Gravitational waves: Opening a new window on the universe

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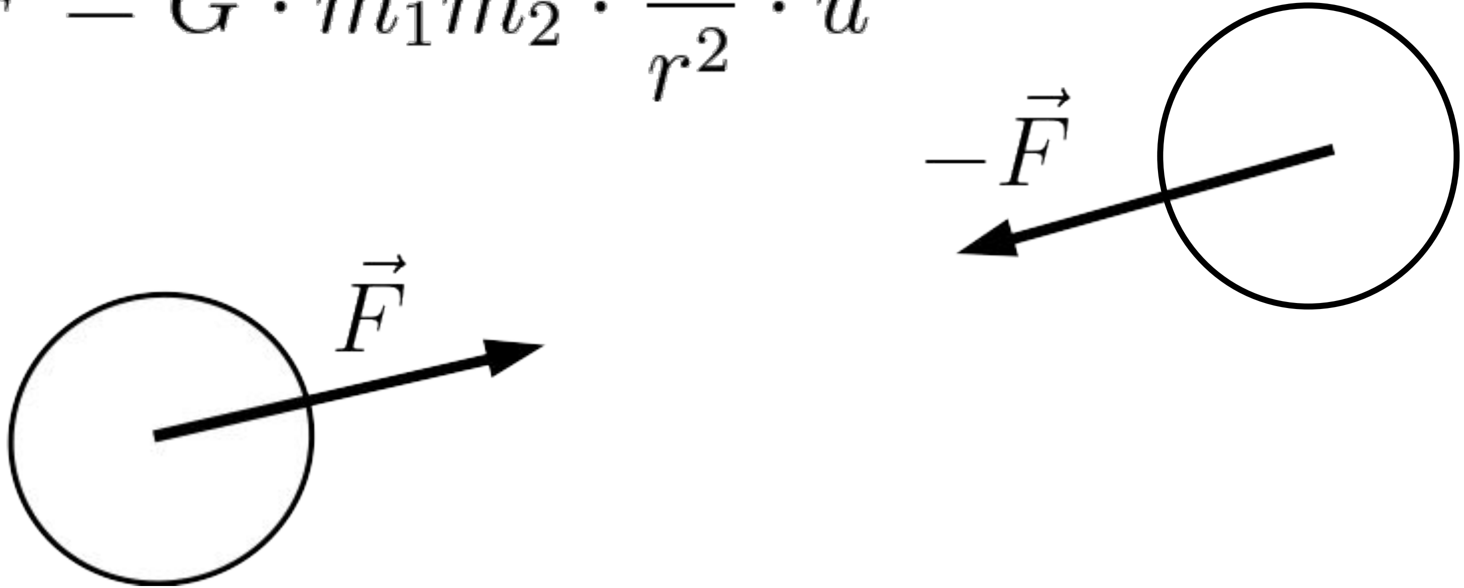
Contents

Gravitation: the classical theory

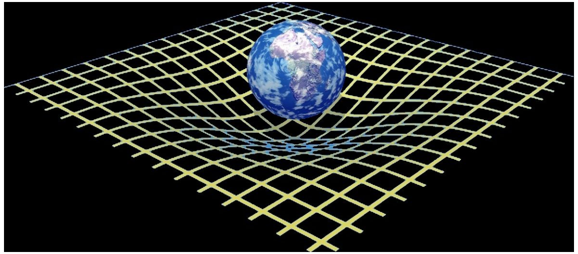
- Flat space, absolute time
- Instantaneous interaction between distant masses

$$\vec{F} = G \cdot m_1 m_2 \cdot \frac{1}{r^2} \cdot \vec{u}$$



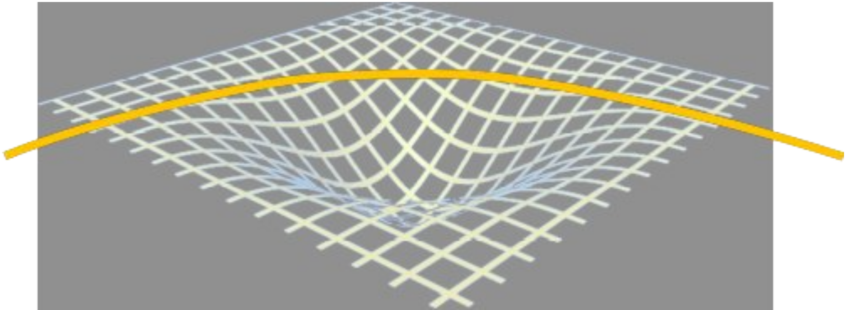
Gravitation: the modern theory

- Theory of General Relativity (GR)
- Einstein 1915-1918 : geometric theory of gravitation
- A mass "bends" and "deforms " space-time



- The trajectory of an object is influenced by the curvature of space-time

J. A. Wheeler : **“Space tells matter how to move and matter tells space how to curve”**



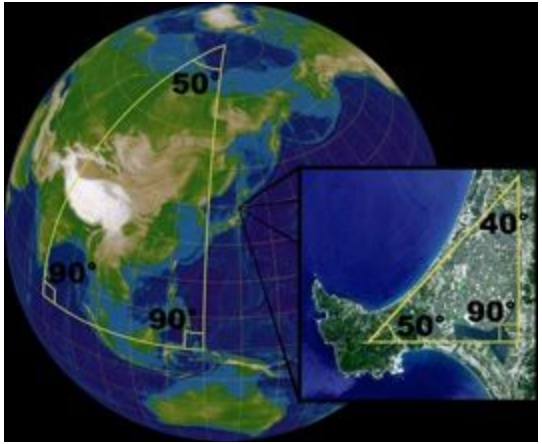
Theoretical piece: curved space

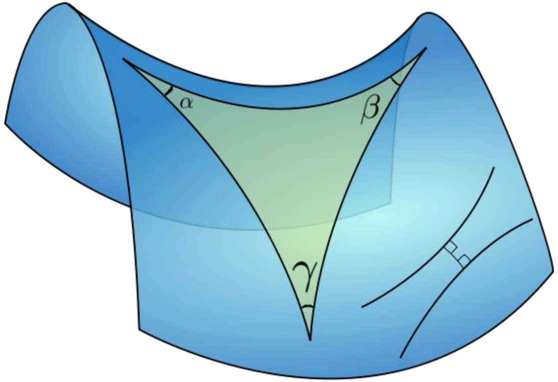
- What is a curved space ? (= "manifold")
    - examples : sphere, saddle
- Can we measure curvature ?
    - we cannot see our space from "outside"
    - but we can measure angles
    - the sum of the angles of a triangle is not always equal to π!
- positive curvature

$$\sum \text{angles} = \alpha + \beta + \gamma > \pi$$

- negative curvature

$$\sum \text{angles} = \alpha + \beta + \gamma < \pi$$





Theoretical piece: curved space-time

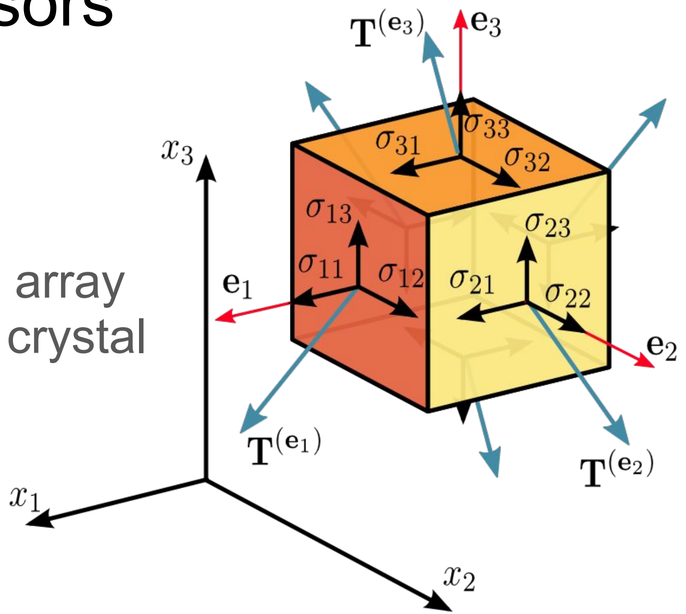
- In General Relativity
    - spacetime is curved and locally flat
    - one cannot go "out" to see the curvature
    - "intrinsically" curved space

      => intrinsic curvature

- go straight (free fall) = follow a "geodesic"
- note that the time is also curved !
- as a first approximation, finds the results (trajectories) of newtonian mechanics

Theoretical piece: tensors

- Tensor = mathematical object
- Does not depend on the coordinate system
- Extends the notion of vector
- In a specific coordinate system, multidimensional array
- Example: electrical conductivity of an anisotropic crystal



$$j^i = \sigma^i_j E^j$$

- Note : summation is implicit over repeated indices (Einstein convention)

$$\sigma^i_j E^j \equiv \sum_j \sigma^i_j E^j$$

Special Relativity

- In space-time (ST), need to measure:
    - the distance between two points;
    - the angle between two vectors;

- The interval: $$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

- Which can be written : $$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$
    - with:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad dx^0 = dt, \quad dx^1 = dx, \quad dx^2 = dy, \quad dx^3 = dz$$

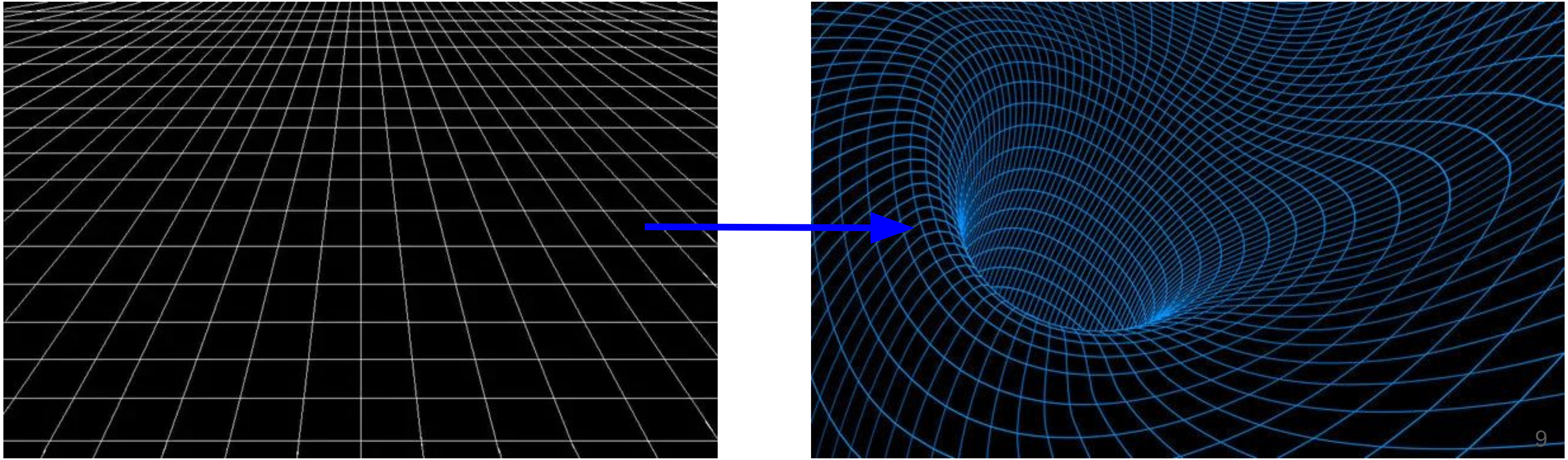
Minkowski metric

General Relativity

- In space-time, describe by pseudo-riemannian manifold:

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}(x)$$

- And the metric is symmetric, and torsion-less



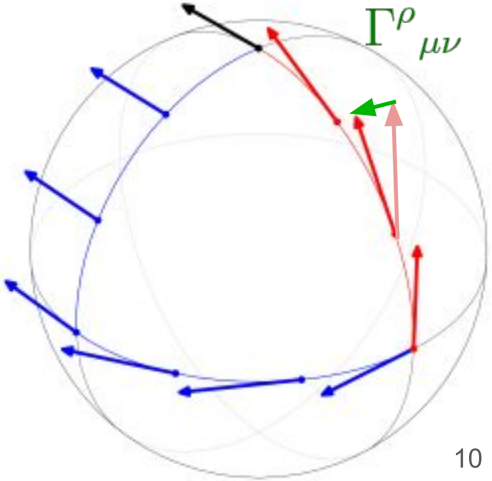
General Relativity

- To translate a vector, need to connect ST region with different metric:

$$D_\mu e_\nu = \Gamma^\rho{}_{\mu\nu} e_\rho$$

Covariant derivative **Christoffel symbols or Connexion**

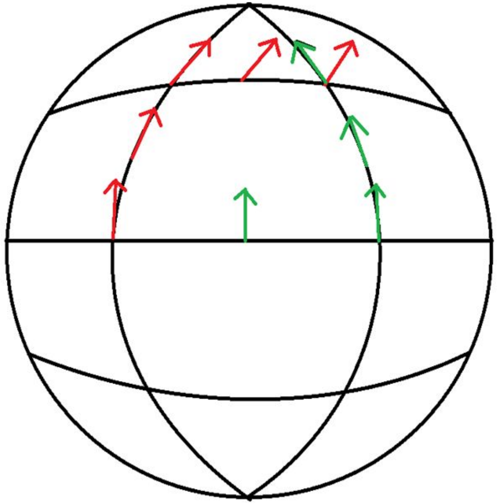
Basis vectors vary



- The connection is written as :

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right)$$

Curved Space-Time



- The commutator of cov. derivatives exhibit the curvature :

$$[D_\mu, D_\nu]\, v^\alpha = R^\alpha{}_{\mu\nu\beta} v^\beta$$

Ricci Identity

- With the Riemann curvature tensor or Riemann tensor :

$$R^\alpha{}_{\mu\nu\beta} = \partial_\nu \Gamma^\alpha{}_{\beta\mu} - \partial_\beta \Gamma^\alpha{}_{\nu\mu} + (\Gamma^\alpha{}_{\nu\rho}\Gamma^\rho{}_{\beta\mu} - \Gamma^\alpha{}_{\beta\rho}\Gamma^\rho{}_{\nu\mu})$$

- The difference from Minkowski metric from the curvature tensor :

$$R_{\mu\nu} = R^\lambda{}_{\mu\nu\lambda}$$

Ricci tensor

The Einstein Field Equation

- To tell how matter curve spacetime and how the curved spacetime modify matter trajectory:

$$\underbrace{\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)}_{\text{curvature term}} + \underbrace{\Lambda g_{\mu\nu}}_{\text{cosmological constant}} = \frac{8\pi G}{c^4}\underbrace{T_{\mu\nu}}_{\text{energy-momentum term}}$$

Non-linear equations

From Einstein Field Equations to Gravitational Waves

- Describe the perturbation from flat space: $h^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu} - \eta^{\mu\nu}$

- Einstein equation with h:

$$\Box h^{\mu\nu} = \frac{16\pi G}{c^4} \tau^{\mu\nu}$$

- with :

$$\tau^{\mu\nu} \equiv (-g) T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}$$

stress-energy tensor **Non-linearities**

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Wave Equation !!!

- with :

$$\tau^{\mu\nu} \equiv (-g) T^{\mu\nu} + \frac{c^4}{16\pi G} \Lambda^{\mu\nu}$$

stress-energy tensor

Non-linearities

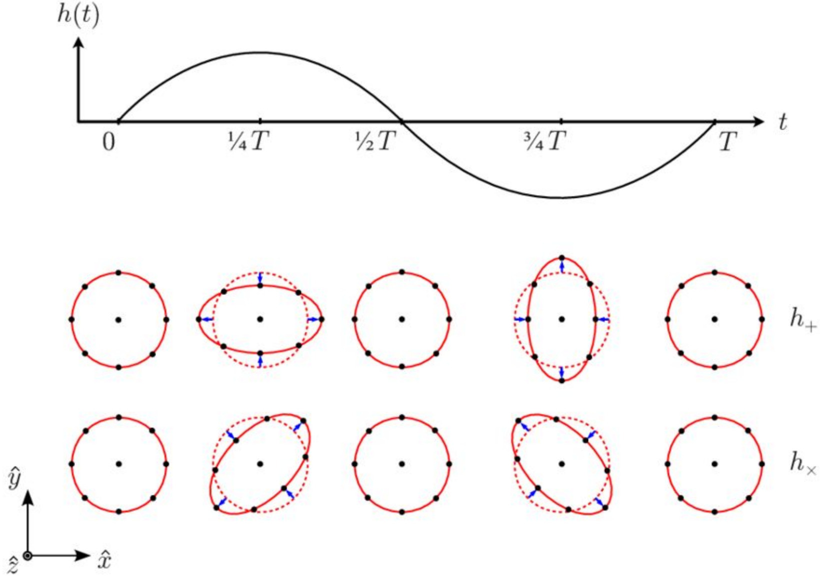
From Einstein Field Equations to Gravitational Waves

- Solution for linearized ($\Lambda_{\mu\nu} = 0$) theory in vacuum ($T_{\mu\nu} = 0$): $\Box h^{\mu\nu} = 0$

$$h_{\mu\nu} = A_{\mu\nu} \cdot e^{-i(\vec{k}\cdot\vec{x} - \omega\cdot t)}$$

and :

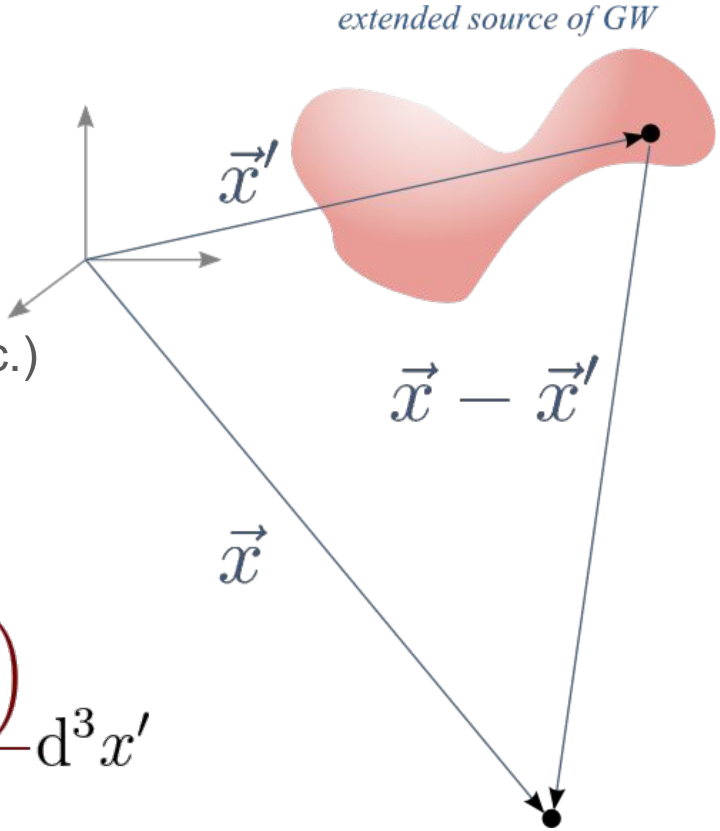
$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



Gravitational Waves from a Source

- Solution with a source ($T_{\mu\nu} \neq 0$):
- Use Green functions
  - Solution of the wave equation
    in the presence of a point source (delta func.)
- Retarded potential

$$h_{\mu\nu}(\vec{x}, t) = -\frac{4G}{c^4} \int_{source} \frac{\tau_{\mu\nu}\left(t - \frac{|\vec{x}-\vec{x}'|}{c}, \vec{x}'\right)}{|\vec{x} - \vec{x}'|} \mathrm{d}^3 x'$$



Gravitational Waves from a Source

- Approximations :
  - isolated source;
  - compact source;
  - observer far from the source $R = |\vec{x} - \vec{x}'|$;
- Taylor development of the stress-energy pseudo-tensor:

$$\int_{source} \frac{\mathrm{d}^3 x'}{|\vec{x} - \vec{x}'|} \tau_{\mu\nu}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}'\right) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} \left(\frac{\partial}{c\partial t}\right)^n \tilde{\int} \mathrm{d}^3 x' |\vec{x} - \vec{x}'|^{n-1} \tau_{\mu\nu}(t, \vec{x}')$$

- It's a multipolar moment expansion of the retarded potential
- At the lowest order (quadrupolar moment) :

$$\bar{h}_{ij}(t) = \frac{2G}{Rc^4} \frac{d^2 I_{ij}}{d\,t^2}\left(t - \frac{R}{c}\right)$$

$\frac{G}{c^4} \approx 8.24 \times 10^{-45}\ \mathrm{s}^2 \cdot \mathrm{m}^{-1} \cdot \mathrm{kg}^{-1}$

I_{ij} = reduced quadrupolar moment of the source $= \int_{source} d\vec{x}\ x_i x_j\ T_{00}(t, \vec{x})$

Orders of magnitude

- Amplitude:

$$h \approx \frac{G}{c^4} \cdot \frac{\ddot{I}}{R}$$

- Example with orbiting objects: a binary system
  - M = total mass, r=distance between the components
  - R=observer-system distance
  - $I \approx M.r^2$ hence $\ddot{I} \approx M \cdot v_{NS}^2 \approx E_c^{NS}$
    - where NS is the part of the source motion without spherical symmetry

- Hence

$$h \approx \frac{G}{c^4} \cdot \frac{E_c^{NS}}{R}$$

Orders of magnitude

- Luminosity: $$L_{GW} \approx \frac{G}{c^5} \cdot \dddot{I}^2$$

- Reminder: $\ddot{I} \approx E_c^{NS}$ hence $\dddot{I} \approx E_c^{NS}/T$
  - T = characteristic time of energy-momentum (or mass) motion from one side of the system to the other
- In case of a transient, violent event

$$L_{GW} \approx \frac{G}{c^5} \cdot \dddot{I}^2 \approx \frac{G}{c^5} \cdot \left(\frac{E_c^{NS}}{T}\right)^2$$

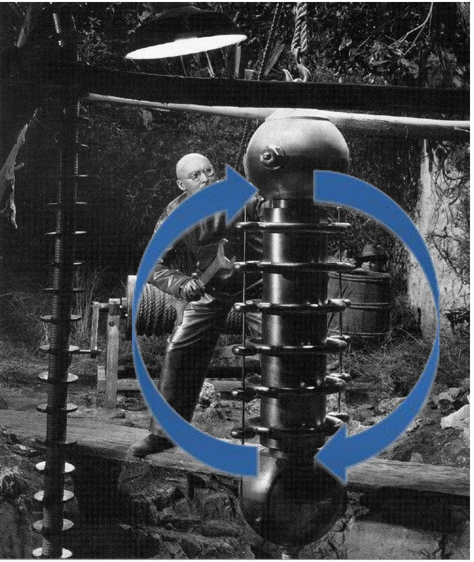
- For a quasi-stationary dynamics

$$L_{GW} \approx \frac{G}{c^5} \cdot \dddot{I}^2 \approx \frac{c^5}{G} \cdot \left(\frac{GM}{c^2 R}\right)^2 \cdot \left(\frac{v_{NS}}{c}\right)^6$$

where one introduces the Schwarzschild radius $R_S = \frac{2GM}{c^2}$

Orders of magnitude

- Mass distribution : needs a quadrupolar moment



- Examples for a binary system

$$h \approx 32\pi^2 \cdot \frac{G}{c^4} \cdot \frac{1}{R} \cdot M \cdot r^2 \cdot f_{orb}^2$$

- M = 1000 kg, r = 1 m, f = 1 kHz, R = 300 m
  - h ~ 10^{-35}
- M = 1.4 $M_\odot$, r = 20 km, f = 400 Hz, R = 1023 m (15 Mpc = 48,9 Mlyr)
  - h ~ 10^{-21}

Doing it in a lab ? No way !

Astrophysical sources

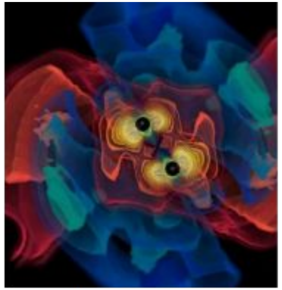
- Need high masses and velocities : astrophysical sources

- Binary system
  - Need to be compact to be observed by ground based detectors
  - → Neutron stars, black holes
  - Signal well modeled but rates not well known… yet

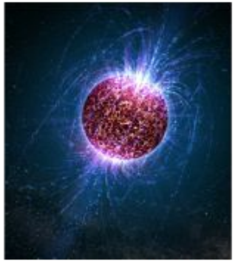
- Spinning neutron stars
  - Nearly monotonic signals
  - Long duration
  - Strength not well known

- Asymmetric explosion
  - Ex: core collapse supernovae
  - « burst » transient
  - Not well modeled

- Gravitational wave background
  - First type : superposition of many faint sources
  - Second type : Residue of the Big Bang or Inflation
  - Stochastic in nature

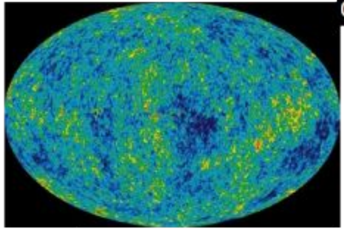


Credit: AEI, CCT, LSU



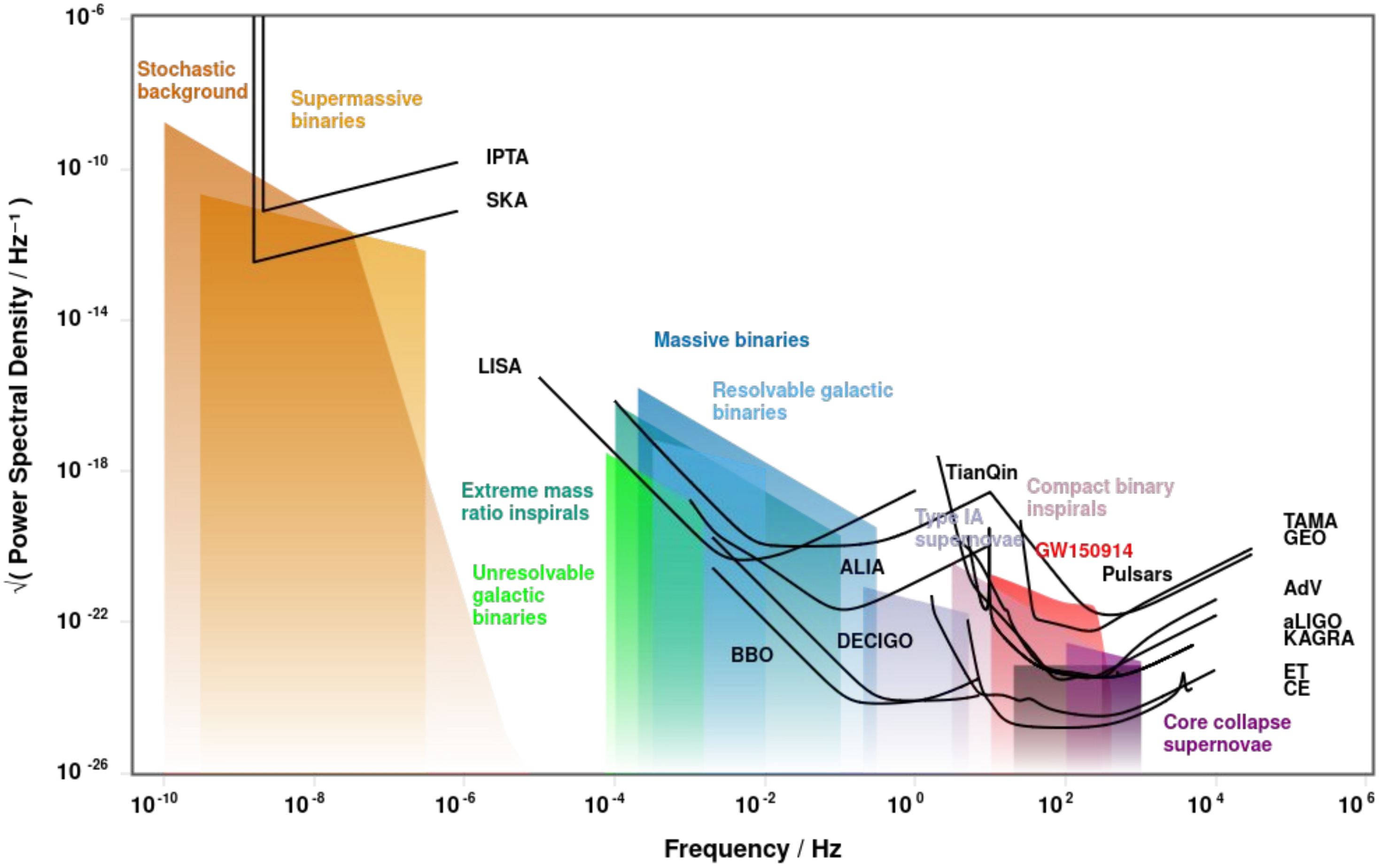
Casey Reed, Penn State

Credit: Chandra X-ray Observatory

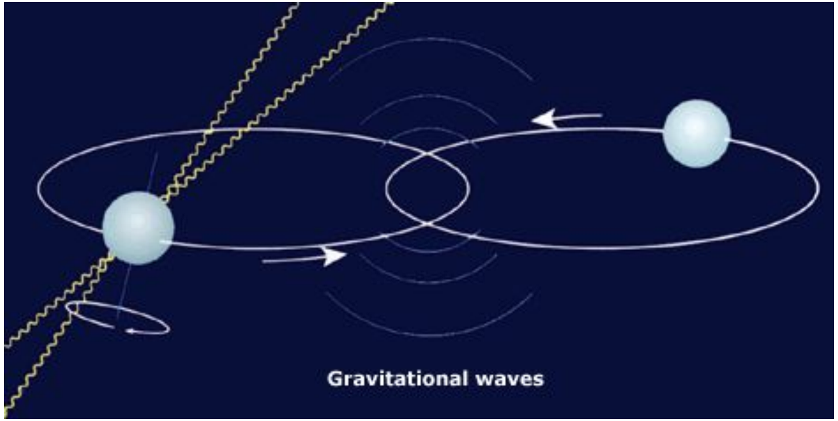


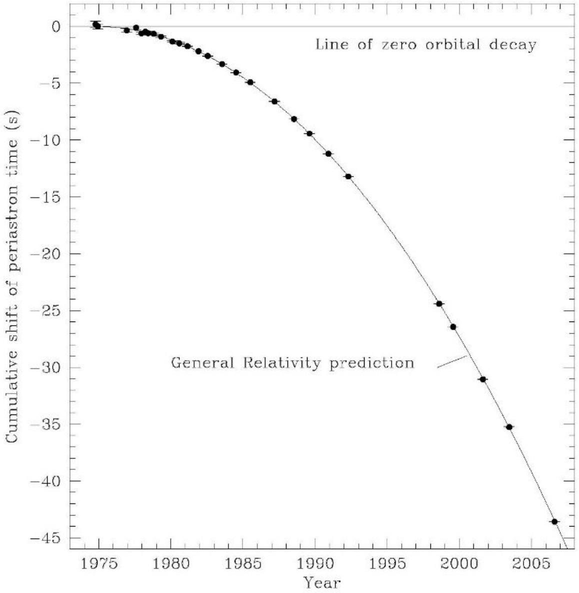
NASA/WMAP Science Team

Astrophysical sources



Indirect evidence: PSR 1913+16



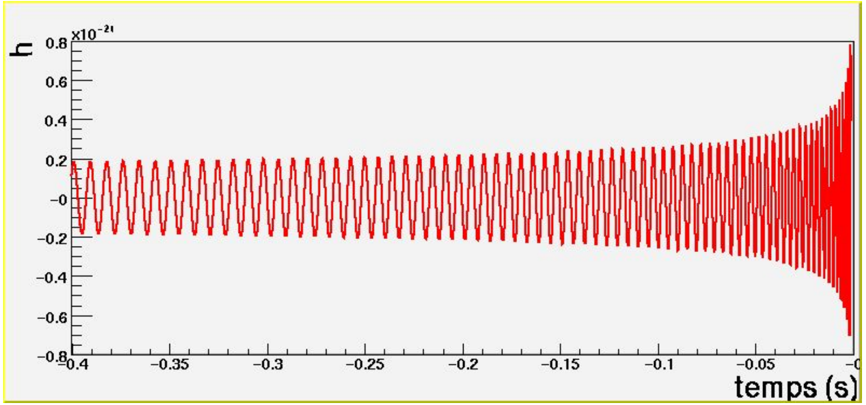


- Binary system of neutron stars
- One neutron star is a radio pulsar
- Discovered in 1975 by Hulse and Taylor
- Studied by Taylor, Weisberg and co.
- Decay of the orbital period compatible with GW emission
- Frequency of GW emitted by PSR 1913+16: ~ 0.07 mHz
    - Undetectable by ground-based detectors (bandwidth 10 Hz- 10 kHz)

$$\dot{P}_{observe} / \dot{P}_{predit} = 1.0013 \pm 0.0021$$

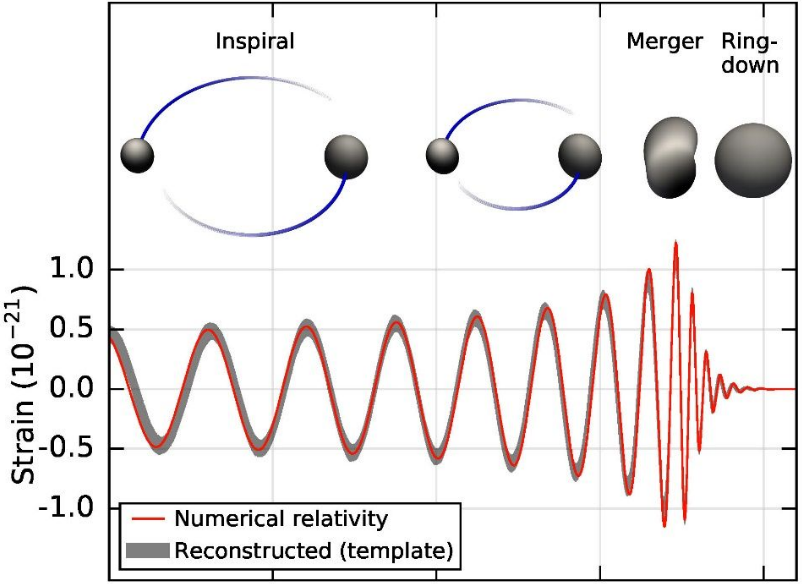
Compact Binary Coalescence

- Binary systems of compact stars at the end of their evolution
  - Neutron stars (NS) and/or black holes (BH)
- Very rare : a few events per million year per galaxy
- Typical amplitude at the detectors:
  - $h \approx 10^{-22}$ at 20 Mpc
- Very distinctive waveform



Compact Binary Coalescence

- System may be binary neutron stars (BNS), binary black holes (BBH) or NSBH
- Phases of the coalescence
- **Inspiral**:
  - Masses m_1 and m_2 orbit each other
  - GW emission -> system looses energy
  - => Frequency ↗, amplitude ↗
  - Waveform characterised by a « chirp mass »



$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{c^3}{G}\left[\frac{5}{96}\pi^{-8/3} f^{-11/3} \dot{f}\right]^{3/5}$$

- **Merger**: computed numerically (numerical GR)
- **Ringdown**: quasi-normal modes decomposition

Compact Binary Coalescence

- At Newtonian order:
  - Amplitude and phase evolve with time ($\tau = t_{coal} - t$) :

$$\begin{cases} A(\tau) = \left(\frac{G\mathcal{M}}{c^2}\right)^{5/4} \left(\frac{5}{c\tau}\right)^{1/4} \\ \Phi(\tau) = \Phi_0 - 2\left[\left(\frac{5G\mathcal{M}}{c^3}\right)^{-1} \tau\right]^{5/8} \end{cases}$$

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  - And the two polarization components of the wave :

$$\begin{cases} h_+(\tau) = A(\tau)\frac{1+\cos^2(\iota)}{2r}\cos(\Phi(\tau)) \\ h_\times(\tau) = A(\tau)\frac{\cos(\iota)}{r}\sin(\Phi(\tau)) \end{cases}$$

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  - And the two polarization components of the wave :

With redshift

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  - And the two polarization components of the wave :

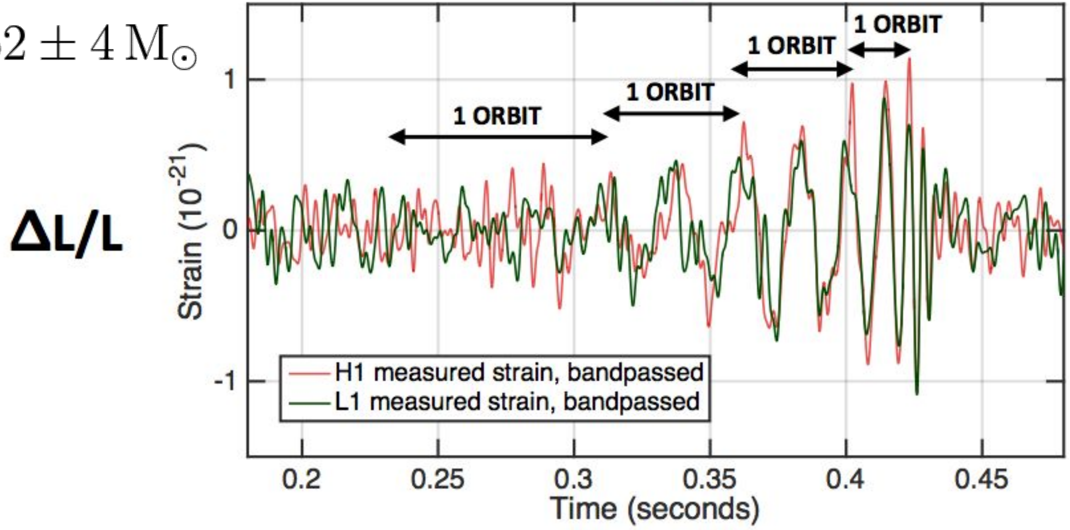
With redshift

$$\begin{cases} h_+(\tau) = A(\tau)\frac{1+\cos^2(\iota)}{2r}\cos(\Phi(\tau)) \\ h_\times(\tau) = A(\tau)\frac{\cos(\iota)}{r}\sin(\Phi(\tau)) \end{cases} \longrightarrow \begin{cases} h_{ob,+}(\tau) = A_{ob}(\tau)\frac{1+\cos^2(\iota)}{2r}\cos(\Phi_{ob}(\tau)) \\ h_{ob,\times}(\tau) = A_{ob}(\tau)\frac{\cos(\iota)}{r}\sin(\Phi_{ob}(\tau)) \end{cases}$$

Compact Binary Coalescence

- First detection : GW150914
  - $D_L = 410^{+160}_{-180}\,\mathrm{Mpc}$
  - $36^{+5}_{-4}\,\mathrm{M}_\odot + 29 \pm 4\,\mathrm{M}_\odot \rightarrow 62 \pm 4\,\mathrm{M}_\odot$

one orbit
=
two GW cycles

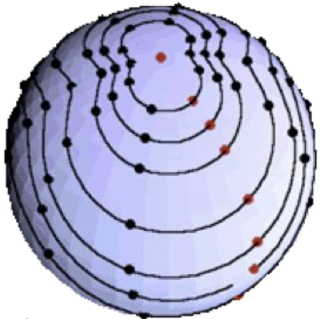


Continuous Waves

- Rotating neutron stars
- Not perfectly spherical

“Mountain” or assymmetries

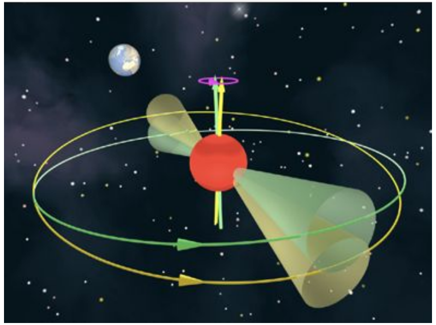
$$\nu \sim 1 - 10^3\ \mathrm{Hz}$$



Oscillation modes

$$h \sim 10^{-25}\ \text{at}\ 3\ \text{kpc}$$

precession



$$h_0 = \frac{4\pi^2 G}{c^4}\frac{I_{zz}\,\varepsilon\, f_{gw}^{\,2}}{d}$$

I_{zz} Moment of inertia along the rotation axis

$$\varepsilon = \frac{I_{xx} - I_{yy}}{I_{zz}}$$

Ellipticity in the equatorial plane

- I_{zz} and ε very poorly known
- Motion and orientation of the detector around the sun
  - Doppler modulation of the signal

End of the first part