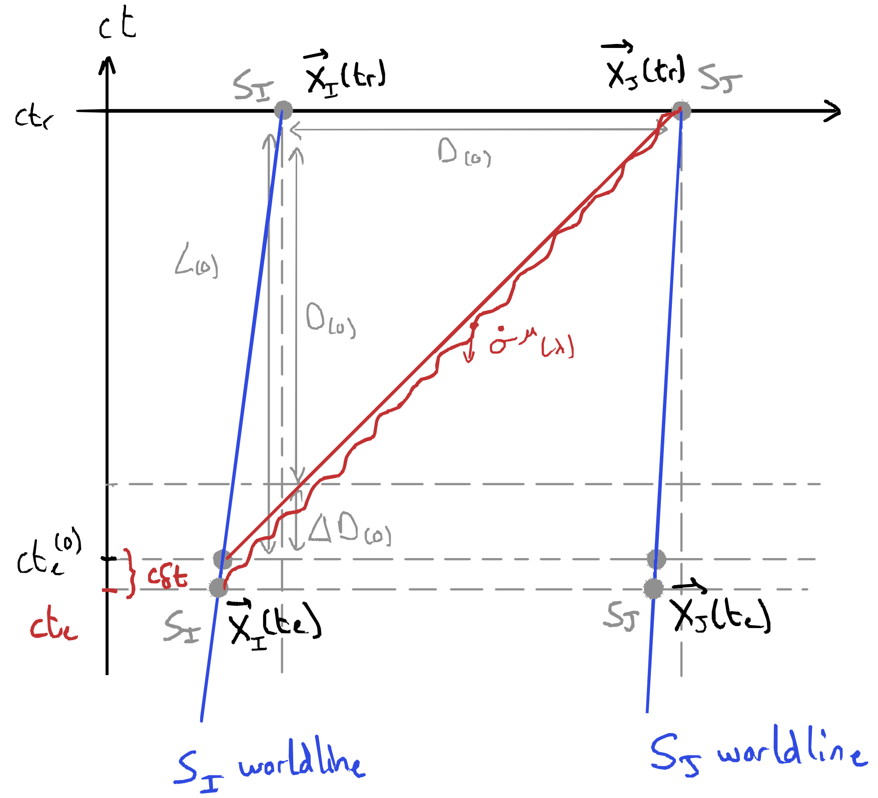
1. Observable: Doppler shifts of frequencies received from distant S/C.

a. Physical set-up



We work from the point of view of a local spacecraft S_J, receiving laser (frequency) from a distant spacecraft S_I.
We assume this laser frequency is measured at ν_0 by an observer:

a. At rest w.r.t the laser source
b. On a Minkowski background (i.e. far from any gravitational source)

From the point of view of the emitter, this frequency is measured as ν_e.

From the point of view of the receiver S/C i, as ν_r.

Between emission by S/C I and reception by S/C J, a number of effects distort the frequency along its path:

1. Relative motion between emitter and receiver
   → orbital Doppler shifts (out-of-band)

2. GW lensing of the photon geodesics in $O(h)$ (in-band)

b. Observed frequency

It can be shown (cf. Caroll, Eq. 5.100) that the frequency of a photon received by a free-falling observer N reads:

$$\nu_N = - g_{\mu\nu} U_N^\mu P^\nu$$

where U_N^μ is the 4-velocity of the observer, and P^ν is the 4-wave vector of the photon.
The 4-wave vector of the photon is proportionnal to the tangent vector $\frac{d\sigma^\nu}{d\lambda} = \dot{\sigma}^\nu$ of the null geodesic joining the two S/C I and J.

Thus one gets :

$$\boxed{\nu_N = -\frac{\nu_0}{c}\left(\eta_{\mu\nu} + \overset{\text{GW}}{h_{\mu\nu}}\right) U^\mu_N \, \dot{\sigma}^\nu} \tag{1.1}$$

$N = \{I, J\}$

2. The TT gauge

a. Metric expression in the TT gauge

When describing GW perturbation $h_{\mu\nu}$, the TT gauge provides a convenient subset of coordinate systems for which :

- Time-like observers see their spatial coordinate unaffected by the wave.

$$X^{(i)}_I(t_e) = X^{(i)}_J(t_r) = 0$$

(Note the time-component is impacted though)

- The GW perturbation $h_{\mu\nu}$ restricts to its spatial, transverse components, the other $h_{\mu\nu}$ components vanishing.

Noting such coordinate system : $\{ct, u, v, k\}$ along polarization vectors $\hat{u}, \hat{v}$ and GW propagation vector $\hat{k}$, we have

$$h_{tt} = h_{tj} = h_{it} = 0$$
$$h_{ik} = h_{kj} = 0$$
only $h_{uu}, h_{uv}, h_{vu}, h_{vv} \neq 0$

In such coordinate systems, the full metric $g_{\mu\nu}$ reads:

$$g_{\mu\nu} dx^\mu dx^\nu =$$
$$-c\,dt^2 + \left[1 + h_+(t,\vec{x})\right] du^2 + \left[1 - h_+(t,\vec{x})\right] dv^2 + 2h_\times(t,\vec{x})\, du\, dv + dk^2$$

Because the GW astrophysical source are typically extremely far away from the observer, the plane wave approximation is relevant:

$$h(t,x) \sim h(ct - k)$$

Then it's even more convenient to work with the light-ray coordinates

$$\xi = ct - k \quad \text{and} \quad \eta = ct + k$$

which put the $g_{\mu\nu}$ metric in the form:

(2.1)
$$\boxed{g_{\mu\nu} dx^\mu dx^\nu = -d\xi\, d\eta + \left[1 + h_+(\xi)\right] du^2 + \left[1 - h_+(\xi)\right] dv^2 + 2h_\times(\xi)\, du\, dv}$$

where it is visible that the metric explicitly depends on a single coordinate only, which is ϕ. This important symmetry feature will be used later

b. Using the TT gauge to write ν_N

Projecting Eq 1.1 in the TT gauge one gets:

because only $h_{ij} \neq 0$

$$\nu_N = -\frac{\nu_0}{c}\left(\eta_{\mu\nu} U^\mu \dot{\sigma}^\nu + h_{ij} \gamma c \beta^i_N \dot{\sigma}^j\right)$$

where $\vec{\beta} = \frac{\vec{v}}{c}$ is the newtonian velocity of S/C N and γ is the associated Lorentz factor $\sqrt{\frac{1}{1-\beta^2}}$.

3. Perturbative treatment of the tangent vector $\dot{\sigma}^\nu$ of the photon null geodesics

One can decompose the null geodesics of the photon into a background value labelled (0) and a first order perturbation $O(h)$ labelled by (1).

$$\dot{\sigma}^\nu = \dot{\sigma}^\nu_{(0)} + \dot{\sigma}^\nu_{(1)} + O(h^2)$$

The perturbation of the photon path due to the wave h is all contained in the term $\dot{\sigma}^{\nu}_{(1)} + O(h^2)$

Plugging this into ν_N expression,

and neglecting terms of order $O(\beta h)$,

β being typically small i.e. $\sim 10^{-4}$,

one gets:

$$\nu_N = \frac{-\nu_0}{c}\Big[\eta_{tt}\,\gamma c\,\dot{\sigma}^t_{(0)} + \eta_{tt}\,\gamma c\,\dot{\sigma}^t_{(1)} + \eta_{ij}\,\gamma\,\beta^i\,\dot{\sigma}^j_{(0)} + \underbrace{\cancel{h_{ij}\,\gamma\,\beta^i\,\dot{\sigma}^j_{(0)}}}_{\sim O(\beta h)} + \underbrace{\cancel{h_{ij}\,\gamma\,\beta^i\,\dot{\sigma}^j_{(1)}}}_{\sim O(\beta h^2)}\Big]$$

D.C. Doppler term

$$\boxed{\nu_N = \gamma\frac{\nu_0}{c}\Big[\dot{\sigma}^t_{(0)} + \underbrace{\dot{\sigma}^t_{(1)}}_{f(h)} - \overbrace{\beta_j\,\dot{\sigma}^j_{(0)}}\Big]}$$

The only term depending on h is $\dot{\sigma}^t_{(1)}$. This the one we need to calculate to evaluate the impact of the GW on the observed frequencies ν_N.

4. Finding the null geodesics function $\sigma^\mu(\lambda)$

The 4-function $\sigma^\mu(\lambda)$ is parametrized by an arbitrary, affine parameter λ, tracking the position of the photon along its null geodesics.

Working perturbatively in h, we have:

$$\sigma^\mu(\lambda) = \underbrace{\sigma^\mu_{(0)}(\lambda)}_{\text{geodesics on unperturbed bkg}} + \underbrace{\sigma^\mu_{(1)}(\lambda)}_{\text{deviation due to } h \text{ at 1st order}} + O(h^2)$$

The symmetry properties made obvious by the TT gauge lead to the finding of 3 killing vectors:

$$K_1^\mu = \delta^\mu_u \,, \quad K_2^\mu = \delta^\mu_v \,, \quad K_3^\mu = \delta^\mu_\eta$$

associated with conserved kinematic quantities along the geodesic:

$$\alpha_i = g_{\mu\nu} K_i^{\mu} \dot{\sigma}^{\nu}(\lambda)$$

As seen in Eq. 2.1, the metric components in the $\{\varphi, \upsilon, v, \eta\}$ coordinate system are:

$$g_{\mu\nu} = \begin{bmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & (1+h_+) & h_\times \\ 0 & 0 & h_\times & (1-h_+) \end{bmatrix}$$

From which one can find expressions for the 3 constants of motion $\alpha_{i=\{1,2,3\}}$:

$$\alpha_1 = g_{\mu\nu} \delta_\upsilon^\mu \dot{\sigma}^\nu = g_{\upsilon\nu} \dot{\sigma}^\nu$$

$$= (1+h_+) \dot{\sigma}^\upsilon + h_\times \dot{\sigma}^v$$

$$\alpha_2 = g_{v\nu} \dot{\sigma}^\nu = (1-h_+) \dot{\sigma}^v + h_\times \dot{\sigma}^v$$

$$\alpha_3 = g_{\eta\nu} \dot{\sigma}^\nu = -1/2 \, \dot{\sigma}^\varphi$$

To these 3 constraints, one can add a fourth one coming from the null condition of the geodesic:

$$g_{\mu\nu}\,\dot{\sigma}^{\mu}\dot{\sigma}^{\nu} = 0$$

$$\Rightarrow -\dot{\sigma}^{\eta}\dot{\sigma}^{\varphi} + (1+h_+)\dot{\sigma}^{u\,2} + (1-h_+)\dot{\sigma}^{v\,2} + 2h_\times \dot{\sigma}^{u}\dot{\sigma}^{v} = 0$$

which can be simplified using the expressions for α_1, α_2 and α_3:

$$2\alpha_3\dot{\sigma}^{\eta} + \alpha_1\dot{\sigma}^{u} + \alpha_2\dot{\sigma}^{v} = 0$$

(4.1) $(1+h_+)\dot{\sigma}^{u} + h_\times\dot{\sigma}^{v} = \alpha_1$

(4.2) $(1-h_+)\dot{\sigma}^{v} + h_\times\dot{\sigma}^{u} = \alpha_2$

(4.3) $-1/2\,\dot{\sigma}^{\varphi} = \alpha_3$

(4.4) $2\alpha_3\dot{\sigma}^{\eta} + \alpha_1\dot{\sigma}^{u} + \alpha_2\dot{\sigma}^{v} = 0$

a. Zeroth order geodesic

Decomposing the constraint equations above in perturbation in h, one gets at zeroth order:

(4a.1) $\dfrac{d\sigma^{u}_{(0)}}{d\lambda} = \alpha_1^{(0)}$ (4a.2) $\dfrac{d\sigma^{v}}{d\lambda} = \alpha_2^{(0)}$

(4a.3) $\dfrac{d\sigma^{\varphi}_{(0)}}{d\lambda} = \alpha_3^{(0)}$

This just reflects that at zeroth order the photon travels along straight lines, between

• two boundaries $\sigma^{\mu}_{(0)}(\lambda_e) = X^{\mu}_{I}(t_e^{(0)})$

and $\sigma^{\mu}_{(0)}(\lambda_r) = X^{\mu}_{J}(t_r^{(0)})$

whose spatial component are unperturbed by the GW:

$$\sigma^{i}_{(1)}(\lambda_e) = \sigma^{i}_{(1)}(\lambda_r) = 0$$

(in the TT gauge)

with $X^{\mu}_{I}(t_e^{(0)}) = (ct_e^{(0)}, u_e^{(0)}, v_e^{(0)}, h_e^{(0)})$

$X^{\mu}_{J}(t_r^{(0)}) = (ct_r^{(0)}, u_r^{(0)}, v_r^{(0)}, h_r^{(0)})$

The equations 4a.i are easy to integrate, e.g.:

$$d_1^{(0)}(\lambda_r - \lambda_e) = \int_{X_I^u}^{X_J^u} d\sigma^{u}_{(0)} = u_r^{(0)} - u_e^{(0)} = \hat{u}\cdot\hat{n}^{(0)} L_{(0)}$$

with $\vec{n}_{(0)}$ being the line of sight at 0th order and $L_{(0)}$ the distance between emission and reception.

Treating $d_2^{(0)}$ and $d_3^{(0)}$ similarly, one gets:

$$\dot{\sigma}^{u}_{(0)} = \frac{\hat{n}^{(0)}\cdot\hat{u}}{\lambda_r - \lambda_e} L_{(0)}$$

$$\dot{\sigma}^{v}_{(0)} = \frac{\hat{n}^{(0)}\cdot\hat{v}}{\lambda_r - \lambda_e} L_{(0)}$$

$$\dot{\sigma}^{\varphi}_{(0)} = \frac{\left(\overbrace{1}^{\dot{\sigma}^{t}_{(0)}} - \overbrace{\hat{n}_{(0)}\cdot\hat{h}}^{\dot{\sigma}^{h}_{(0)}}\right)}{\lambda_r - \lambda_e} L_{(0)} \quad \boxed{= \text{cst} \;,}$$

Using Eq 4.4, we can also deduce: $\boxed{\dot{\sigma}^t_{(0)} = \frac{L_{(0)}}{\lambda_r - \lambda_e}}$

b. 1st order geodesics equation

At linear order, Eq. 4.1-3 become:

$$(4b.1) \quad \dot{\sigma}^u_{(1)} + h_+ \dot{\sigma}^u_{(0)} + h_\times \dot{\sigma}^v_{(0)} = \alpha_1^{(1)}$$

$$(4b.2) \quad \dot{\sigma}^v_{(1)} - h_+ \dot{\sigma}^v_{(0)} + h_\times \dot{\sigma}^u_{(0)} = \alpha_2^{(1)}$$

$$(4b.3) \qquad -1/2\, \dot{\sigma}^\xi_{(1)} = \alpha_3^{(1)}$$

We need to evaluate $\dot{\sigma}^t_{(1)} = 1/2\left(\dot{\sigma}^\eta_{(1)} + \dot{\sigma}^\xi_{(1)}\right)$

From (4.b.3) one has:

$$\frac{d\sigma^\xi_{(1)}}{d\lambda} = -2\alpha_3^{(1)}$$

which after integration gives:

$$2\alpha_3^{(1)}(\lambda_r - \lambda_e) = \int_{\lambda_e}^{\lambda_r} \dot{\sigma}^t_{(1)}\, d\lambda - \underbrace{\int_{\lambda_e}^{\lambda_r} \dot{\sigma}^k_{(1)}\, d\lambda}_{0 \text{ in TT gauge}}$$

$$\Rightarrow \quad 2\alpha_3^{(1)}(\lambda_r - \lambda_e) = c\left(t_r^{(1)} - t_e^{(1)}\right)$$
$$= c\,\delta t$$

Now integrating Eq. 4.b.1 :

$$\ldots \int^{\lambda_r} \ldots d\sigma^u_{(0)} \ldots \int^{\lambda_r} \ldots d\sigma^v_{(0)} \ldots$$

$$(\lambda_r - \lambda_e)\, \alpha_1 = \underbrace{\int_{\lambda_e} h_+(\xi) \underbrace{\frac{d\xi}{d\lambda}}_{d_1^{(0)}} d\lambda}_{I_{uu}} + \underbrace{\int_{\lambda_e} h_\times(\xi) \underbrace{\frac{d\xi}{d\lambda}}_{d_2^{(0)}} d\lambda}_{I_{uv}}$$

$$I_{uu} = \frac{\Delta u_{(0)}}{\lambda_r - \lambda_e} \int_{\varphi_e^{(0)}}^{\varphi_r^{(0)}} h_+(\varphi) \underbrace{\left(\frac{d\varphi}{d\lambda}\right)^{-1}_{(0)}}_{= \text{ constant } = \frac{\lambda_r - \lambda_e}{1 - \vec{k}\cdot\hat{n}_{(0)}}} d\varphi$$

which gives $I_{uu} = \dfrac{\hat{n}^u_{(0)}}{1 - \vec{k}\cdot\hat{n}_{(0)}} H_{uu}$

with $H_{ij} = \displaystyle\int_{\varphi_e^{(0)}}^{\varphi_r^{(0)}} h_{ij}(\varphi)\, d\varphi$

Proceeding similarly for $d_1^{(1)}$ and $d_2^{(1)}$, one finds:

$$d_1^{(1)} = \frac{\eta^{ui}\, \hat{n}^j_{(0)}}{(\lambda_r - \lambda_e)(1 - \hat{k}\cdot\hat{n}_{(0)})} H_{ij}$$

$$d_2^{(1)} = \frac{\eta^{vi}\, \hat{n}^j_{(0)}}{(\lambda_r - \lambda_e)(1 - \hat{k}\cdot\hat{n}_{(0)})} H_{ij}$$

and finally, derived from the null condition:

$$\boxed{c\,\delta t = \frac{1}{2}\,\frac{n^l_{(0)}\, n^m_{(0)}}{1-\hat{k}\cdot\hat{n}_{(0)}}\, H_{lm}}$$

After some extra algebra, one can find:

$$\boxed{\begin{aligned}\dot{\sigma}^t_{(1)} = {} & \overbrace{\frac{\left(1-\frac{1}{2}\,\vec{k}\cdot\hat{n}_{(0)}\right) n^l_{(0)}\, n^m_{(0)}}{(\lambda_r-\lambda_e)\,\left(1-\hat{k}\cdot\hat{n}_{(0)}\right)^2}\, H_{lm}}^{\text{cst along geodesic}} \\ & \underbrace{-\frac{1}{2}\,\frac{L_{(0)}\,\hat{n}^l_{(0)}\,\hat{n}^m_{(0)}}{(\lambda_r-\lambda_e)\,\left(1-\hat{k}\cdot\hat{n}_{(0)}\right)}\, h_{lm}}_{\text{variable piece}}\end{aligned}}$$

5. Combining the frequencies

$$\nu_N = \gamma\,\frac{\nu_0}{c}\left[\dot{\sigma}^t_{(0)} + \dot{\sigma}^t_{(1)} - \beta_j\,\dot{\sigma}^j_{(0)}\right]$$

Measurement based on comparison between emitted and received frequencies:

$$\Delta\nu = \nu^r - \nu^e$$

The fractional relative frequency is the observable:

$$y = \frac{\nu^r - \nu^e}{\nu^e}$$

$$y \simeq \frac{\nu^r - \nu^e}{\underbrace{\gamma \frac{\nu_0}{c} \dot{\sigma}^t_{(0)}(\lambda_e)}_{\bar{\nu}_e} + \underbrace{\delta\nu_e}_{\sim O(\beta) + O(h)}}$$

$$y \simeq \overbrace{\frac{\nu^r - \nu^e}{\bar{\nu}_e}}^{O(\beta)+O(h)} \left(1 - \cancel{\underbrace{\delta\nu_e}_{O(\beta)+O(h)}}\right)$$

$$\nu^r - \nu^e = \Big[\overbrace{\dot{\sigma}^t_{(0)}(\lambda_r) - \dot{\sigma}^t_{(0)}(\lambda_e)}^{0 \text{ since } \ddot{\sigma}^t_{(0)} = 0} \underbrace{}_{\text{the constant terms vanish}}$$

$$+ \dot{\sigma}^t_{(1)}(\lambda_r) - \dot{\sigma}^t_{(1)}(\lambda_e)$$

$$- \beta^J_i \dot{\sigma}^i_{(0)}(\lambda_r) + \beta^I_i \dot{\sigma}^i_{(0)}(\lambda_e) \Big]$$

$$= \gamma \frac{\nu_0}{c} \frac{L_0}{\lambda_e - \lambda_r} \Big[-1/2 \frac{\hat{n}^l_{(0)} \hat{n}^m_{(0)}}{1 - \hat{k} \cdot \hat{n}_{(0)}} \big(h_{lm}(\lambda_r) - h_{lm}(\lambda_e) \big)$$

$$+ \Delta\beta_i \left(\dot{\sigma}^i_{(0)}(\lambda_r) - \dot{\sigma}^i_{(0)}(\lambda_e) \right) \Big]$$

Normalizing by $\nu_e = \gamma \frac{\nu_0}{c} \hat{\sigma}^{b}_{(0)}$

one gets finally :

$$\boxed{y = -\frac{1}{2} \frac{\hat{n}^{l}_{(0)} \hat{n}^{m}_{(0)}}{1 - \hat{k} \cdot \hat{n}_{(0)}} \left[h_{lm}(\varphi_r) - h_{lm}(\varphi_e) \right] + \Delta\vec{\beta}_{IJ} \cdot \hat{n}_0}$$

where we used that at zeroth order $\varphi \propto \lambda$