Foundations of Gravitational Waves

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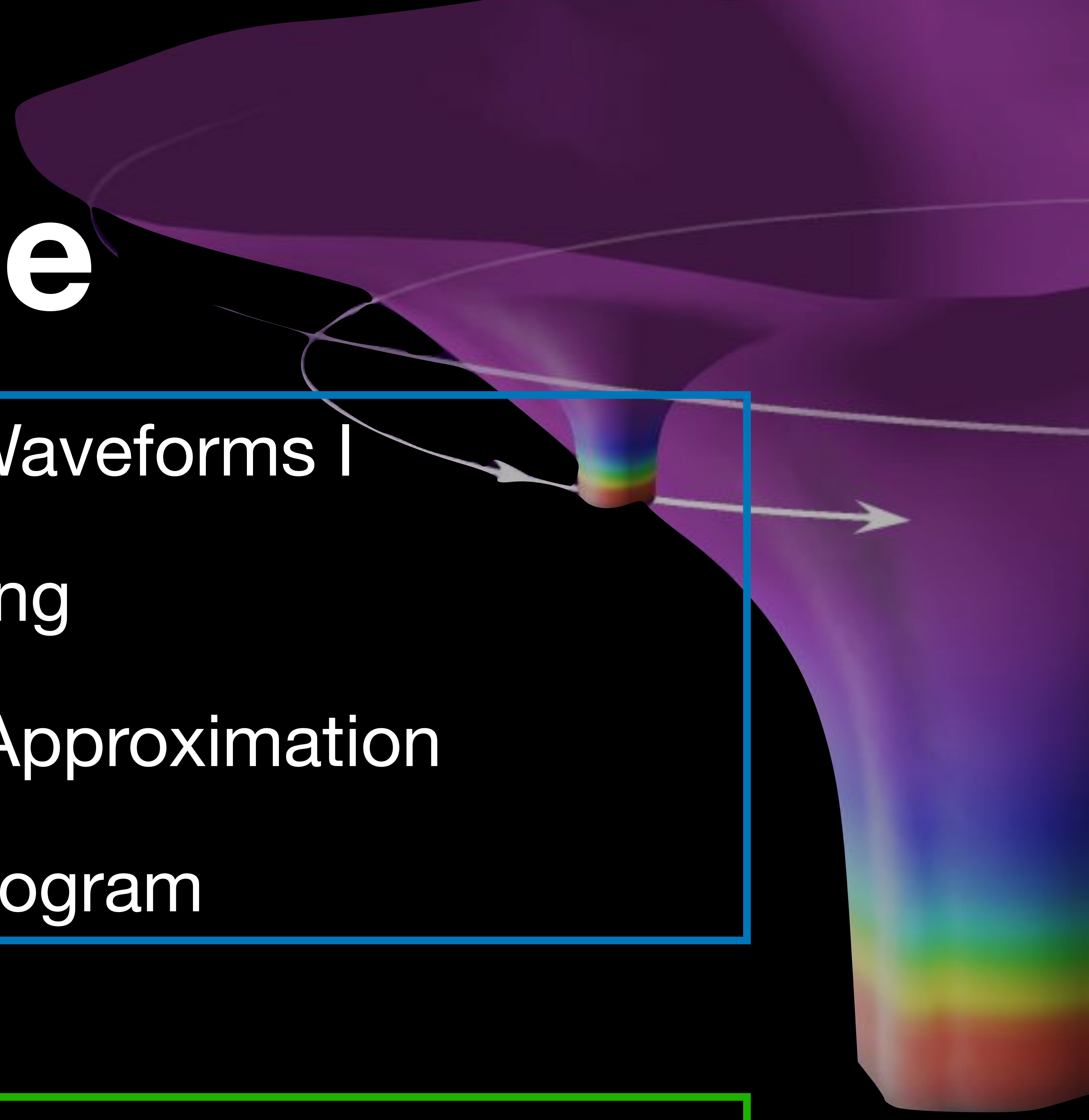
White Paper: arXiv:2311.01300
Redbook: arXiv:2402.07571

Les Houches LISA School for Early Career Scientists, October, 2025

Gravitational Wave(forms) Outline



Today: Foundations

- General Overview
  - GW150914: Structure of a Waveform
  - Complexities of Waveforms
  - Parameter Space
  - LISA Science Objectives
- The Mathematics
  - Einstein Field Equations
  - Linearised gravity

Wednesday 10:30: Waveforms I

Perturbation Modelling

- Post-Newtonian Approximation
- The Self-Force Program

Wednesday 17:00: Waveforms II

Merger - Ringdown Modelling

- Quasi-Normal Modes (QNMs)
- Numerical Relativity

Thursday 17:00: Waveforms III

Waveform Modelling

Saturday 10:15 Waveforms

Hands-on Session

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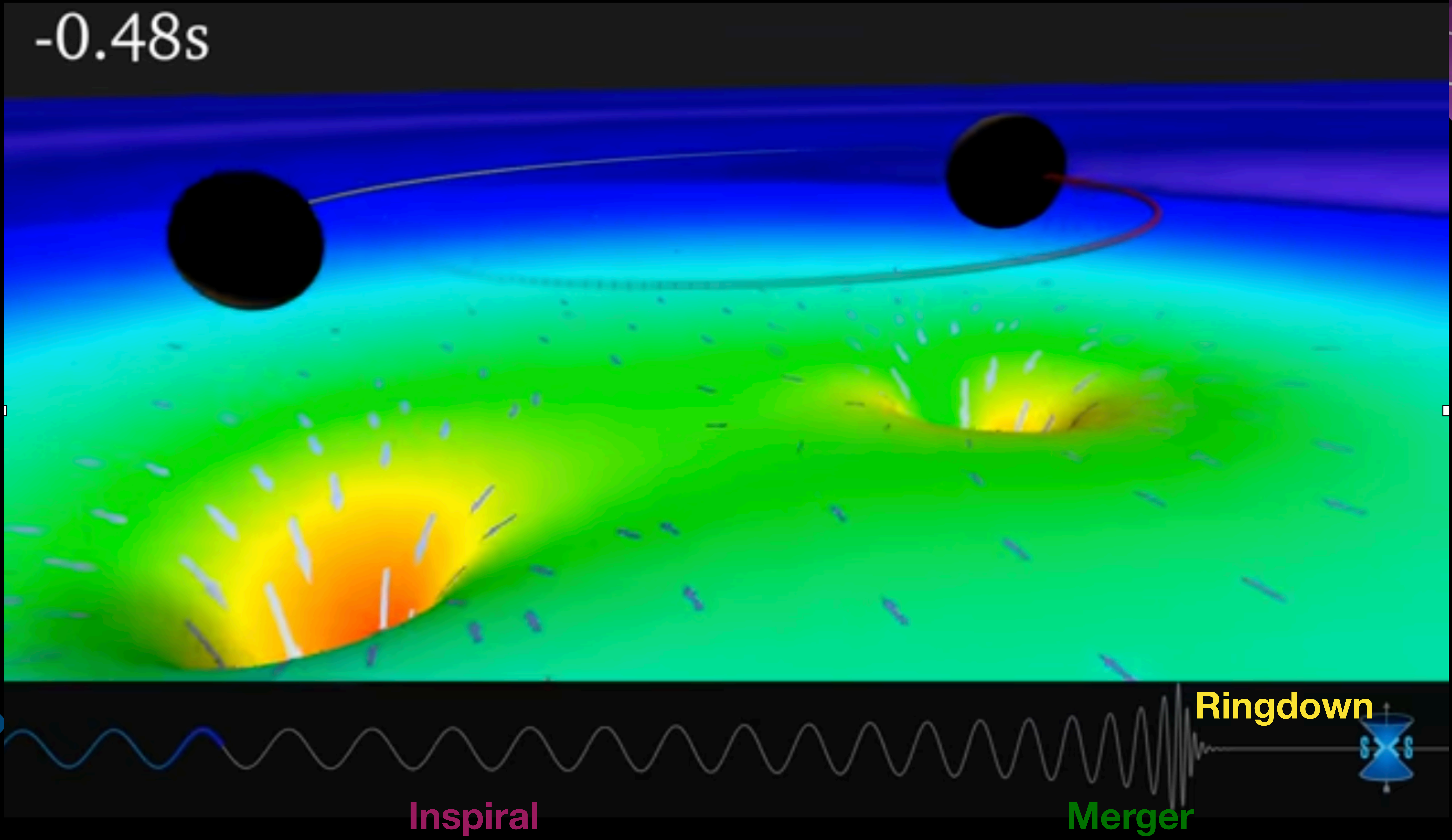
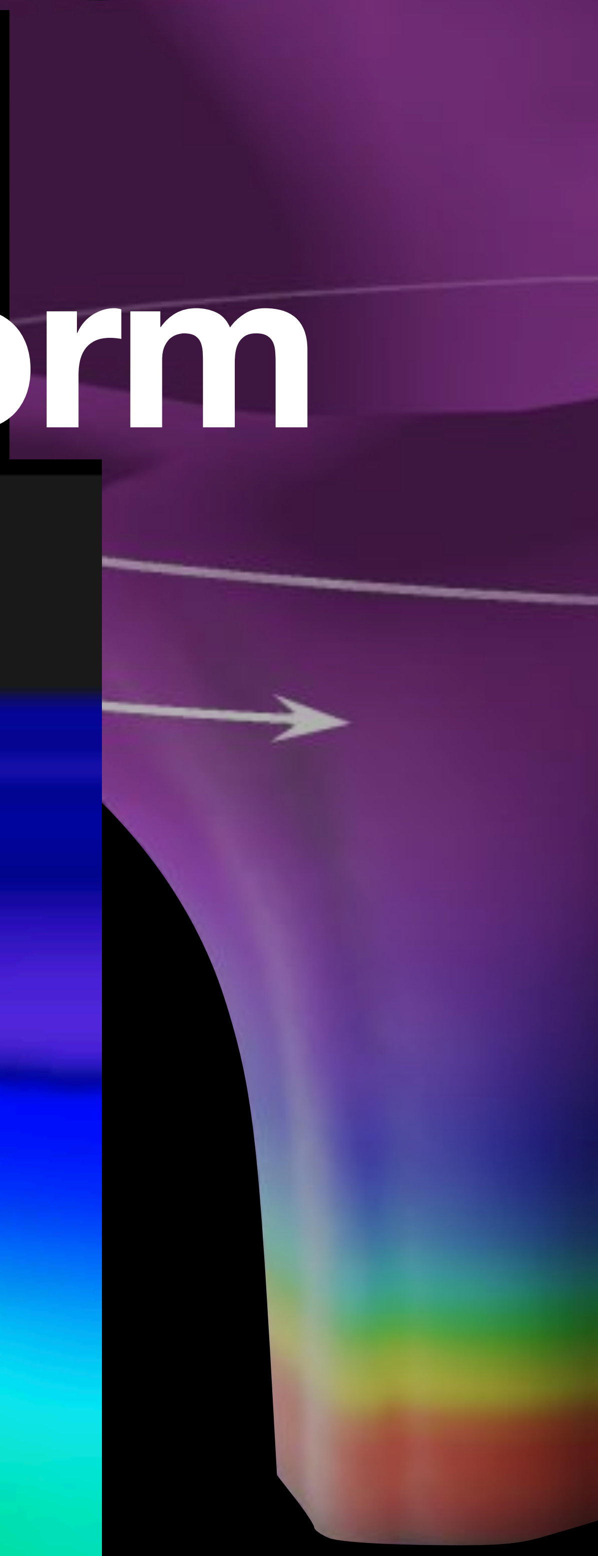
GW150914: The First Gravitational Waveform

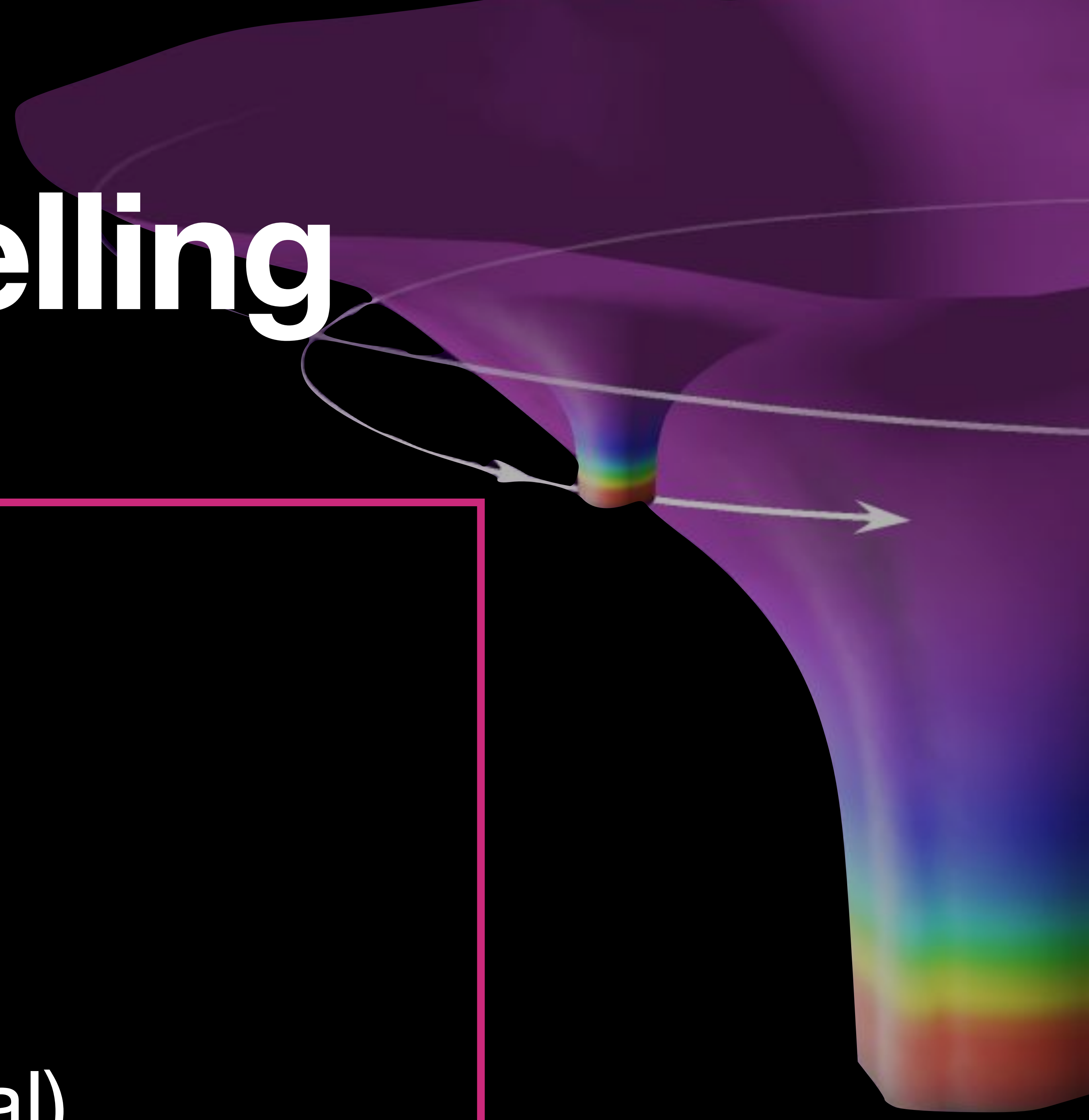




IMR

Complexities of Waveform Modelling

Build from ‘simple’ towards generic



GW150914

- Comparable Masses, Quasi-circular orbit, no / small spin

Building

- Non-spinning -> Spinning (speeds up / slows down inspiral)
- (Anti)-aligned spin -> Non-aligned (spin-precession effects)
- Quasi-circular -> Eccentric Orbits (beats)
- Comparable masses -> Large mass ratios (more complex waveforms)

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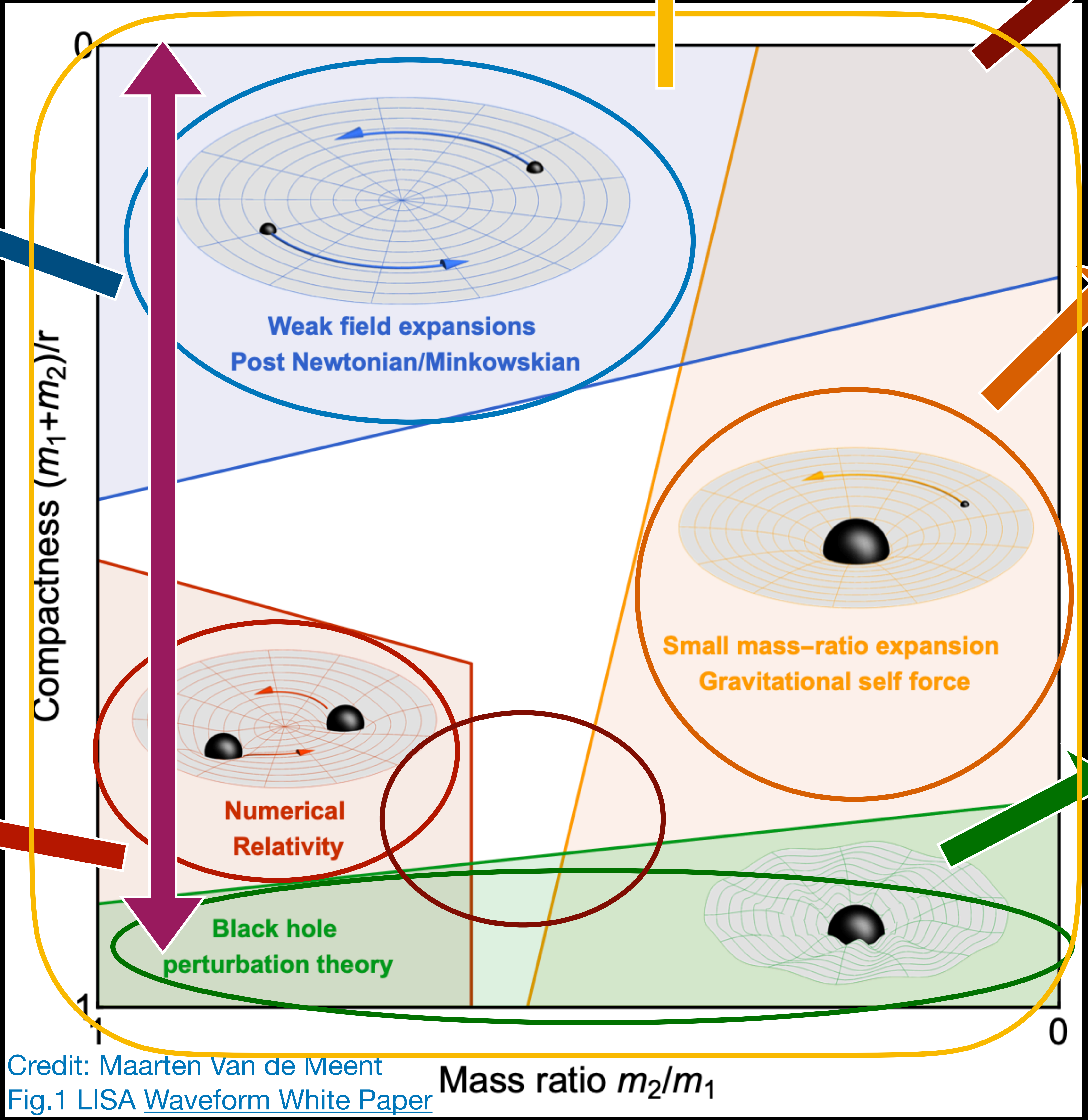
Parameter Space

Post-Newtonian / Minkowskian
- Perturbation from flat spacetime
$$g_{ab} = \eta_{ab} + \epsilon h_{ab} + \mathcal{O}(\epsilon^2),$$
- Post-Newtonian (PN): system velocity $\Rightarrow \epsilon \sim v^2/c^2,$
- Post-Minkowskian: Gravitational Potential $\Rightarrow \epsilon \sim G,$
- Application: Inspiral, all masses
- See Waveforms I (Wed 10:30)

Hybrids
- Stitching together of long PN inspires to NR simulations

Numerical Relativity
- Numerically solve Einstein's field equations,
$$R^{ab} - \frac{1}{2} g^{ab} R = \frac{8\pi G}{c^4} T^{ab},$$
- 3+1 Decomposition
- Application: Comparable masses, late inspiral, merger, ringdown
- See Waveforms II (Wed 17:00)

Smooth Parameter Space Coverage
- Phenomenological Waveforms
- Effective One-Body
- NR Surrogate



Credit: Maarten Van de Meent
Fig.1 LISA Waveform White Paper

PN / SF Comparisons
- Use gauge invariants
- SF used to read off unknown PN coefficients

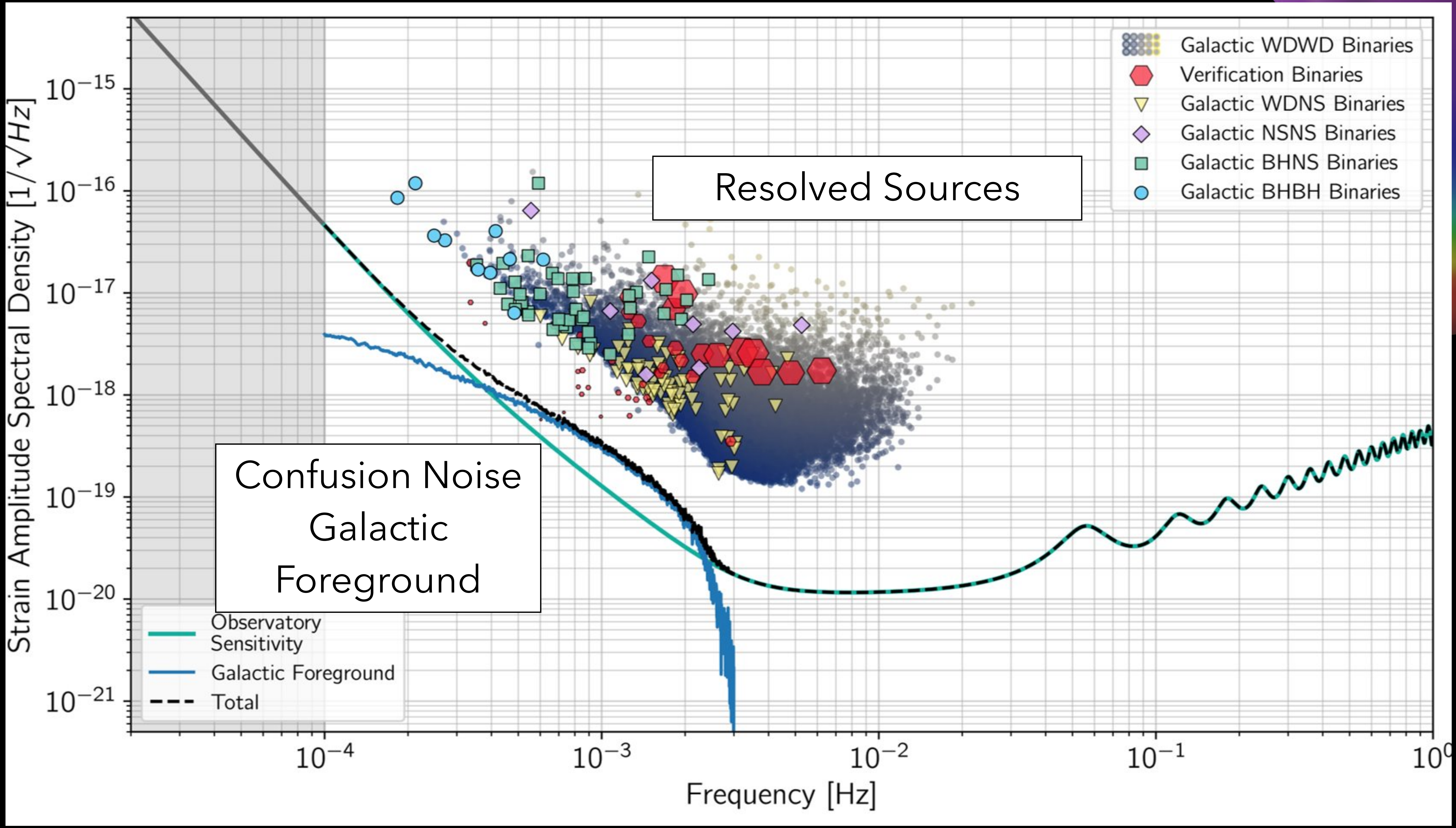
Self-Force (SF)
- Perturbation in the mass-ratio from vacuum curved spacetime (ideally Kerr blackhole)
$$\mathfrak{g}_{ab} = g_{ab} + \epsilon h^{(1)}_{ab} + \mathcal{O}(\epsilon^2),$$
$$\epsilon \sim m_2/m_1,$$
- Mainly numerical but recent progress in semi-analytical methods
- Application: Extreme mass ratios, inspiral, plunge, merger, ringdown
- See Waveforms I (Wed 10:30)

Blackhole Perturbation / Quasi-normal Modes
- Perturbation from vacuum curved spacetime (ideally Kerr blackhole)
$$\mathfrak{g}_{ab} = g_{ab} + \epsilon h^{(1)}_{ab} + \mathcal{O}(\epsilon^2),$$
$$h_{lmn} = A_{lmn} e^{i\omega_{lmn} t},$$
- Application: Ringdown
- See Waveforms II (Wed 17:00)

LISA Science Objectives

OBJ1: Study the formation and evolution of compact binary stars and the structure of the Milky Way Galaxy

- 10^4 detectable binaries
- 10^7 form foreground
- Undetectable from EM
- Continuous signals
- Mostly white dwarfs but ~100 with NS or BH
- Formation / evolution via kicks and common envelope
- Galaxy mass distribution
- Mass transfer and tides



From LISA Redbook: arXiv:2402.07571, M.Colpi et al.

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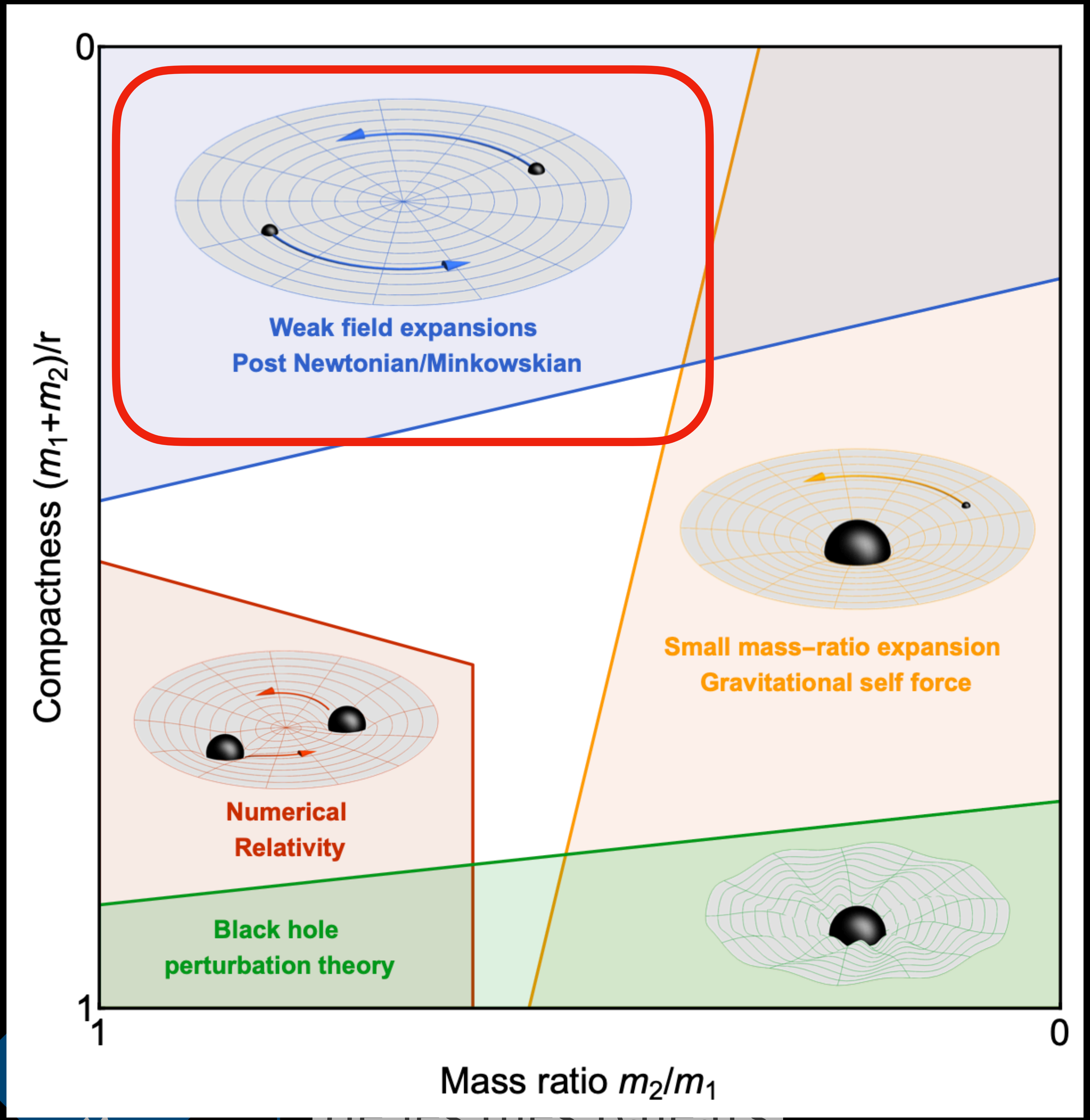

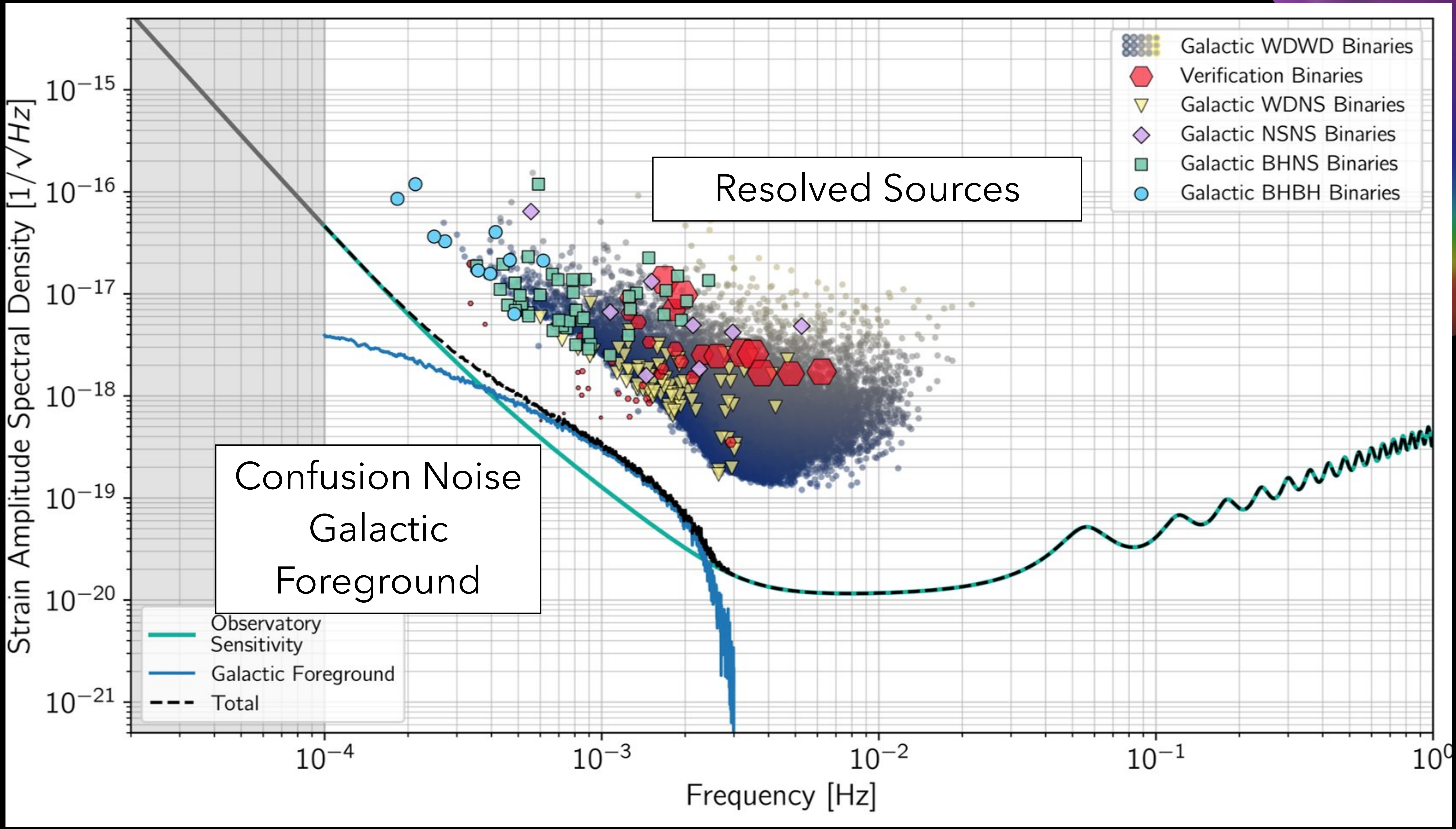

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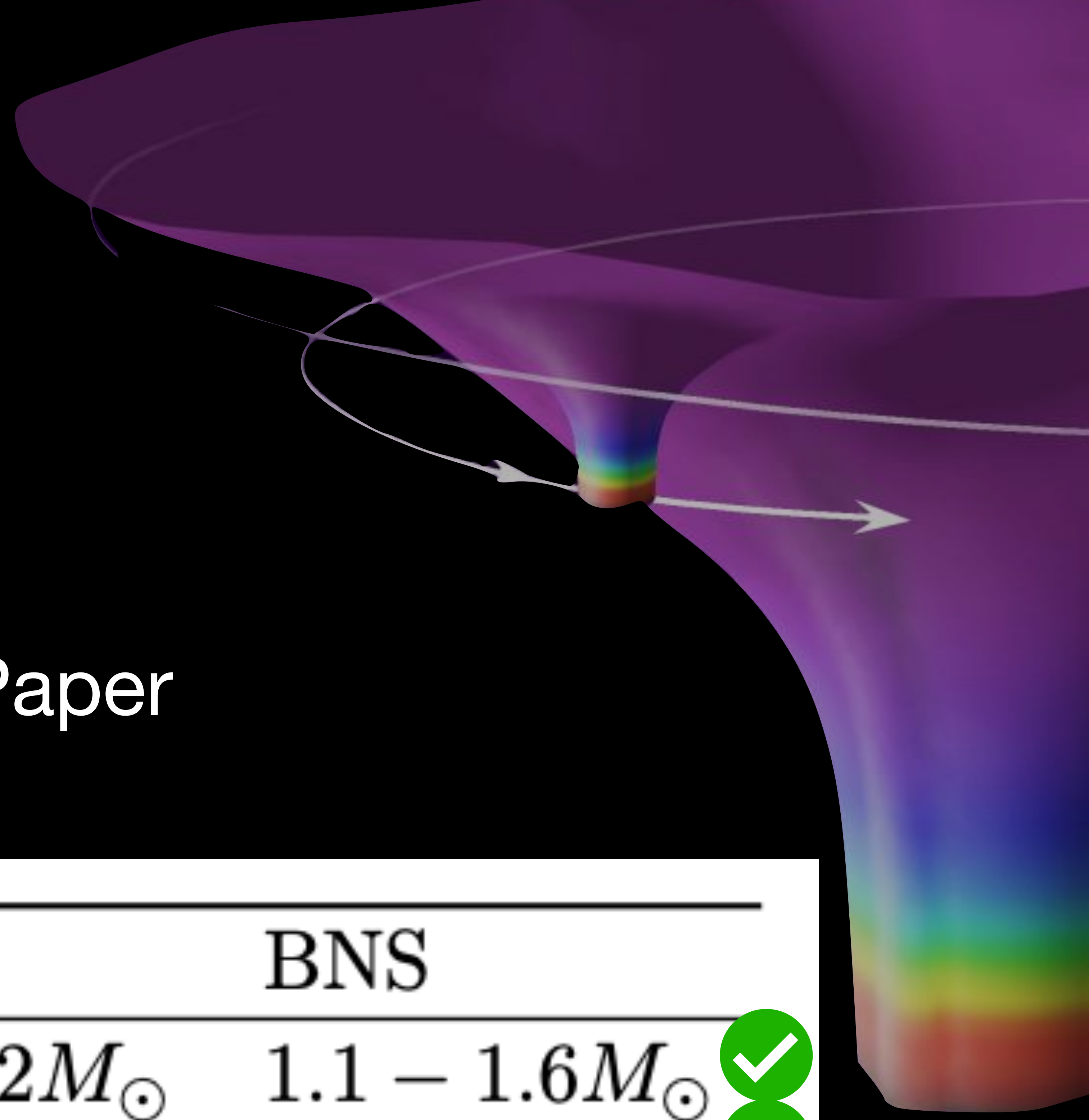
Fig.1 Waveform White Paper, arXiv:2311.01300

Science Objectives

Status of Waveforms for MBHBs



Galactic Binaries: Table 6 of Waveform White Paper

Parameter	Notation	BWD	WDNS	BNS
Total Chirp Mass	M	$0.1 - 1 M_\odot$	$0.4 - 1.2 M_\odot$	$1.1 - 1.6 M_\odot$
Mass Ratio (> 1)	q	$1 - 10$	$1 - 5$	$1 - 1.6$
Eccentricity in LISA band	$e_{\rm init}$	0	$0 - 1$	$0 - 1$
Signal to Noise Ratio	SNR	< 1000	< 1000	< 1000

Caveats:
- Mass transfer
- Tidal effects

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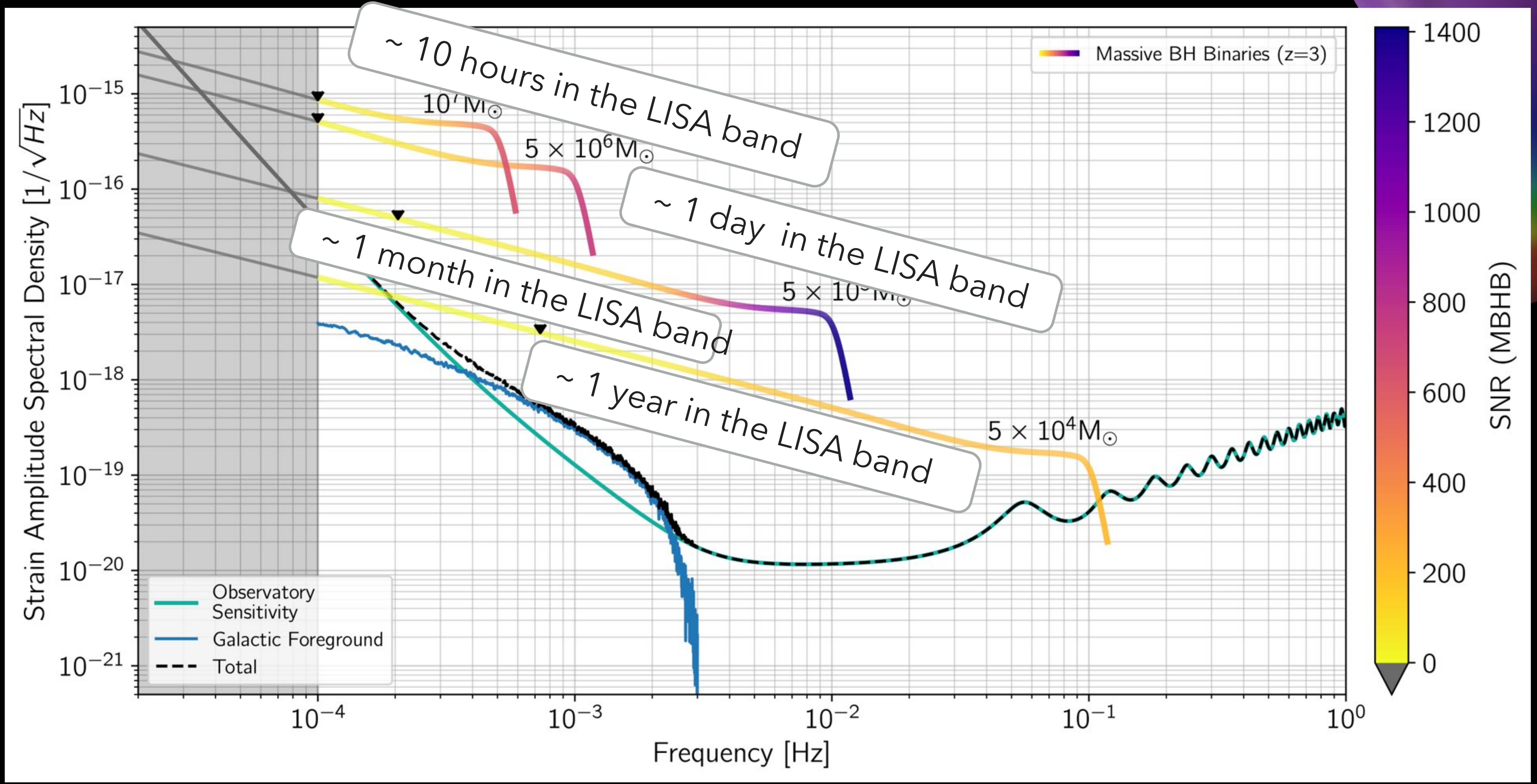
LISA Science Objectives

OBJ2: Trace the origins, growth and merger histories of massive Black Holes across cosmic epochs

- Between 10^4 and 10^7 solar masses
- Detection would Inform hierarchial growth
- Few per year
- Mass ratio up to 1000 (bulk <10)
- Expected high eccentricities
- Population census



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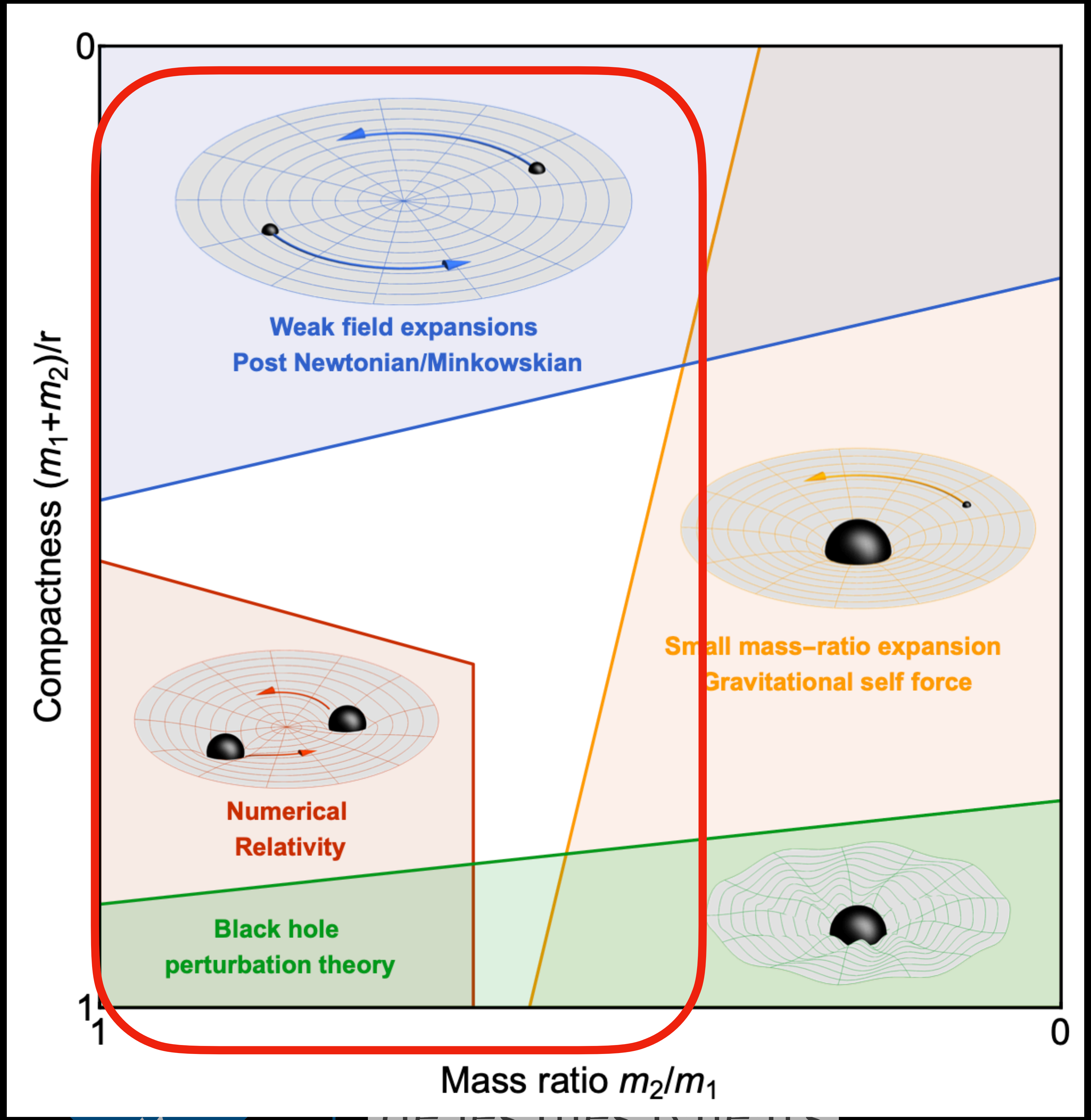


Fig.1 Waveform White Paper, arXiv:2311.01300

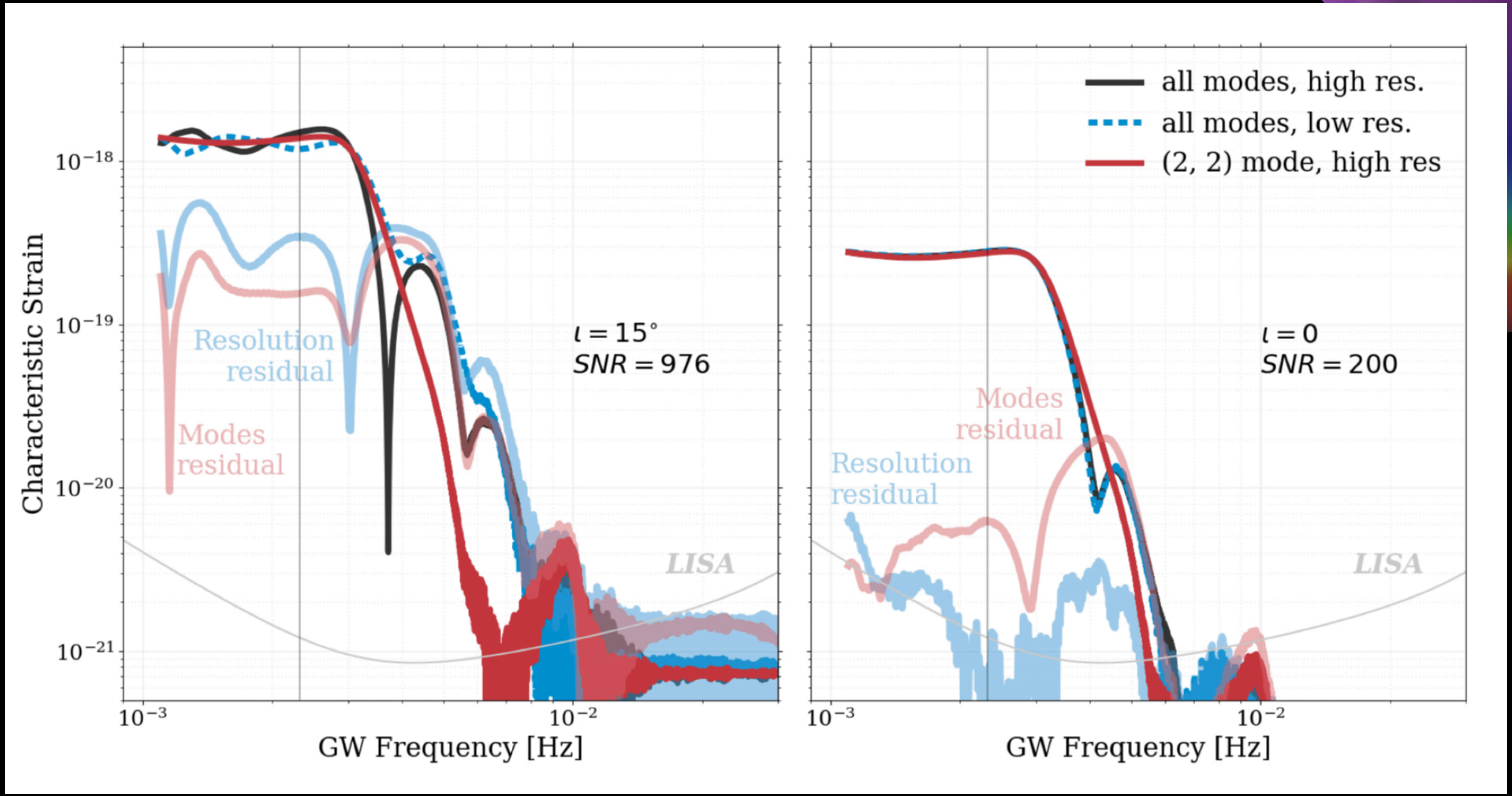
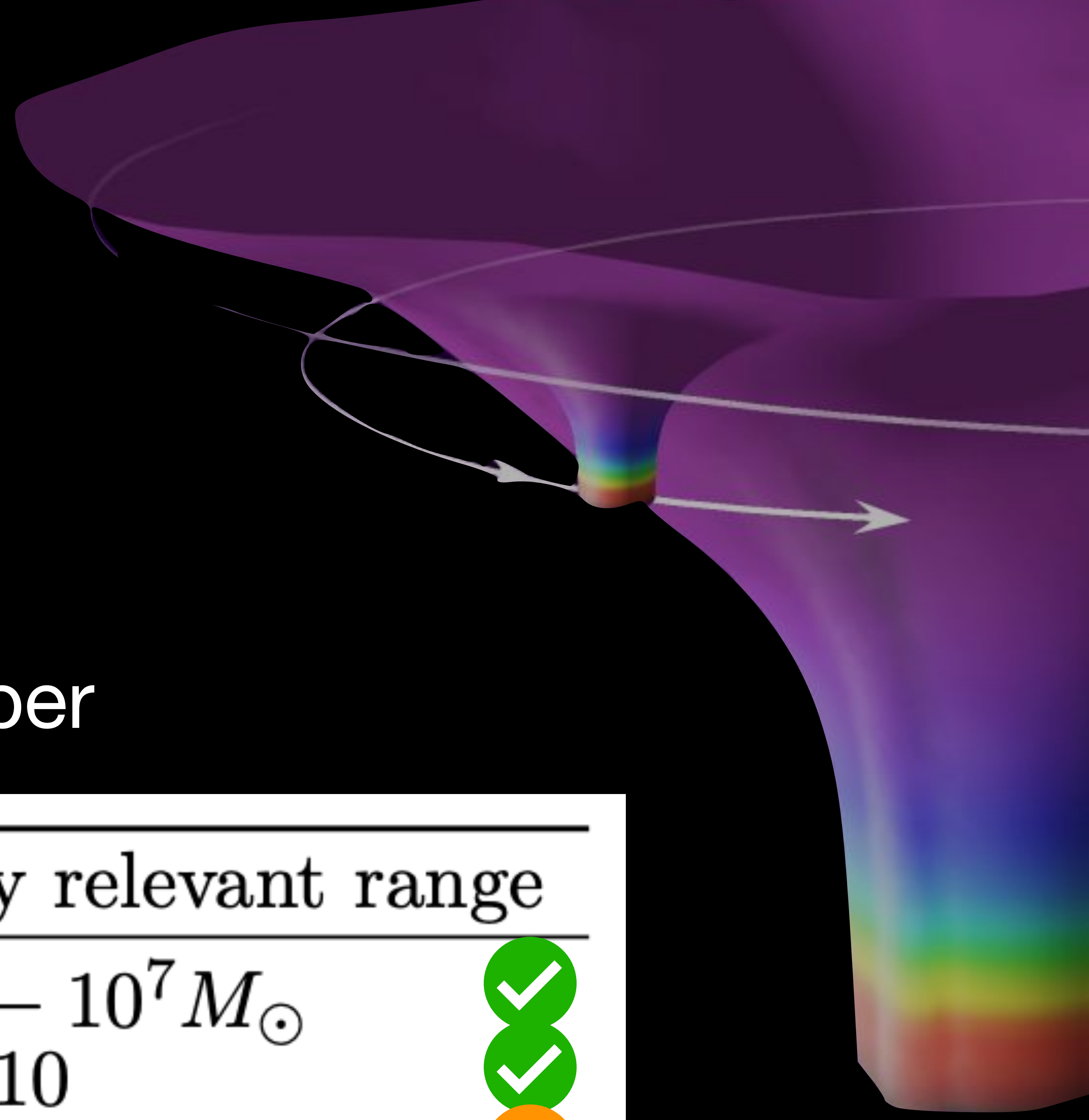


Fig.4 Waveform White Paper, arXiv:2311.01300 (LISA Waveform WG)

From LISA Redbook: arXiv:2402.07571, M.Colpi et al.

Science Objectives

Status of Waveforms for MBHBs



Massive BHBs: Table 3 of Waveform White Paper

Parameter	Notation	Astrophysically relevant range
Total mass in the detector frame	M	$10^5 - 10^7 M_\odot$
Mass ratio (> 1)	q	$1 - 10$
Dimensionless spin	$\max\lvert\chi_i\rvert$	$0 - 0.998$
Eccentricity entering LISA band	$e_{\rm init}$	$0 - 0.99$
Eccentricity at last stable orbit	$e_{\rm merge}$	< 0.1
Signal to noise ratio	SNR	$10 - 10^4$

Generic: Mis-aligned spins **and** eccentric

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LISA Science Objectives

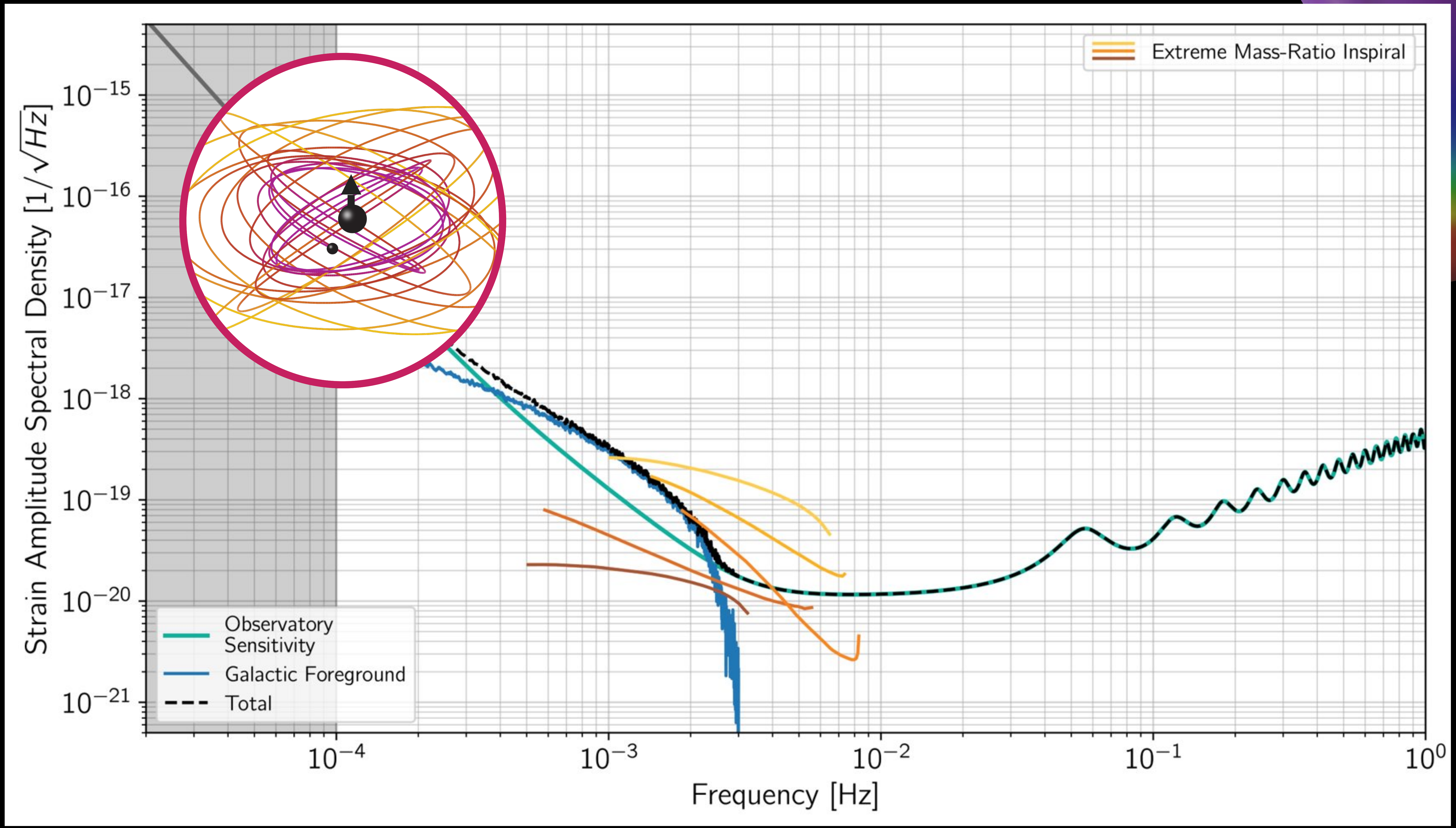
OBJ3: Probe the properties and immediate environments of Black Holes in the local Universe using extreme mass-ratio inspirals and intermediate mass-ratio inspirals

EMRIs: mass ratio 10^5 -10^8

- Environmental info
- Galaxy and BH evolution
- Eccentricity and spin => formation channel
- 1-1000 per year
- SNR ~100 => sky location 0.05deg^2

IMRIs: mass ratio 10^2 -10^4

- IMBH (10^2 -10^4 solar M)
- Light (multi band), Heavy (secondary spin)



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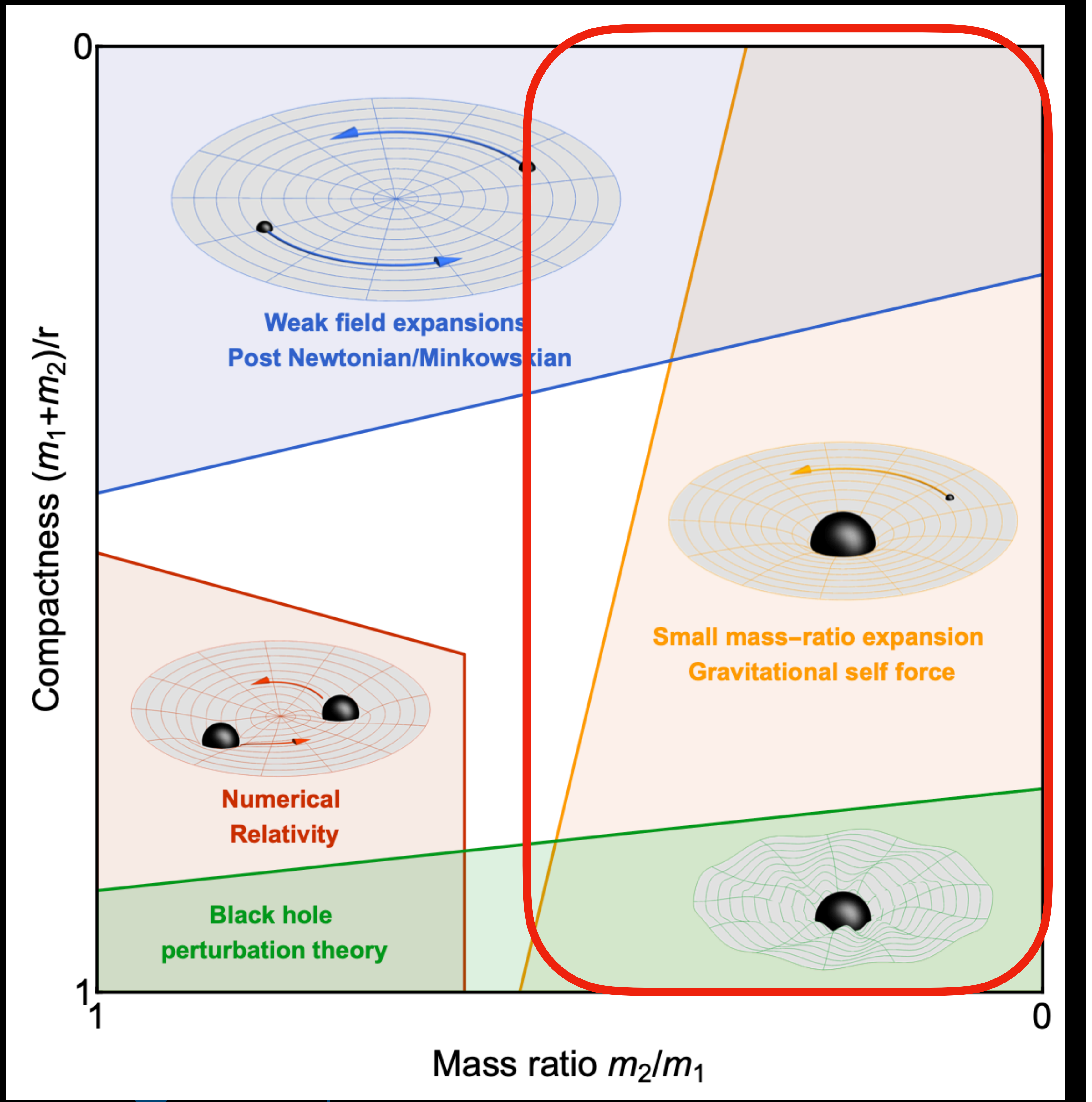
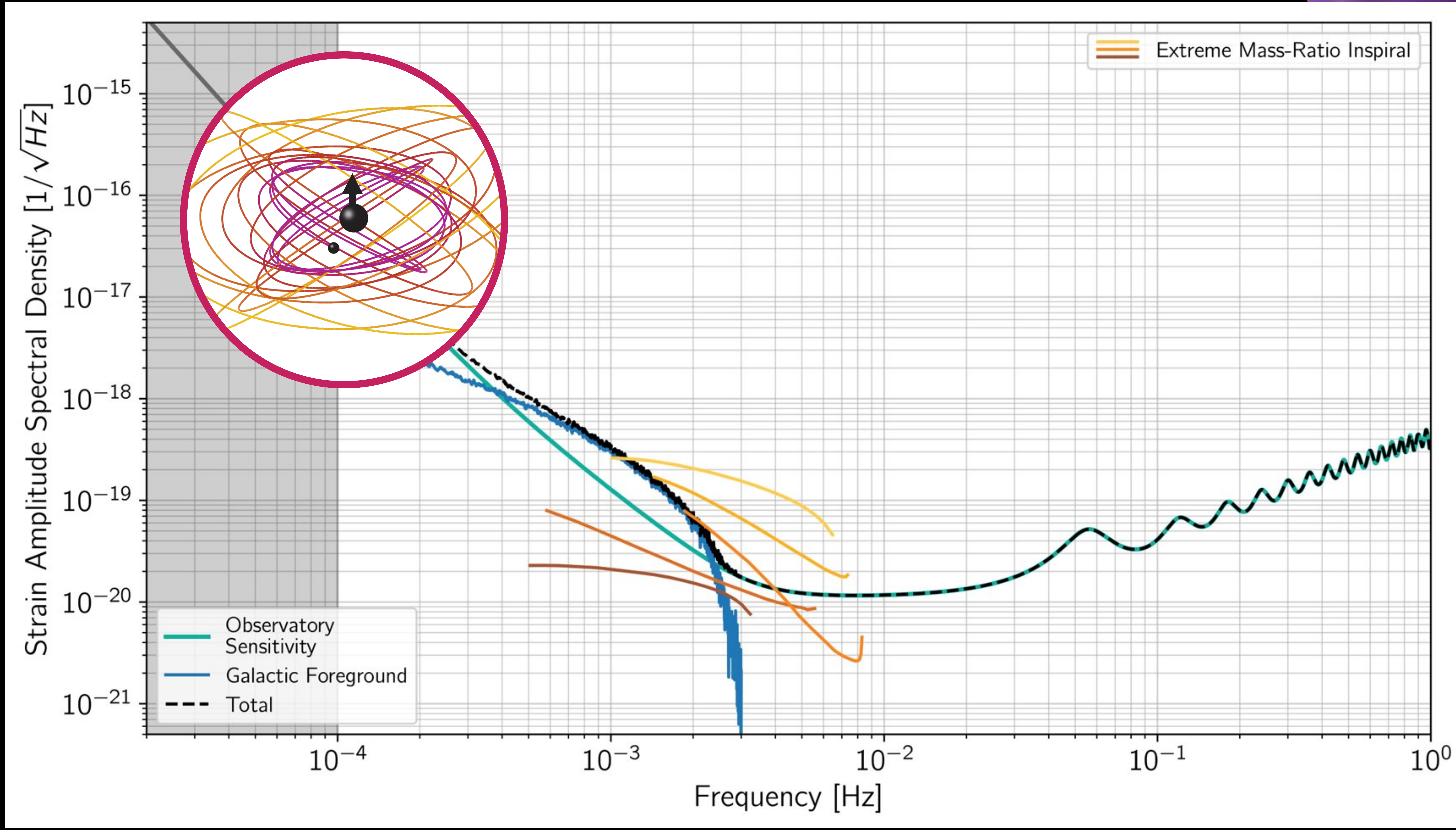


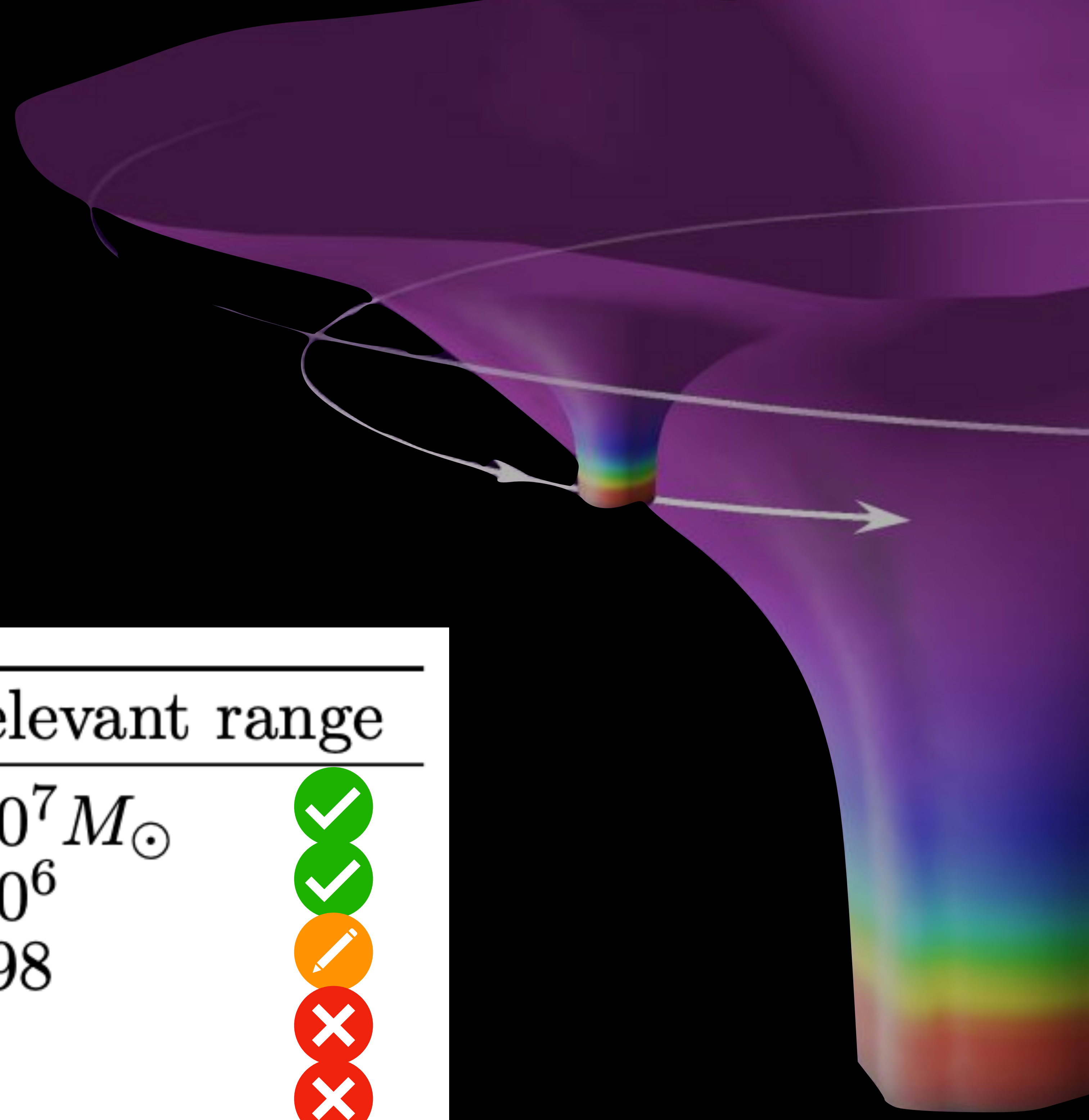
Fig.1 Waveform White Paper, arXiv:2311.01300



From LISA Redbook: arXiv:2402.07571, M.Colpi et al.

Science Objectives

Status of Waveforms for EMRIs \ IMRIs



EMRIs: Table 4 of Waveform White Paper

Parameter	Notation	Astrophysically relevant range
Total mass in the detector frame	M	$10^5 - 10^7 M_\odot$
Mass ratio (> 1)	q	$10^4 - 10^6$
Dimensionless spin	$\max\lvert\chi_i\rvert$	$0 - 0.998$
Eccentricity entering LISA band	$e_{\rm init}$	$0 - 0.8$
Eccentricity at last stable orbit	$e_{\rm merge}$	$0 - 0.2$
Signal to noise ratio	SNR	$20 - 100$

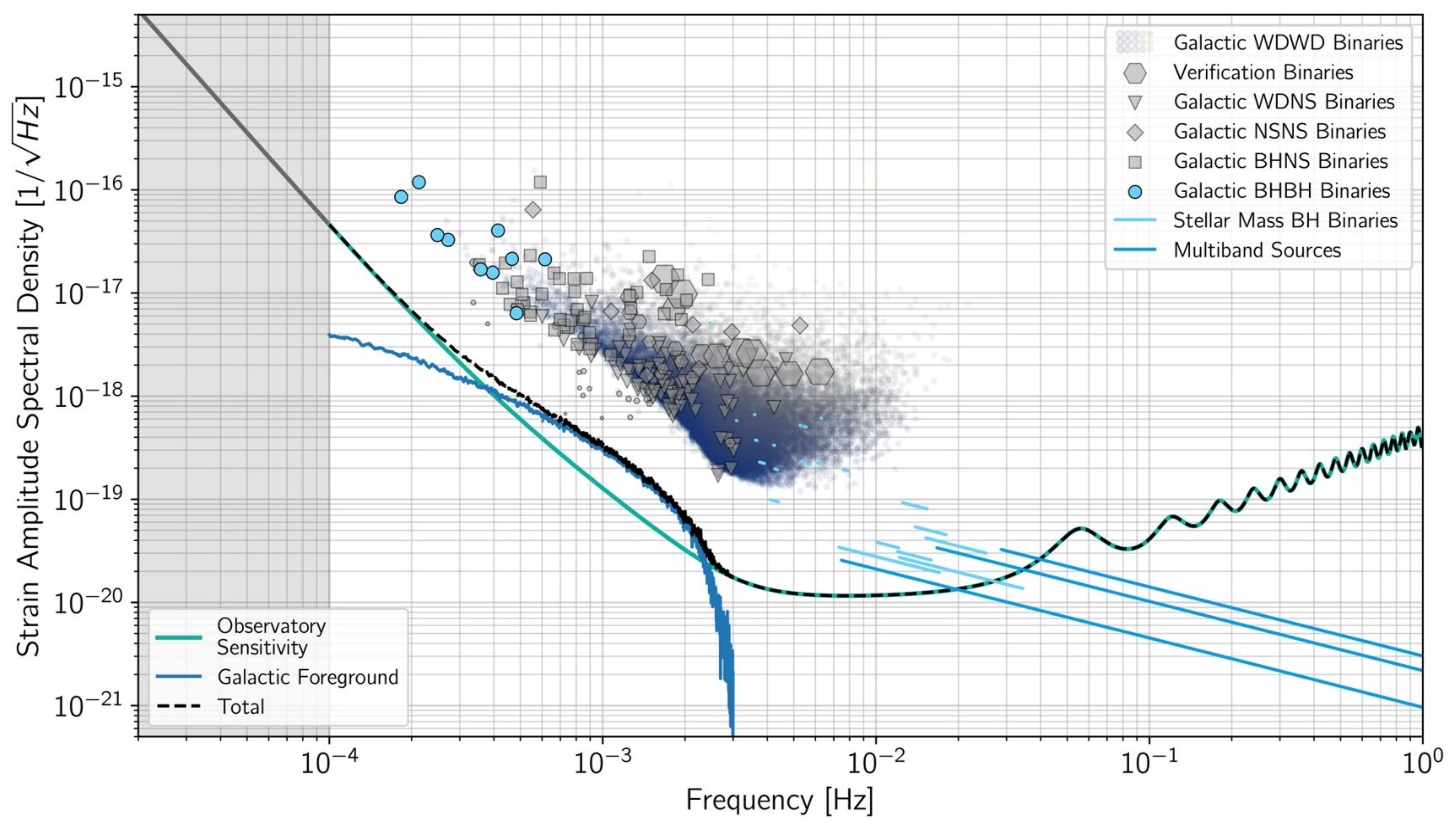
IMRIS: Table 5 of Waveform White Paper

Parameter	Notation	Heavy IMRIs	Light IMRIs
Binary mass	M	$10^4 - 10^7 M_\odot$	$10^2 - 10^4 M_\odot$
Mass ratio (> 1)	q	$10 - 10^4$	$10 - 10^4$
Dimensionless spin	$\max\lvert\chi_i\rvert$	$0 - 0.998$	$0 - 0.998$
Eccentricity entering LISA band	$e_{\rm init}$	$0 - 0.9995$	$0 - 0.9995$
Eccentricity at last stable orbit or leaving LISA band	$e_{\rm merge}$	$0 - 0.9$	$0 - 0.9$
Signal to noise ratio	SNR	$10 - 10^2$	$10 - 10^3$

LISA Science Objectives

OBJ4: Understand the astrophysics of stellar-mass Black Holes

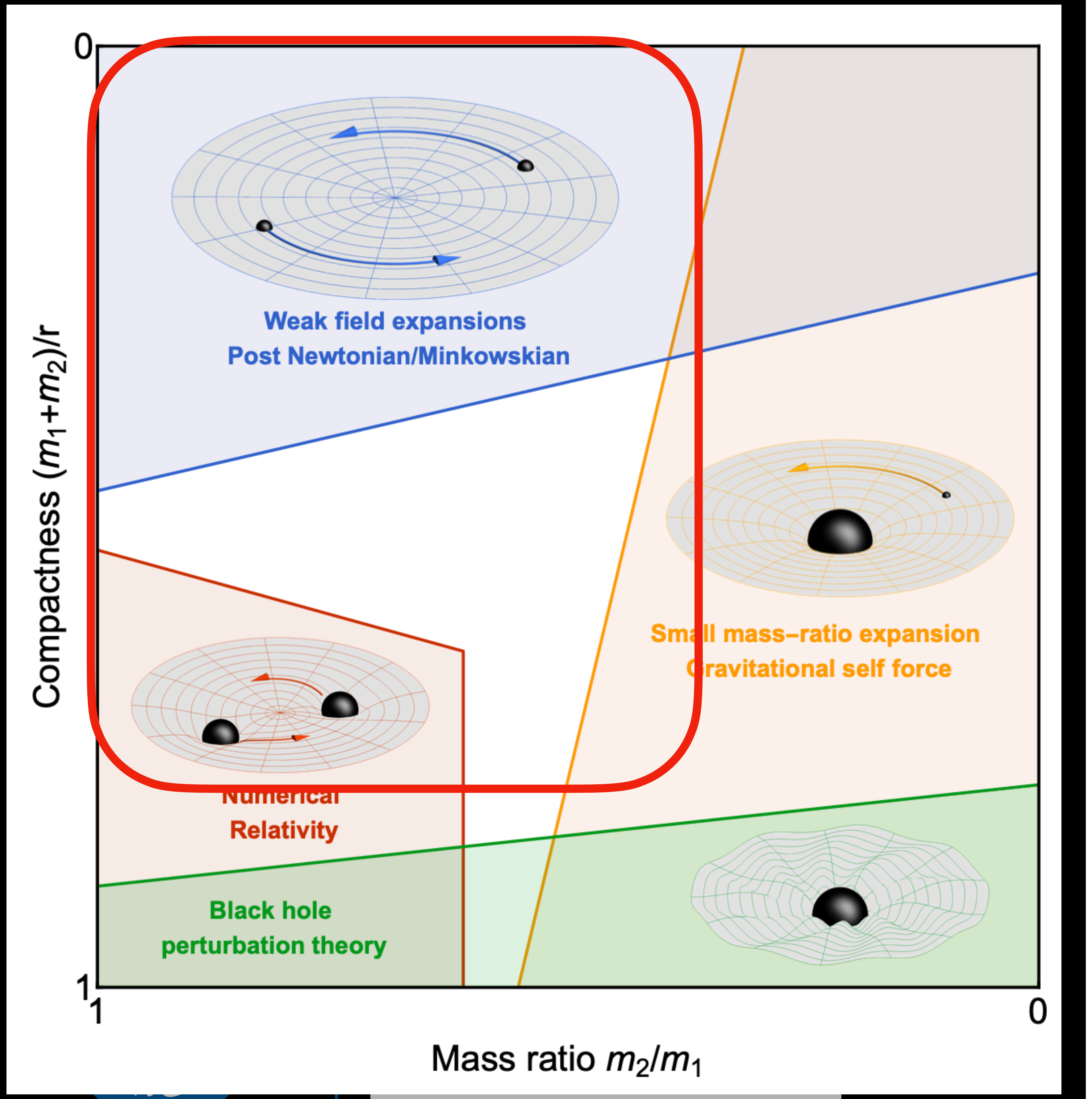
- BHs of 50-100 solar masses and 10^5 cycles
- Eccentricity will infrom formation channels
- Environment / centre of mass acceleration inform pair instability mass gap
- Possible multi-band astrophysics



From LISA Redbook: arXiv:2402.07571, M.Colpi et al.

LISA Science Objectives

OBJ4: Understand the astrophysics of stellar-mass Black Holes



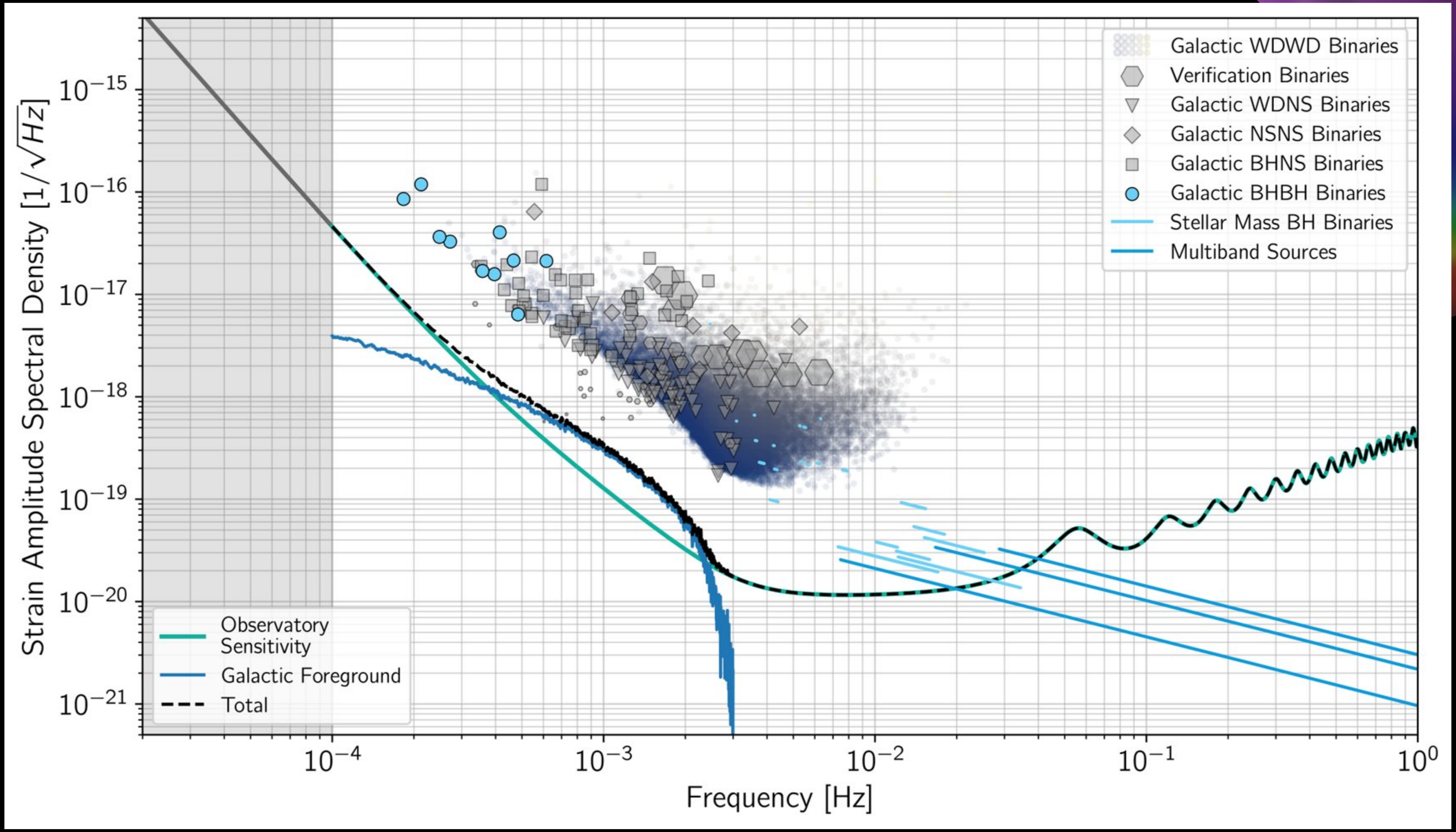


Fig.1 Waveform White Paper, arXiv:2311.01300

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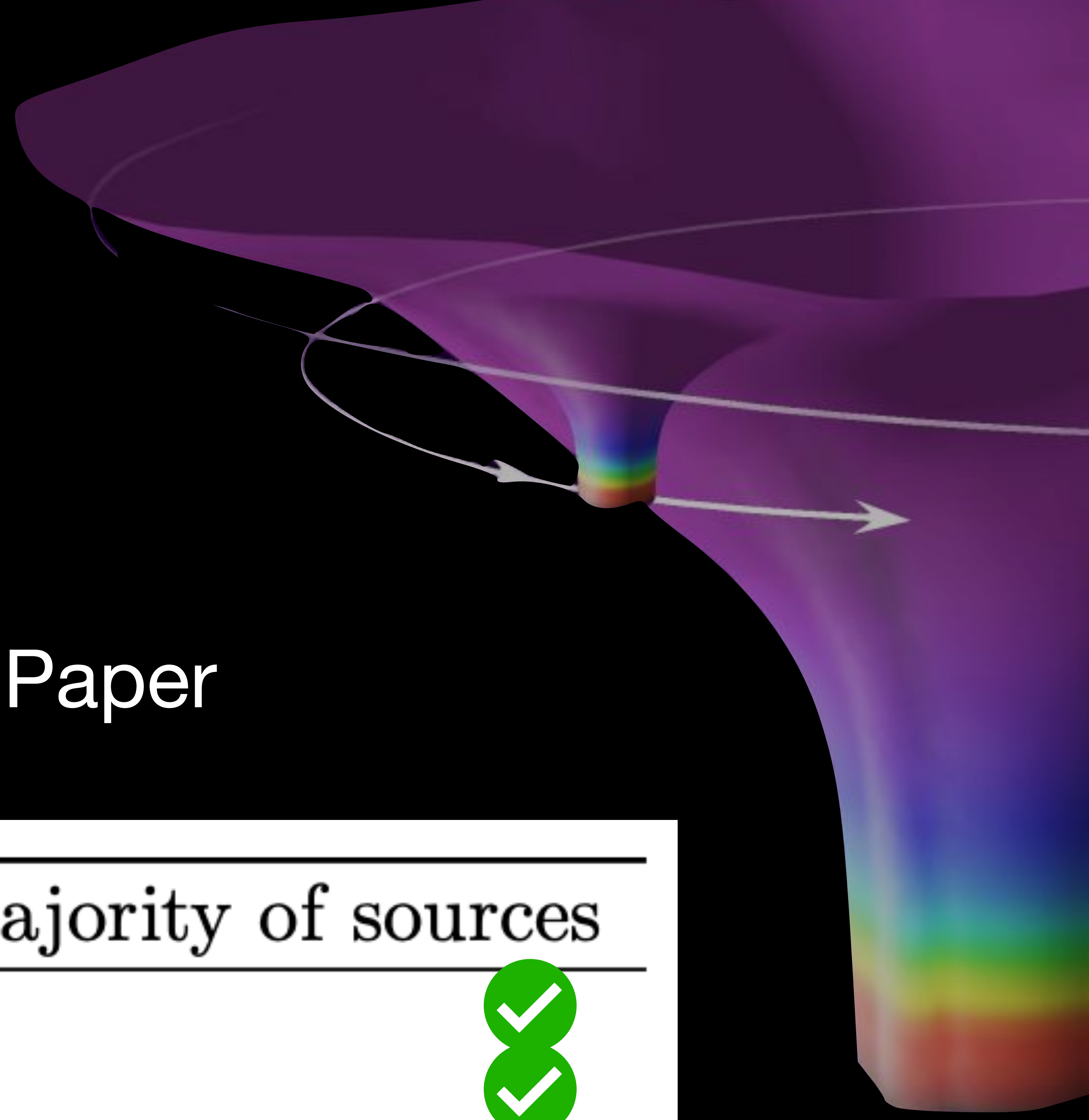
Science Objectives

Status of Waveforms for MBHBs



Stellar Origin BHBs: Table 7 of Waveform White Paper

Parameter	Notation	range for majority of sources
Chirp Mass	$\mathcal{M}$	5–40
Mass Ratio (>1)	q	1–3.3
Dimensionless Spin	$\max\lvert\chi_i\rvert$	0–0.3
Eccentricity entering LISA band	e_{init}	0–1
Eccentricity at last stable orbit	e_{merge}	out of band
Signal to Noise Ratio	SNR	< 50

Generic: Mis-aligned spins **and** eccentric

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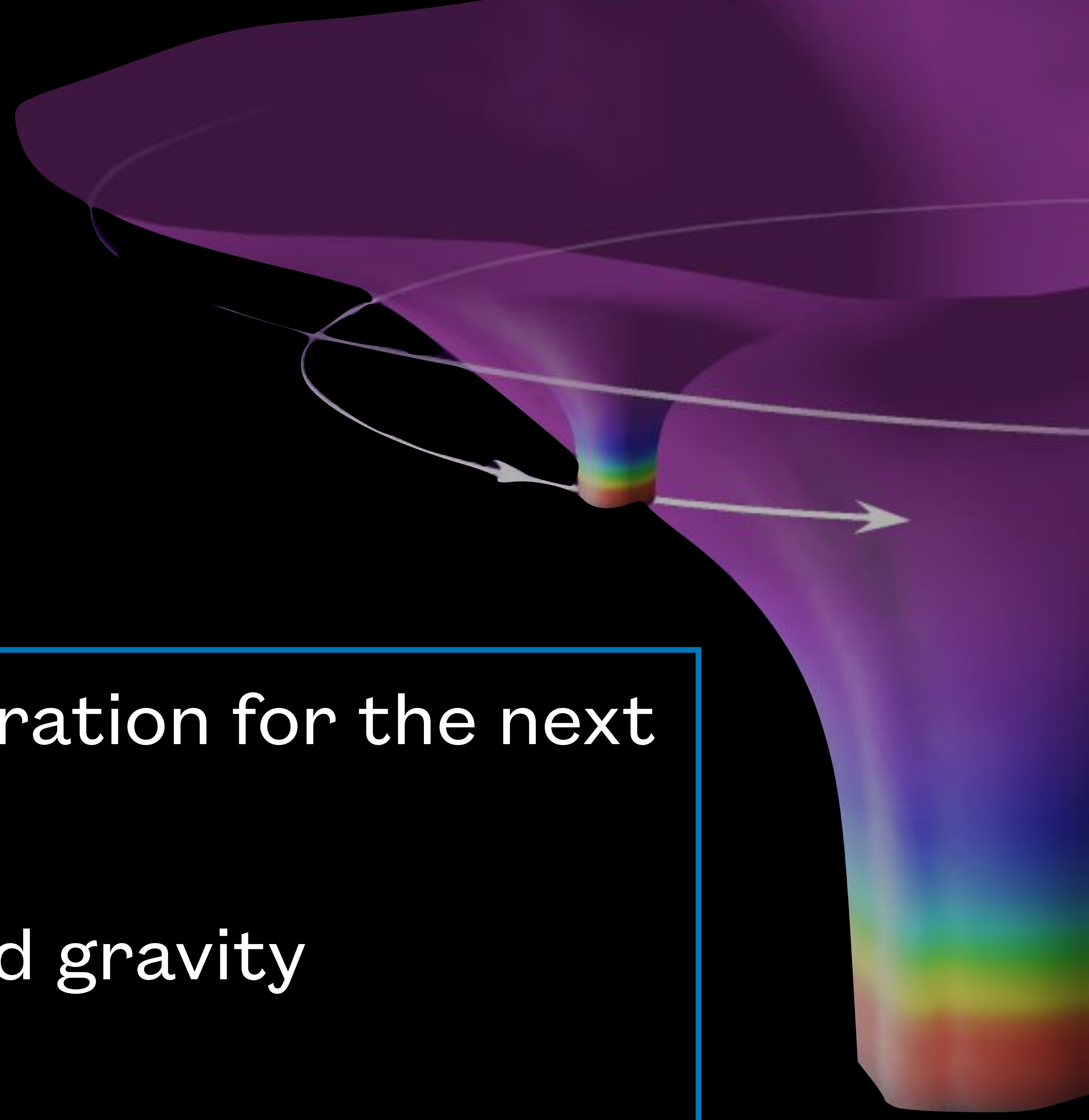
Today's Plan

The Mathematics



- Some recap of key mathematical objects and equations (in preparation for the next lectures)
- How we can derive the propagation equation for GWs in linearised gravity
- Intuition building for the case of circular orbital motion

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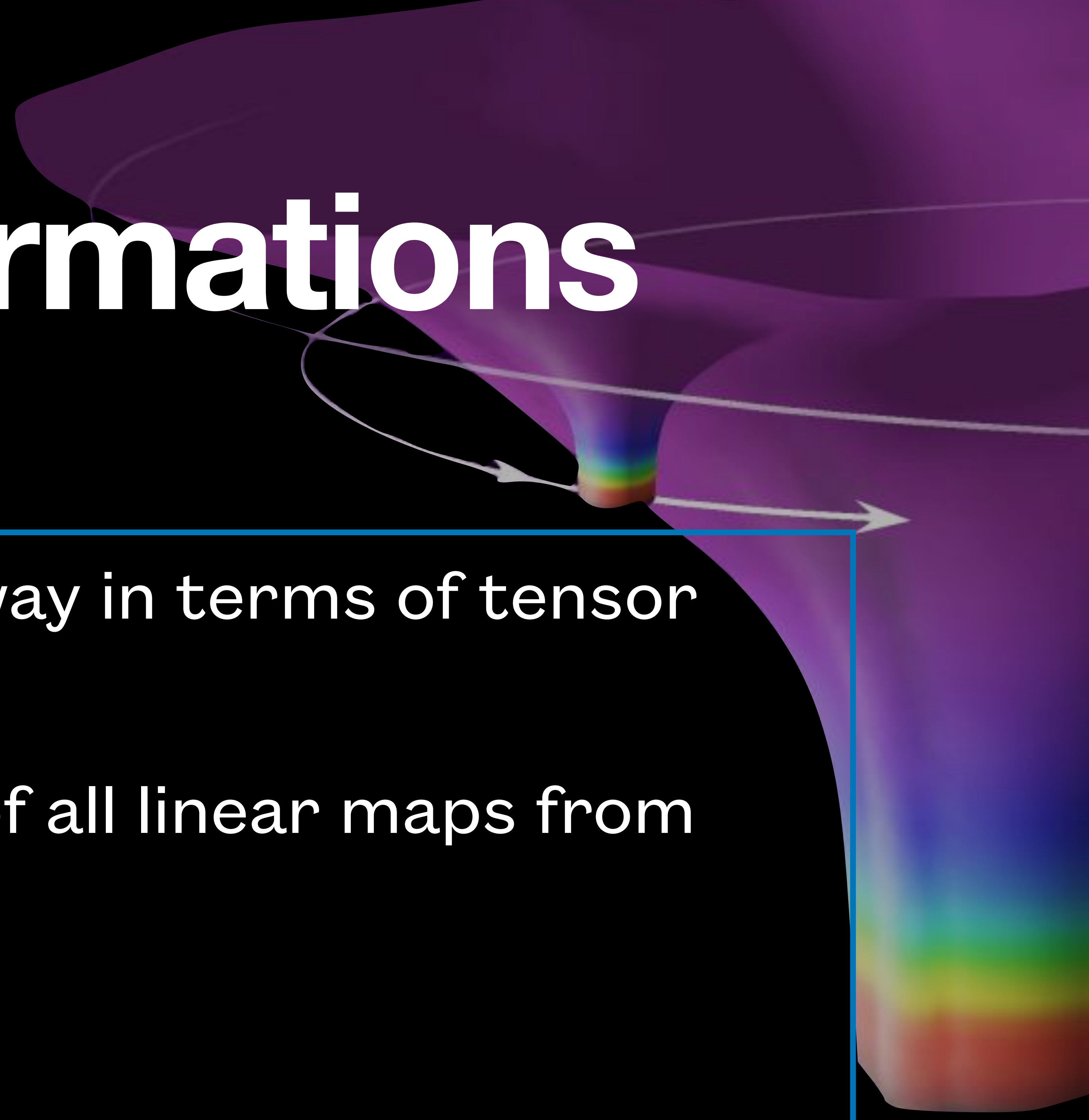
Tensors and Coordinate Transformations



- The laws of physics in GR are formulated in a coordinate-independent way in terms of tensor equations
- Tensors are generalizations of vectors and dual vectors (i.e. the space of all linear maps from vectors to real numbers)
- Multilinear map from a collection of dual vectors and vectors to $\mathbb{R}$

$$T : \underbrace{V^* \times \cdots \times V^*}_{m} \times \underbrace{V \times \cdots \times V}_{n} \to \mathbb{R}$$

- Under a change of coordinates $x^\alpha \to x^{\alpha'}$, tensors transform as

$$T^{\alpha'_1 \ldots \alpha'_m}{}_{\beta'_1 \ldots \beta'_n} = \frac{\partial x^{\alpha'_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x^{\alpha'_m}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial x^{\beta'_1}} \cdots \frac{\partial x^{\beta_n}}{\partial x^{\beta'_n}} T^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}$$

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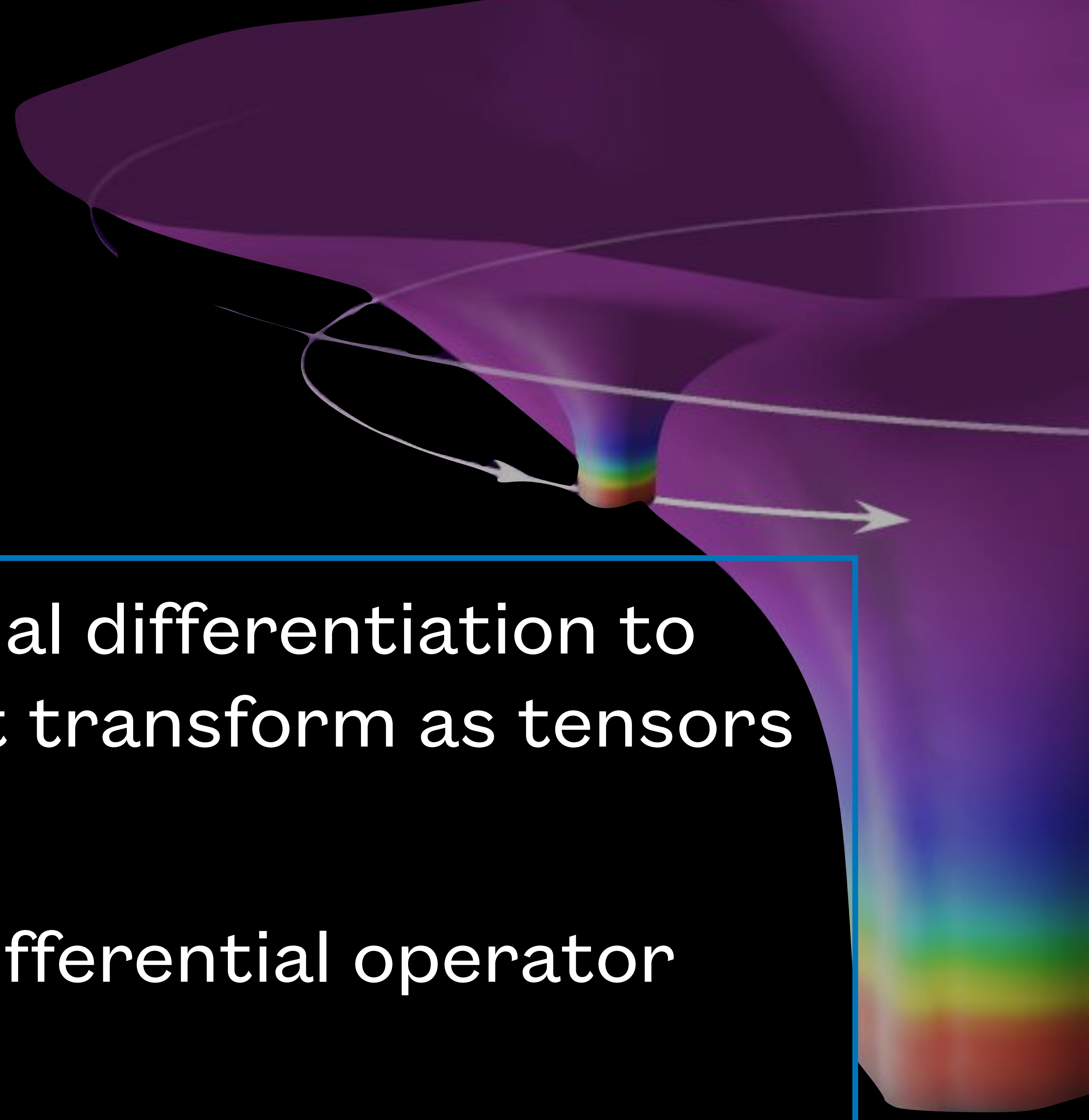

Covariant Derivative

- Intuitively, the covariant derivative ∇_μ extends the concept of partial differentiation to curved spacetime (partial derivatives of tensor components do not transform as tensors in curved spacetime)

- By introducing connection coefficients, we can define a covariant differential operator

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma^{\beta}_{\mu\nu} A_\beta$$

- ∇_μ transforms tensors of (k, m) dimension into one of $(k, m + 1)$

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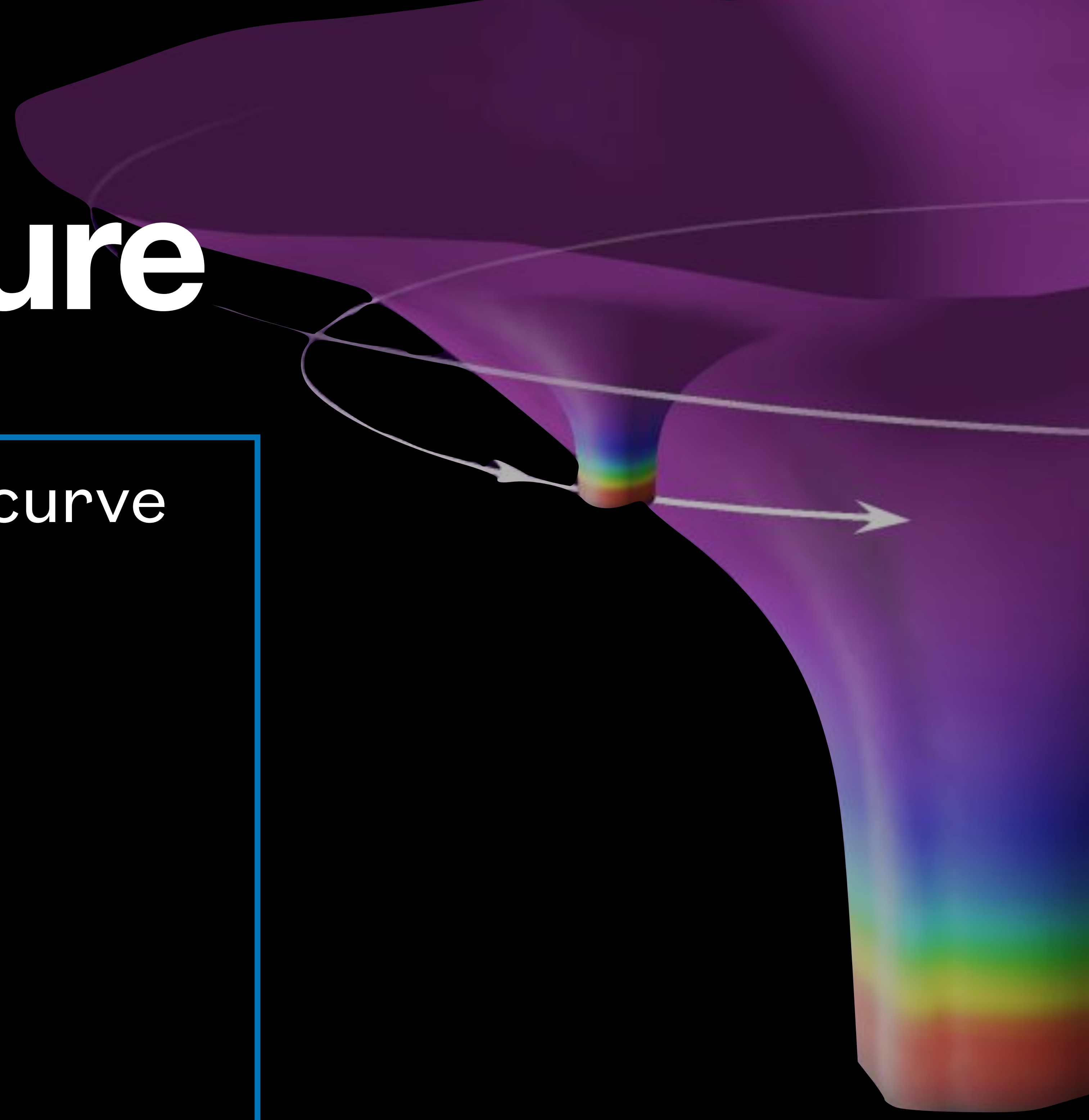
Covariant Derivative and Curvature



- Change in a vector when parallel-transported around a small closed curve
- Curvature can be defined in terms of lack of commutativity of differentiations
- The Riemann tensor precisely encodes this

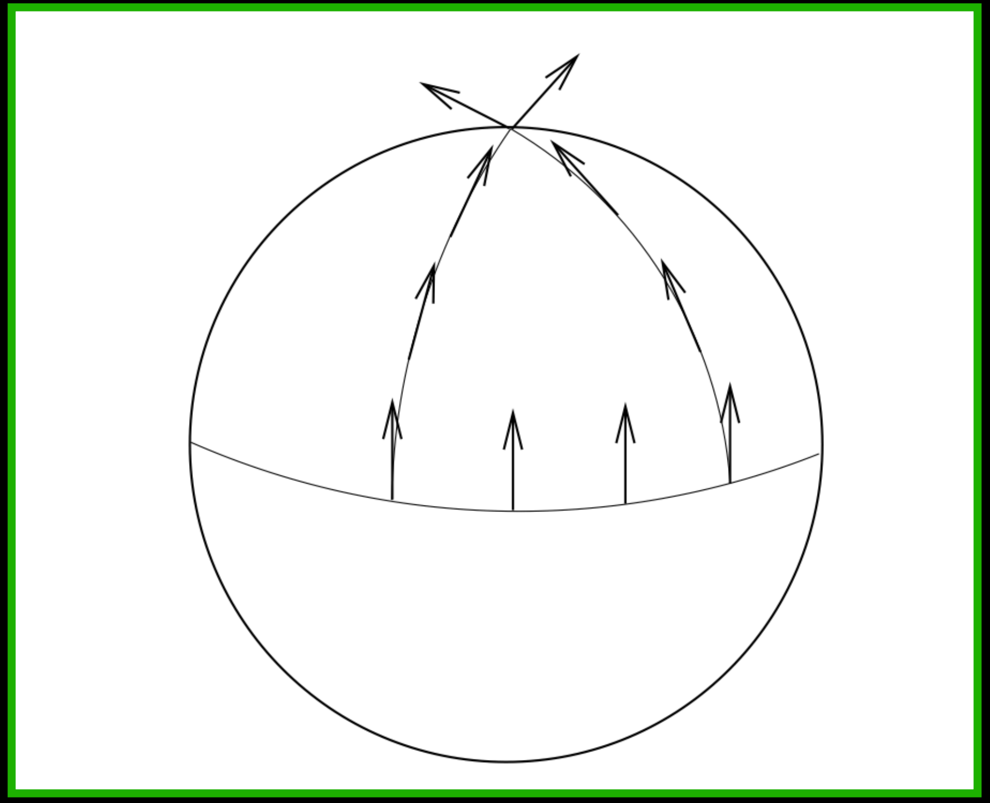
$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) t^\mu = R^\mu{}_{\gamma\alpha\beta} t^\gamma$$

- A vector W^μ is said to be parallel-transported along the curve with tangent vector V^μ if

$$V^\nu \nabla_\nu W^\mu = 0$$

- A geodesic is a curve whose tangent vector u^μ is parallel-transported along itself:

$$u^\nu \nabla_\nu u^\mu = 0$$



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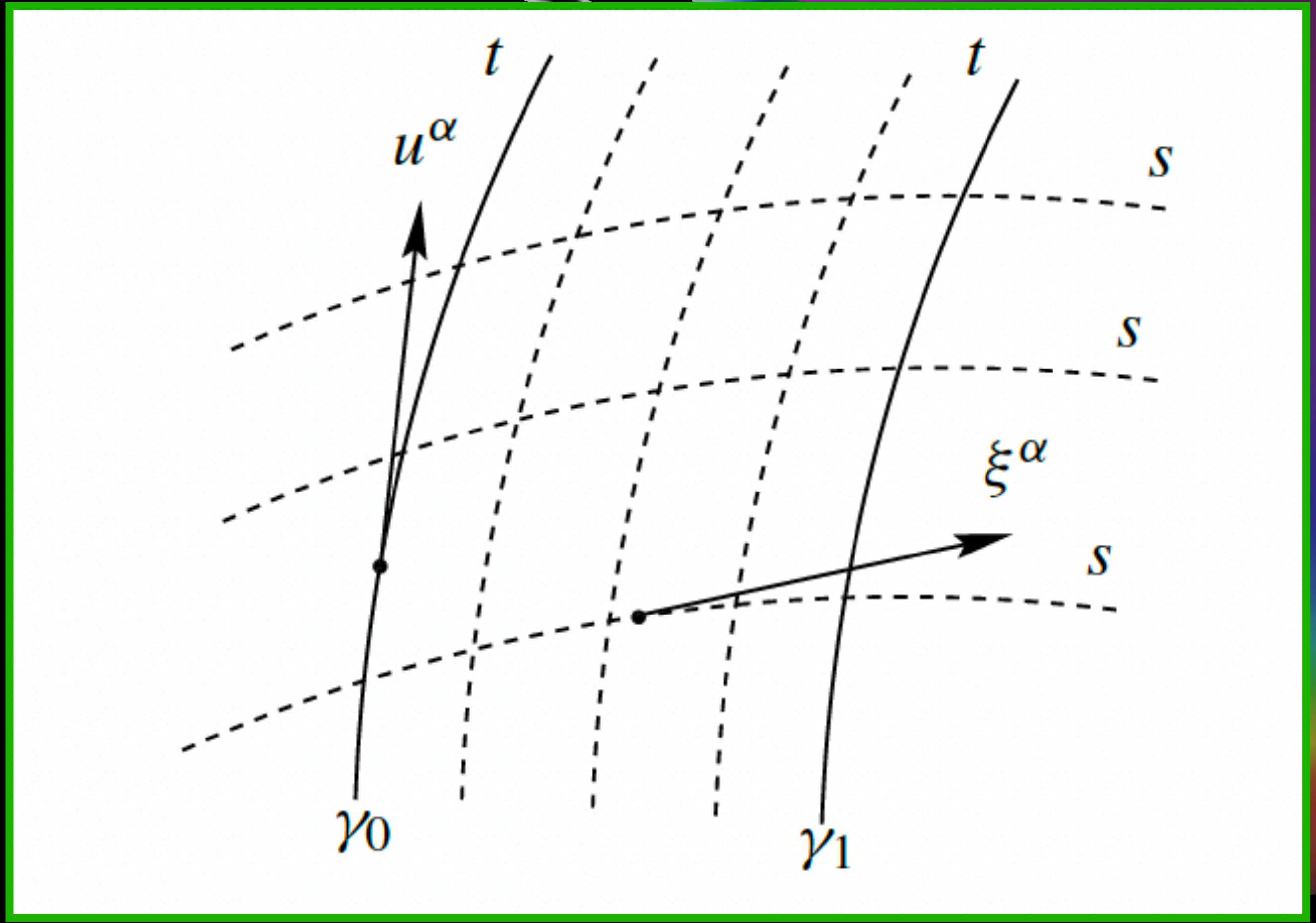
Geodesic Deviation

- Effect of curvature is also encapsulated in the geodesic deviation equation

$$\frac{D^2\xi^\mu}{d\tau^2} = -R^\mu_{\nu\rho\sigma}\xi^\rho\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$$

- Curvature produces a relative acceleration between two neighbouring geodesics, e.g. neighbouring observers in free fall
- Inhomogeneities in the gravitational field create some tidal forces, which are measured by the Riemann tensor
- The same concept underlies the detection of GWs: when a GW passes through a ring of test masses, produces the characteristic “stretch and squeeze” pattern.
- Geodesic deviation provides both geometric and experimental insight into gravity as curvature



Poisson, A Relativist’s toolkit

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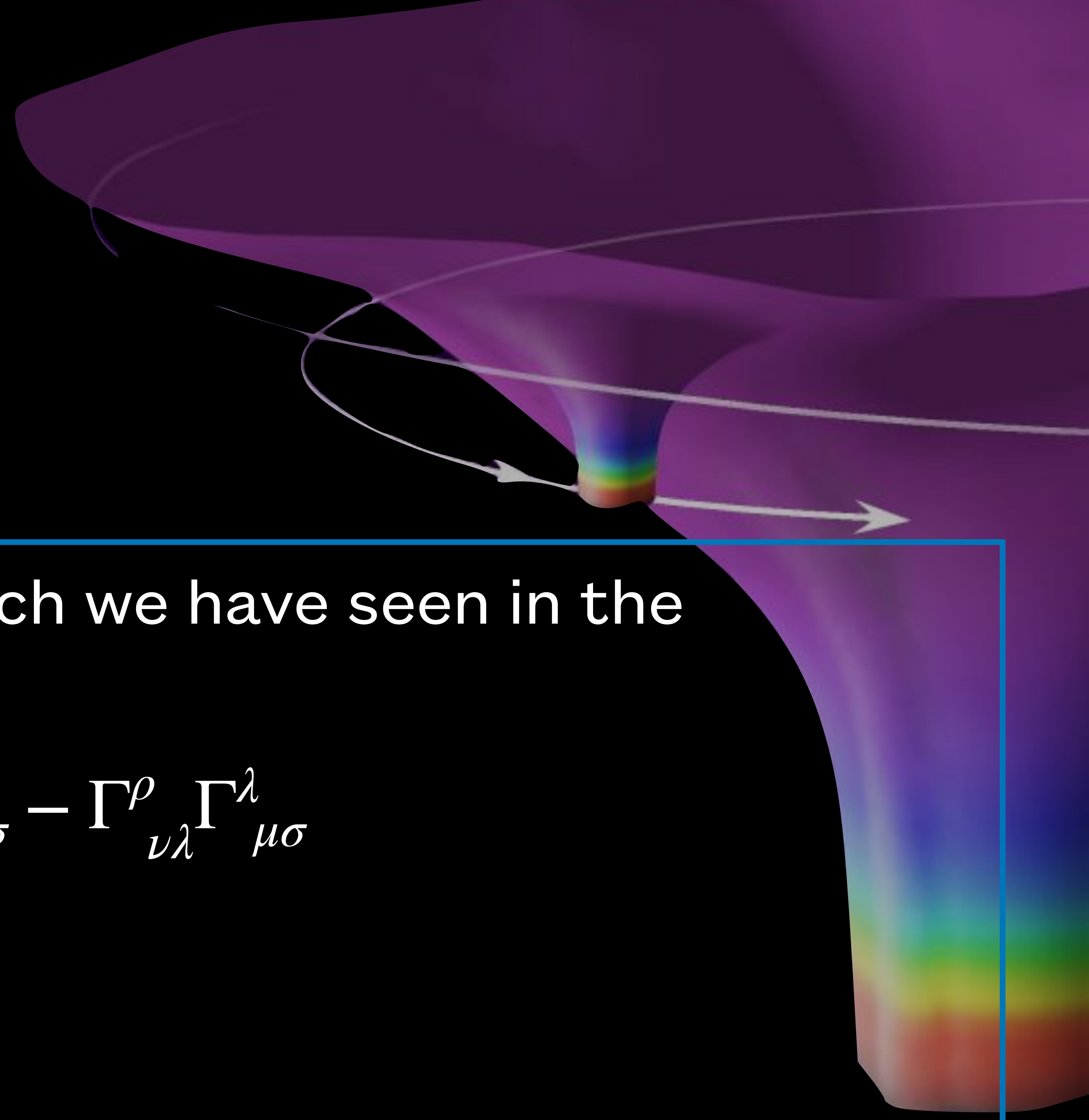
The Riemann Tensor



- The Riemann tensor can be written explicitly in terms Christoffel symbols, which we have seen in the definition of covariant derivative

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}{}_{\mu\sigma} + \Gamma^{\rho}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\sigma}$$

- Contractions of the Riemann tensor give us the Ricci tensor

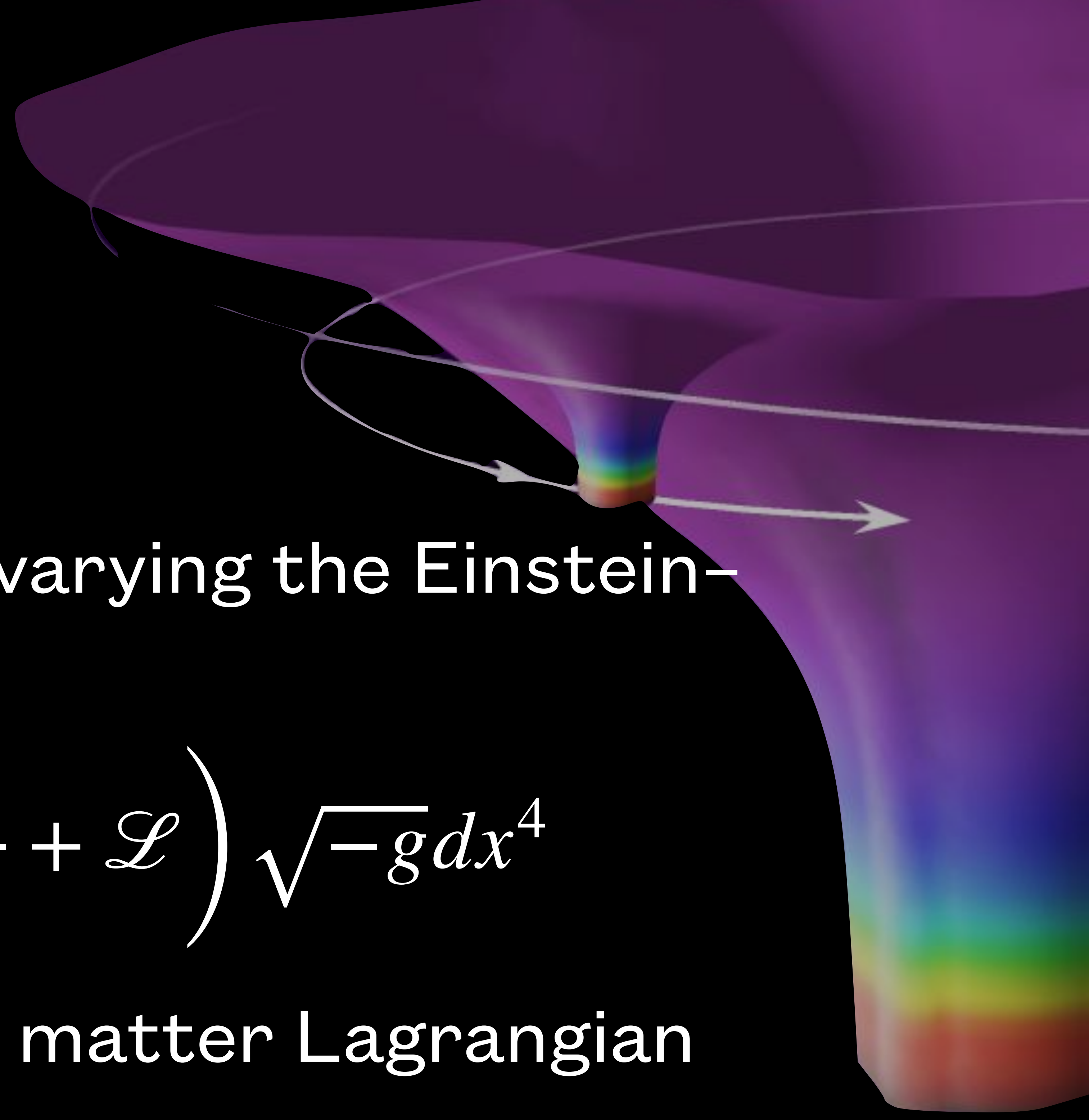
$$R_{\beta\delta} = R^{\alpha}_{\beta\alpha\delta}$$

and Ricci scalar

$$R = R^{\alpha}_{\alpha}$$

which are the tensors appearing in the Einstein's field equations

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$



Einstein field equations

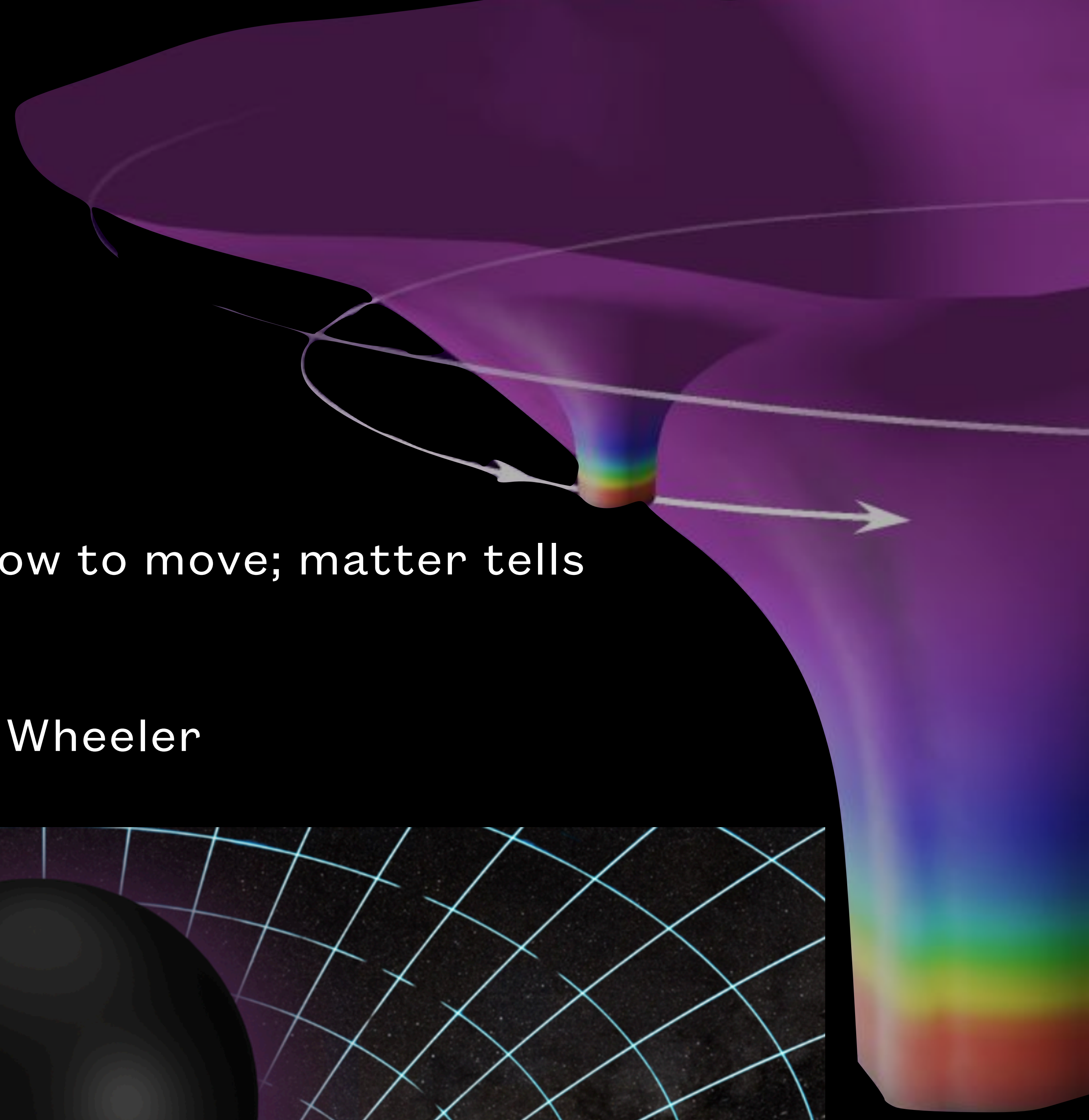
- Einstein field equations can be derived from a variational principle, varying the Einstein–Hilbert action, plus a term including the matter contribution

$$S = S_H + S_M = \alpha \int \left(\frac{R}{16\pi} + \mathscr{L} \right) \sqrt{-g}\, dx^4$$

with respect to $g_{\alpha\beta}$, where g is the metric determinant and $\mathscr{L}$ is the matter Lagrangian density, which is related to to the stress energy tensor via

$$T_{\alpha\beta} = -2 \frac{\partial \mathscr{L}}{\partial g^{\alpha\beta}} + \mathscr{L} g_{\alpha\beta}$$

Einstein field equations

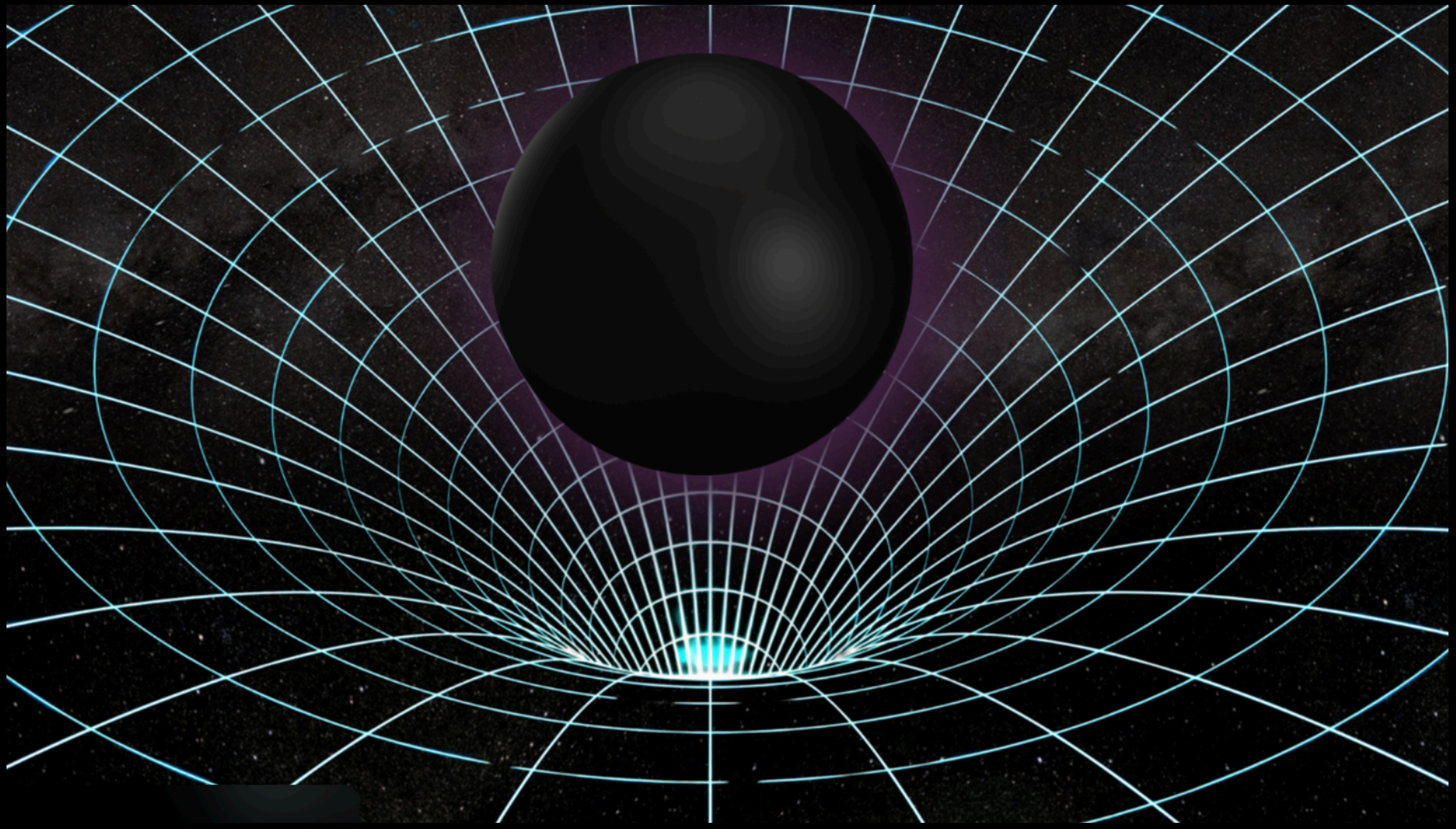


Spacetime tells matter how to move; matter tells spacetime how to curve.

J. Wheeler

The result of this calculation gives us the field equations relating to spacetime geometry to the matter distribution

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$



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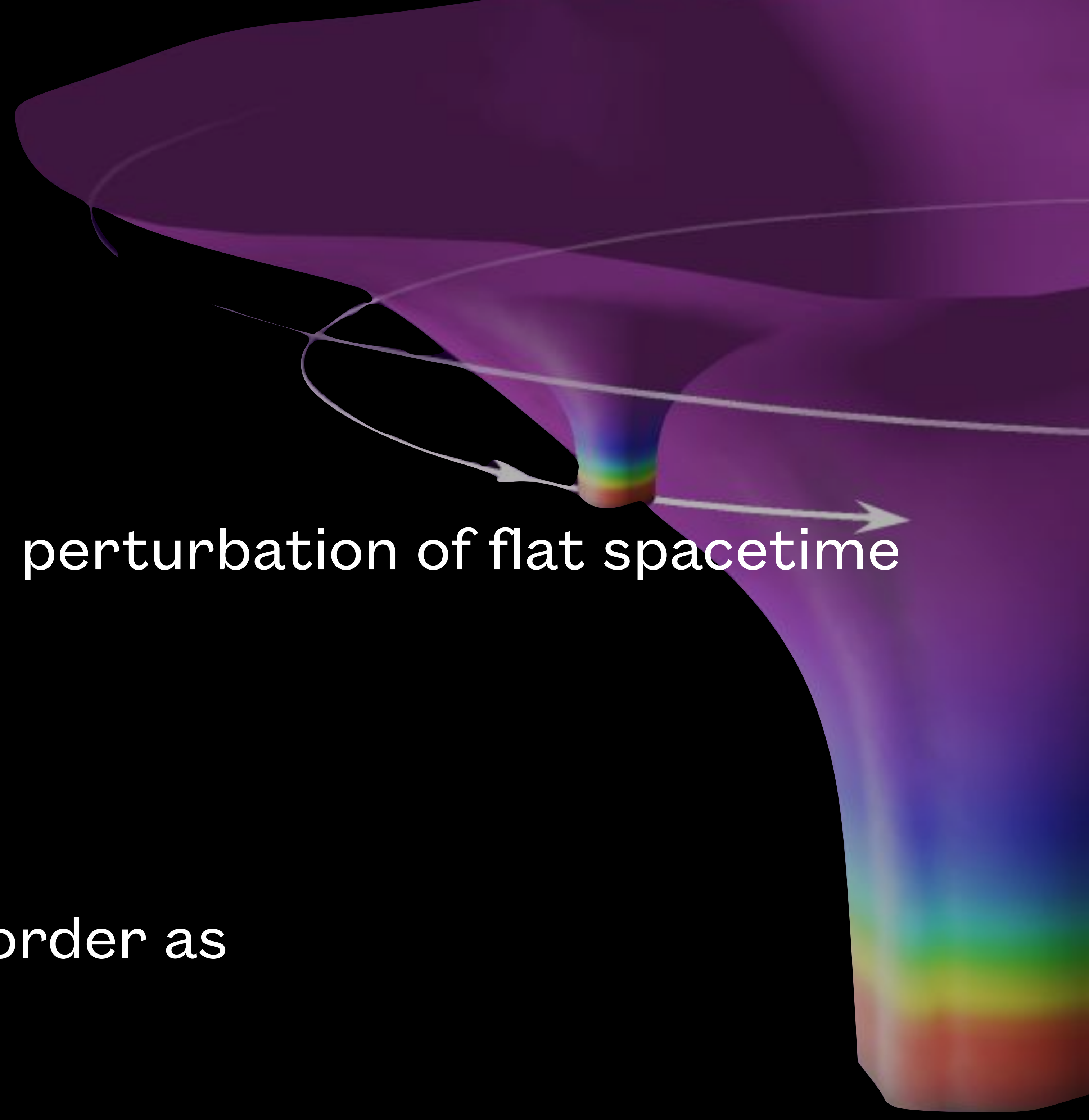
Linearised Gravity

Metric Perturbations



- We will now work in the weak-field approximation, and expand the metric as a small perturbation of flat spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $h_{\mu\nu}$ is in some sense "small".

- Under a coordinate transformation $x^{\mu'} = x^{\mu} + \xi^{\mu}$, the metric transforms at linear order as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Remember: $g'_{\mu'\nu'} = \dfrac{\partial x^\mu}{\partial x^{\mu'}} \dfrac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$

Plugging the linearised metric into the definition of the Riemann tensor, one gets

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}\right)$$

and hence the Ricci tensor (scalar) via contraction with the inverse metric $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

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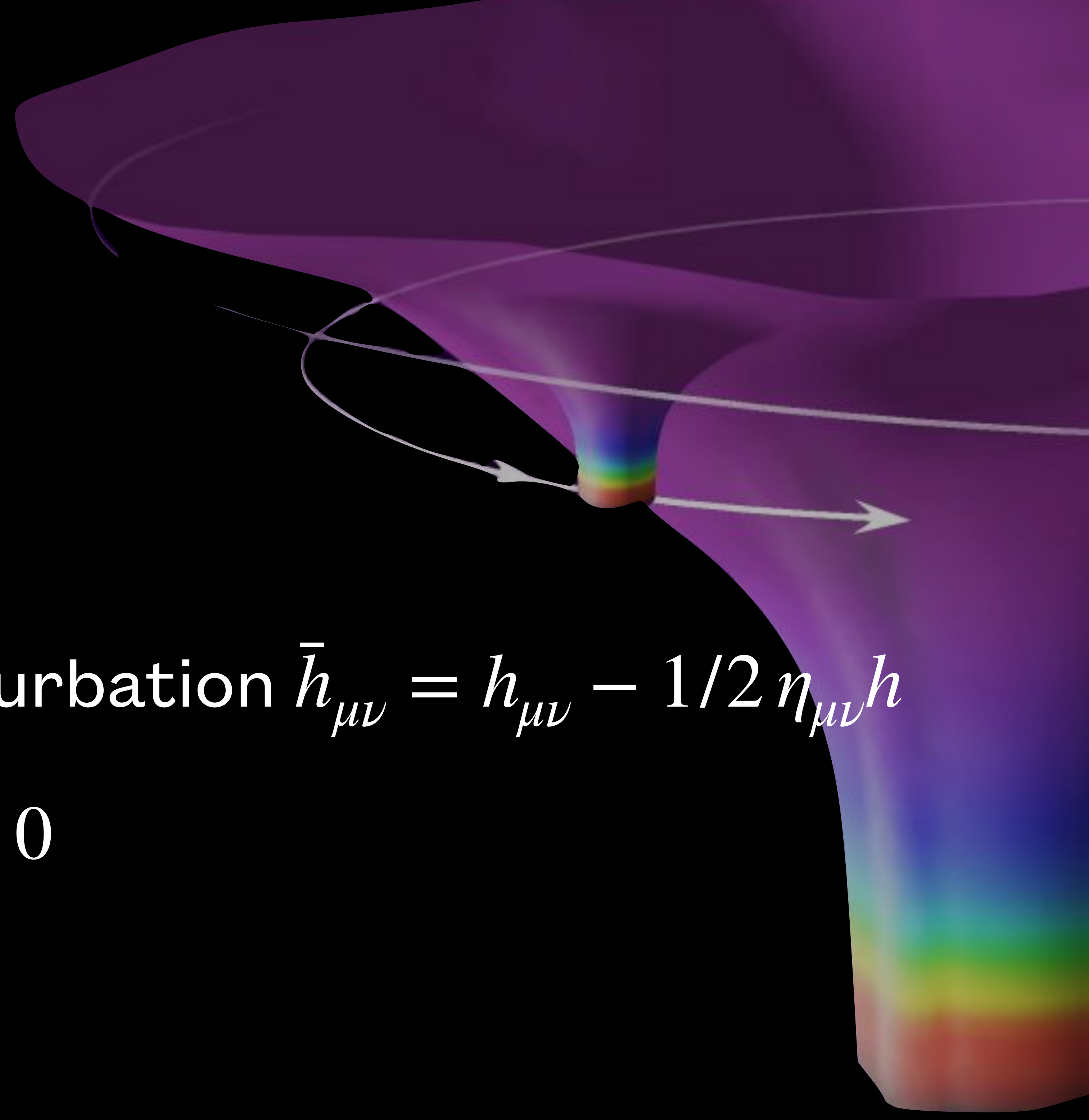
Linearised Gravity

The sourced wave equation



Then we can

- 1) re-express quantities in terms of the trace-reversed metric perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - 1/2\,\eta_{\mu\nu}h$
- 2) use gauge freedom to impose the Lorenz gauge condition $\partial^{\mu}\bar{h}_{\mu\nu} = 0$

… arriving at a wave equation for the trace-reversed metric perturbation

$$\partial_{\rho}\partial^{\rho}\bar{h}_{\mu\nu} = \Box\,\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

- This equation is particularly nice because it is a hyperbolic PDE, hence represents a well-posed problem: solutions exist, are unique and depend smoothly on initial data

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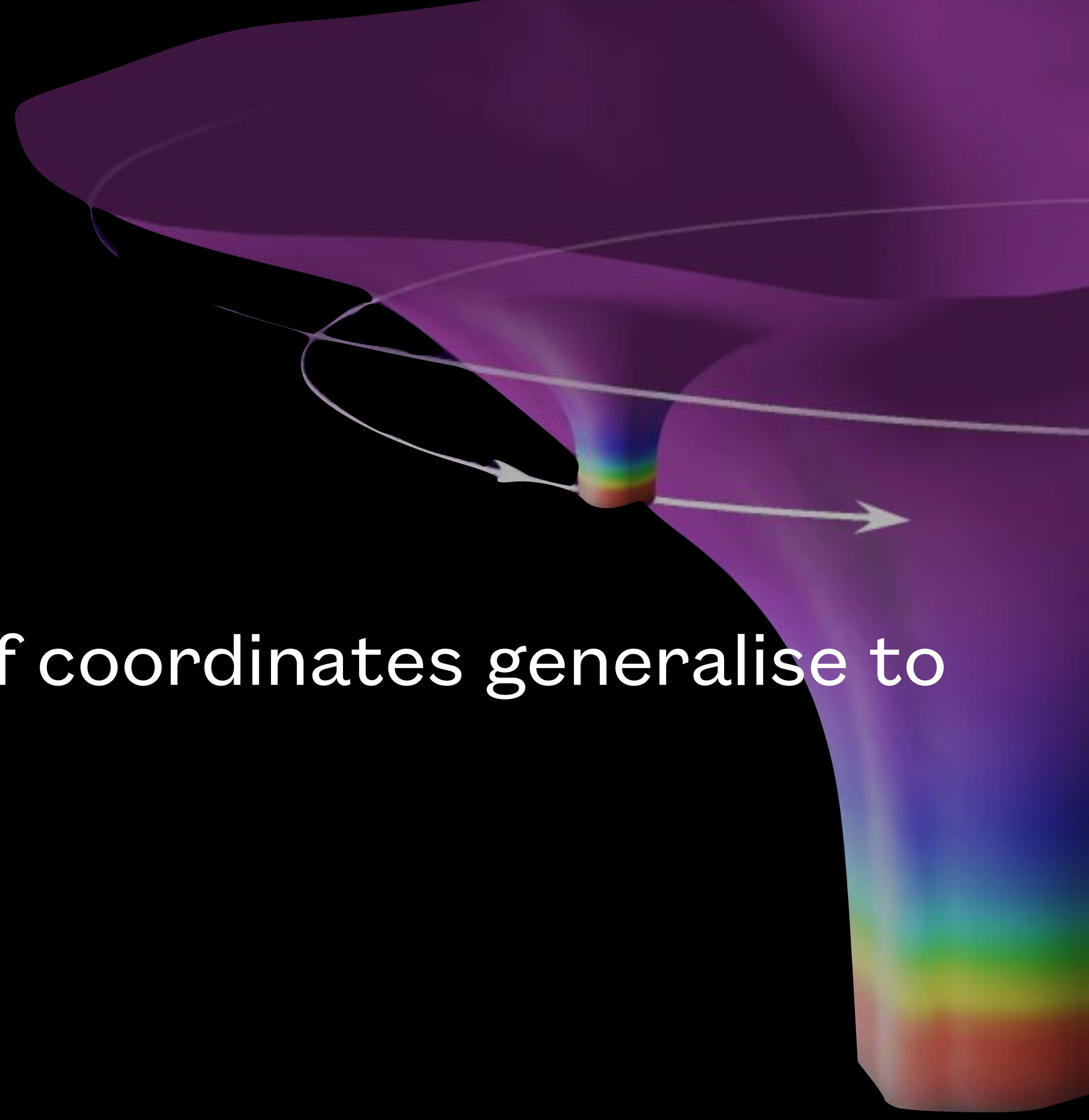
Linearised Gravity

On a curved background



- On a **curved background**, metric transformations under changes of coordinates generalise to

$$h_{\mu\nu} \to h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$$

$$g'_{\mu\nu}(x') = g_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi^\alpha \, g_{\alpha\nu} - \partial_\nu \xi^\beta \, g_{\mu\beta} + \mathcal{O}(\xi h, \xi^2)$$

but also $g'_{\mu\nu}(x') = g'_{\mu\nu}(x) + \xi^\lambda \partial_\lambda g_{\mu\nu}$

- Wave equation becomes

$$\nabla^\sigma \nabla_\sigma h_{\mu\nu} + 2R^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = \Box \, h_{\mu\nu} + 2R^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

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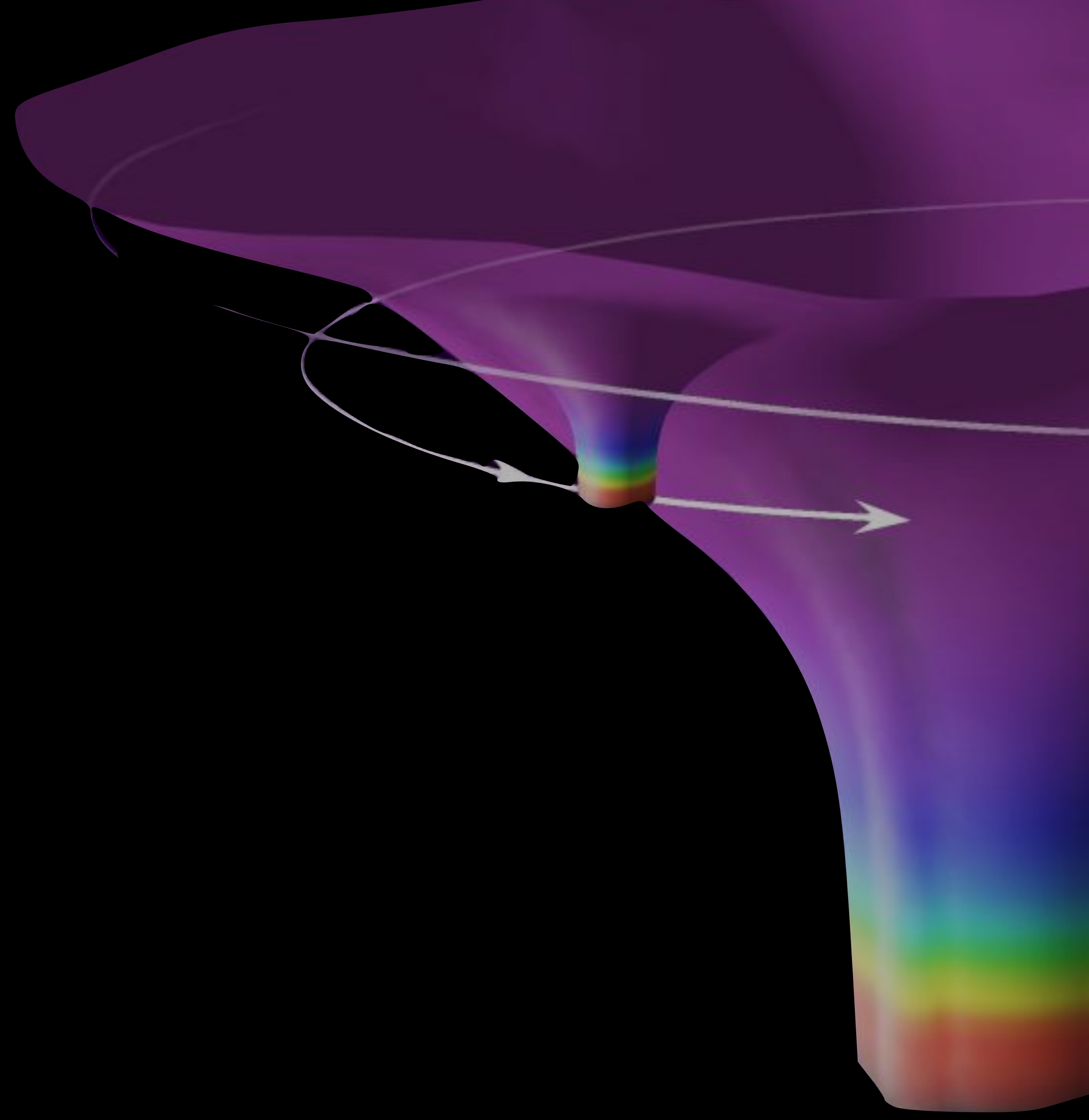
Linearised Gravity

Gravitational Waves



$$\Box\,\bar{h}_{\mu\nu} = (-c^{-2}\partial_t^2 + \nabla^2)\bar{h}_{\mu\nu}$$

implies GWs travel at the speed of light.

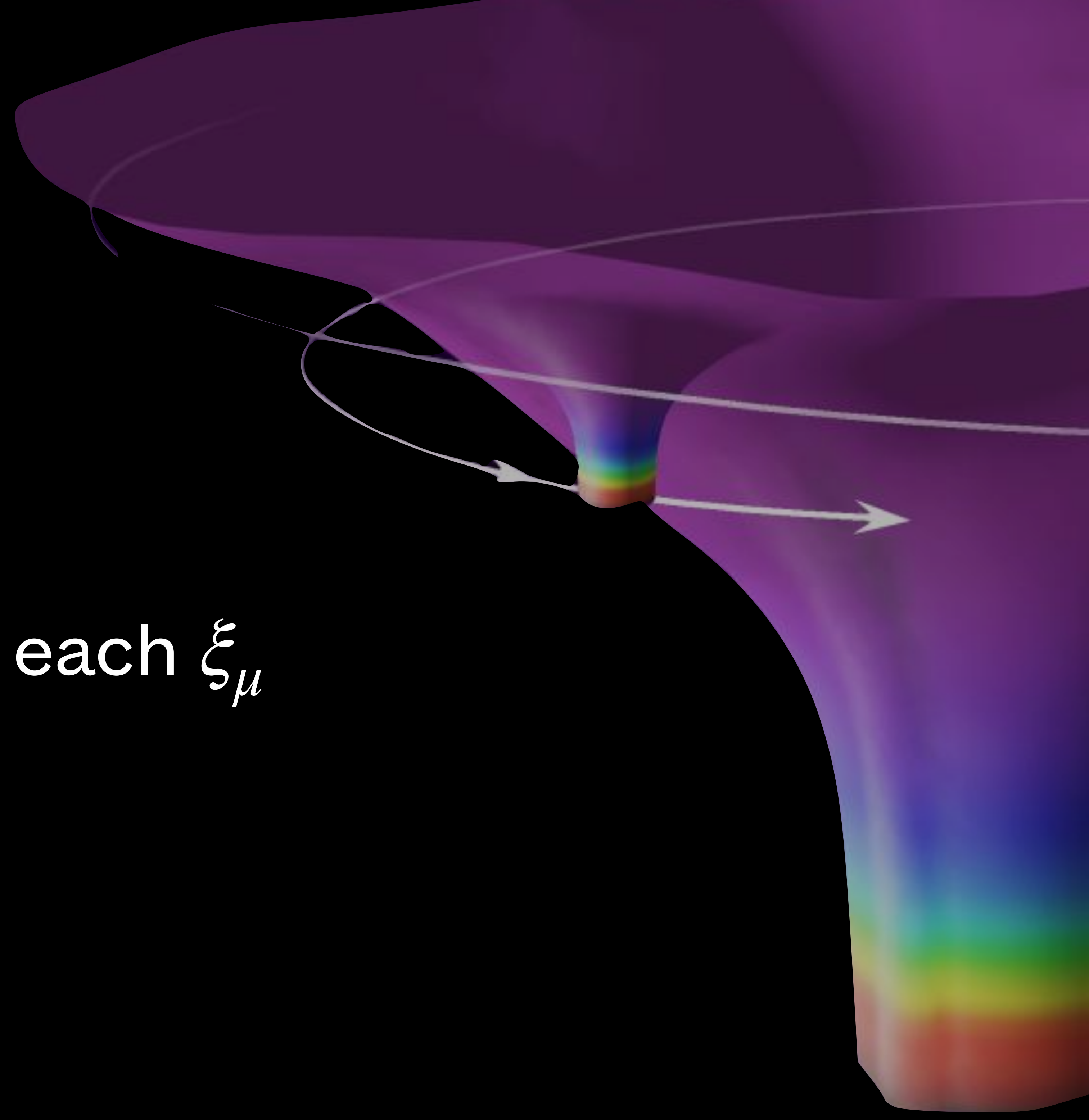
Coincident EM-GW observations are a way to test this!

- Under coordinate changes we have $\partial^{\nu}\bar{h}_{\mu\nu} \to \partial^{\nu}\bar{h}_{\mu\nu} + \Box\xi_{\nu}$, since

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\partial_{\rho}\xi^{\rho}$$

- Then we can always perform a further coordinate transformation within LG if $\Box\,\xi_{\nu} = 0$

Linearised Gravity
Gravitational Waves



A further coordinate transformation eliminates 4 extra d.o.f., one for each ξ_μ

We can then choose to work in a transverse-traceless gauge where

$$\bar{h} = 0,\ \bar{h}^{0i} = h^{0i} = 0$$

Because of Lorenz gauge condition $\partial_\mu h^{\mu 0} = 0 \rightarrow = h^{00} = \mathrm{const}$

We will see that at leading order this term is related to the Newtonian potential of the source

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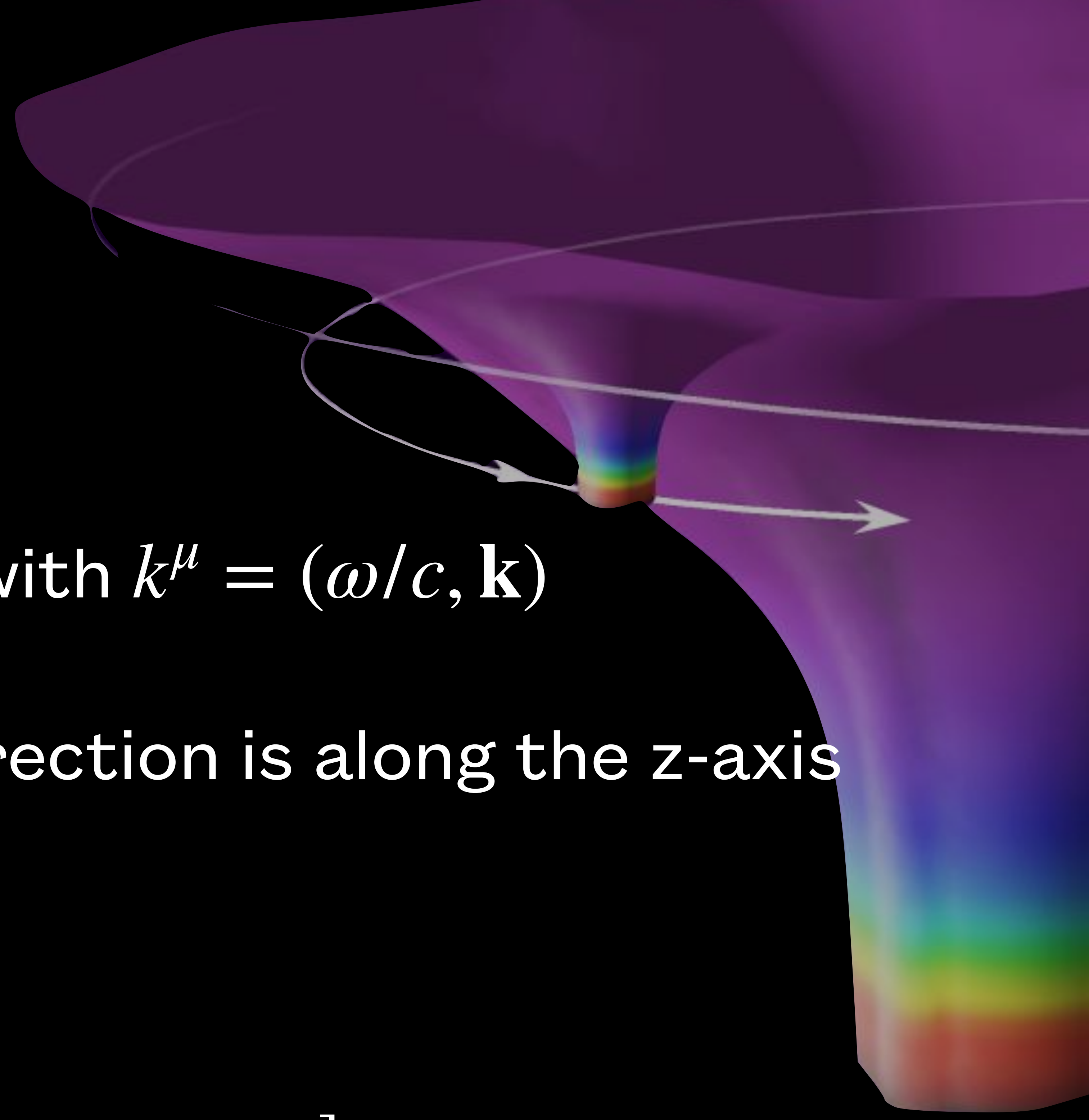
Linearised Gravity

Gravitational Waves



The wave equation admits plane-wave solutions $h_{ij}^{TT} = Re \int d^3k \ \epsilon_{ij} e^{i(k_\mu x^\mu)}$, with $k^\mu = (\omega/c, \mathbf{k})$

- Using Lorenz gauge condition $\partial^i h_{ij} = 0$ and assuming the propagation direction is along the z-axis

$$h_{ij}^{\mathrm{TT}}(t, z) = \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix}_{ij} \cos\left[\omega\left(t - z/c\right)\right]$$

- In GR there exist only two polarizations, that are transverse to the direction of propagation

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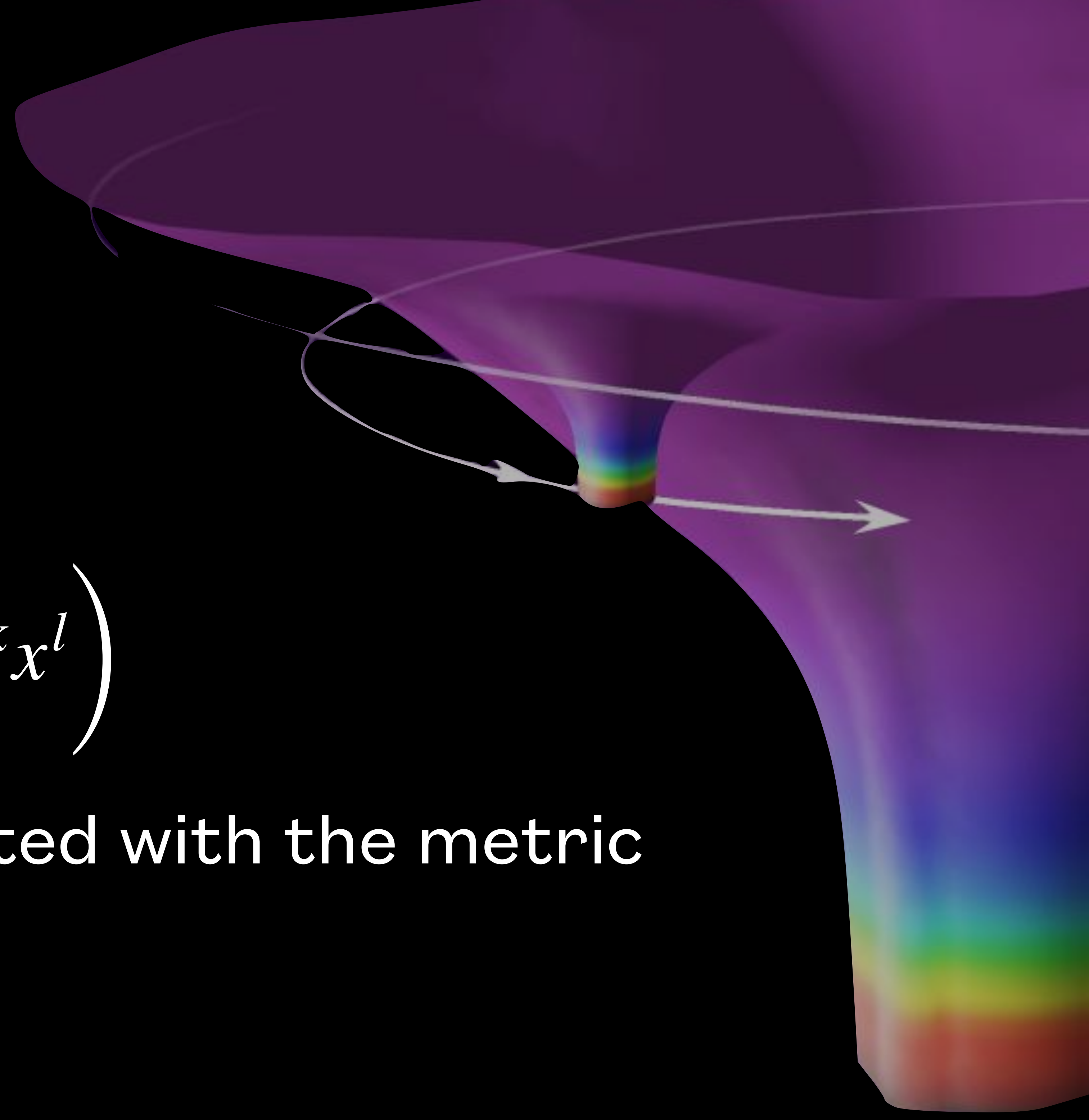
Effect of GWs on test particles



- In a free-falling detector frame
$$ds^2 \sim -c^2 dt^2 \left(1 + R_{0i0j} x^i x^j\right) - 2c\, dt dx^i \left(\frac{2}{3} R_{0jik} x^j x^k\right) + dx^i dx^j \left(\delta_{ij} - \frac{1}{3} R_{ikjl} x^k x^l\right)$$
- Corrections to the flat metric are $O(x^2/L^2)$, where L is the scale associated with the metric perturbation
- Assuming $\frac{dx^i}{d\tau} \ll c\frac{dt}{d\tau}$, we can write the geodesic deviation equation as
- $$\frac{d^2\xi^i}{d\tau} \approx \frac{d^2\xi^i}{dt} := \ddot{\xi}^i = -c^2 R^i_{0j0}\xi^j \left(\frac{dt}{d\tau}\right)^2 \approx -c^2 R^i_{0j0}\xi^j = \frac{1}{2}\ddot{h}^{ij}_{TT}\xi_j$$

since $R_{i0j0} = \frac{1}{2}\left(\partial_0\partial_0 h_{ij} - \partial_0\partial_j h_{i0} - \partial_i\partial_0 h_{0j} + \partial_i\partial_j h_{00}\right)$

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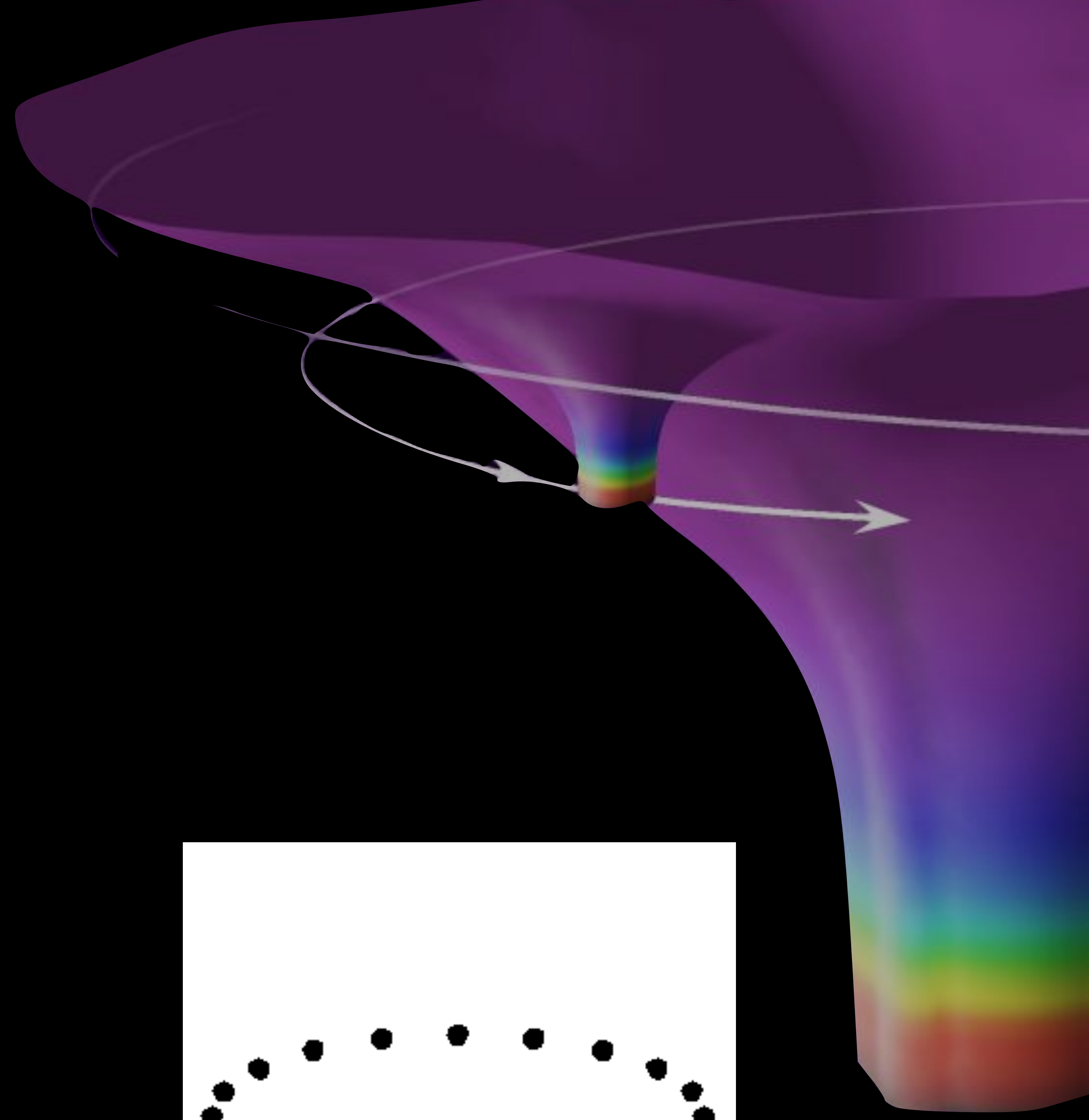
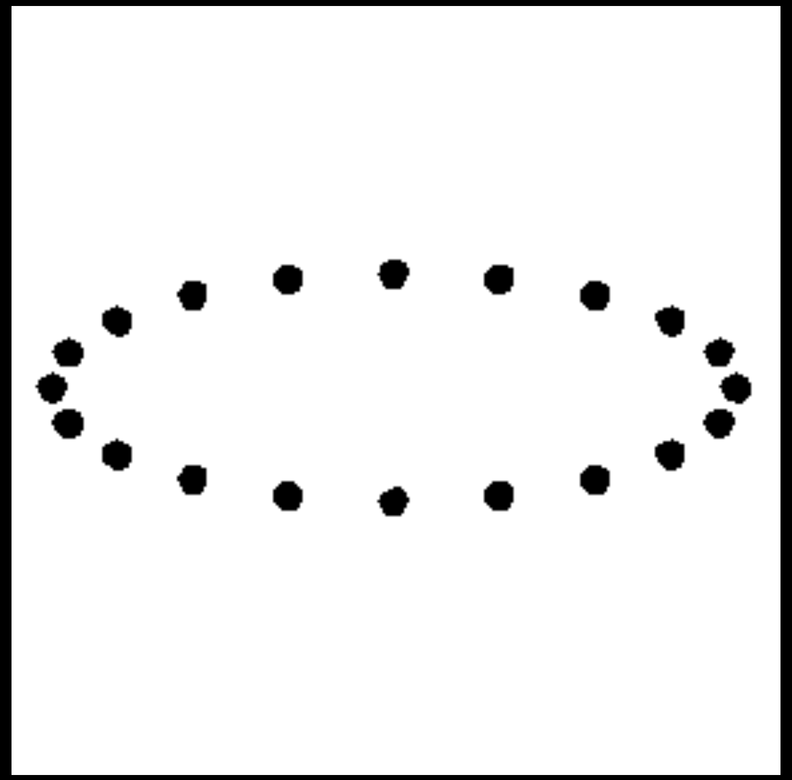
Effect of GWs on test particles



- $\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{ij}_{TT}\xi_j$
- Derive time-dependence of a small displacement around the unperturbed positions $\xi^i = (x_0 + \delta x(t), y_0 + \delta y(t), 0)$
- Assume, e.g. a linearly polarised wave $h^{\mathrm{TT}}_{ab}(t) = h_+ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \sin(\omega t)$

$$\delta x(t) = \frac{h_+}{2} x_0 \sin \omega t, \quad \delta y(t) = -\frac{h_+}{2} y_0 \sin \omega t$$



So particles describe ellipses with time-dependent semi-major(minor) axes

$$\frac{x(t)^2}{R^2\left(1 + h_+ \sin \omega t\right)} + \frac{y(t)^2}{R^2\left(1 - h_+ \sin \omega t\right)} = 1$$

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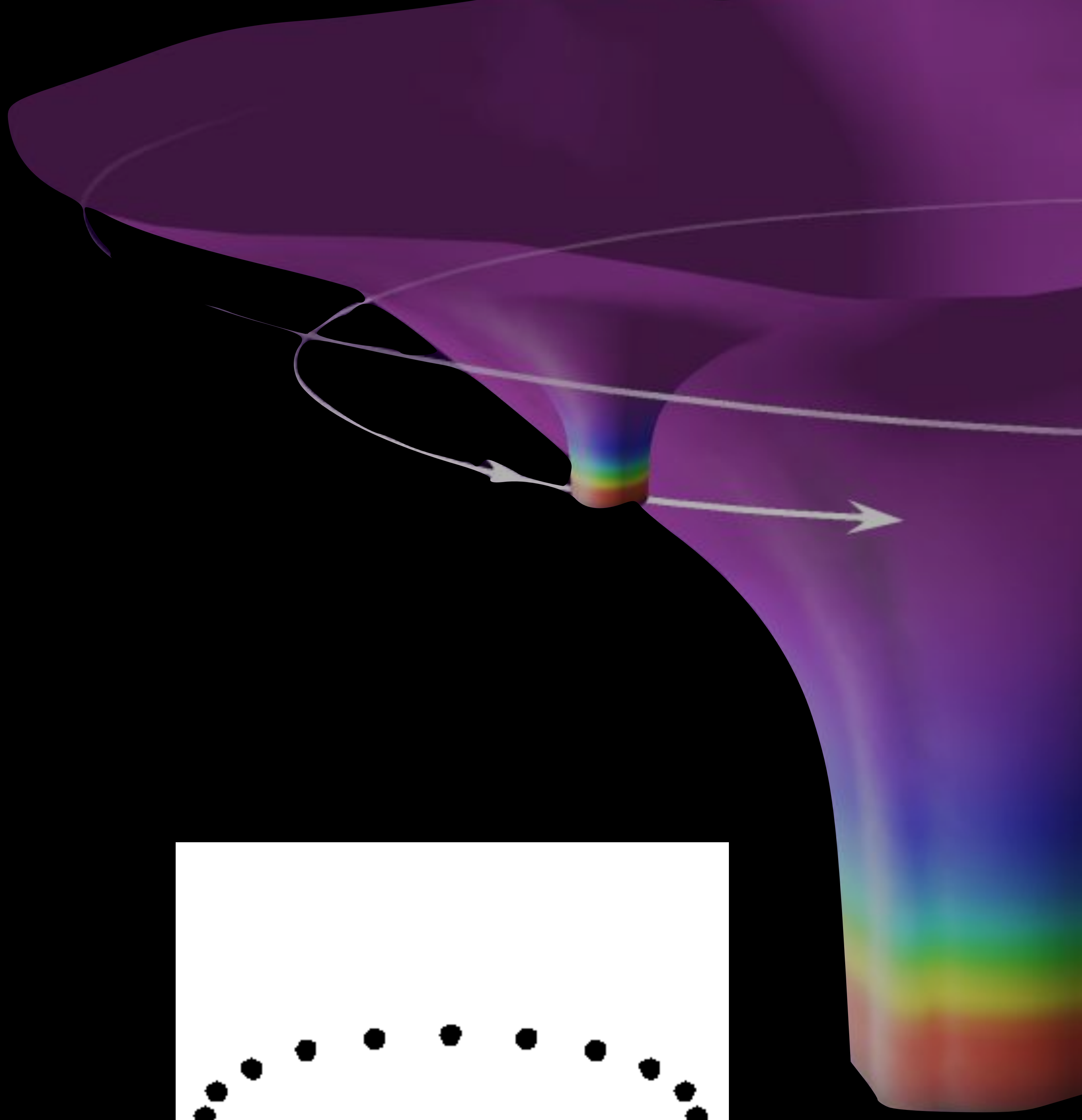
Effect of GWs on test particles



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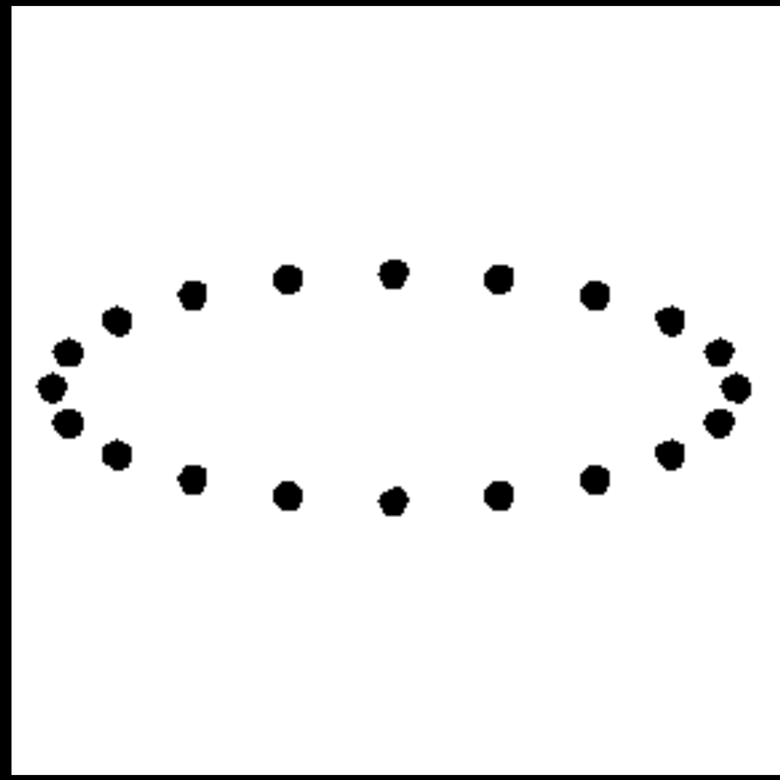

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Extra GW polarizations

Generic metric theories of gravity admit up to 6 polarisations, some of which longitudinal

- 2 tensorial (TT)
- 2 vectorial (LL)
- 2 scalars (TL)

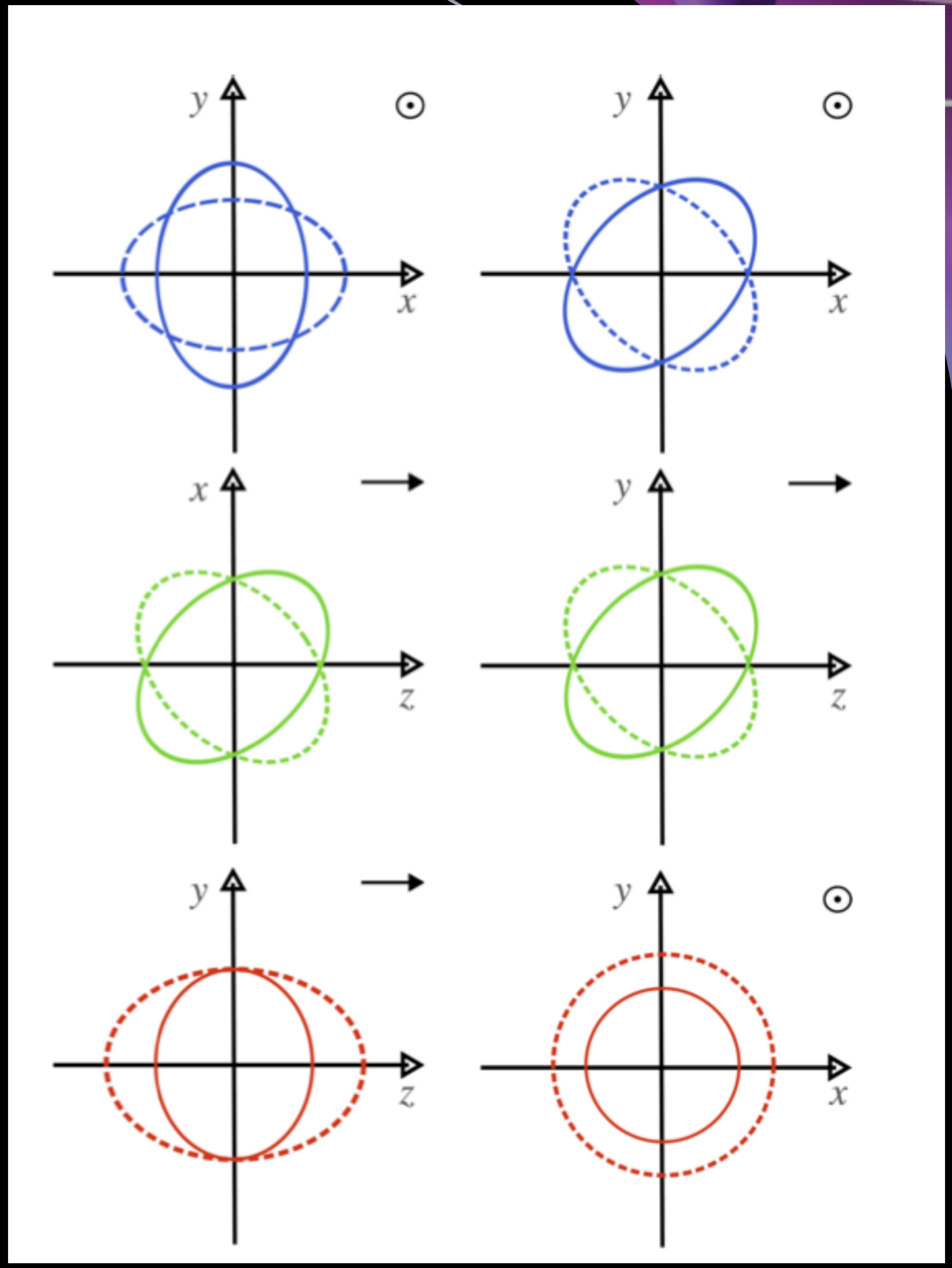
$$S^{jk} = \begin{pmatrix} A_S + A_+ & A_\times & A_{V1} \\ A_\times & A_S - A_+ & A_{V2} \\ A_{V1} & A_{V2} & A_L \end{pmatrix}$$

$$d^i(t) = n^i(t) + F^i_a(t, \hat{\Omega}_s, \psi) h^a(t)$$

Tensor

Vector

Scalar



Will, LRR, Vol 17, 4, (2014)

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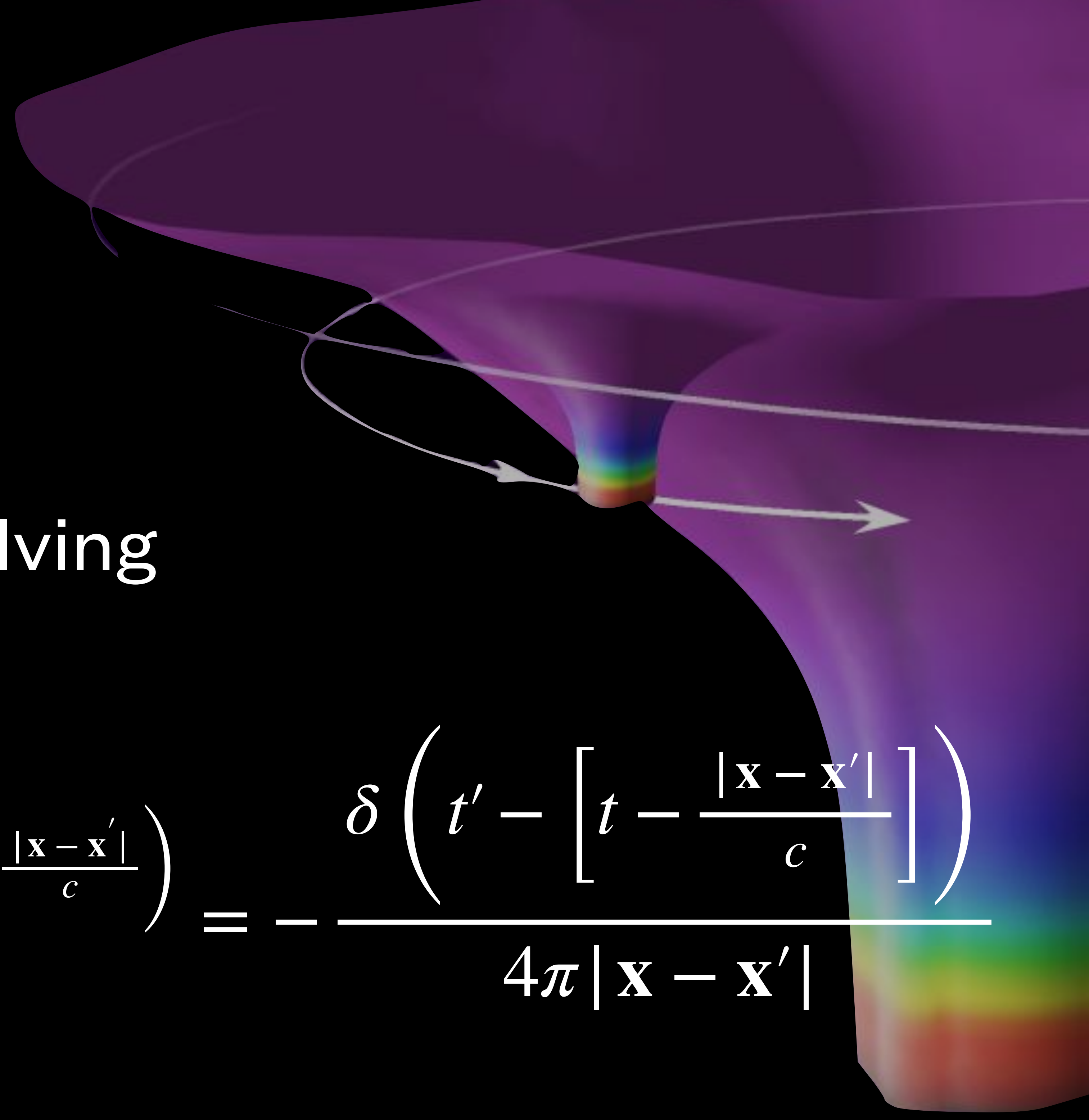
Sourced GW equation



We can solve the wave equation using retarded Green's functions, solving $\Box G(x-x') = -\delta^{(4)}(x-x')$,

$$G(t-t',\mathbf{x}-\mathbf{x}') = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega(t-t')}\frac{e^{i\frac{\omega}{c}|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} = -\frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega\left(t-t'-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)} = -\frac{\delta\left(t'-\left[t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right]\right)}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

and

$$\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\int d^4x' G(t-t',\mathbf{x}-\mathbf{x}')T_{\mu\nu}(\mathbf{x}') = \frac{4G}{c^2}\int d^3x'\frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{\mu\nu}\left(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)$$

- Now suppose our source is confined to a compact sphere of radius ϵ and we are observing GWs from a much larger distance: $r \gg \epsilon$

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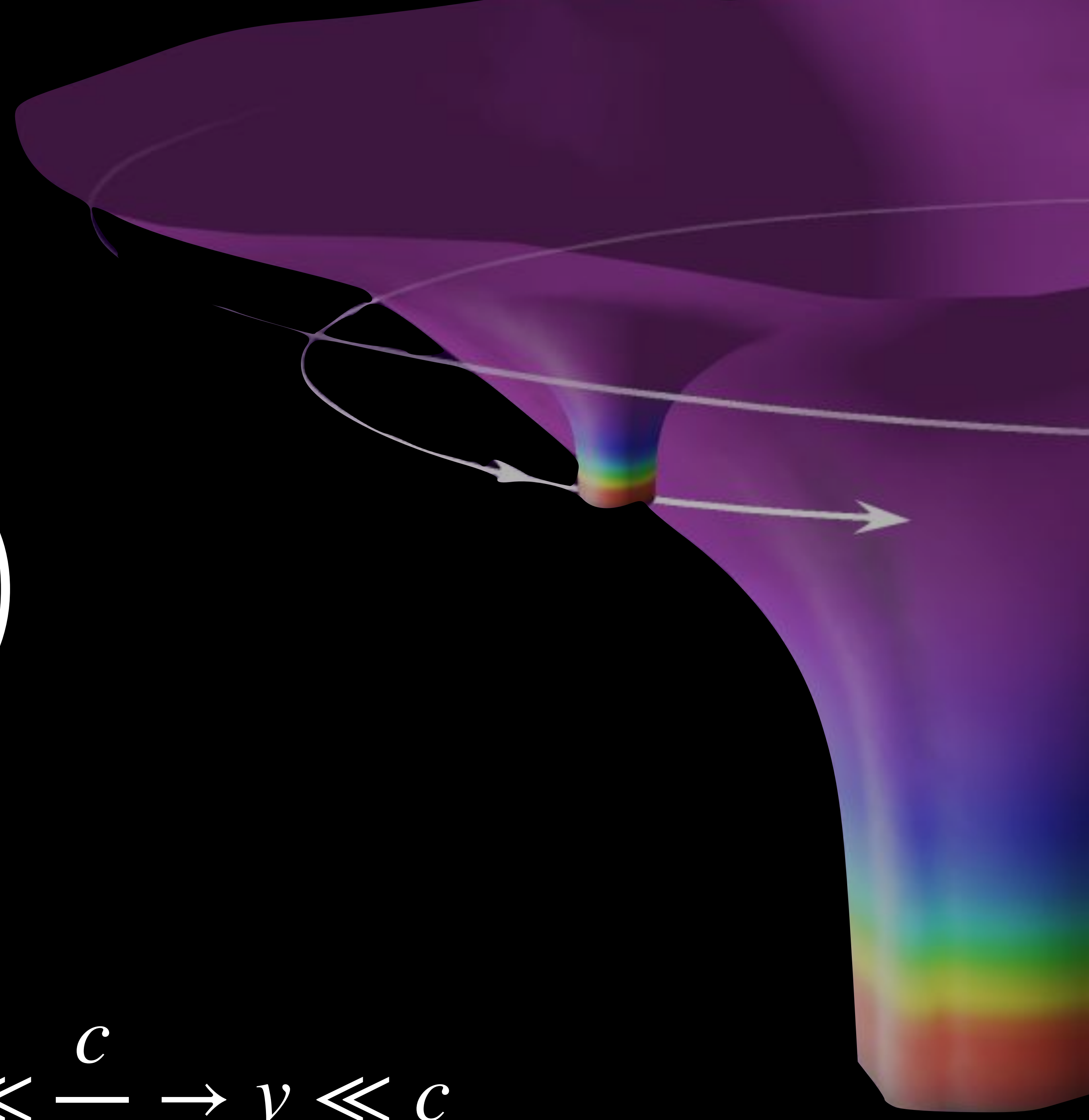
Sourced GW equation



$$\bar{h}_{\mu\nu} = \frac{4G}{c^4}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|} T_{\mu\nu}\left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, x'\right)$$

- Now suppose:
  - our source is confined to a compact region of radius ϵ
  - we are observing GWs from a much larger distance: $r \gg \epsilon$ and $\epsilon \ll \frac{c}{\omega} \to v \ll c$
- Then $|\mathbf{x}-\mathbf{x}'| \sim \mathbf{r}$ (since $|x'| < \epsilon$) and the above reduces to

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4 r}\int d^3x' \, T_{\mu\nu}\left(t - r/c, x'\right)$$

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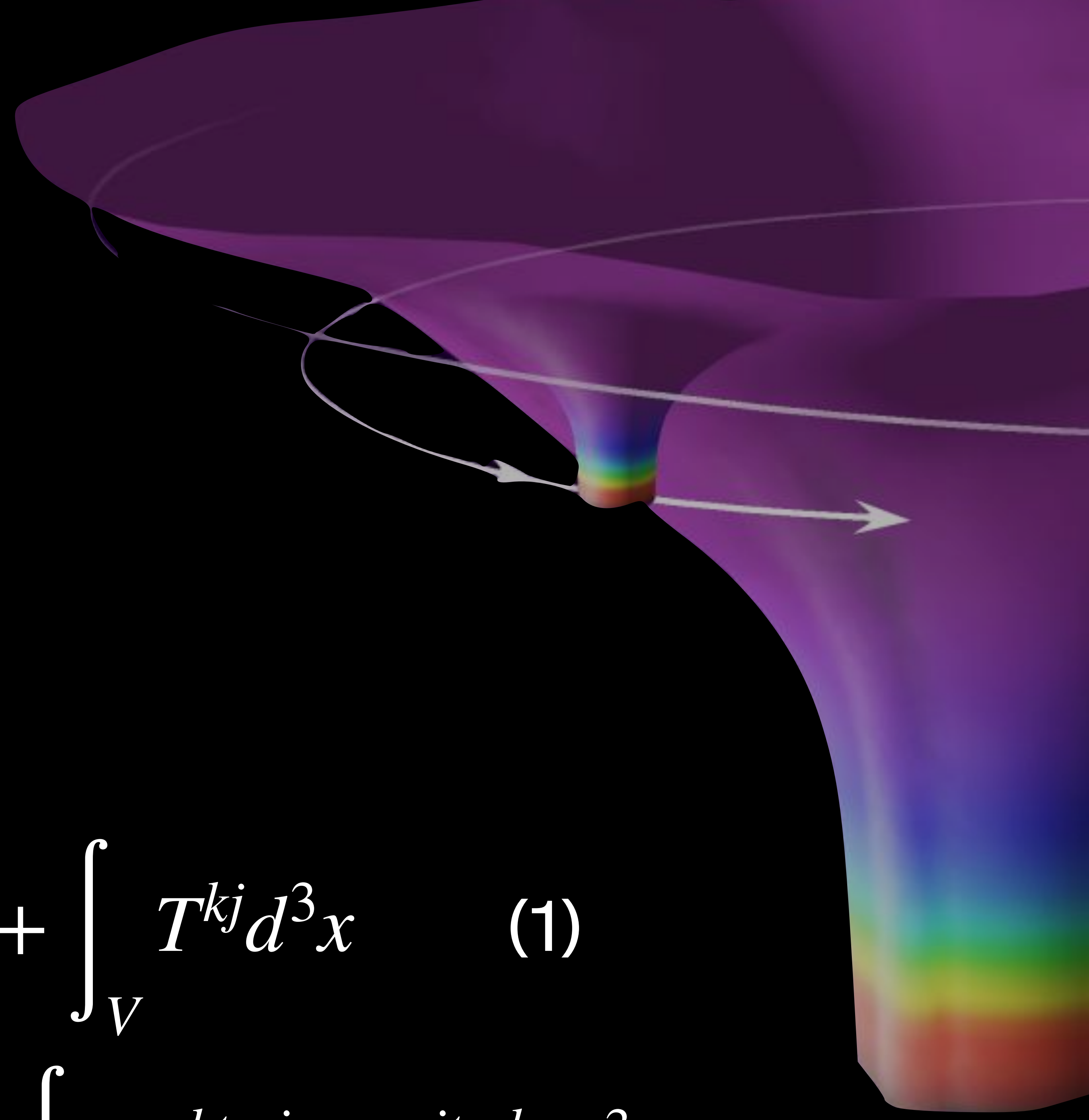
The quadrupole formula



We can now use conservation of $T_{\mu\nu}$

$$\partial_\mu T^{\mu j} = c^{-1}\partial_t T^{tj} + \partial_i T^{ij} = 0$$

$$\partial_\mu T^{\mu t} = c^{-1}\partial_t T^{tt} + \partial_i T^{it} = 0$$

and Gauss theorem

$$c^{-1}\partial_t \int_V T^{tj} x^k d^3x = -\int_V \partial_i T^{ij} x^k d^3x = -\cancel{\int_{\partial V} T^{ij} x^k dS_i} + \int_V T^{kj} d^3x \qquad (1)$$

$$c^{-1}\partial_t \int_V T^{tt} x^j x^k d^3x = -\int_V \partial_i T^{it} x^j x^k d^3x = -\cancel{\int_{\partial V} \ldots} + \int_V (T^{kt}x^j + T^{jt}x^k) d^3x \qquad (2)$$

$$\partial_t^2 Q^{jk} = c^{-2}\partial_t^2 \int_V T^{tt} x^j x^k d^3x = 2\int_V T^{kj} d^3x$$

$$\rightarrow \bar{h}^{kj}(t, \mathbf{x}) = \frac{2G}{c^4 r}\partial_t^2 Q^{kj}(t - r/c) \quad \text{where } Q^{jk} = c^{-2}\int_V T^{tt} x'^j x'^k d^3x'$$

$$\boxed{\bar{h}_{\mu\nu} = \frac{4G}{c^4 r}\int d^3x' \, T_{\mu\nu}\left(t - r/c, x'\right)}$$

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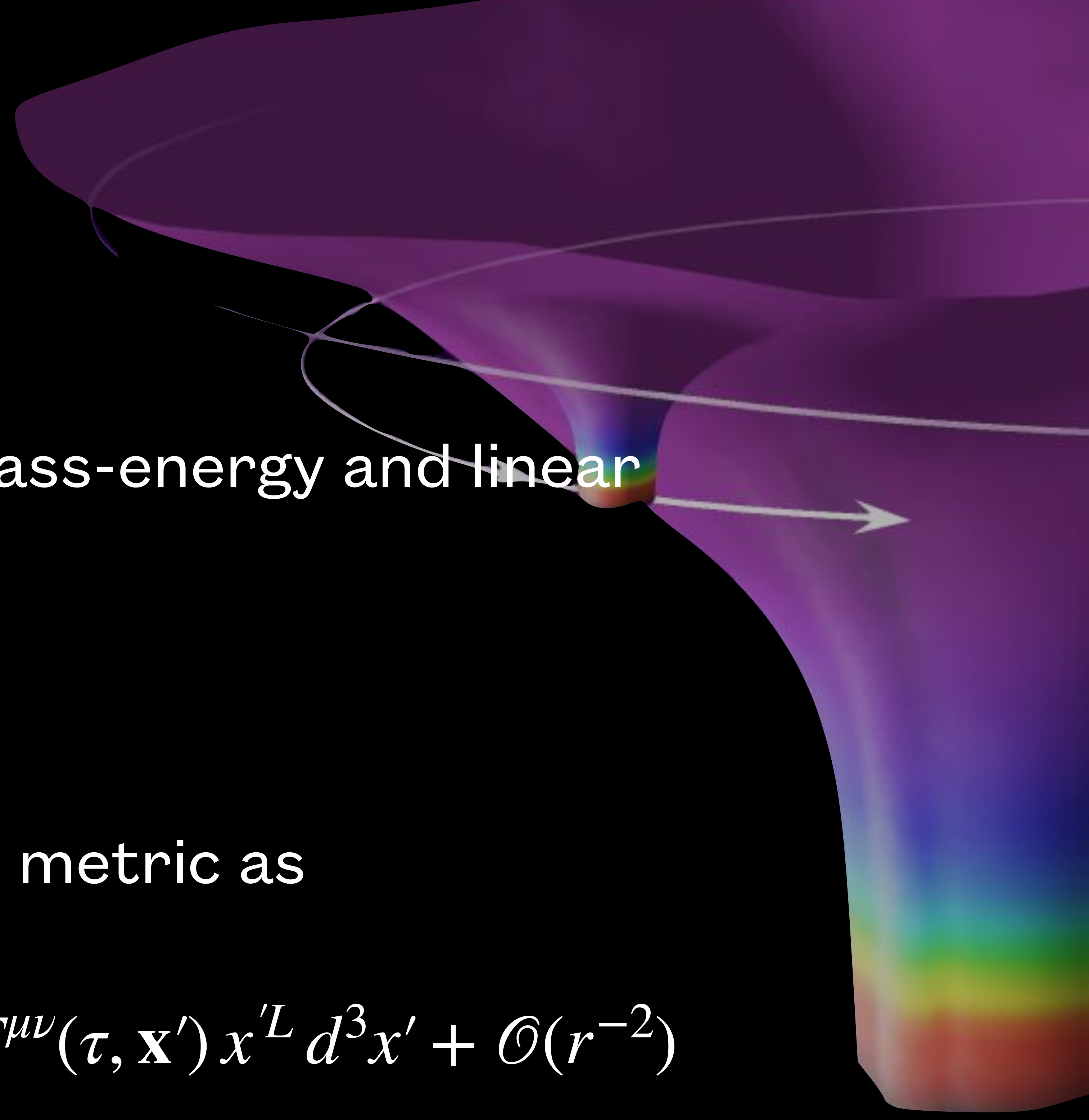
The quadrupole formula



In GR, GW radiation has quadrupolar nature: no monopole or dipole radiation (mass-energy and linear momentum conservation)

$$\frac{dE}{dt} = \frac{c^3 r^2}{32\pi G} \int d\Omega < \dot{h}^{TT} ij \,|\, \dot{h}^{TT} ij >$$

More generally, far away from the source, defining $\tau := t - r/c$, we can expand the metric as

$$\bar{h}^{\mu\nu} = \frac{4G}{c^4 r} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell\,!} \int \partial_L T^{\mu\nu}(\tau, \mathbf{x}')\, x'^L\, d^3x' + \mathcal{O}(r^{-2}) = \frac{4G}{c^4 r} \sum_{\ell=0}^{\infty} \frac{n_L}{\ell\,! c^\ell} \left(\frac{d}{d\tau}\right)^\ell \int T^{\mu\nu}(\tau, \mathbf{x}')\, x'^L\, d^3x' + \mathcal{O}(r^{-2})$$

where $n^j = x^j/r$ and L is a multi-index notation i.e. $x'^L = x'^{i_1} \ldots x'^{i_L}$

- Different radiative multipole moments contribute to the metric perturbation, with the quadrupole being the dominant one
- Higher orders are suppressed by inverse powers of c

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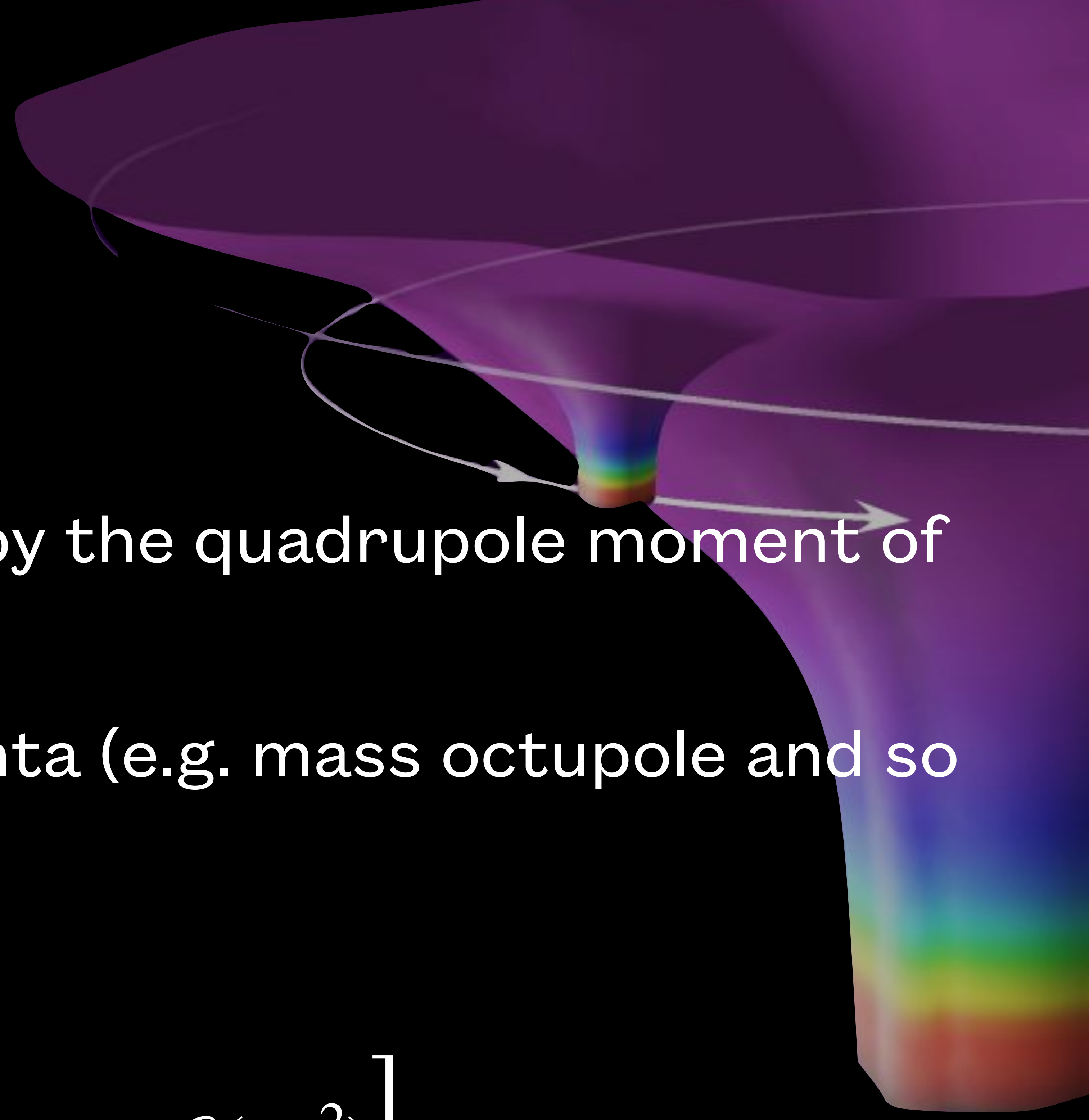
Multipole expansion



- The time-dependent part of the metric perturbation is dominated by the quadrupole moment of the mass distribution
- At higher orders in the expansion we find higher-order mass momenta (e.g. mass octupole and so on) as well current multipoles.
- E.g. the next order reads

$$h_{TT}^{jk} = \frac{2G}{c^4 R}\left[\ddot{I}^{\langle jk\rangle} + \frac{1}{3c}\left(\dddot{I}^{\langle jkn\rangle} + 2\epsilon^{mnj}\ddot{J}^{\langle mk\rangle} + 2\epsilon^{mnk}\ddot{J}^{\langle mj\rangle}\right) n_n + \mathcal{O}(c^{-2})\right]$$

with $I^{<L>}$ being symmetric trace-free mass multipole moments and $J^{<L>}$ being STF current multipole moments obtained from integrals of the type

$$J^{ij} = c^{-1}\int d^3x\, T^{0i}(t,\mathbf{x})\, x^j$$

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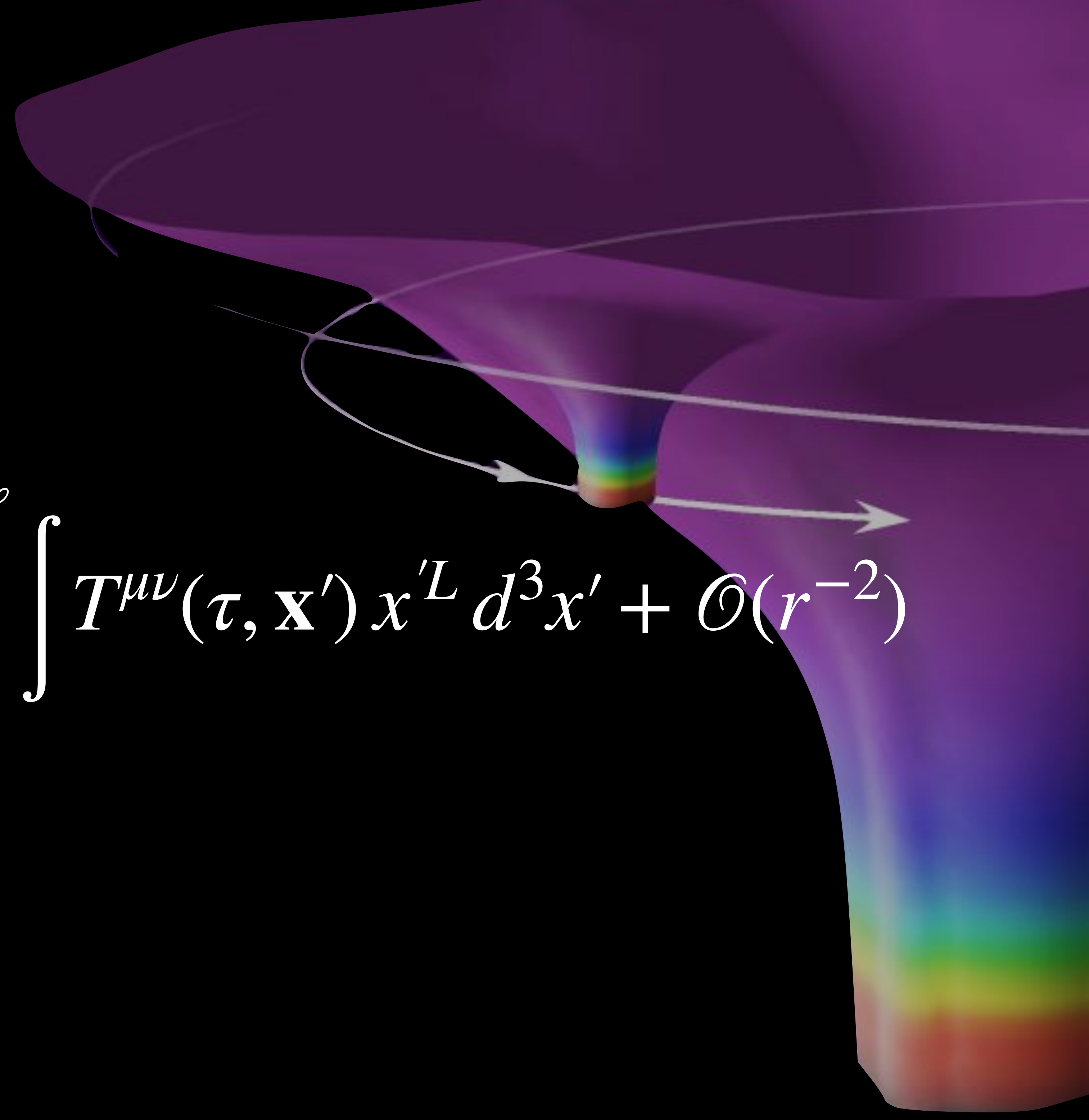
Multipole expansion



$$\bar{h}^{\mu\nu} = \frac{4G}{c^4 r} \sum_{\ell=0}^{\infty} \frac{n_L}{\ell\,! c^\ell} \left(\frac{d}{d\tau}\right)^\ell \int T^{\mu\nu}(\tau, \mathbf{x}')\, x'^L\, d^3x' + \mathcal{O}(r^{-2})$$

- The multipoles $\ell = 0{,}1$ do no radiate
- E.g.

$$M = c^{-2} \int T^{00}(t, x') d^3x'$$

gives a Newtonian potential like term in $\bar{h}^{00} = \dfrac{4GM}{c^2 r} + \dots$

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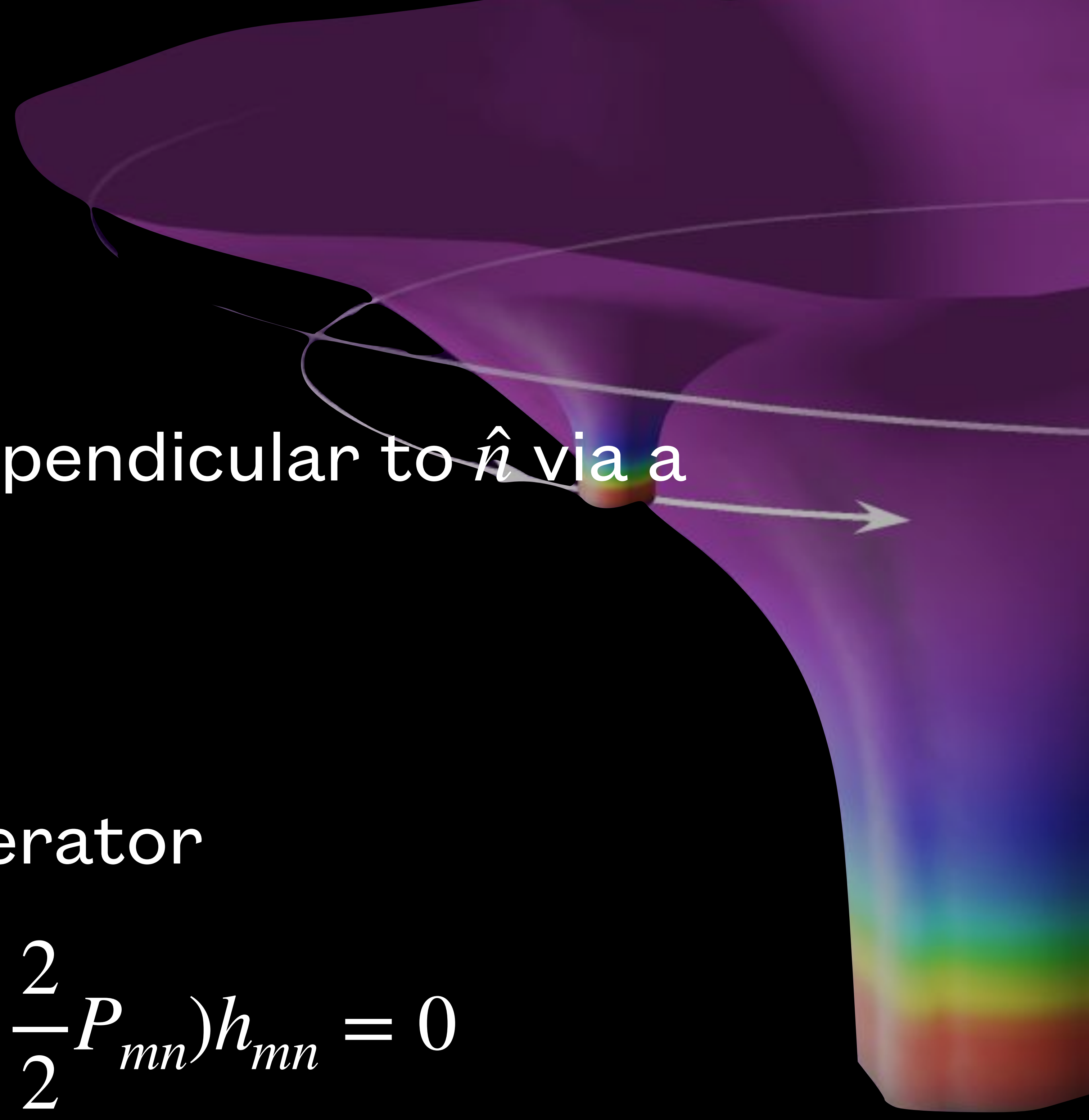
Projection on the TT gauge



To recover TT-gauge quantities, project the tensor in the direction perpendicular to $\hat{n}$ via a projection operator

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j$$

- P_{ij} is manifestly transverse. We can now build a traceless projection operator

$$P_{jkmn}(\hat{n}) = P_{jm}P_{kn} - \frac{1}{2}P_{jk}P_{mn} \qquad \delta^{ij}h_{ij} = \delta^{ij}P_{ijmn}h_{mn} = (P_{mn} - \frac{2}{2}P_{mn})h_{mn} = 0$$

To define the multipoles uniquely, we can impose the trace-free condition on the source moments before projecting, e.g.

$$Q_{ij}^{STF} = Q_{ij} - \frac{1}{3}\delta^{ij}Q_k^k$$

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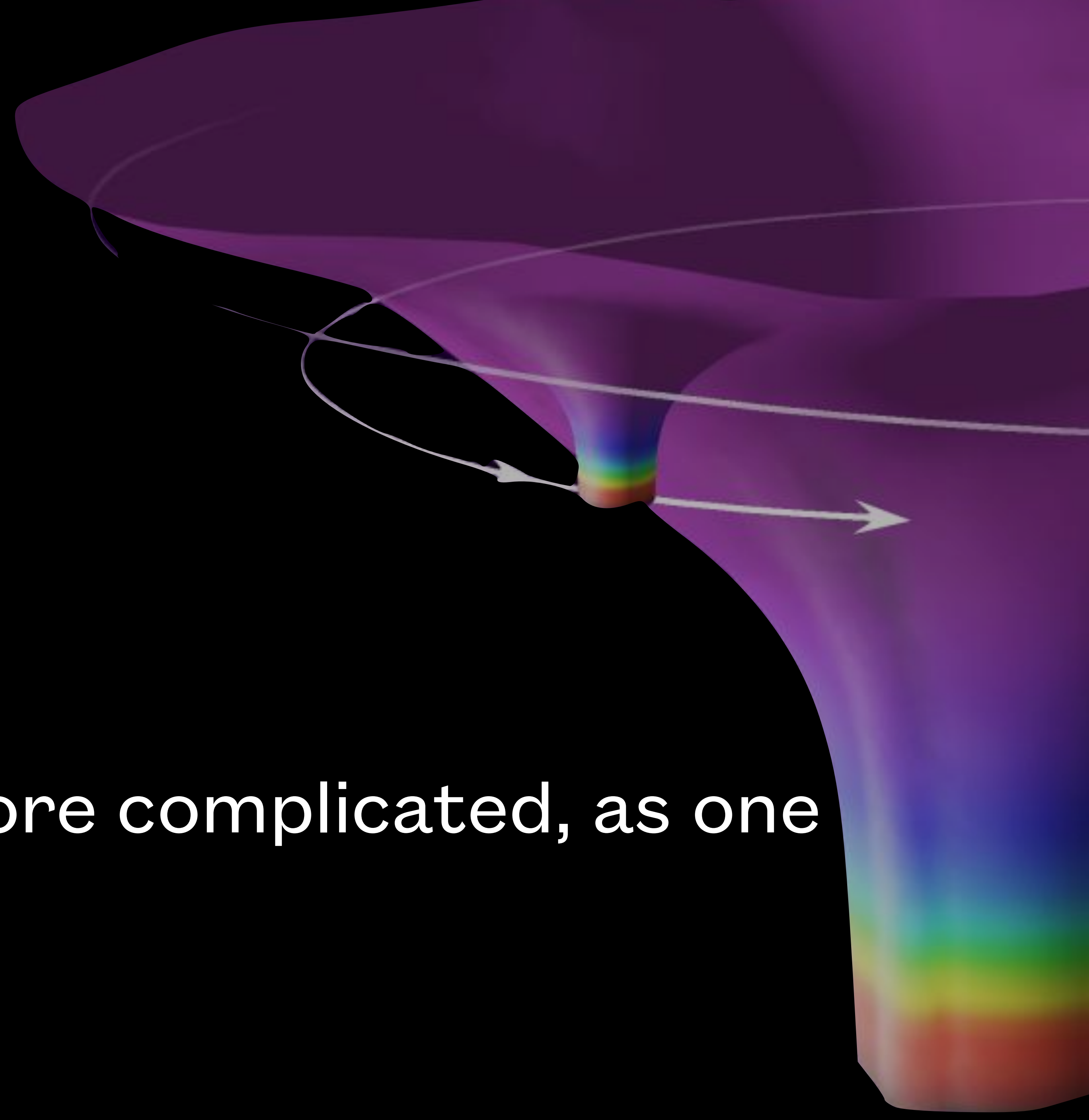
Higher-order source terms



- We have worked at first order in linearised gravity
- At higher orders, the source term in the field equations becomes more complicated, as one needs to account for
    - nonlinear self-interactions of the gravitational field
    - gauge effects (gauge condition needs to be satisfied at higher orders)
- Matter, nonlinear and gauge contributions can be expressed via a stress-energy pseudotensor. E.g. in harmonic gauge

$$\tau^{\mu\nu} = (-g)\left(T^{\mu\nu} + t^{\mu\nu}_{LL} + t^{\mu\nu}_{H}\right)$$

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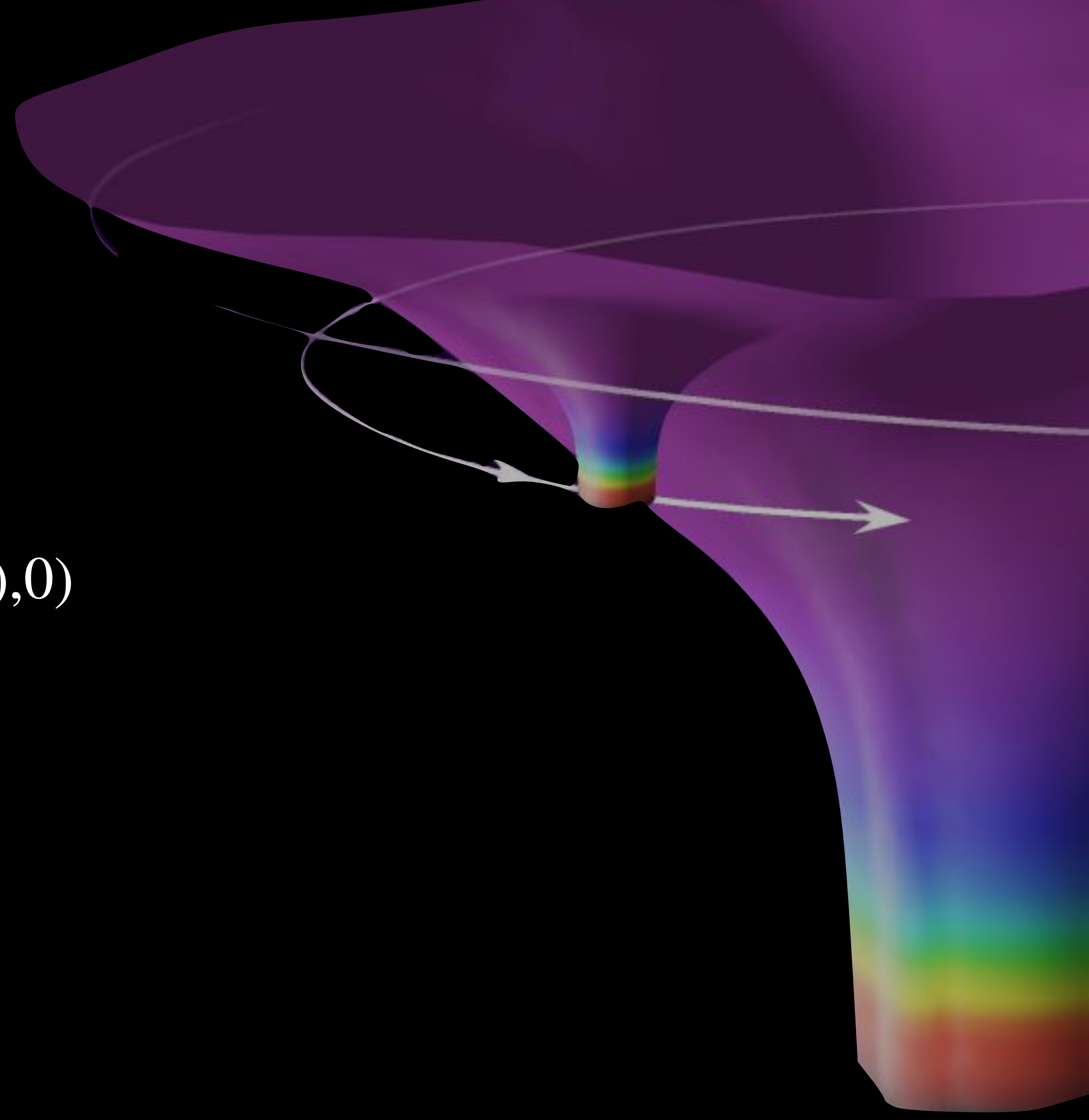
Masses on a circular orbit



- Two masses with reduced mass $\mu = \dfrac{m_1 m_2}{m_1 + m_2}$
- CM frame where $c^{-2}T^{00} = \rho = \mu\delta^{(3)}(\mathbf{x} - \mathbf{x}_0(t))$ and assume trajectory $\mathbf{x}_0(t) = R(\cos(\omega t), \sin(\omega t), 0)$
- The traceless trace-free part of the mass quadrupole moment is

Then apply definition $Q^{ij} = c^{-2}\displaystyle\int_V T^{00}(x'^i x'^j - \frac{1}{3}\delta^{ij}r^2)d^3x'$

$$\rightarrow Q^{ij} = \mu\left(x_0^i(t)x_0^j(t) - \frac{1}{3}R^2\delta^{ij}\right)$$

$$\rightarrow \begin{aligned} \ddot{Q}^{xx}(t) &= -2\mu R^2\omega^2\cos(2\omega t) \\ \ddot{Q}^{yy}(t) &= 2\mu R^2\omega^2\cos(2\omega t) \\ \ddot{Q}^{xy}(t) &= \ddot{Q}^{yx}(t) = -2\mu R^2\omega^2\sin(2\omega t) \\ \ddot{Q}^{yz}(t) &= \ddot{Q}^{zy}(t) = 0 \end{aligned}$$

At this point we can get the polarisations via $h_+(t) = h_{ij}^{\mathrm{TT}}e_+^{ij}(\hat{n}), \quad h_\times(t) = h_{ij}^{\mathrm{TT}}e_\times^{ij}(\hat{n})$

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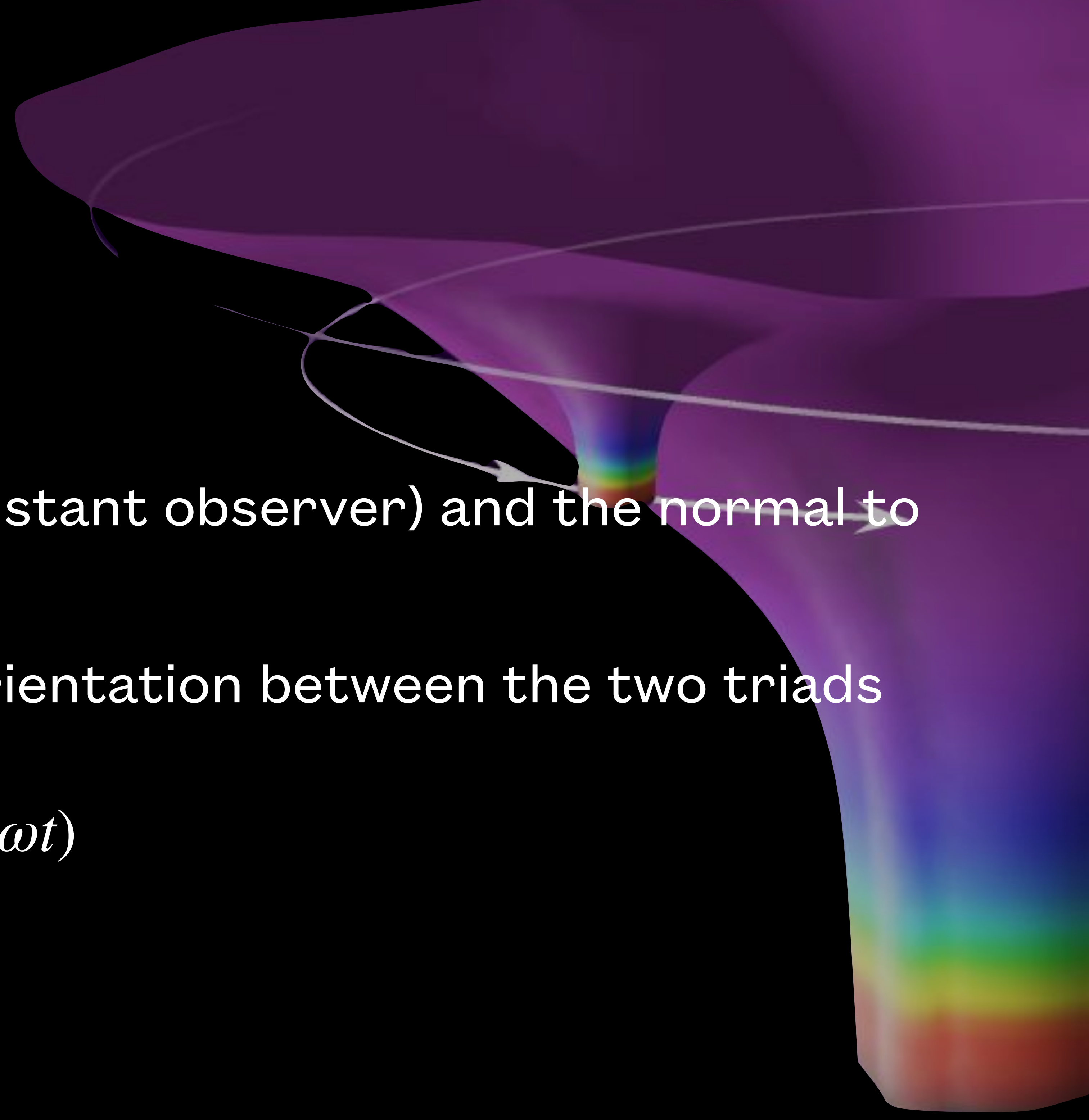


GW from a circular orbit

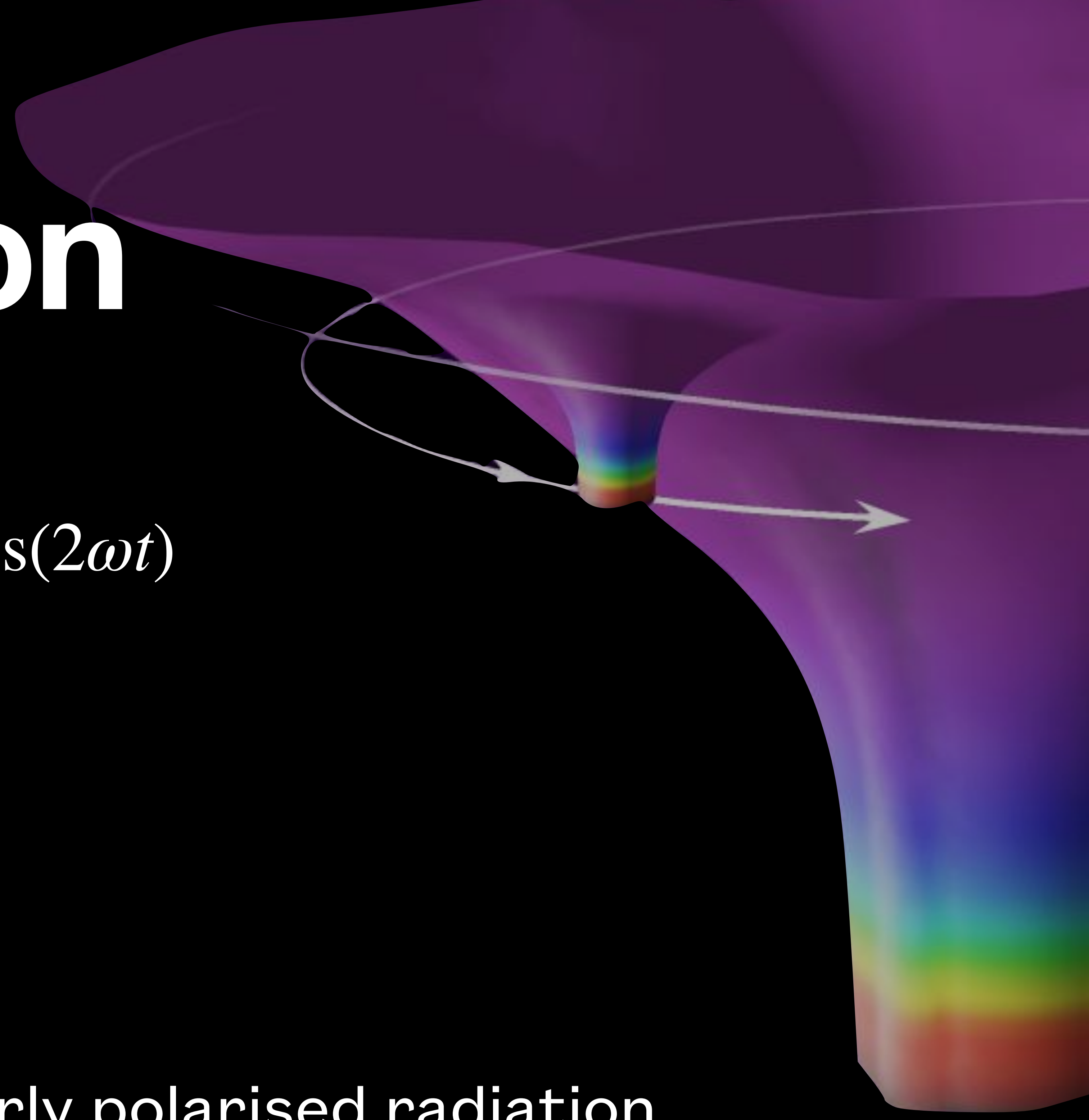
- Consider a polarization frame (x', y', z') where ι is the angle between $\hat{z}'$ (pointing towards a distant observer) and the normal to the orbital plane $\hat{z}$
- Then one can write the polarization tensors in terms of the angles specifying the relative orientation between the two triads

$$h_+(t) = \frac{4G}{c^4} \frac{\mu R^2 \omega^2}{r} \frac{(1 + \cos^2 \iota)}{2} \cos(2\omega t)$$

$$h_\times(t) = \frac{4G}{c^4} \frac{\mu R^2 \omega^2}{r} \cos \iota \sin(2\omega t)$$

- At leading order, radiation is emitted at twice the orbital frequency
- Higher orders in the expansion of radiative multipoles (e.g. current-type quadrupole moment, mass octupole etc…) contribute terms with characteristic frequencies $\omega_n = n\omega$
- In terms of an expansion into spin-weighted spherical harmonics, these contributions are collectively referred to "higher harmonics"

Polarization and orbital inclination



$$h_+(t) = \frac{4G}{c^4}\frac{\mu R^2\omega^2}{r}\frac{(1+\cos^2\iota)}{2}\cos(2\omega t)$$

$$h_\times(t) = \frac{4G}{c^4}\frac{\mu R^2\omega^2}{r}\cos\iota\sin(2\omega t)$$

- In this approximation:
  - a face-on source contains both polarizations with the same amplitudes: circularly polarised radiation
  - edge-on sources, $h_\times(t) = 0$, hence radiation is linearly polarised
  - in general, radiation is elliptically polarised
- If the orbital plane precesses, $\iota = \iota(t)$ and hence changes in the polarizations amplitudes encode information about the orbital dynamics

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Spin-weighted spherical harmonics

- The complex GW strain is often written in terms of an expansion onto spin-weighted spherical harmonics

$$h(t) = h_+ - ih_\times = \sum_{\ell \geq 2} \sum_{-\ell \leq m \leq \ell} h_{\ell m}(t) \, {}_{-2}Y_{\ell,m}(\theta, \phi)$$

- This decomposition, as we saw implies a choice of the wave frame (to define the two polarizations) and a choice of the radiation/source frame for the definition of θ, ϕ
- $(2, \pm 2)$ tends to be dominant, but other modes can give significant contribution, this contribution depends
    - On the intrinsic amplitudes of the complex functions $h_{\ell m}$
    - On the orientation of the source, via $Y_{\ell,m}(\theta, \phi)$
- Bias caused by incomplete waveform mode content widely studied for LISA

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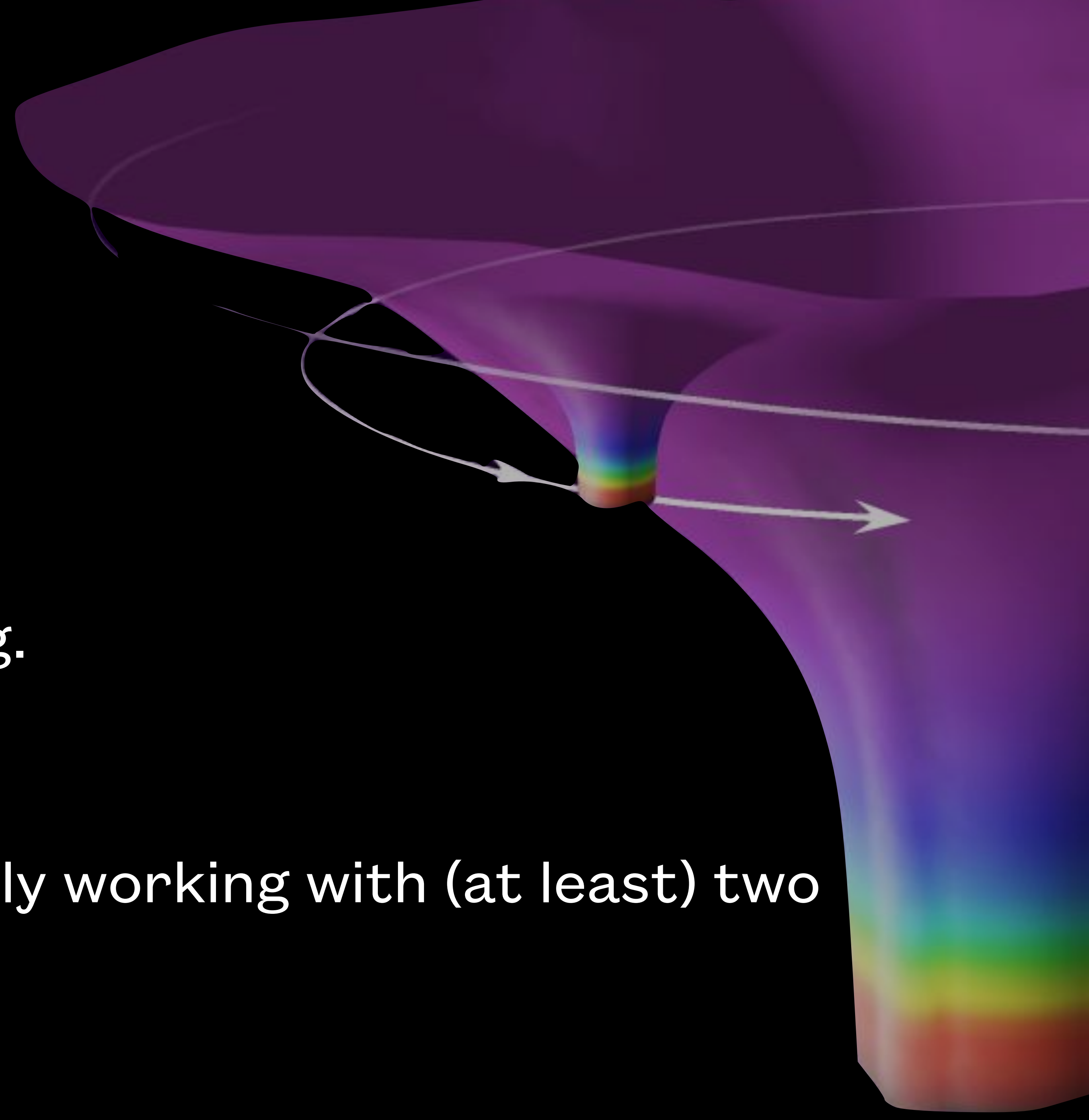
Interaction with detectors



- The effect of a GW on a detector can be encoded in the antenna pattern functions, e.g.

$$H(t) = \mathscr{F}^{+}(t)h_{+}^{\mathrm{SSB}} + \mathscr{F}^{\times}(t)h_{\times}^{\mathrm{SSB}}$$

- One first need to express the polarizations in a given frame, so waveforms are typically working with (at least) two frames

    Wave frame (with $\hat{z}$ pointing towards distant observers)

    Source frame: source frame is adjusted to specific modelling needs, for instance it might be adjusted to some symmetries/physically meaningful reference vectors of the problem

- E.g. for LISA source-frame polarizations are transformed from source-frame to the SSB, which is used to describe the positions of the LISA spacecrafts and astro sources

$$\begin{pmatrix} h_{+}^{\mathrm{SSB}} \\ h_{\times}^{\mathrm{SSB}} \end{pmatrix} = \begin{pmatrix} \cos 2\psi & -\sin 2\psi \\ \sin 2\psi & \cos 2\psi \end{pmatrix} \begin{pmatrix} h_{+}^{\mathrm{src}} \\ h_{\times}^{\mathrm{src}} \end{pmatrix}$$

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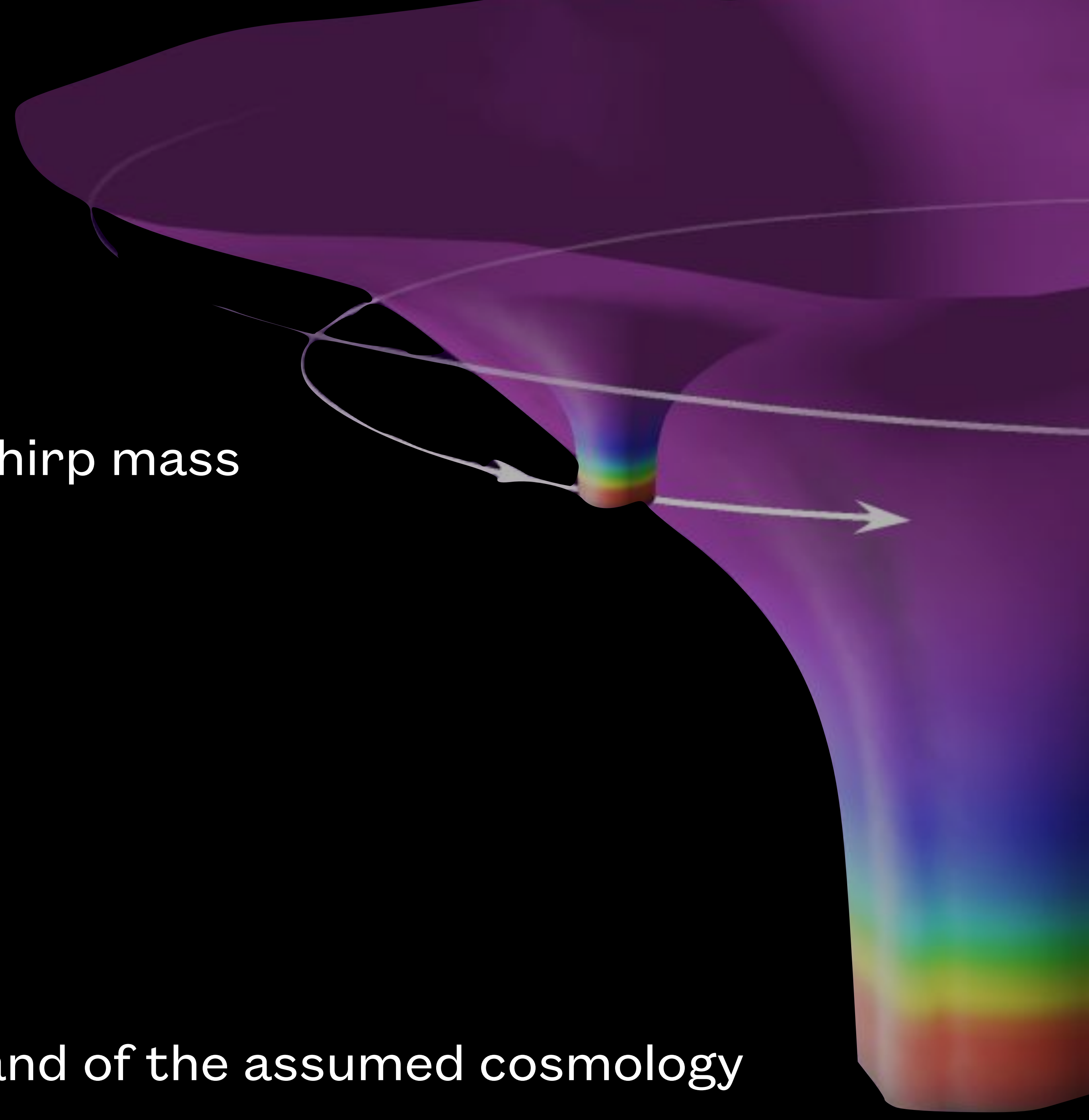
GW in an expanding universe



- In astrophysical applications, the leading-order mass dependence is expressed in terms of the chirp mass

$$\mathcal{M} = \mu^{3/5} M^{2/5}$$

- where M is the total mass of the binary. Hence, $h_+(t) \propto \mathcal{M}^{5/3}$
- In an expanding universe:

$$h_+(t) \propto \frac{1}{d_L(z)} \left(\frac{G\mathcal{M}_{det}(z)}{c^2} \right)^{5/3} f_{gw}^{2/3}$$

- The GW strain amplitude depends on the luminosity distance d_L, which is a function of redshift and of the assumed cosmology

$$d_L(z) = (1+z) \frac{c}{H_0} \int_0^z \frac{dz'}{\sqrt{\Omega_M (1+z')^3 + \Omega_\Lambda}}$$

- The observed mass parameters are not source-frame ones, e.g. $\mathcal{M}_{det} = (1+z)\mathcal{M}$ and so on
- To get the source-frame mass, must assume a cosmology (e.g., ΛCDM) and infer redshift from luminosity distance.

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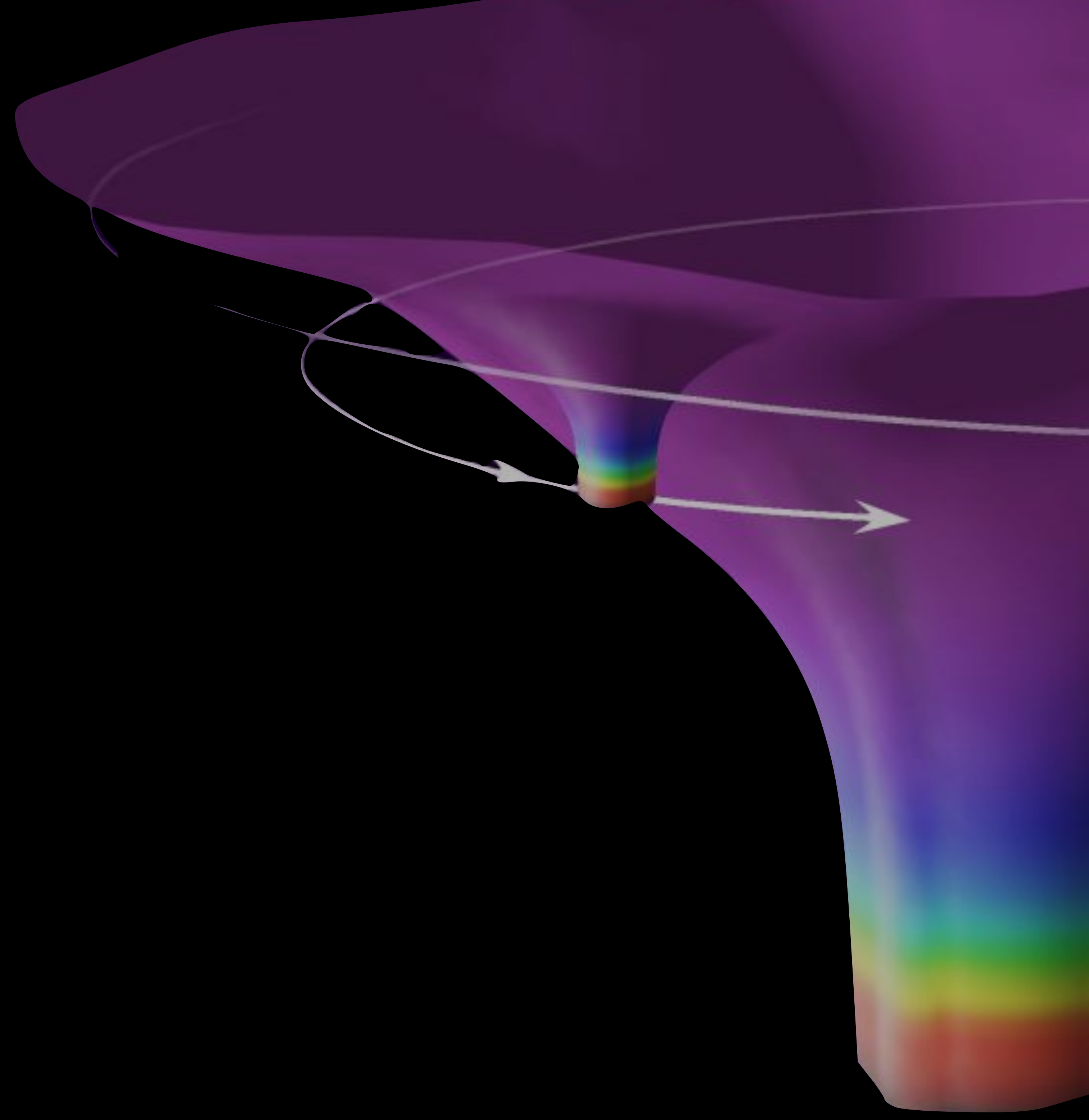
GW in an expanding universe



- In an expanding universe:

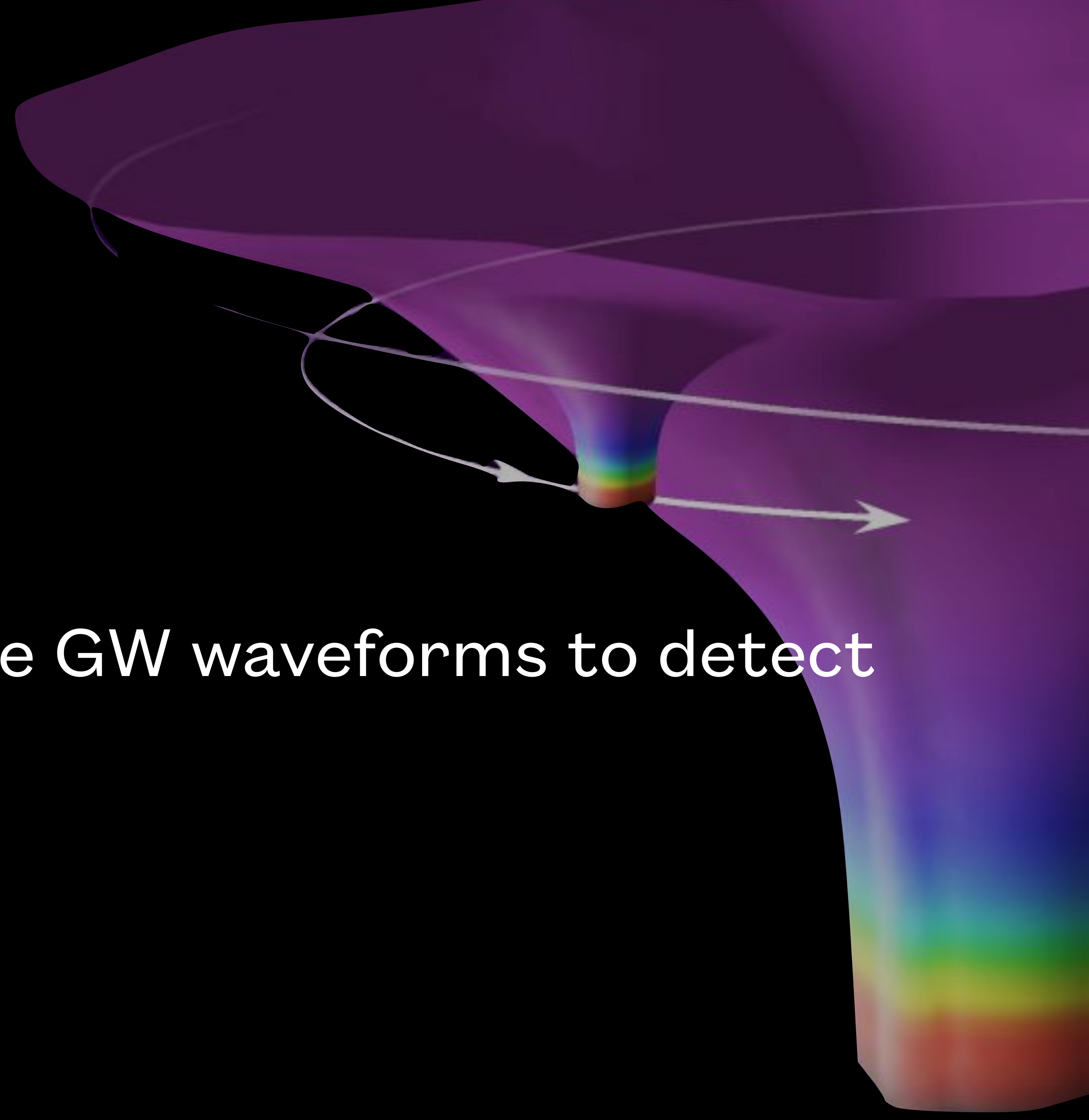
$$h_+(t) \propto \frac{4}{d_L(z)} \left(\frac{G\mathcal{M}_{det}(z)}{c^2}\right)^{5/3} f_{gw}^{2/3}$$

- Furthermore from flux-balance arguments:

$$\dot{f}_{gw} \propto \mathcal{M}_{det}^{5/3} f_{gw}^{11/3}$$

- From the combined measurement of amplitude and phasing evolution, one can infer d_L
- GW sources can probe the distance- redshift relation if redshift is measured or statistically inferred from observations

What you will see next

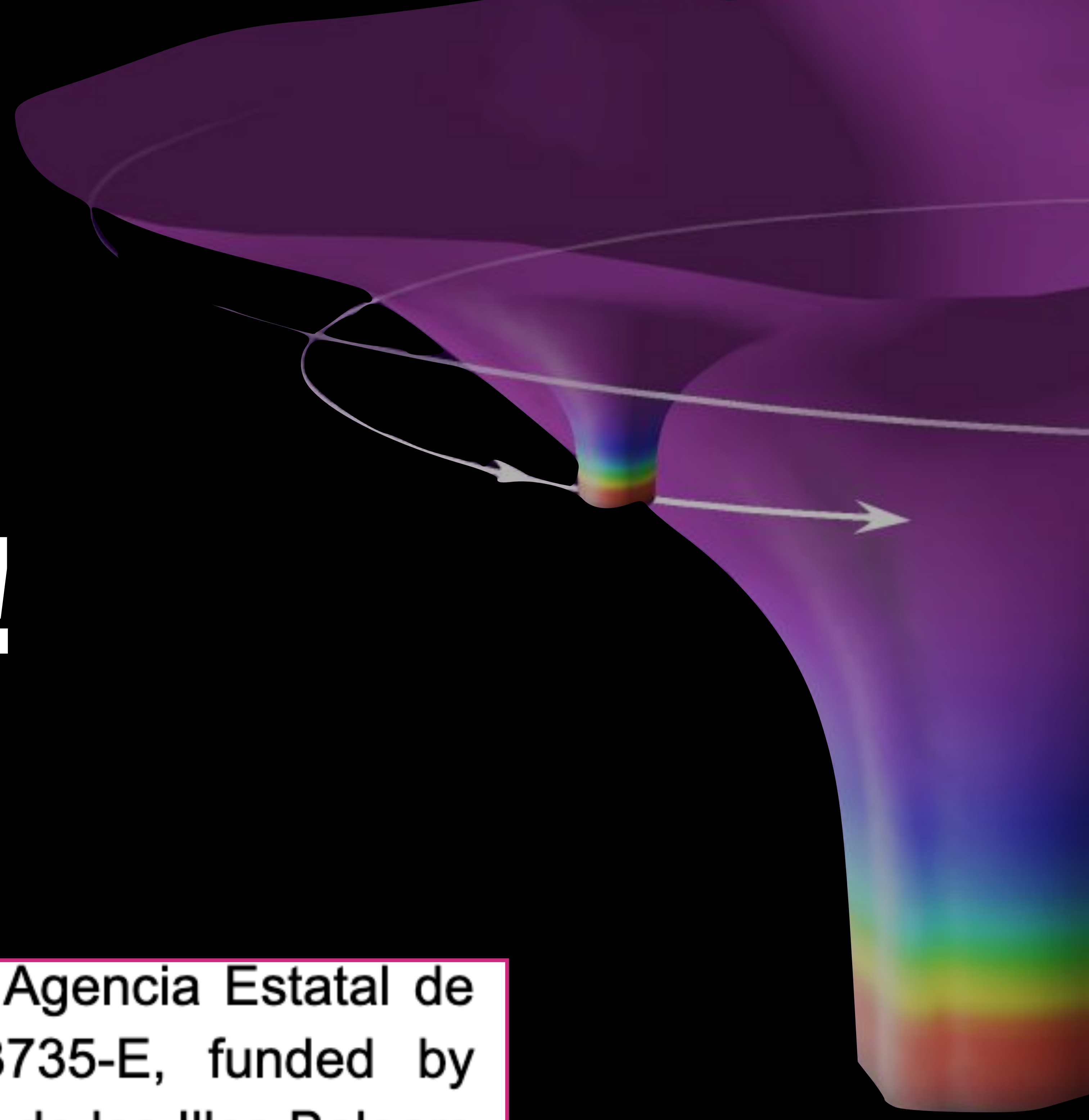


LISA will detect a rich spectrum of astrophysical sources

Some sources spurred the development of specific frameworks to compute GW waveforms to detect them and characterise them (e.g. EMRIS->gravitational self force)

Over the rest of the week you will see lectures about the

- Post-Newtonian framework
- Gravitational self-force (GSF)
- Black-hole perturbation theory
- Numerical Relativity
- Inspiral-merger-ringdown (IMR) waveform models



Thanks !!

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