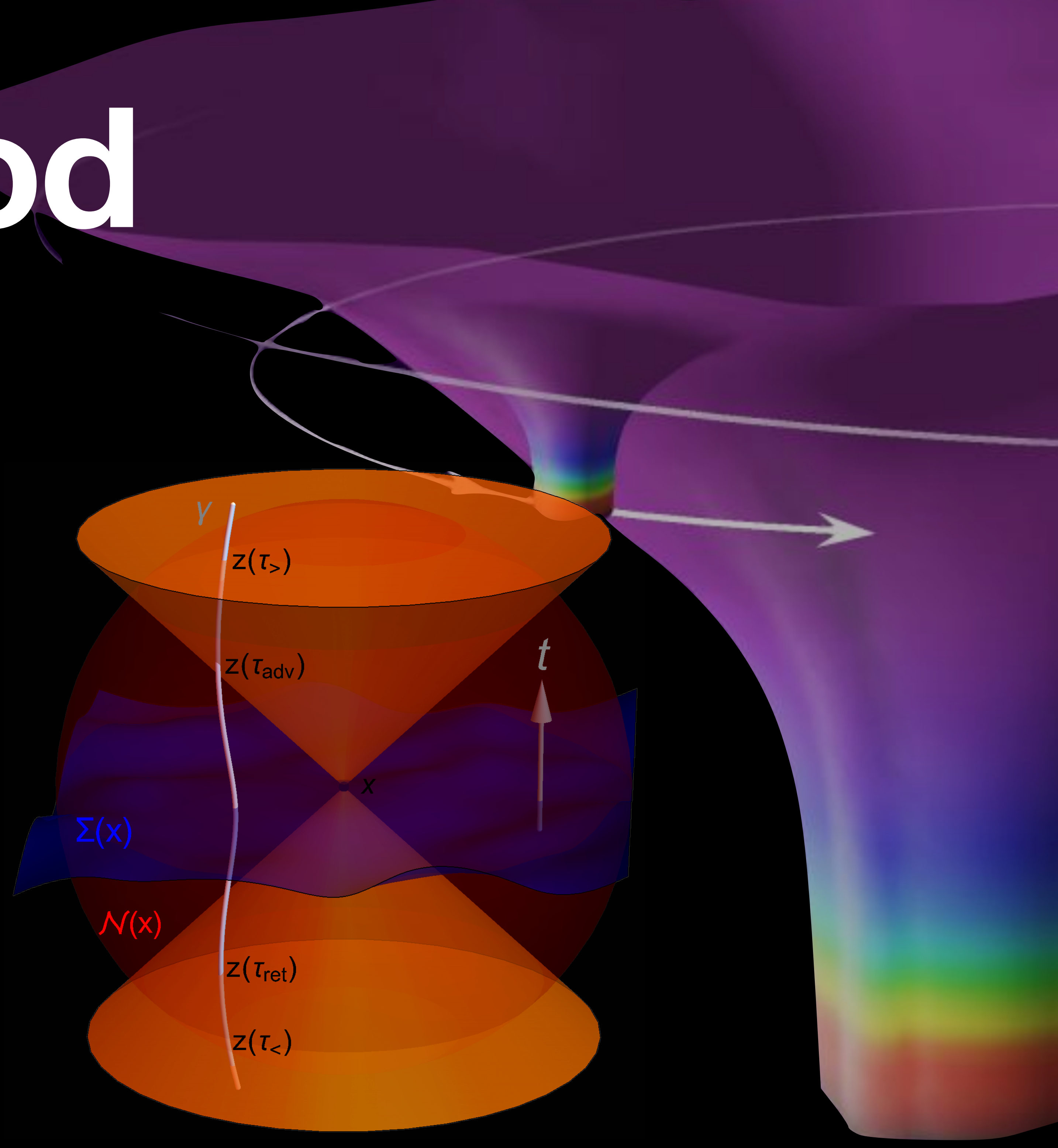
Self-Force Semi-Analytical Method

Green's Functions

- Matched Expansions: Casals et al. (arXiv:1306.0884) Schwarzschild Scalar

$$f_a(x) = q^2 \nabla_a \int_{\tau_<}^{\tau_+} G_{(R)}(x, x(\tau'))d\tau' = -\, q^2 \nabla_a \left[\int_{\tau_<}^{\tau_+} V(x, x(\tau'))d\tau + \int_{-\infty}^{\tau_<} G^+(x, x(\tau'))d\tau \right]$$





Agreement in Kerr with mode-sum self-field:

$$\phi = -\,0.00366 q/M$$

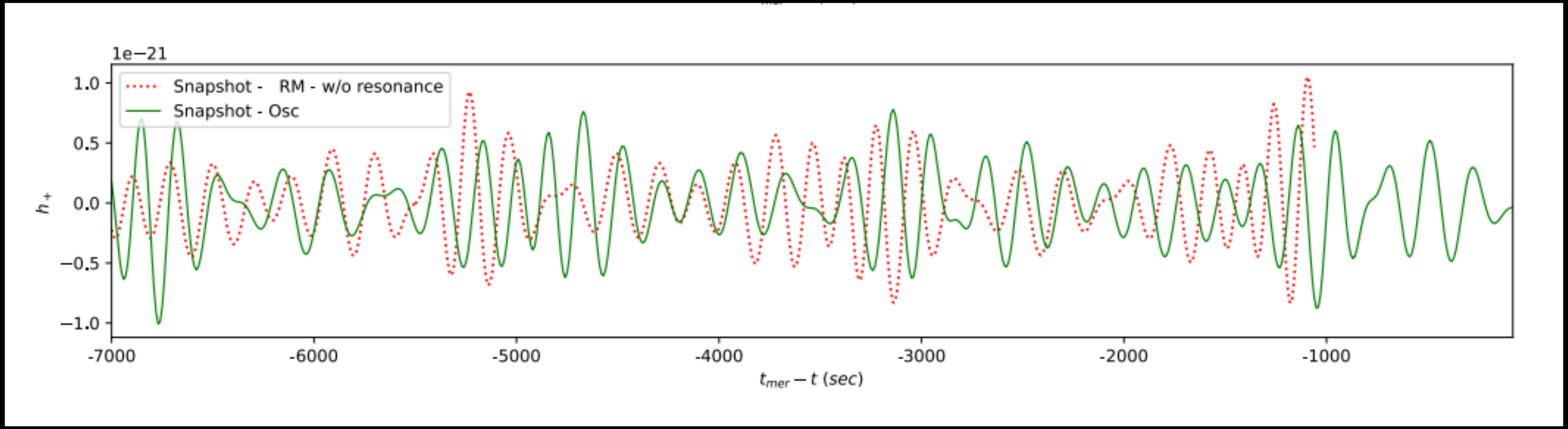
On the horizon … Challenges

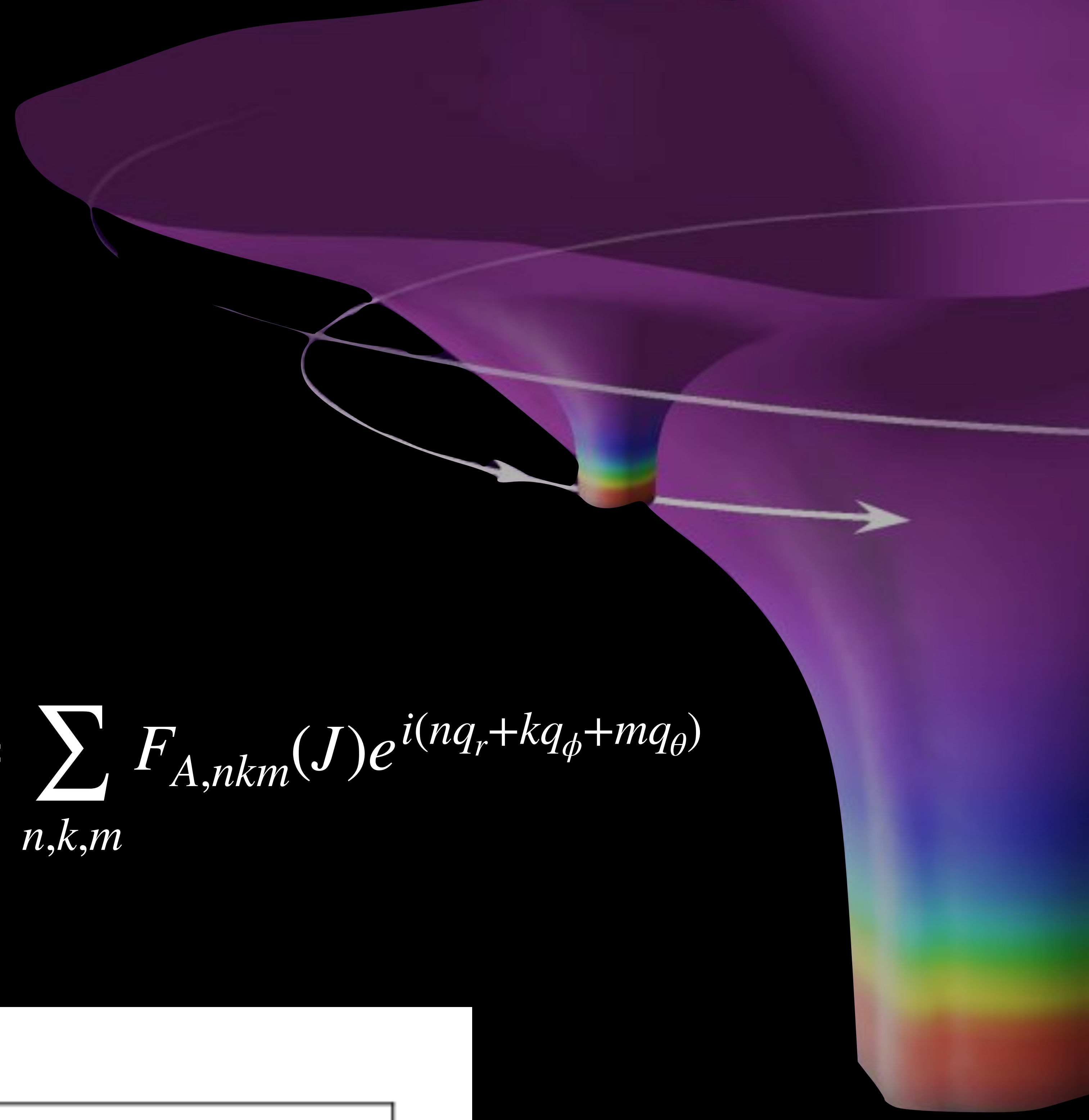
Resonances by a third body

- I lied to you earlier,

$$\frac{d\mathcal{J}_A}{dt} = \epsilon \left[F^{(0)}_A(\mathcal{J}_B) \boxed{+ qF^{(1)}_{A,d}(\mathcal{J}_C) +} \epsilon F^{(1)}_A(\mathcal{J}_B) + \mathcal{O}(\epsilon^2) \right], \qquad q \sim M^2 M_* x_* / R^3, \qquad F^{(1/2)}_A(\mathcal{J}_B) = \sum_{n,k,m} F_{A,nkm}(J) e^{i(nq_r + kq_\phi + mq_\theta)}$$

Resonances occur when $nq_r + kq_\phi + mq_\theta = 0$





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Taken from P.Gupta et al., arXiv:2205.04808

Status Summary

Challenges & Prospects

Good News

- 1st order Kerr generic SF has been solved (slowly),
  - Several avenues of improved methods en route
  - Crosschecks between different methods common
- 0PA Kerr FAST EMRI waveforms (100ms) now available for equatorial eccentric orbits
- 1PA Schwarzschild circular waveforms available via bhptoolkit.

Challenges

- 2nd order Kerr generic is extremely difficult,
  - Separate pathways to Kerr and eccentric en route
- Schwarzschild scalar offline grid for fast waveforms used 3 million CPUs, Kerr generic most likely 3 billion.
- Environment (including N-bodies) and beyond Einstein have not been included in any EMRI waveform tools (but on the way - currently not included in DDPC)

Exciting Prospects

- Transition to Plunge and Plunge methods are emerging
- New PN-SF hybrid methods show promise in Kerr
- Worldtube excision methods allow full power of NR applied to high-mass ratios (Schwarzschild scalar only)
- New semi-analytical methods for 1GSF showing promise (Kerr scalar only)

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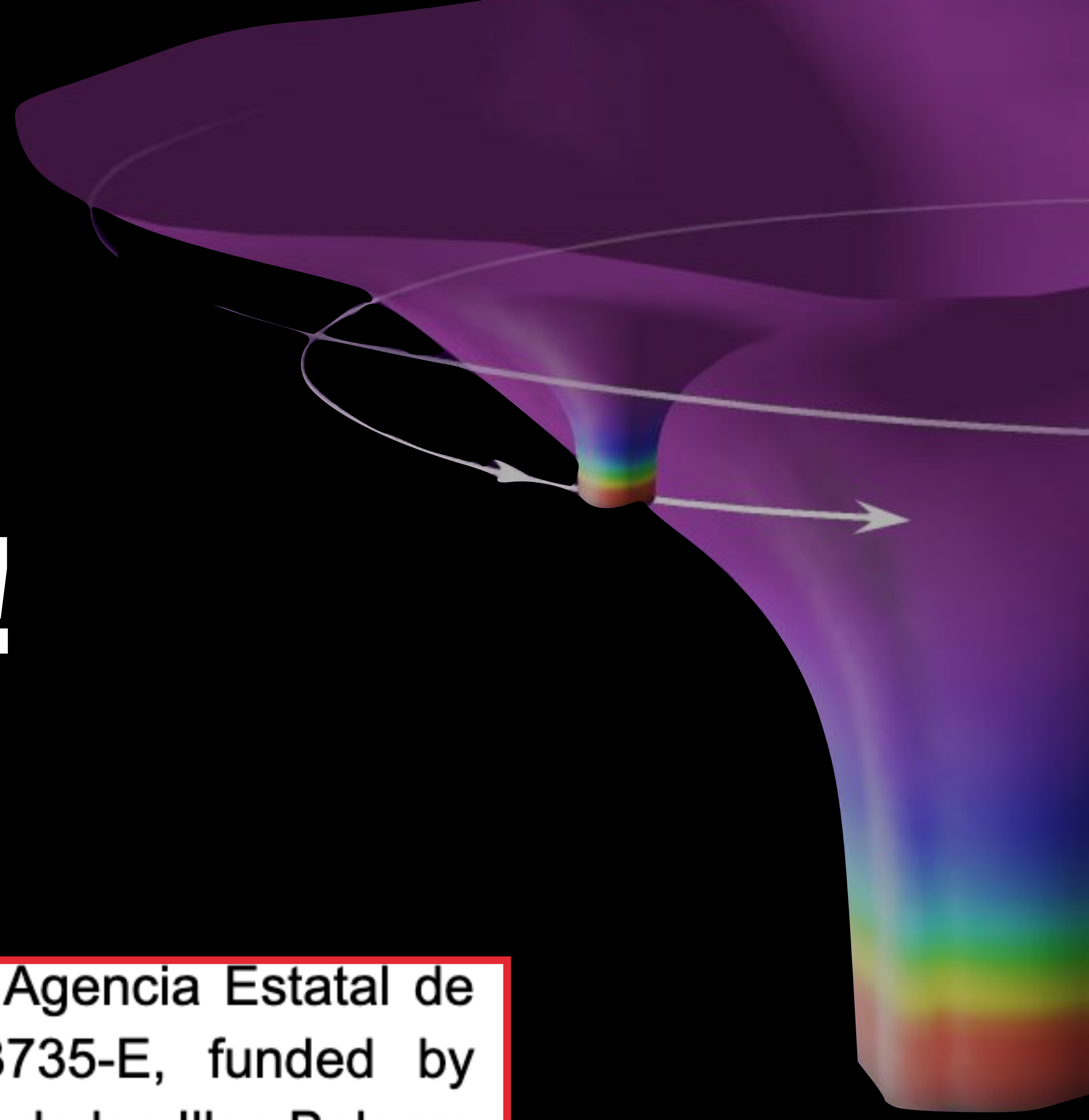

Thanks !!

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