STANDARD MODEL

(and beyond ?)

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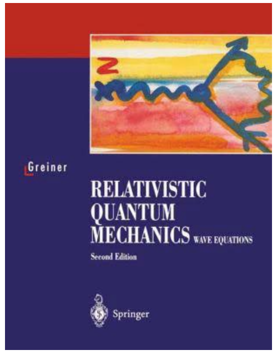
Instructions from the organisers

- parler rapidement de l'eqn de Dirac, les spineurs, les matrices gamma
- la notion de symmétrie de jauge (p.ex. en QED)
- et agiter un peu les mains vers la fin pour dire "il se trouve que les autres interactions fondamentales du SM suivent le même modèle que l'EM, basées sur les symmétries de jauge aussi, mais utilisant des groupes un peu plus complexes p.ex. SU(2)"

Donc pas mal condensé, tout en restant pédagogique

Selective bibliography



My favorites to prepare a lecture

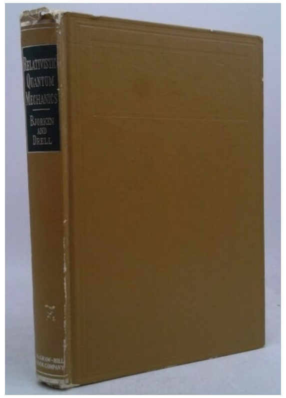
Walter Greiner lectures, Springer

- W. Greiner, **Relativistic Quantum Mechanics**
- W. Greiner et J. Reinhardt, **Quantum Electrodynamics**
- W. Greiner et J. Reinhardt, **Field Quantization**
- W. Greiner, S Schramm er E. Stein, **Quantum Chromodynamics**
- W. Greiner et B. Muller, **Gauge Theory of Weak Interactions**

In french (and very similar to this lecture)

Physique des particules, B.C., Dunod

Mécanique quantique relativiste, B.C., Dunod



The old classics

Relativistic Quantum Mechanics, J.D. Bjorken, S.D. Drell, McGraw-Hill Book Company

Relativistic Quantum Fields, J.D. Bjorken, S.D. Drell, McGraw-Hill Book Company

Introduction to High Energy Physics, Donald H. Perkins, Addison-Wesley

Quarks and Leptons, Francis Halzen et Alan D. Martin, Wiley

For quantum field theory (many books o nthe subject)

Théorie quantique des champs, Jean-Pierre Derendinger, Presses polytechniques et universitaires romandes

Introduction to Quantum Field Theory, S.J. Chang, World Scientific Publishing

An Introduction to Quantum Field Theory , M.E. Peskin and D.V. Schroeder Westview Press Inc.

Many mores : Itzykson-Zuber, Weinberg, Landau-Lifshitz …

Quick introduction to the SM

Energy, length…

$$\text{Energy, } E \quad \underset{E=\hbar\omega}{\overset{\hbar}{\longleftrightarrow}} \quad \text{Pulsation, } \omega \sim \frac{1}{T}$$

$$c \quad \updownarrow \quad E = pc \qquad\qquad c \quad \updownarrow \quad \lambda = \frac{c}{\omega}$$

$$\text{Momentum, } p \quad \underset{p=\frac{\hbar}{\lambda}}{\overset{\hbar}{\longleftrightarrow}} \quad \text{Wave length, } \lambda$$

Quantum mechanics

Energy and Frequency are of the same nature

Special relativity

Time and Space are of the same nature

Natural units

c and $\hbar$ are only conversion factors depending on a choice of units : (Joule, meter, second,…)

$\hbar = c = 1$ (dimensionless)

All quantities have the dimension of a power of energy

$[length] = [time] = [energy]^{-1}$ $\quad [mass] = [momentum] = [energy]$ $\quad [action] = [speed] = 1$

Usual energy unit : electron-volt $\quad 1\text{eV} = 1.6 \times 10^{-19}\text{J}$

A few numbers : $\hbar c = 197\ \text{MeV.fm}$

$m_e = 511\ \text{keV (or keV/c}^2)$ $\quad m_{nucleon} = 940\ \text{MeV}$

… and elementarity

Compton wavelength

$E = \frac{\hbar c}{2\pi\lambda}$: Energy scale ↔ Length scale

To probe a system of size L

if $\lambda \gg L$: punctual system = like an **elementary particle**

if $\lambda \ll L$: sensitive to a sub-structure

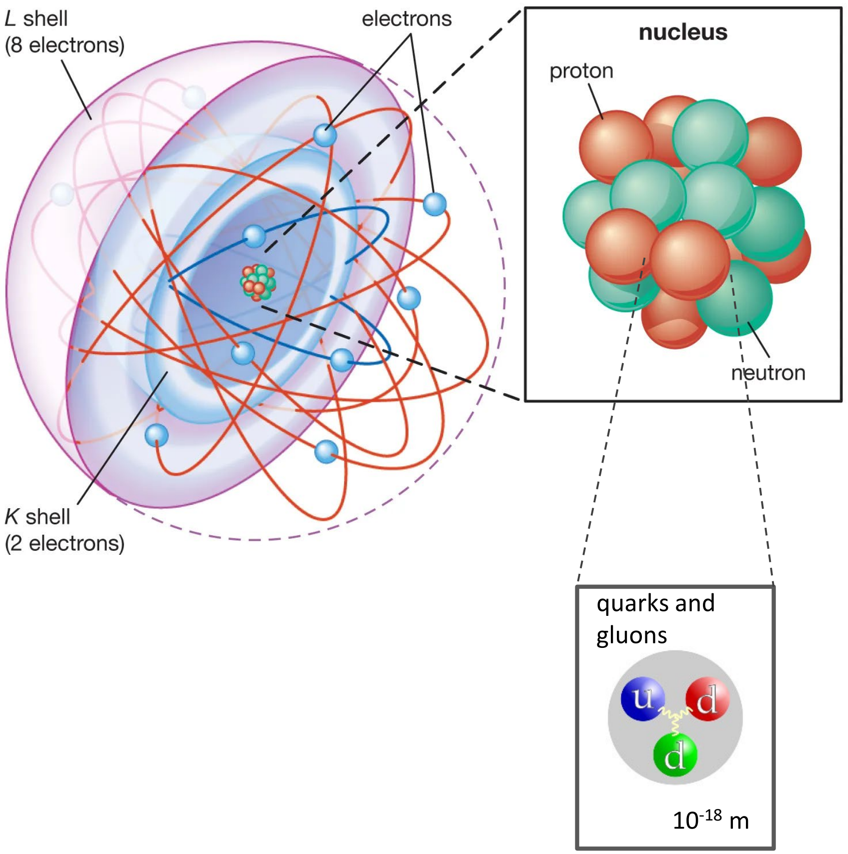
Small length ↔ Large energies

Atom radius : $\lambda \sim \text{Å}\ (10^{-10} m) \rightarrow E \sim 1\ \text{keV (X-ray)}$

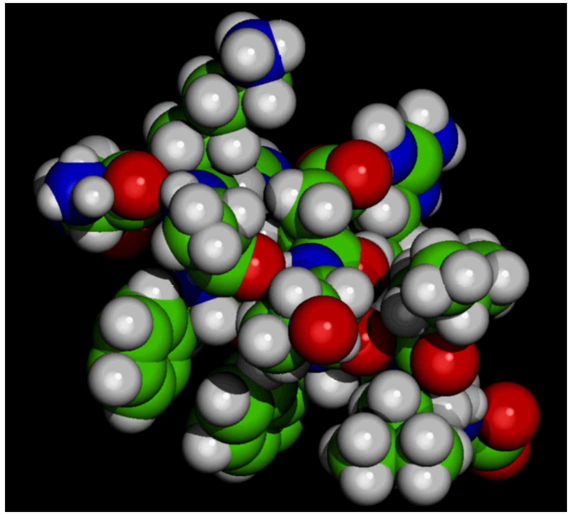
Proton radius : $\lambda \sim \text{Å}\ (10^{-15} m) \rightarrow E \sim 100\ \text{MeV (QCD scale)}$

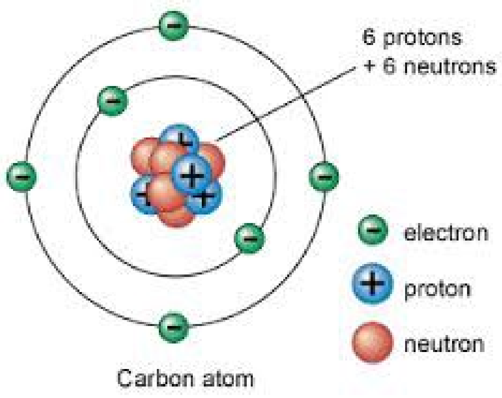
LHC : $E \sim 10\ \text{TeV} \rightarrow \lambda \sim 10^{-21}\ \text{m}\ (\ll \text{quark size})$

No evidence of electron or quark substructure



Every day matter





Electromagnetic interaction

Electrons and **nuclei** in **atoms**

Atomic physics, condensed matter, thermodynamics macro and micro-scopic, chemistry…

Nuclear forces

Nucleus = protons+neutrons

Nucleons = 3 **quarks up or down**

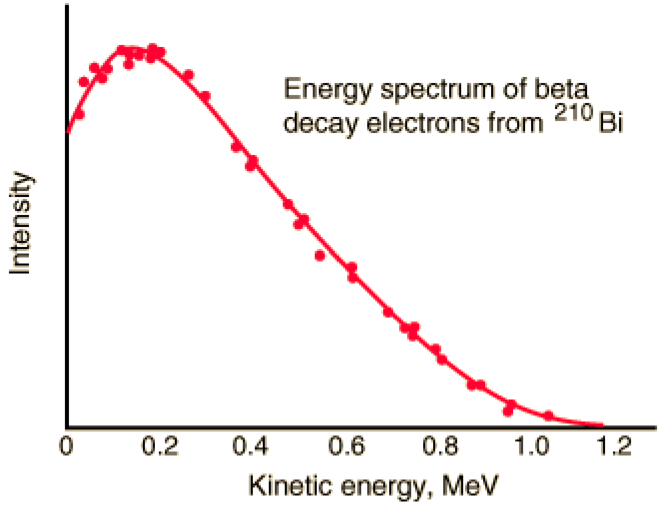
Strong interaction : maintains the cohesion of nucleons an nuclei despite of Coulomb repulsion

Weak interaction : only interaction that modify the nature of particules : $n \leftrightarrow p, u \leftrightarrow d$, $e^- \leftrightarrow \nu_e$

Necessitates a new kind of particle

$\rightarrow$ the **neutrino**

No large scale effects



Continuous electron energy spectrum in β decay

3 body interaction : $n \rightarrow p + e^- + \overline{\nu}_e$

Theoretical tools

Neutron decay

$$\boldsymbol{n} \rightarrow \boldsymbol{p} + \boldsymbol{e}^{-} + \overline{\boldsymbol{\nu}}_{\boldsymbol{e}} \qquad m_n = 938{,}3\ MeV \qquad m_p = 939{,}6\ MeV$$

Released energy : $\Delta m \sim 1\ \mathrm{MeV}$

Subatomic scales $\rightarrow$ **Quantum mechanics**

Kinetic energies of electron and neutrino of the order or larger than mass energy

$\rightarrow$ **Special relativity**

Relativistic Quantum Mechanics

Variable number and nature of particles between initial and final state

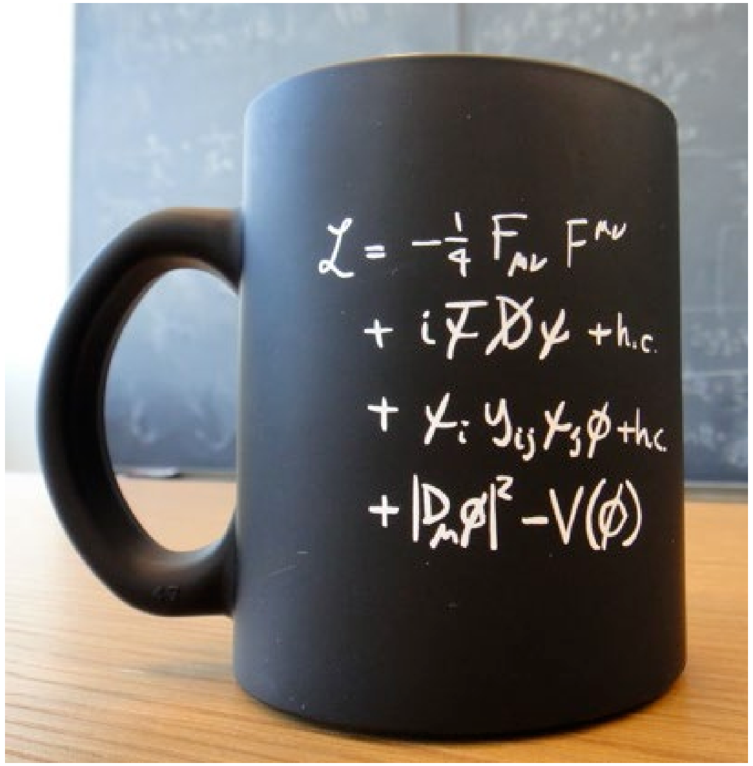
$\rightarrow$ **Second quantificaction, creation/annihilation operators**

Relativistic Quantum Field Theory

The Standard Model

Theoretical description of the interactions of fundamental fermions : quark and leptons

Based on **Gauge invariance** in **Quantum Field Theory**



Interaction processes are described using : the **SM Lagrangian** + **Scattering theory** (Feynman diagrams)

The Standard Model Lagrangian

$$\mathcal{L}_{SM} = \sum_{\ell=e,\mu,\tau} i\bar{\psi}_\ell\gamma^\mu\partial_\mu\psi_\ell + \sum_{\ell'=\nu_e,\nu_\mu,\nu_\tau} i\bar{\psi}_{\ell'}\gamma^\mu\partial_\mu\psi_{\ell'} + \sum_i^3\sum_{q=u,c,t} i\bar{\psi}_{q_i}\gamma^\mu\partial_\mu\psi_{q_i} + \sum_i^3\sum_{q'=d,s,b} i\bar{\psi}_{q'_i}\gamma^\mu\partial_\mu\psi_{q'_i} - \frac{1}{2}(\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)(\partial^\mu W^{-\nu} - \partial^\nu W^{-\mu}) - \frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu)$$

$$-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{4}\sum_{a=1}^{8}(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu)(\partial^\mu G^{a\nu} - \partial^\nu G^{a\mu}) + \frac{1}{2}\partial_\mu h\partial^\mu h - \sum_{\ell=e,\mu,\tau}\frac{\lambda_\ell v}{\sqrt{2}}\bar{\psi}_\ell\psi_\ell - \sum_i^3\sum_{q=u,c,t}\frac{\lambda_q v}{\sqrt{2}}\bar{\psi}_{q_i}\psi_{q_i} - \sum_i^3\sum_{q'=d,s,b}\frac{\lambda_{q'} v}{\sqrt{2}}\bar{\psi}_{q'_i}\psi_{q'_i}$$

$$-\left(\frac{gv}{2}\right)^2 W^+_\mu W^{-\mu} - \frac{1}{2}\left(\frac{gv}{2\cos\theta_W}\right)^2 Z_\mu Z^\mu - \frac{1}{2}\left(-2m^2\right)^2 h^2 + \frac{g}{4\cos\theta_W}\left(\sum_{\ell=e,\mu,\tau}\bar{\psi}_\ell\gamma^\mu(4\sin^2\theta_W - 1 + \gamma^5)\psi_\ell Z_\mu + \sum_{\ell'=\nu_e,\nu_\mu,\nu_\tau}\bar{\psi}_{\ell'}\gamma^\mu(1-\gamma^5)\psi_{\ell'}Z_\mu\right)$$

$$+\frac{g}{4\cos\theta_W}\left(\sum_i^3\sum_{q=u,c,t}\bar{\psi}_{q_i}\gamma^\mu(1-\frac{8}{3}\sin^2\theta_W - \gamma^5)\psi_{q_i}Z_\mu + \sum_i^3\sum_{q'=b,s,b}\bar{\psi}_{q'_i}\gamma^\mu(\frac{4}{3}\sin^2\theta_W - 1 + \gamma^5)\psi_{q'_i}Z_\mu\right) + \frac{g}{2\sqrt{2}}\left(\sum_{\ell=e,\mu,\tau}\bar{\psi}_{\nu_\ell}\gamma^\mu(1-\gamma^5)\psi_\ell W^+_\mu + \sum_{\ell=e,\mu,\tau}\bar{\psi}_\ell\gamma^\mu(1-\gamma^5)\psi_{\nu_\ell}W^-_\mu\right)$$

$$+\frac{g}{2\sqrt{2}}\left(\sum_i^3\sum_{\substack{q=u,c,t\\q'=d,s,b}} V_{qq'}\bar{\psi}_q\gamma^\mu(1-\gamma^5)\psi_{q'_i}W^+_\mu + \sum_i^3\sum_{\substack{q=u,c,t\\q'=d,s,b}} V^*_{qq'}\bar{\psi}_{q'_i}\gamma^\mu(1-\gamma^5)\psi_{q_i}W^-_\mu\right) + g_{em}\left(-\sum_{\ell=e,\mu,\tau}\bar{\psi}_\ell\gamma^\mu\psi_\ell A_\mu + \frac{2}{3}\sum_{q=u,c,t}\bar{\psi}_{q_i}\gamma^\mu\psi_{q_i}A_\mu - \frac{1}{3}\sum_{q'=d,s,b}\bar{\psi}_{q'_i}\gamma^\mu\psi_{q'_i}A_\mu\right)$$

$$+g_s\left(\sum_{i,j}^3\sum_a^8\sum_{q=u,c,t}\bar{\psi}_{q_j}\gamma^\mu\psi_{q_i}G^a_\mu T^a_{ij} + \sum_{i,j}^3\sum_a^8\sum_{q'=d,s,b}\bar{\psi}_{q'_j}\gamma^\mu\psi_{q'_i}G^a_\mu T^a_{ij}\right) - \sum_{\ell=e,\mu,\tau}\frac{\lambda_\ell}{\sqrt{2}}\bar{\psi}_\ell\psi_\ell h - \sum_i^3\sum_{q=u,c,t}\frac{\lambda_q}{\sqrt{2}}\bar{\psi}_{q_i}\psi_{q_i}h - \sum_i^3\sum_{q'=d,s,b}\frac{\lambda_{q'}}{\sqrt{2}}\bar{\psi}_{q'_i}\psi_{q'_i}h$$

$$+\frac{g}{2\sqrt{2}}\left(\sum_i^3\sum_{\substack{q=u,c,t\\q'=d,s,b}} V_{qq'}\bar{\psi}_q\gamma^\mu(1-\gamma^5)\psi_{q'_i}W^+_\mu + \sum_i^3\sum_{\substack{q=u,c,t\\q'=d,s,b}} V^*_{qq'}\bar{\psi}_{q'_i}\gamma^\mu(1-\gamma^5)\psi_{q_i}W^-_\mu\right) + g_{em}\left(-\sum_{\ell=e,\mu,\tau}\bar{\psi}_\ell\gamma^\mu\psi_\ell A_\mu + \frac{2}{3}\sum_{q=u,c,t}\bar{\psi}_{q_i}\gamma^\mu\psi_{q_i}A_\mu - \frac{1}{3}\sum_{q'=d,s,b}\bar{\psi}_{q'_i}\gamma^\mu\psi_{q'_i}A_\mu\right)$$

$$+g_s\left(\sum_{i,j}^3\sum_a^8\sum_{q=u,c,t}\bar{\psi}_{q_j}\gamma^\mu\psi_{q_i}G^a_\mu T^a_{ij} + \sum_{i,j}^3\sum_a^8\sum_{q'=d,s,b}\bar{\psi}_{q'_j}\gamma^\mu\psi_{q'_i}G^a_\mu T^a_{ij}\right) - \sum_{\ell=e,\mu,\tau}\frac{\lambda_\ell}{\sqrt{2}}\bar{\psi}_\ell\psi_\ell h - \sum_i^3\sum_{q=u,c,t}\frac{\lambda_q}{\sqrt{2}}\bar{\psi}_{q_i}\psi_{q_i}h - \sum_i^3\sum_{q'=d,s,b}\frac{\lambda_{q'}}{\sqrt{2}}\bar{\psi}_{q'_i}\psi_{q'_i}h$$

$$+ig_{em}\left[\partial_\mu A_\nu W^{-\mu}W^{+\nu} + \partial_\mu W^+_\nu W^{-\nu}A^\mu + \partial_\mu W^-_\nu W^{+\mu}A^\nu - \partial_\mu A_\nu W^{-\nu}W^{+\mu} - \partial_\mu W^+_\nu W^{-\mu}A^\nu - \partial_\mu W^-_\nu W^{+\nu}A^\mu\right]$$

$$+ig\cos\theta_W\left[\partial_\mu Z_\nu W^{-\mu}W^{+\nu} + \partial_\mu W^+_\nu W^{-\nu}Z^\mu + \partial_\mu W^-_\nu W^{+\mu}Z^\nu - \partial_\mu Z_\nu W^{-\nu}W^{+\mu} - \partial_\mu W^+_\nu W^{-\mu}Z^\nu - \partial_\mu W^-_\nu W^{+\nu}Z^\mu\right] + \frac{g^2 v}{2}W^+_\mu W^{-\mu}h + \frac{g^2 v}{4\cos^2\theta_W}Z_\mu Z^\mu h - \lambda v h^3$$

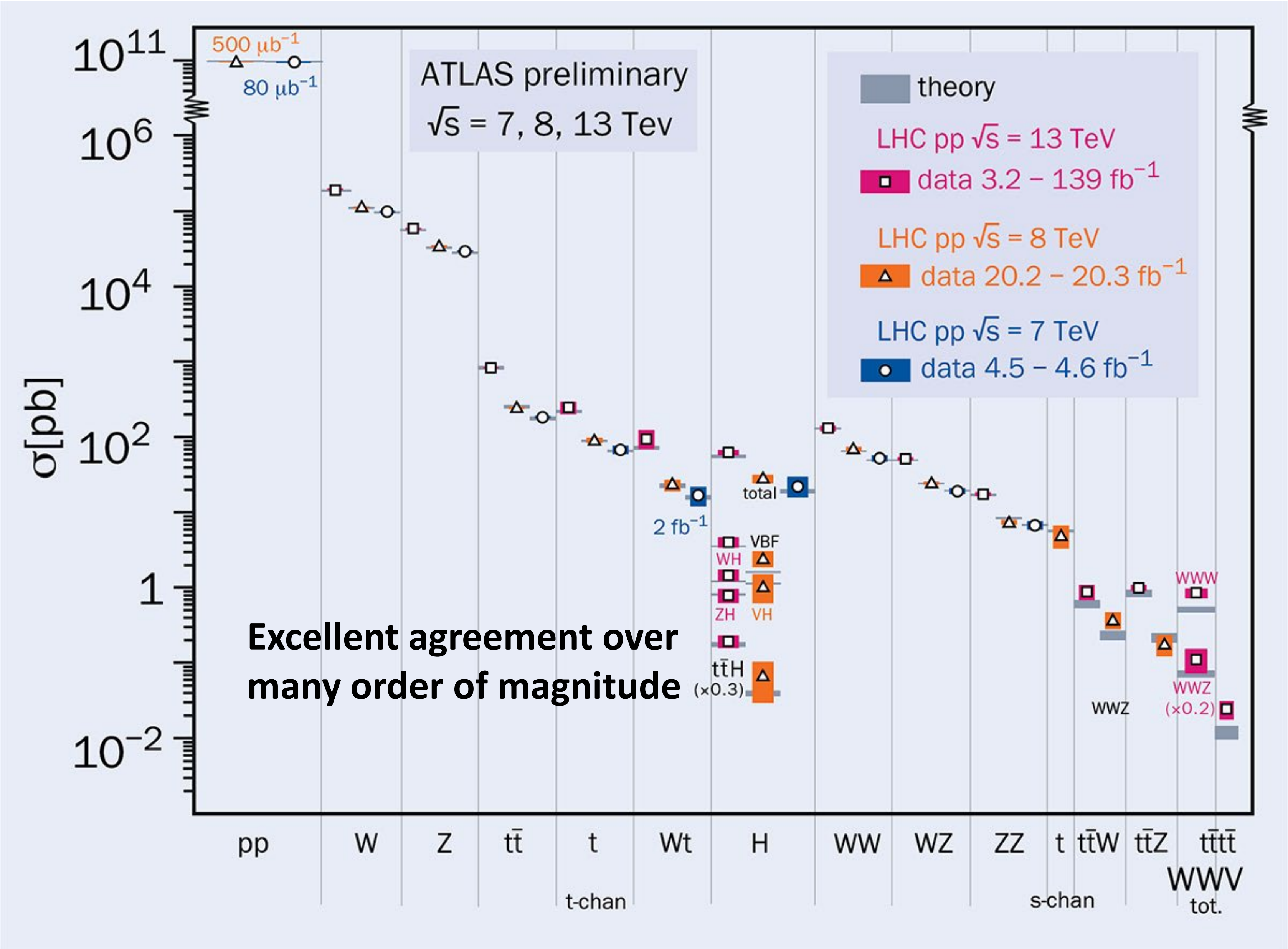
$$+g^2_{em}\left[W^+_\nu W^{-\mu}A_\nu A^\mu - W^+_\mu W^{-\mu}A_\nu A^\nu\right] + g^2\cos^2\theta_W\left[W^+_\nu W^{-\mu}Z_\nu Z^\mu - W^+_\mu W^{-\mu}Z_\nu Z^\nu\right] + g^2\cos\theta_W\sin\theta_W\left[2W^+_\mu W^{-\mu}Z_\nu A^\nu - W^+_\mu W^{-\nu}A_\nu Z^\mu - W^+_\mu W^{-\nu}A^\mu Z_\nu\right]$$

$$+\frac{g^2}{2}\left[W^-_\mu W^{-\mu}W^+_\nu W^{+\nu} - W^-_\mu W^{+\mu}W^-_\nu W^{+\nu}\right] + \frac{g^2}{4}W^+_\mu W^{-\mu}h^2 + \frac{g^2}{8\cos^2\theta_W}Z_\mu Z^\mu h^2 - \frac{\lambda}{4}h^4 - \frac{g_s}{2}\sum_{a,b,c}^{8} f^{abc}(\partial_\mu G^{a\nu} - \partial_\nu G^a_\mu)G^{\mu b}G^{\nu c} - \frac{g_s^2}{4}\sum_{\substack{a,b,c\\d,e,f}}^{8} f^{abc}f^{ade}G^b_\mu G^c_\nu G^{\mu d}G^{\nu e}$$

$$g_{em} = g\sin\theta_W,\ v^2 = \frac{-m^2}{\lambda}\ (m^2 < 0, \lambda > 0),\ m_\ell = \frac{\lambda_\ell v}{\sqrt{2}},\ m_q = \frac{3\lambda_q v}{\sqrt{2}},\ m_W = \frac{gv}{2},\ m_Z = \frac{gv}{2\cos\theta_W},\ m_h = \sqrt{-2m^2}$$

Exercice : find the typo(s)

Cross sections in the SM



A bit of theoretical framework

(don’t worry too much about the details of the equations)

Beyond classical mechanics

Classical mechanics

Time evolution of discrete coordinates $x_i(t)$

Euler-Lagrange equations $\frac{\partial \mathcal{L}}{\partial x_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = 0, \quad p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}_i}$

Poisson brackets $\{x_i, p_j\} = \delta_{ij}, \quad \{x_i, x_j\} = \{p_i, p_j\} = 0$

Quantum physics

Schrödinger equation : $i\partial_0|\psi\rangle = -\frac{\vec{\nabla}^2}{2m}|\psi\rangle + \hat{V}|\psi\rangle$

Time evolution of wavefunction ψ(x,t)

|ψ(x)|^2 : presence probability

Observables become operators, canonical commutations

$$[x_i, p_j] = i\hbar\delta_{ij}, \qquad [x_i, x_j] = [p_i, p_j] = 0$$

Relativistic field theory (Maxwell EM)

Space-time evolution of a field $\varphi_i(x)$

Lagrangian density is a function of fields :

$$\mathcal{L}(\{\varphi_i\}, \{\partial_k\varphi_i\}, \{x_k\})$$

Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \varphi_i} - \partial_k \frac{\partial \mathcal{L}}{\partial \partial_k \varphi_i} = 0, \quad \pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}_i}$$

Quantum field theory

Fields become operators : $\hat{\varphi}_i(x)$

$\hat{\varphi}_i^+\hat{\varphi}_i$: particle+antiparticles density operator

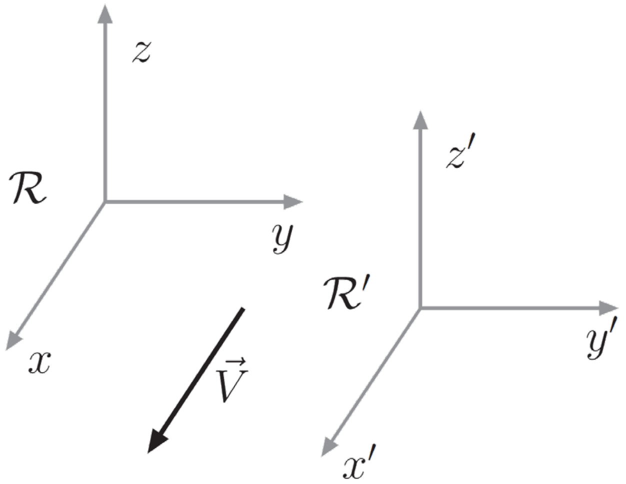
Observables are functionials on fields

Canonical commutations

$$[\hat{\varphi}_i(t,x), \hat{\pi}_j(t,x')] = i\hbar\delta_{ij}\delta(x-x'), \qquad [\hat{\varphi}_i(t,x), \hat{\varphi}_j(t,x')] = [\hat{\pi}_i(t,x), \hat{\pi}_j(t,x')] = 0$$

Relativity 101

How do coordinates transform between two inertial referential ?



Relativity principle

The law of physics are the same in all inertial referentials

"Time is absolute"

Galileo Transformation :

$$\begin{cases} t' = t \\ x' = x - Vt \\ y' = y \\ z' = z \end{cases}$$

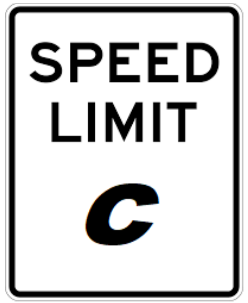
Speeds are additive quantities

$\dot{x} = \dot{x}' + V$

No limit to speed

"There is a speed «*c*» that is the same in all referentials"

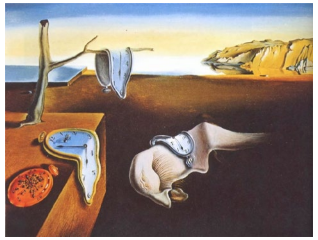
Lorentz Transformation :



$$\begin{cases} ct' = \gamma ct - \gamma\beta x \\ x' = -\gamma\beta ct + \gamma x \\ y' = y \\ z' = z \end{cases} \qquad \beta = \frac{V}{c}, \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Speeds are no longer additive

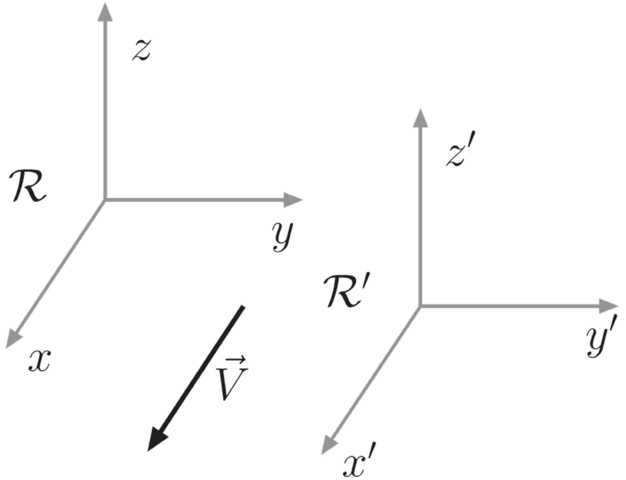
Time changes from one referential to another !!!



The quantity $s^2 = (ct)^2 - x^2 - y^2 - z^2$
$= (ct')^2 - x'^2 - y'^2 - z'^2$ is invariant in a referential change.

Relativity 101

How do coordinates transform between two inertial referential ?



Relativity principle

The law of physics are the same in all inertial referentials

Motivated by electromagnetism

EM wave equation :

$$\mu_0 \varepsilon_0 \frac{\partial^2 \Phi}{\partial t^2} - \Delta \Phi = 0, \qquad \Phi = U, \vec{A}, \vec{E}, \vec{B}$$

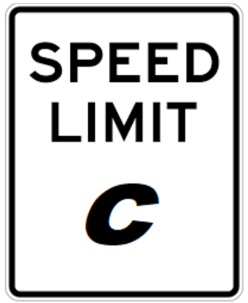
Wave propagating at speed $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

independently of the speed of the source !

Same light velocity

"There is a speed «*c*» that is the same in all referentials"

Lorentz Transformation :



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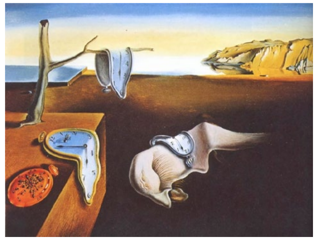


The quantity $s^2 = (ct)^2 - x^2 - y^2 - z^2$
$= (ct')^2 - x'^2 - y'^2 - z'^2$ is invariant in a referential change.

Relativity 101: 4-vectors

4-vector

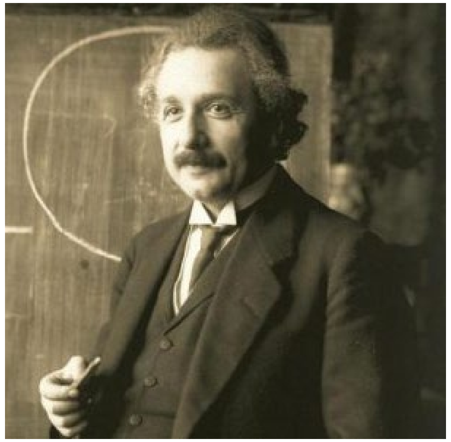
$$A = \left(A^0, \vec{A}\right) = (A^0, A^1, A^2, A^3)$$

$$\begin{cases} A'^0 = \gamma A^0 - \beta A^1 \\ A'^1 = \beta A^0 - \gamma A^1 \\ A'^2 = A^2 \\ A'^3 = A^3 \end{cases}$$ Same Lorentz transform

Einstein sum convention

$$g_{\mu\nu} x^\nu = \sum_\nu g_{\mu\nu} x^\nu$$

Repeated upper/lower index implies sum !!!



Metric Tensor

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Contravariant, covariant and norm

$$A_\mu = g_{\mu\nu} A^\nu$$

$A^2 = A_\mu A^\mu = (A^0)^2 - \left|\vec{A}\right|^2$ is a Lorentz invariant

$A.B = A_\mu B^\mu = A^\mu B_\mu$ is also Lorentz invariant

4-momentum

$$p^\mu = \gamma m \partial_0 x^\mu \qquad p \xrightarrow[\gamma\to 1]{} (m, m\vec{v})$$

$$p = (E, \vec{p})$$

$$p^2 = p_\mu p^\mu = E^2 - |\vec{p}|^2 = m^2 \Rightarrow \boldsymbol{E^2 = m^2 + |\vec{p}|^2}$$

Total 4-momentum is conserved in interactions

Relativistic Lagrangian and Hamiltonian

$S = \int \mathcal{L} dt$ must be Lorentz invariant, i.e.

$$\mathcal{L}dt \propto ds = \sqrt{dt^2 - d\vec{x}^2} \Rightarrow \mathcal{L} = \alpha\sqrt{1-\beta^2}$$

$$p^i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = -\alpha\gamma\dot{x}^i = \gamma m \dot{x}^i \Rightarrow \alpha = -m$$

$$H = \vec{p}.\dot{\vec{x}} - L = \gamma m = p^0$$

4-gradiant

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \qquad \partial = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$$

$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \Delta = \Box$ (d'Alembertian)

Relativity 101: Summary

Time and space are not independent

The speed of light (in a vacuum) is the maximum speed of a particle

Lorentz invariance builds quantities that are the same in all referentials : Scalar products of 4-vectors

Laws of physics does not depends on referential : Action and Lagrangian must be Lorentz Invariants

Quantum mechanics 101

Classical mechanics

1 parameter : time
Generalized coordinates : $x_i(t)$
Observables : position, momentum, energy are numerical values
Canonical Poisson brackets :

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = \{p_i, p_j\} = 0$$

Quantum mechanics

1 parameter : time
Wavefuction : $\psi(x,t)$ such as $\int \psi(x,t)\psi^*(x,t)dx = 1$
$\rho = |\psi(x,t)|^2$: presence probability density at position x
Form a vector space (Hilbert space) : state $|\psi(x,t)\rangle$
Observables : position, momentum, energy… becomes operators over the Hilbert space
Canonical commutators

$$[x_i, p_j] = i\hbar\delta_{ij}, \qquad [x_i, x_j] = [p_i, p_j] = 0$$

Warning

There is 2 different concepts of position in QM
- The parameter of the wavefunction $\psi(\boldsymbol{x}, t)$
- The position operator : $\hat{\boldsymbol{x}}$ with $\hat{\boldsymbol{x}}\,|\psi(x)\rangle = x|\psi(x)\rangle$

Deriving operators

Position operator : $\hat{\boldsymbol{x}}_{\boldsymbol{i}} = \boldsymbol{x}_{\boldsymbol{i}}$
Momentum operator : $[x, \partial_x]|\psi(x)\rangle = -|\psi(x)\rangle \Rightarrow \hat{\boldsymbol{p}}_{\boldsymbol{i}} = -\boldsymbol{i\hbar\partial_i}$
Energy operator : Hamilton equation : $\partial_0 p_i = -\partial_i H \Rightarrow i\hbar\partial_i\partial_0|\psi(x)\rangle = \partial_i\hat{H}|\psi(x)\rangle \Rightarrow \hat{\boldsymbol{H}} = \boldsymbol{i\hbar\partial_0}$

Quantizing the dispersion relation

$E = \frac{p^2}{2m} + V \xrightarrow[quantum]{}$ operator $i\hbar\partial_0 = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + \hat{V}$ applied to $|\psi\rangle$: $\boldsymbol{i\hbar\partial_0|\psi\rangle = -\frac{\hbar^2}{2m}\vec{\nabla}^2|\psi\rangle + \hat{V}|\psi\rangle}$ **Schrödinger equation**

The simplest quantum relativistic equation

Klein Gordon equation

$$E \xrightarrow[quantum]{} \hat{H} = i\partial_0 \qquad p \xrightarrow[quantum]{} \hat{p} = -i\partial_i$$

$E^2 = m^2 + p^2 \xrightarrow[quantum]{}$ operator $-\partial_0\partial^0 = m^2 + \partial_i\,\partial^i$ applied to φ

$$\boldsymbol{(\partial_\mu\partial^\mu + m^2)\varphi = 0}$$

Scalar field

φ is a complex numerical function

It can only describe a spin 0 object

Planewave solution (Fourier modes)

$$\varphi_{\vec{k}}^{\pm}(x) = \frac{1}{\sqrt{2E_k}(2\pi)^{\frac{3}{2}}} e^{\pm i k_\mu x^\mu}$$

with $k_0 = \pm E_k = \pm\sqrt{m^2 + \left|\vec{k}\right|^2}$

Negative energy solutions !!!

Antiparticles

Absorb the minus sign into the time : $e^{-i(Et - \vec{k}.\vec{x})} = e^{-i\left((-E)(-t) + (-\vec{k}).\vec{x}\right)}$

Particle with **negative energy**, moving **forward in time**, with **momentum** $-\vec{k}$

KG naïve solution

=

Particle with **positive energy**, moving **backward in time**, with **momentum** $\vec{k}$

Interpetation used in relativistic quantum mechanics

=

Antiparticle with **positive energy**, moving **forward in time**, with **momentum** $\vec{k}$

Effectively observed

Antiparticle : same mass, same spin, opposite chargeS

Conserved current

Conservation equation

Equation of the form : $\vec{\nabla}.\vec{J} = \frac{\partial \rho}{\partial t}$ or $\frac{\partial}{\partial t}\iiint \rho dV = \oiint \vec{J}.d\vec{S}$

Found in fluid mechanics : conservation of matter

in electromagntism : conservation of charge

In covariant formalism : $\partial_\mu j^\mu = 0$, with $j = (\rho, \vec{J})$ a 4-current

In quantum mechanics (Schrödinger)

$$\frac{\partial \psi^* \psi}{\partial t} = \frac{i\hbar}{2m}\vec{\nabla}.[\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*]$$

Probability current conservation

The density part is $\rho = \psi^*\psi$, positive real number

If ψ is normalized, ρ can be interpreted as a probability density

This is one of the **postulates of quantum mechanics**

In relativistic quantum mechanics (Klein-Gordon)

$$\partial_\mu(\varphi^*\partial^\mu\varphi - \varphi\partial^\mu\varphi^*) = 0$$

The density part is $\rho = \varphi^*\partial^0\varphi - \varphi\partial^0\varphi^*$ is no longer always positive,

The probability interpretation is lost

QFT

$$\varphi_{\vec{k}}^{\pm}(x) = \frac{1}{\sqrt{2E_k}(2\pi)^{\frac{3}{2}}} e^{\pm ik_\mu x^\mu}$$

$$\hat{\varphi}_{\vec{k}}^{\pm}(x) = \frac{\hat{a}^{\pm}}{\sqrt{2E_k}(2\pi)^{\frac{3}{2}}} e^{\pm ik_\mu x^\mu}$$

$\hat{a}^{\pm}$: creation/annihilation operators

$\varphi_{\vec{k}}^{\pm *}\varphi_{\vec{k}}^{\pm} = \hat{n}_k$: number of particles and antiparticules wit momentum k

The Dirac equation

Relativistic Schrödinger-like equation

$i\partial^0\psi = \widehat{H}\psi$

To ensure covariance, time and space must have the same power,

$\widehat{H} = -i\alpha_1\partial^1 - i\alpha_2\partial^2 - i\alpha_3\partial^3 + \beta m$

With :

- ψ is a vector of size n
- α_i et β are $n \times n$ matrices
- each component ψ_u is a scalar field : it satisfy the KG equation

Derivation of the Dirac Equation

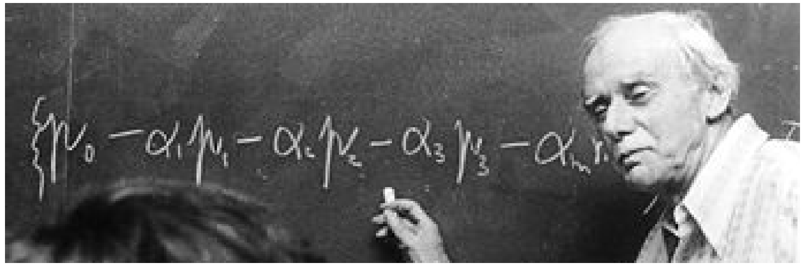
$$-\partial_0\partial^0\psi_u = \partial_i\partial^i\psi_u + m\psi_u$$

$$= \frac{1}{2}(\alpha_i\alpha_j + \alpha_j\alpha_i)\partial_i\partial^j\psi_u - i\frac{1}{2}(\alpha_i\beta + \beta\alpha_i)m\partial^i\psi_u + \beta^2 m\psi_u$$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}\mathbb{I}_n \qquad \{\alpha_i, \beta\} = 0 \qquad \beta^2 = \alpha_i^2 = \mathbb{I}_n$$

$$\partial_0\psi = i\alpha_i\partial^i\psi + \beta m \Rightarrow i\underbrace{\beta}_{\gamma_0}\partial^0\psi + i\underbrace{\beta\alpha_i}_{\gamma_i}\partial^i\psi - m\psi = 0$$

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \qquad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_n$$



The Dirac matrices

Only even n are possible

The first n satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_n$ is $n = 4$

ψ is a 4 components vector $\neq$ 4-vector : spinor

Describes spin ½ particles and antiparticles

One possible set of matrices :

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Instead of $\psi^+ \rightarrow \bar{\psi} = \psi^+\gamma^0$

$\bar{\psi}\gamma_\mu\psi$ is a 4-vector, $\bar{\psi}\psi$, is a Lorentz scalar

Dirac Lagrangian

$$\mathcal{L}_{Dirac}(\psi, \bar{\psi}, \{\partial_\mu\psi\}, \{\partial_\mu\bar{\psi}\}) = \bar{\psi}(i\gamma_\mu\partial^\mu + m)\psi$$

Electromagnetism and gauge

4-Potential

$\vec{\nabla}.\vec{B} = 0$

$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0$

} one can define a 4-vector A

$\vec{B} = \vec{\nabla} \times \vec{A}$ and $\vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t}$

A^0: scalar potential, $\vec{A}$: vector potential

Gauge

The 4-potential is not unique (but $\vec{E}$ and $\vec{B}$ are)

If A is a 4-potential, then

$$A'^{\mu} = A^{\mu} + \partial^{\mu}\xi(x)$$

gives the same fields

$$F'^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} + \partial^{\mu}\partial^{\nu}\xi - \partial^{\nu}\partial^{\mu}\xi$$
$$= \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$

Gauge gives an extra freedom in the choice of potential to simplify some equations

Maxwell Equations

$$\vec{\nabla}.\vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}.\vec{E} = \rho$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}$$

EM strength tensor

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$$

$$F = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Lorenz Gauge

One can always find a function $\xi(x)$such as

$$\partial_{\mu}A'^{\mu} = 0$$

Then : $\partial_{\mu}\partial^{\mu}\xi = \partial_{\mu}A^{\mu}$ which always admit a solution

Vector Field equation

$\vec{\nabla}.\vec{E} = \rho$

$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + \vec{j}$

} $\partial_{\mu}F^{\mu\nu} = j^{\nu}$

$j = (\rho, \vec{j})$: 4-current

$$\partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} = j^{\nu}$$

Vector field free lagrangian

$$\partial^{\mu}A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} = 0 \text{ free}$$

is the Euler-Lagrange equation from lagrangian

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

Free solutions of relativistic equations

Fourier Modes

Klein-Gordon, Dirac, Maxwell…

$$\psi^{\pm}_{\vec{k}} = \frac{1}{\sqrt{2E_k}}\frac{1}{(2\pi)^{\frac{3}{2}}} K e^{\pm i k_\mu x^\mu}$$

$-$: particle solution

$+$: antiparticule solutions

Klein-Gordon

Complex scalar field : spin 0 particles, $K = 1$

Dirac

Complex spinor field : spin ½ particles

$K = u_{k,s}$ or $v_{k,s}$: base of spinors space, $s = 1,2$

u_1: particle of spin -½ u_2: particle of spin +½

v_1: antiparticle of spin -½ v_2: antiparticle of spin +½

$$(k_\mu \gamma^\mu + m)u = 0 \qquad (k_\mu \gamma^\mu - m)v = 0$$

Maxwell (and Proca)

Real vector field : spin 1 particles

$K = \varepsilon_{1,2,(3)}$: polarisation 4-vector, $k_\mu \varepsilon_i^\mu = 0$

Because of gauge symmtery :

2 polarisation for a massless field, 3 for a massive field

Generic solution : Quantum Fields

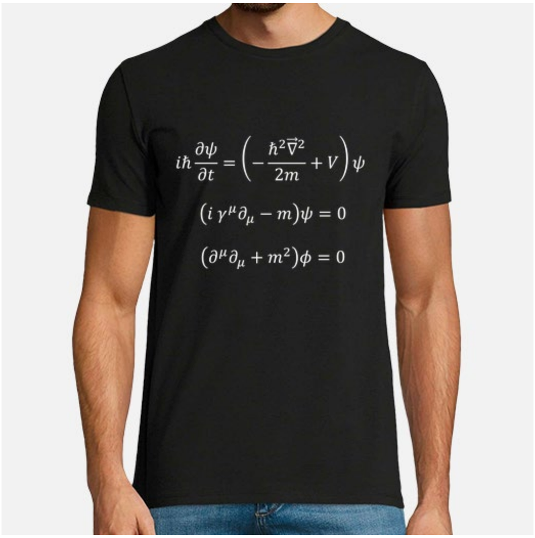
Exemple of the Dirac field :

$$\psi(x) = \sum_s \int \mathrm{d}^3\vec{k} \frac{1}{\sqrt{2E_k}}\frac{1}{(2\pi)^{\frac{3}{2}}}\left(\boldsymbol{b}_{k,s}\boldsymbol{u}_{k,s}\, e^{-ik_\mu x^\mu} + \boldsymbol{d}^{+}_{k,s}\boldsymbol{v}_{k,s}\, e^{ik_\mu x^\mu}\right)$$

$\boldsymbol{b}_{k,s}$: annihilation operator of a particles

$\boldsymbol{d}^{+}_{k,s}$: creation operator of an antiparticle

$\boldsymbol{d}_{k,s}$ and $\boldsymbol{b}^{+}_{k,s}$ are found in the complex conjugate $\bar{\psi}(x)$



Gauge invariance and QED

Quantum Mechanics

Physics is invariant under a change of phase of the wavefunction

$$\psi' = e^{i\theta}\psi \Rightarrow |\psi'|^2 = |\psi|^2$$

Special relativity

Local theory : no instantaneous interaction

Invariance under dephasing must be local

$$\psi' = e^{i\theta(x)}\psi$$

(Non)invariance of the Dirac lagrangian

The space dependent phase does not commute with the derivative

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi$$

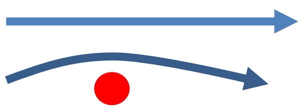
$$\mathcal{L}' = \bar{\psi}'(i\gamma_\mu\partial^\mu - m)\psi' = \bar{\psi}e^{-i\theta(x)}(i\gamma_\mu\partial^\mu - m)e^{i\theta(x)}\psi$$

$$= \bar{\psi}(i\gamma_\mu\partial^\mu - \gamma_\mu\partial^\mu\theta - m)\psi$$

Covariant derivative

Free propagation : $\vec{p} = m\vec{v}$

Interaction : $\vec{p} = m\vec{v} + q\vec{A}$



Absorb the deformation into derivation : make the line straight again

$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$, new vector field A

Full lagrangian invariance

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - q\gamma_\mu A^\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$\mathcal{L}' = \bar{\psi}\left(i\gamma_\mu\partial^\mu - \gamma_\mu q\left(\frac{1}{q}\partial^\mu\theta + A'^\mu\right) - m\right)\psi - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$$

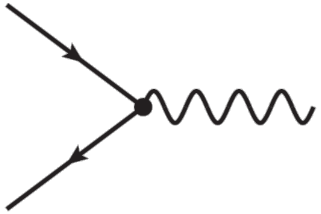
$$= \bar{\psi}(i\gamma_\mu\partial^\mu - \gamma_\mu qA^\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$A^\mu = \frac{1}{q}\partial^\mu\theta + A'^\mu$ is an EM gauge transform, it leaves $F'_{\mu\nu}$invariant

QED lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - q\,\bar{\psi}\gamma_\mu\psi A^\mu$$

Free fermions **Free photons** **coupling**



Euler-Lagrange equations :

Dirac : $(i\gamma_\mu\partial^\mu - m)\psi = q\,\gamma_\mu\psi A^\mu$

Maxwell : $\partial_\mu F^{\mu\nu} = q\bar{\psi}\gamma^\nu\psi = j^\nu$

Yang-Mills theory

Yang-Mills theory

Lagrangian for a free fermion (electron)

↓

Impose a **local symmetry** to the Lagrangian

↓

Introduce a new interacting massless vector (photon) field

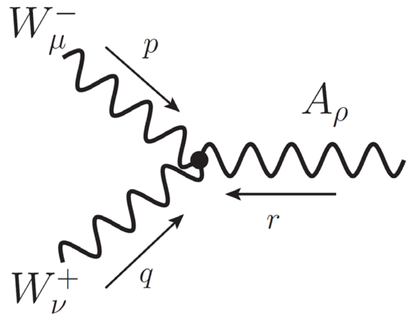
Symmetries and groups

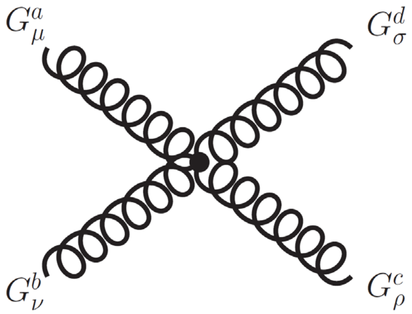
Symmetries = Invariance under set of transformation
→ Group (mathematical structure)

Continuous, differentiable group : **Lie group**
To preserve the normalization of the fields : **Unitary group**

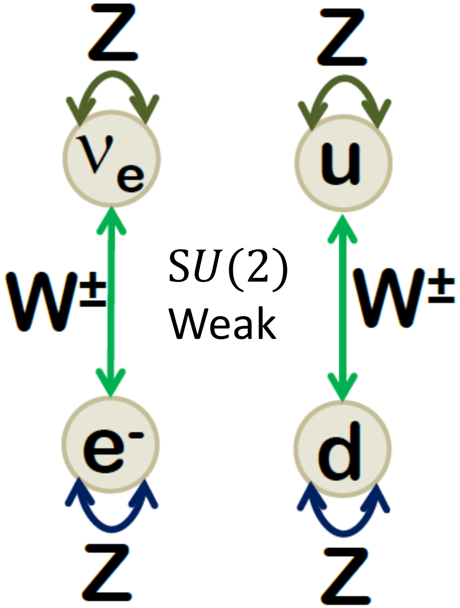
Gauge groups of the Standard Model

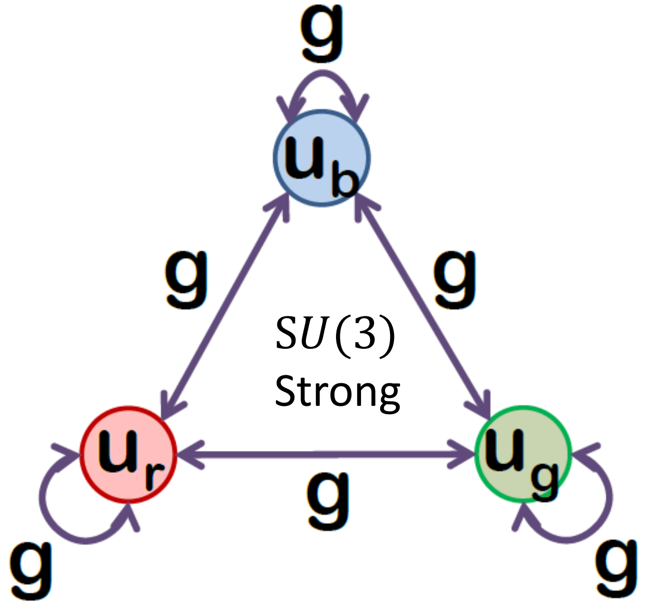
$SU(2) \times U(1)$: electroweakweak interaction
bosons $W^{\pm}, Z, \gamma$
$SU(3)$: strong interaction :
bosons : 8 gluons
Non abelian groups : Self couplings





$U(1)$
Electromagnetism





Higgs mechanism

Massive vector fields

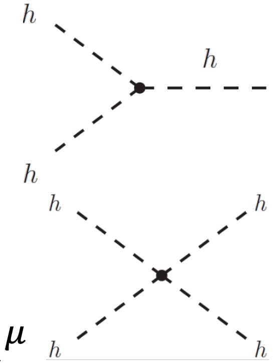
The Maxwell Lagrangian can be extended to include a mass term, analogous to the KG mass

$$\mathcal{L}_{KG} = \frac{1}{2}\left(\partial_\mu \varphi \partial^\mu \varphi^+ - m^2 \varphi \varphi^+\right)$$

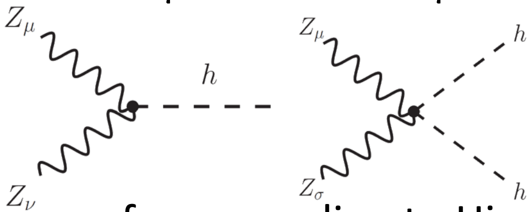
$$\mathcal{L}_{Proca} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} M^2 A_\mu A^\mu$$

But this term breaks gauge symmetry !!!

Total lagrangian



$$\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - q\, \bar{\psi}\gamma_\mu \psi A^\mu$$
$$+\frac{1}{2}\left[\partial_\mu h^+ \partial^\mu h - (2m^2)h^2 - \lambda v h^3 - \lambda h^4\right]$$
$$-\frac{1}{2}(gv)^2 A_\mu A^\mu - g^2 v h A_\mu A^\mu - g^2 h^2 A_\mu A^\mu$$



Mass term for vector boson from coupling to Higgs v.e.v.

Higgs Field

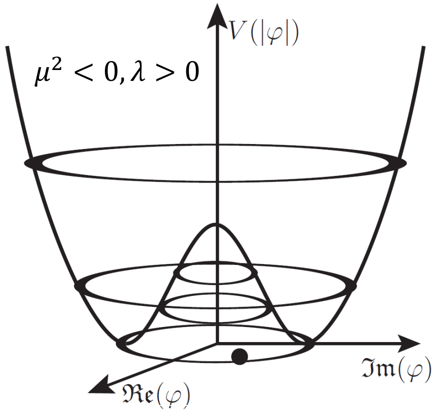
Add a **complex scalar field** to the theory, with a specific potential

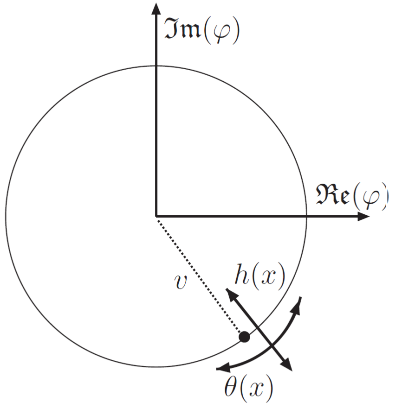
$$\mathcal{L}_{Higgs} = \frac{1}{2}\left(D_\mu \varphi D^\mu \varphi^+ - \mu^2 |\varphi|^2 - \lambda |\varphi|^4\right)$$

$-\boldsymbol{V(\varphi)}$

The potential minimum (vacuum) is degenerate

It takes one specific value $v e^{i\theta(x)} = \sqrt{-\frac{\mu^2}{2\lambda}}\, e^{i\theta(x)}$





And the field can be rewritten as $\varphi(x) = (v + \boldsymbol{h(x)})e^{i\theta(x)}$

$h(x)$: Real scalar field, with « normal » KG dynamics

$\theta(x)$: Goldstone bosons, no physical effects because of gauge invariance !!!

A bit of scattering theory

(keep don’t worrying too much about the details of the equations)

Propagators

Green function

$D_x\psi(x) = \rho(x)$, D_x : differential operator at point x

Admits a solution of the form

$$\psi(x) = \psi_0(x) + \int G(x-y)\rho(y)dy$$

With $\psi_0(x)$, solution of the free equation $D_x\psi(x) = 0$

$$D_x\,\psi(x) = \int D_x G(x-y)\rho(y)dy = \rho(x)$$

$$D_x G(x-y) = \delta(x-y)$$

$G(x-y)$ is the Green function of the differential equation or propagator.

Dirac equation with source

$$(i\gamma_\mu\partial^\mu - m)\psi = q\gamma_\mu A^\mu\psi$$

Admits a solution of the form

$$\boldsymbol{\psi(x) = \psi_0(x) + \int S_F(x-y)q\gamma_\mu A^\mu(y)\psi(y)dy}$$

Integral equation !!!

$$\psi(x) = \psi_0(x)$$
$$+q\int S_F(x-y)\gamma_\mu A^\mu(y)\psi_0(y)dy$$
$$+q^2\int S_F(x-z)\gamma_\mu A^\mu(z)S_F(z-y)\gamma_\nu A^\nu(y)\psi_0(y)dy$$
$$+q^3\int\ldots + q^4\int\ldots + \cdots$$

Propagator in Fourier space

$$S_F(x-y) = \frac{1}{(2\pi)^4}\int S_F(k)d^4k\,,\ S_F(k) = \frac{k_\mu\gamma^\mu + m}{k^2-m^2}$$

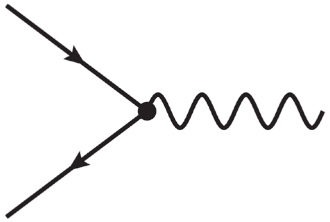
!!!

And $S_F(x-y)\sim\psi\bar{\psi}$

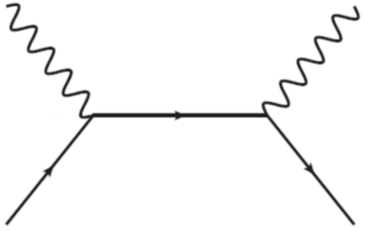
For scalars $\Delta(k) = \frac{1}{k^2-m^2}$

For vectors $D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2}$ or $D_W^{\mu\nu}(k) = \frac{-g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}}{k^2-M^2}$

ψ_0 : free propagation



$S_F\gamma_\mu A^\mu\psi_0$: single vertex interaction

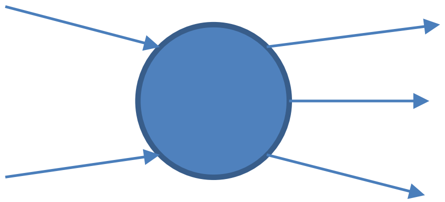


$S_F\gamma_\mu A^\mu\psi_0 S_F\gamma_\nu A^\nu$:
two vertices interaction

Scattering amplitude

Scattering process

Initial state $|i\rangle$ free at $t = -\infty$



Final state $|f\rangle$ free at $t = +\infty$

$1 \to n$ decay : lifetime, decay width $\Gamma = 1/\tau$

$2 \to n$ scattering : cross-section σ

$$\Gamma, \sigma \sim \left|S_{fi}\right|^2$$

Scattering amplitude

$S_{fi} = \langle i|S|f\rangle = \langle \Psi_i|\psi_f\rangle$ overlap over propagated intial state and the final state

$$= \delta_{fi} + q\int \bar{\psi}\gamma^\mu A_\mu \psi dx + q^2 \int \bar{\psi}\gamma^\mu A_\mu S_F \gamma^\nu A_\nu \psi dx dy + \cdots$$

$$S \approx \mathbb{I} + \int \mathcal{L}(x)dx + \int \mathcal{L}(x)\mathcal{L}(y)dxdy + \cdots$$

Perturbative expension in powers of the coupling,

$\mathcal{L}$ is the **interaction Lagrangian**.

QED interaction lagrangian and vertices

$$\mathcal{L}(x) = q\bar{\psi}\gamma^\mu\psi A_\mu$$

$\psi \sim b^+ + d$

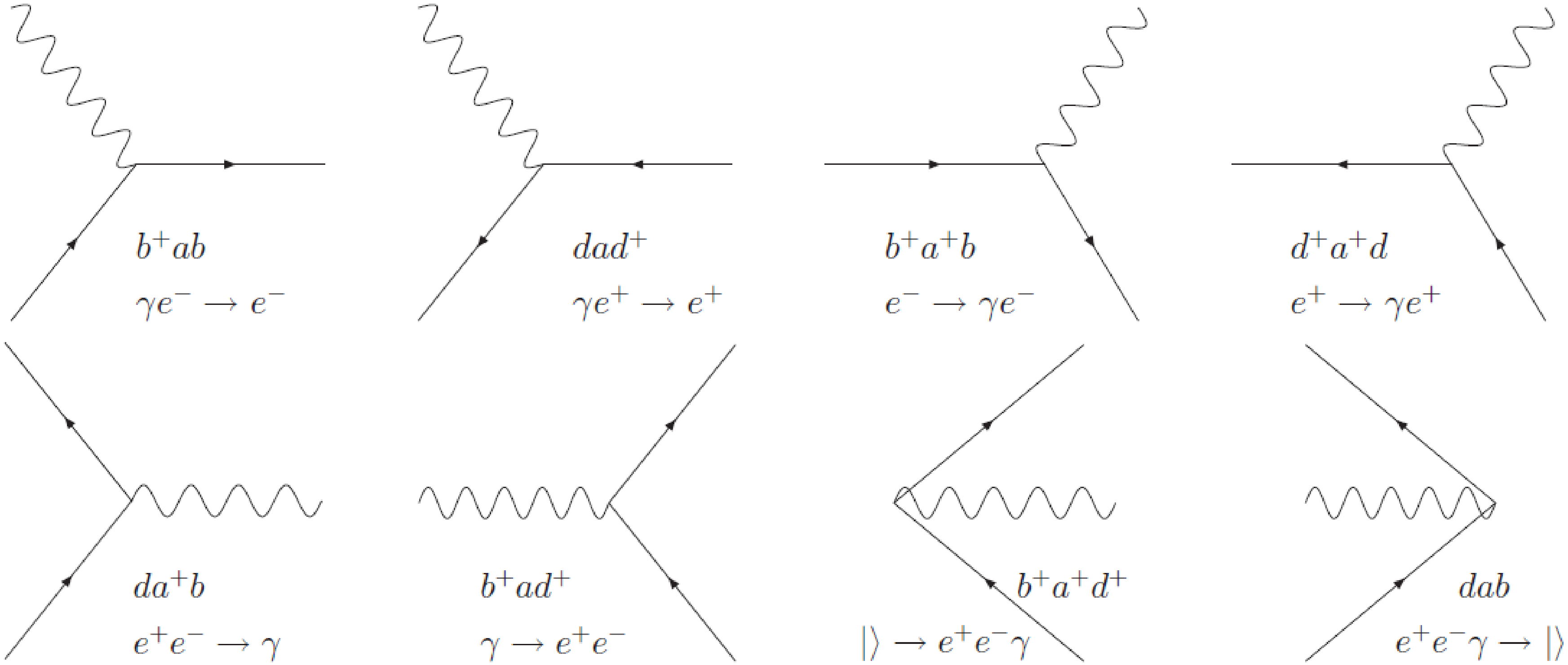
$\bar{\psi} \sim d^+ + b$

$A_\mu \sim a^+ + a$

Lagrangien contains all potential vertices

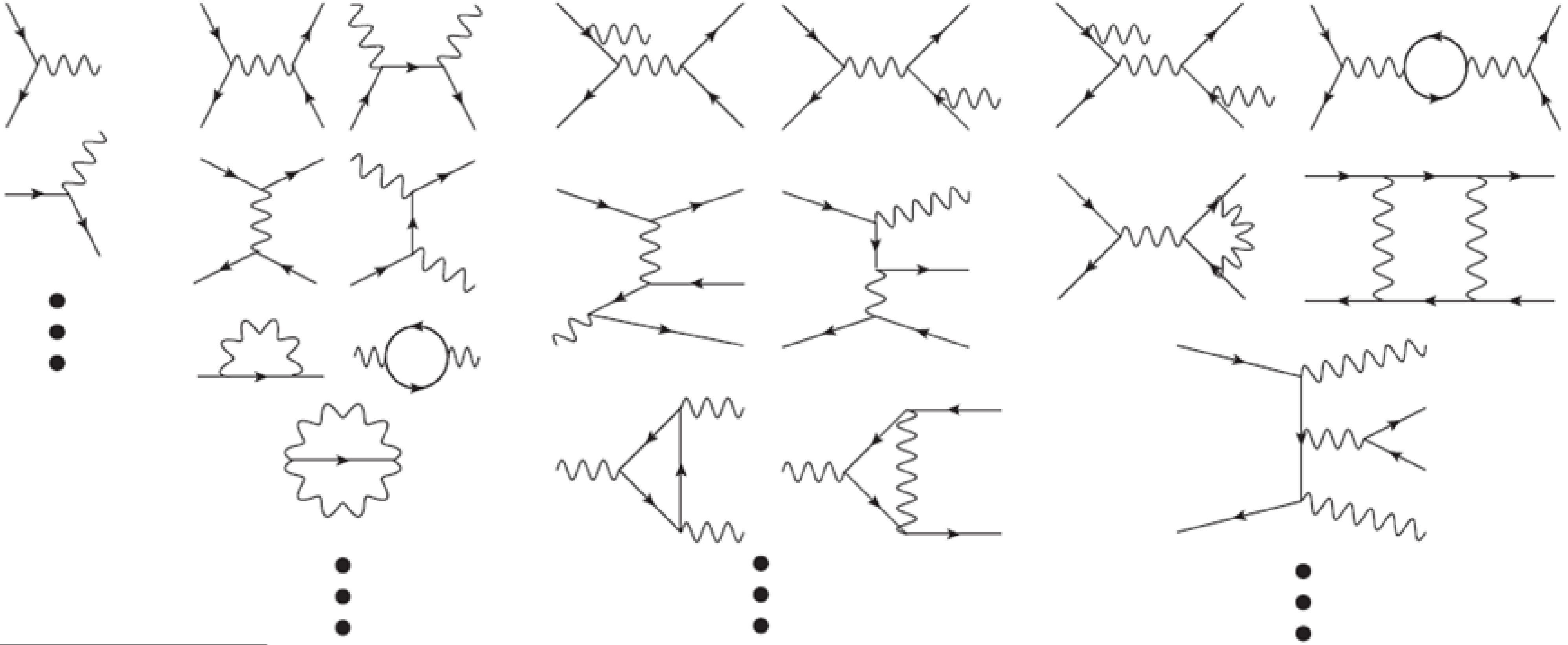
$$\mathcal{L}(x) \propto q \propto \sqrt{\alpha}$$

$$\alpha = \frac{q^2}{4\pi} \approx \frac{1}{137} \ll 1$$



Feynman diagrams

$$S \approx \mathbb{I} + \underbrace{\int \mathcal{L}(x)dx}_{\propto\alpha^{1/2}} + \underbrace{\int \mathcal{L}(x)\mathcal{L}(y)dxdy}_{\propto\alpha} + \underbrace{\int \mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)dxdydz}_{\propto\alpha^{3/2}} + \underbrace{\int \mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)\mathcal{L}(w)dxdydzdw}_{\propto\alpha^{2}} + \cdots$$



1 vertex
Elem. interactions
2 body decays

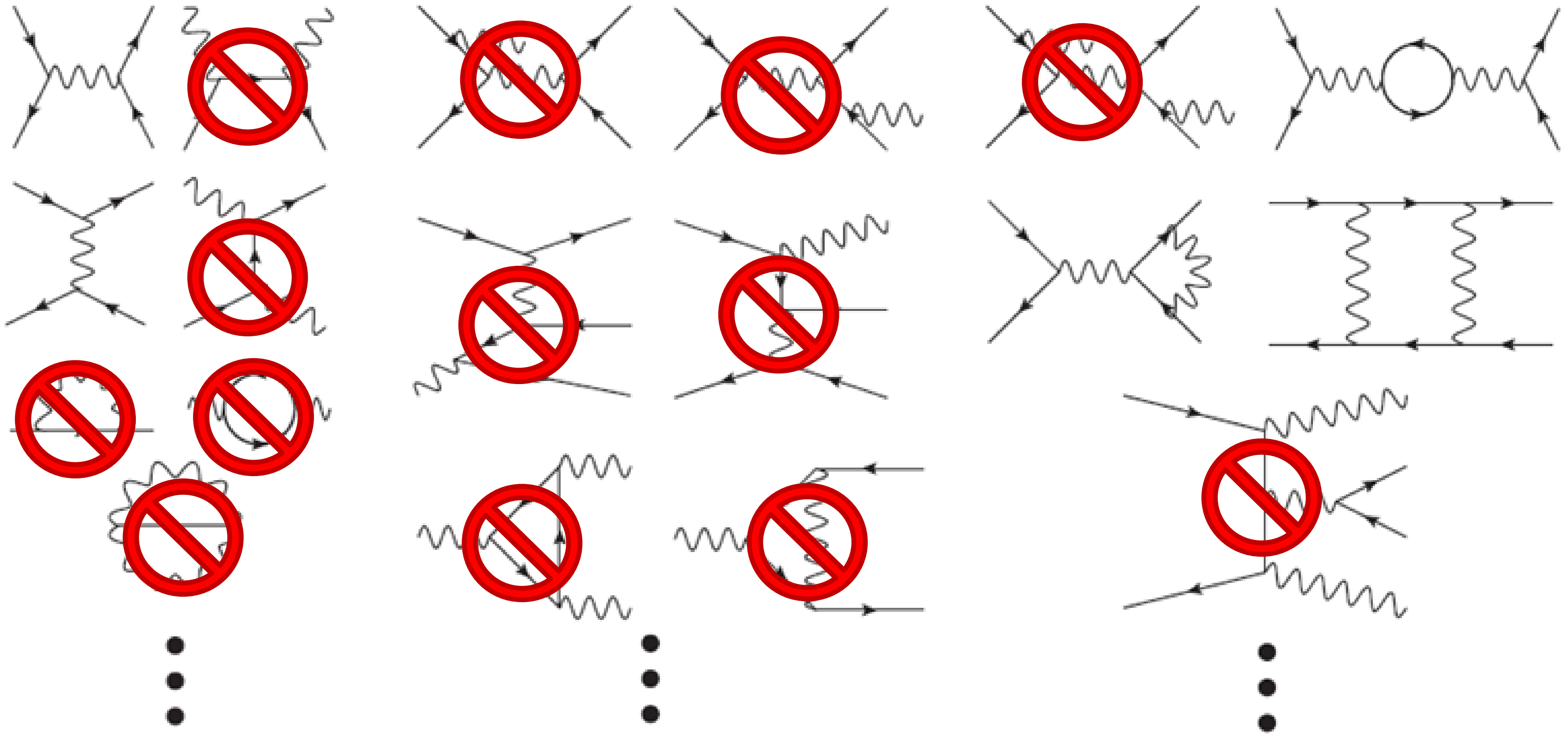
2 vertices
2→2 scattering
Vaccum fluctuations
Propagator corrections

3 vertices
2→3 scattering
Real corrections
Decays with loops

4 vertices
2→4 scattering
2→2 with loop
Box diagrams

Feynman diagrams

$$S \approx \mathbb{I} + \underbrace{\int \mathcal{L}(x)dx}_{\propto \alpha^{1/2}} + \underbrace{\int \mathcal{L}(x)\mathcal{L}(y)dxdy}_{\propto \alpha} + \underbrace{\int \mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)dxdydz}_{\propto \alpha^{3/2}} + \underbrace{\int \mathcal{L}(x)\mathcal{L}(y)\mathcal{L}(z)\mathcal{L}(w)dxdydzdw}_{\propto \alpha^{2}} + \cdots$$



Exemple : $\boldsymbol{e^- e^+ \rightarrow e^- e^+}$

Interaction Lagrangian : contains elementary vertices.

Scattering matrix : contains all possible interactions connecting any intial to any final states,

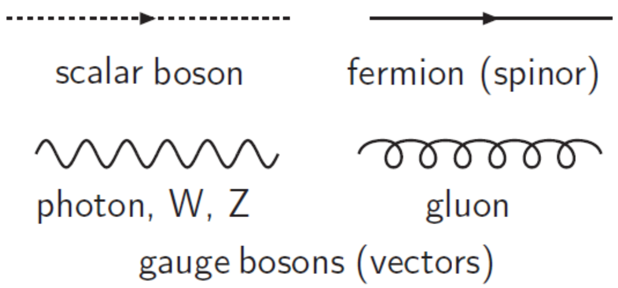
$S_{fi} = \langle i|S|f\rangle$ will select only diagrams connecting the intial state $|i\rangle$ to the final state $|f\rangle$

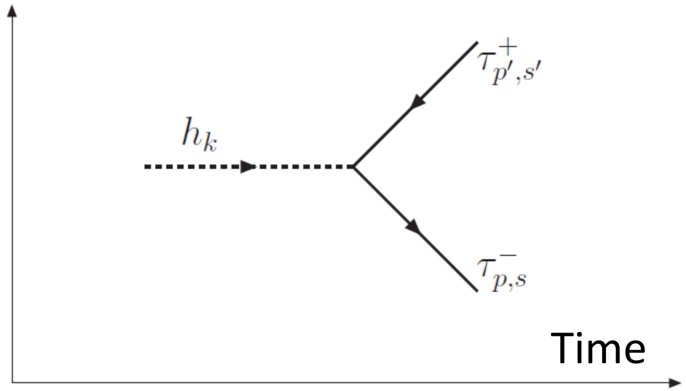
Computing cross-sections

The method

1. Draw all diagrams at a fixed order
2. Translate diagrams into Feynman amplitude using Feynman rules
3. Compute !!!

Drawing convention





From scattering matrix to observable

$$S_{fi} = \underbrace{(2\pi)^4\delta^4(P_f - P_i)\prod \frac{1}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2E_i}}}_{\textit{Phase space Kinematics}} \underbrace{\boldsymbol{\mathcal{M}}_{fi}}_{\textit{Feynman Dynamics}}$$

Width $1 \to 2$: $\frac{d\Gamma}{d\Omega} = \frac{1}{32\pi^2}\frac{|\vec{p}|}{m^2}\left|\boldsymbol{\mathcal{M}}_{fi}\right|^2$,

Cross-section $2 \to 2$: $\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p_f}|}{|\vec{p_i}|}\left|\boldsymbol{\mathcal{M}}_{fi}\right|^2$

Feynman rules

$$i\mathcal{M}_{fi} = \prod \text{Feynman rules}$$

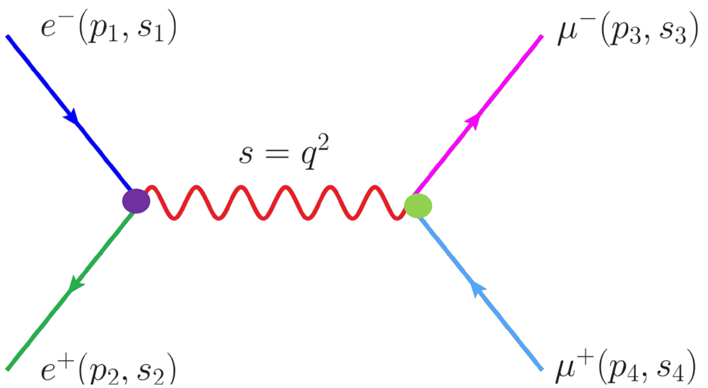
External lines : « Z » factors : $u_{k,s}$ or $v_{k,s}$ for spinors, $\varepsilon^{\mu}_{1,2,(3)}$ for vectors

Internal lines : propagators in fourier space : $\propto \frac{i}{p^2 - m^2}$

Vertices (coupling) : given by the Lagrangian, $iq\gamma^{\mu}$ for EM

More rules for loops, identical particles, relative sign of 2 diagrams…

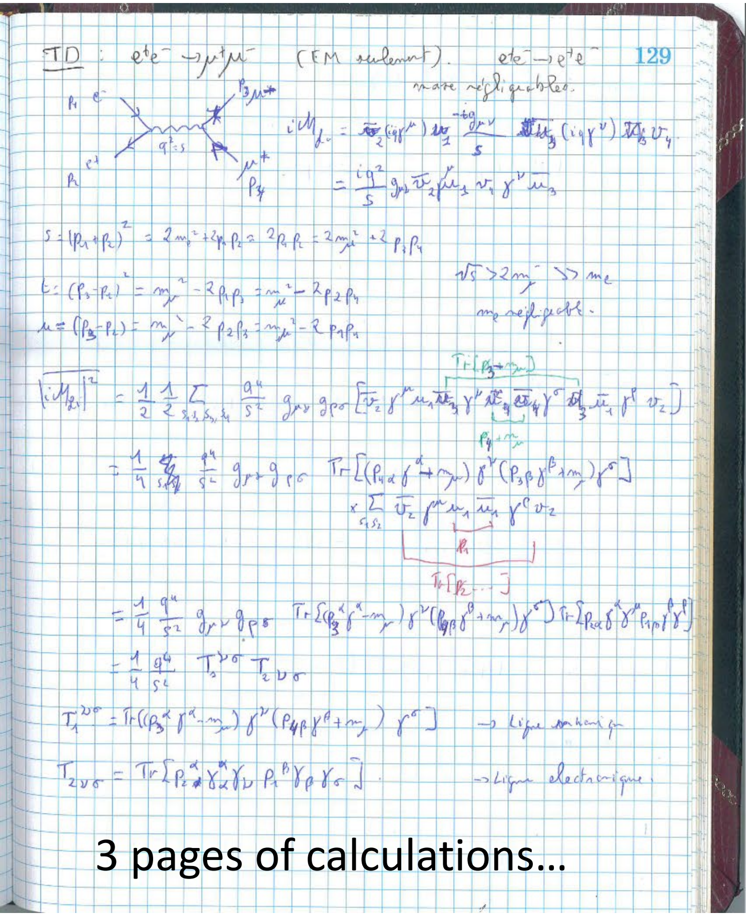
Exemple : $e^- e^+ \to \mu^- \mu^+$



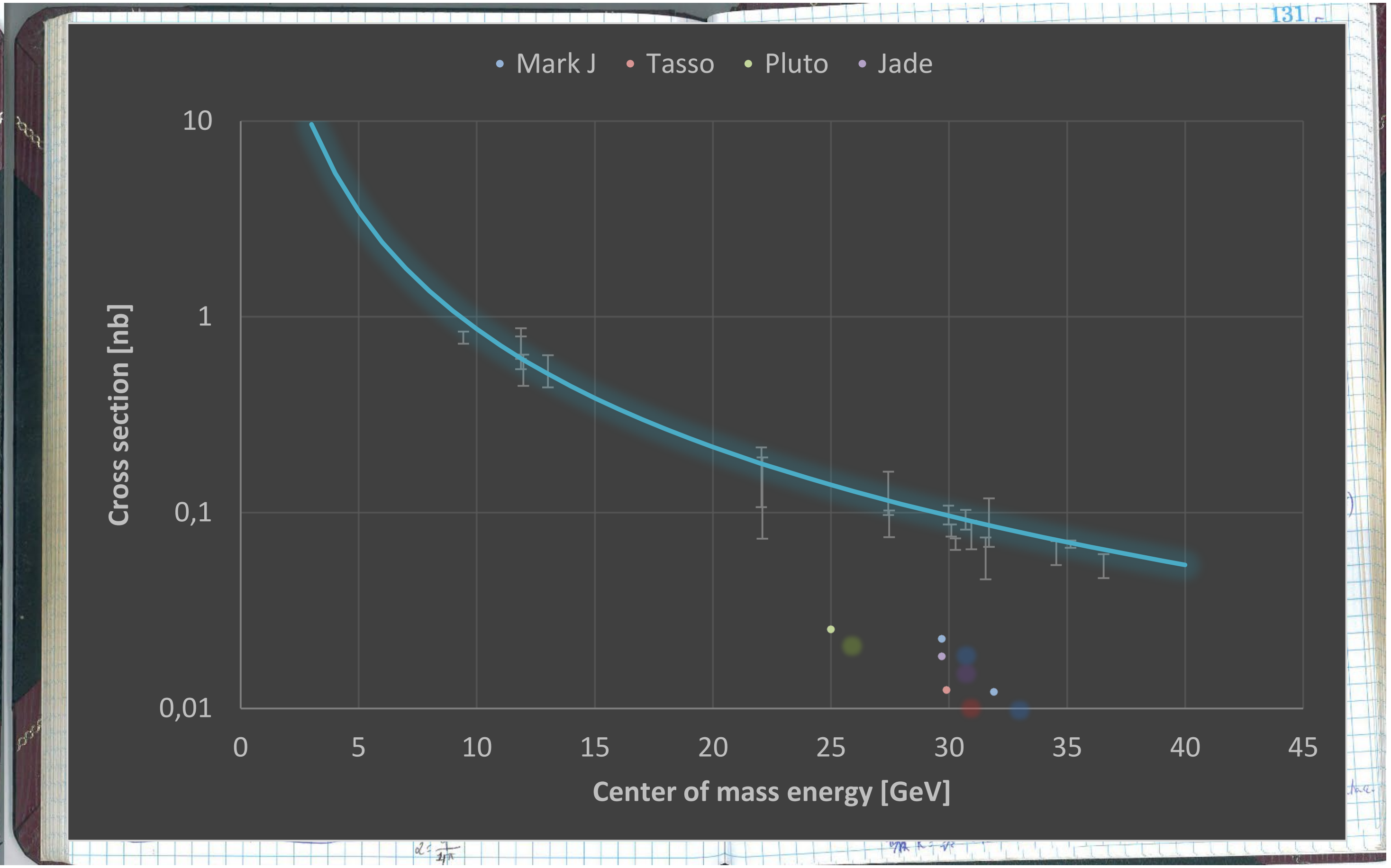
Feynman amplitude from rules

$$i\mathcal{M}_{fi} = \bar{u}_3 \times [-iq\gamma^\mu] \times v_4 \times \frac{-ig_{\mu\nu}}{s} \times \bar{v}_2 \times [-iq\gamma^\nu] \times u_1$$

$$i\mathcal{M}_{fi} = \frac{iq^2}{s} \bar{u}_3 \gamma^\mu v_4 \bar{v}_2 \gamma_\mu u_1 = \frac{ie^2}{s} J_a^\mu J_{b\mu}$$

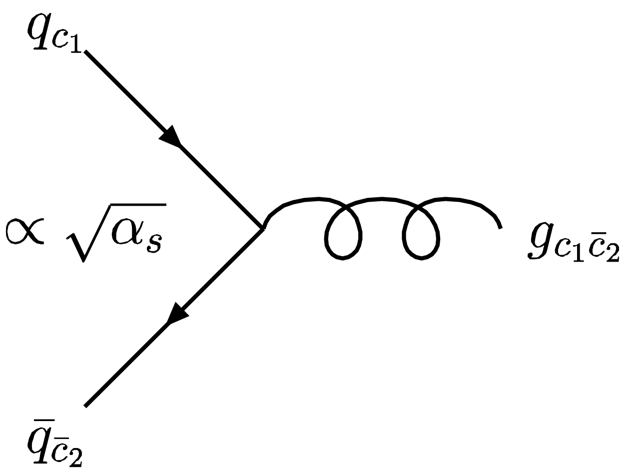


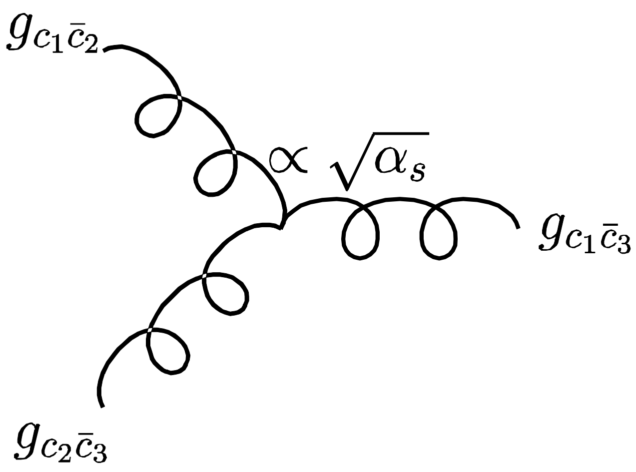
3 pages of calculations…

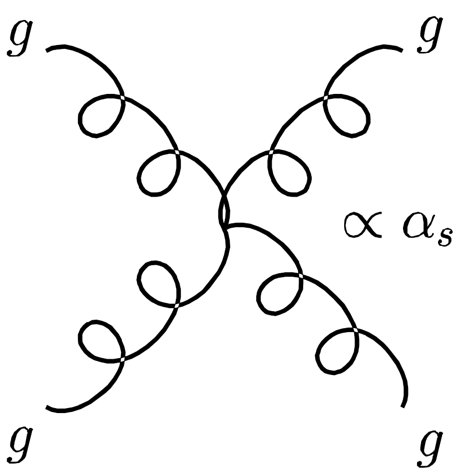


$$\sigma = \frac{4\pi\alpha^2}{3s}$$

SM Vertices : Strong interaction







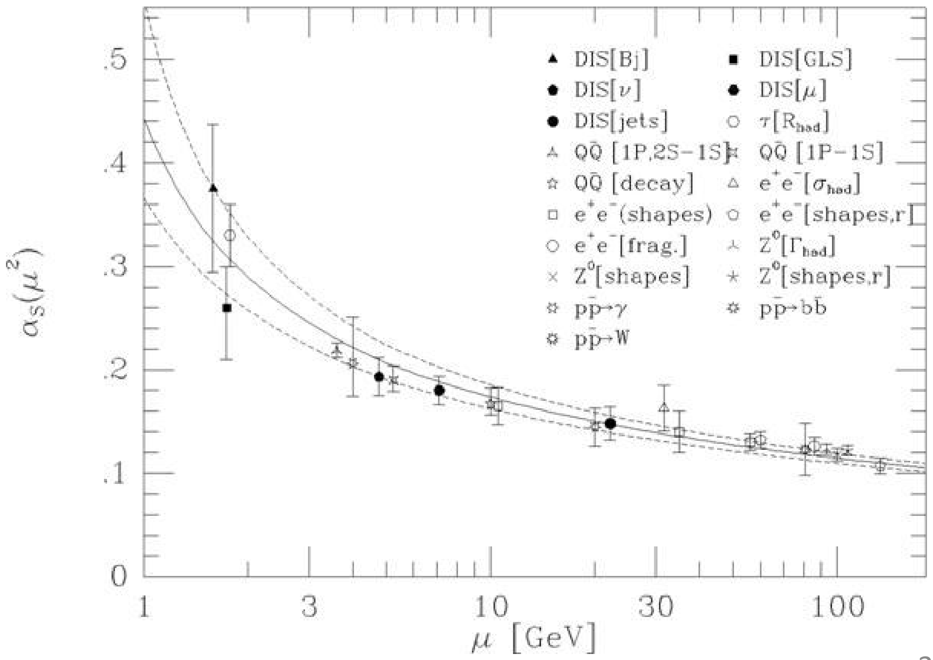
Non-abelian Yang-Mills SU(3)

Each quark exist with one of **three "color" charge** : "red", "green", "blue"
Gluons : change the color charge of quarks

Strong interaction intensity decreases with energy scale

Confinement at low energy (only "white" object exists : qqq or $q\bar{q}$)
Asymptotic freedom at high energy : perturbative expansion is ok

We can use Feynman diagrams to describe what happens at LHC



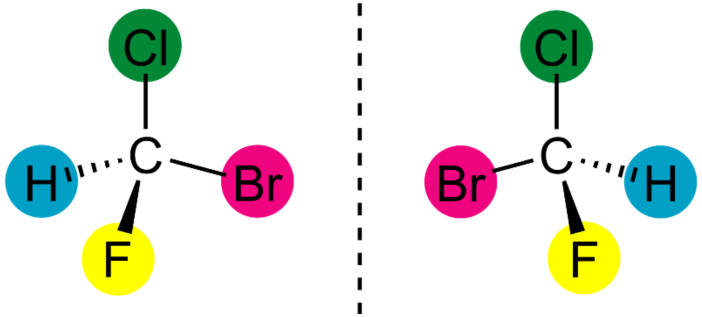
SM Vertices : Electroweak charged currents

Parity violation

P : symmetry $t, \vec{x} \rightarrow t, -\vec{x}$

4-component spinor (Dirac) = 2 2-component spinors

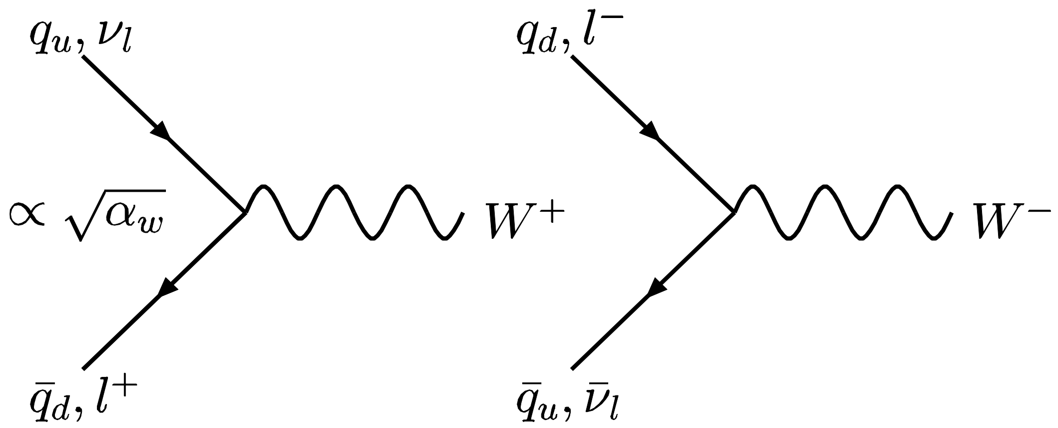
$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \qquad \phi \overset{P}{\leftrightarrow} \chi$$



There are 2 kinds of electrons **"left"** and **"right"**
EM and strong interaction does not make a difference

Weak charged currents ($W^{\pm}$) only couples to left-handed fermions !!!

This forbids a mass term for fermion as well…



Flavour mixing

Charged currents is the only interaction that changes the nature of particles,

Charged currents of quarks do not stay in the same family
Cabibbo-Kobayashi-Maskawa mixing :

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 0.97428 & 0.2253 & 0.00347 \\ 0.2252 & 0.97345 & 0.0410 \\ 0.00862 & 0.0403 & 0.999152 \end{pmatrix}$$

This matrix is complex… allow CP violation

SM Vertices : Electroweak neutral currents and self couplings

Gauge group and mixing

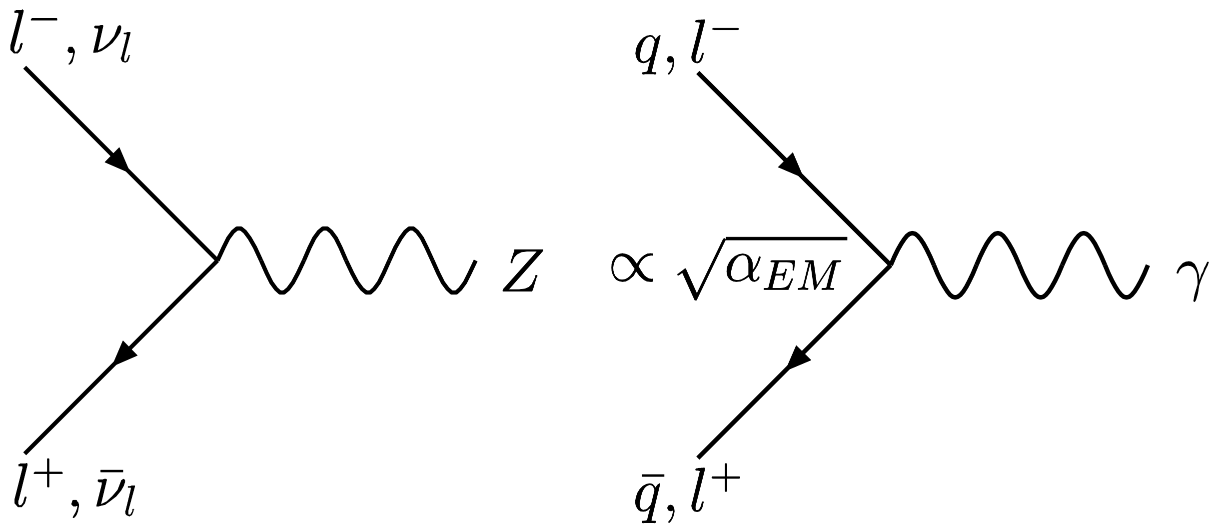
$SU(2)_L$: 3 bosons W^+, W^-, W^0
$U(1)_Y$: 1 boson B

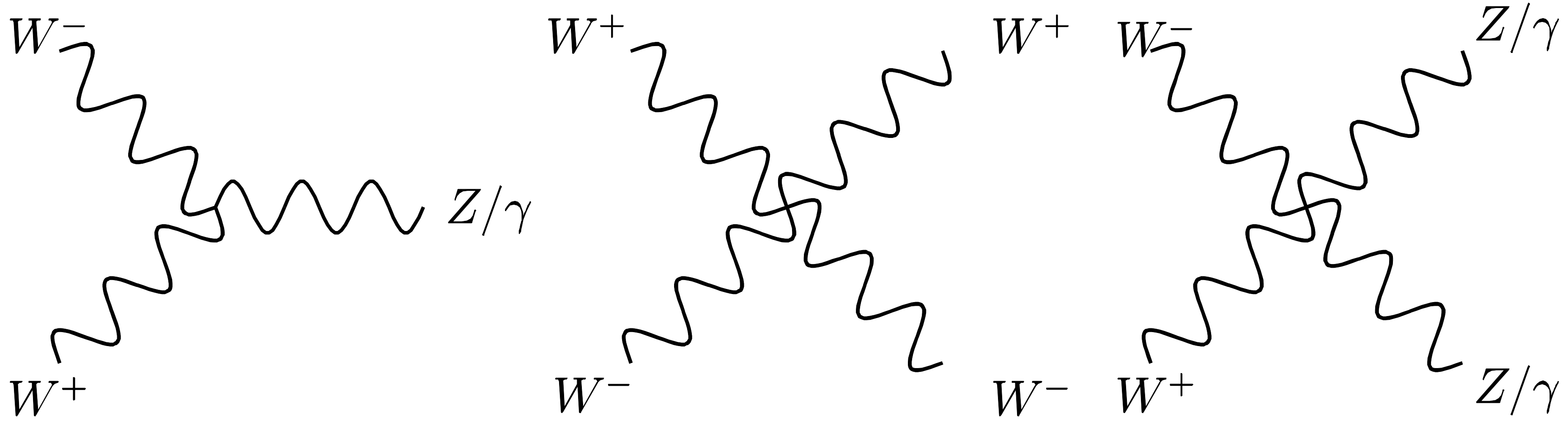
B and W^0 have the same quantum number and can mix

$$\begin{pmatrix} A \\ Z \end{pmatrix} = \begin{pmatrix} cos\theta_W & -sin\theta_W \\ sin\theta_W & cos\theta_W \end{pmatrix} \begin{pmatrix} W^0 \\ B \end{pmatrix}$$

A (the photon) conserves parity and keeps its zero mass after symmetry breaking
$SU(2)_L \times U(1)_Y \rightarrow U(1)_{EM}$

Neutral currents always conserve flavour





SM Vertices : Higgs boson

Spontaneous symmetry breaking of SU(2)

Higgs mechanism gives :
- 1 Higgs boson
- 1 vacuum expectation value

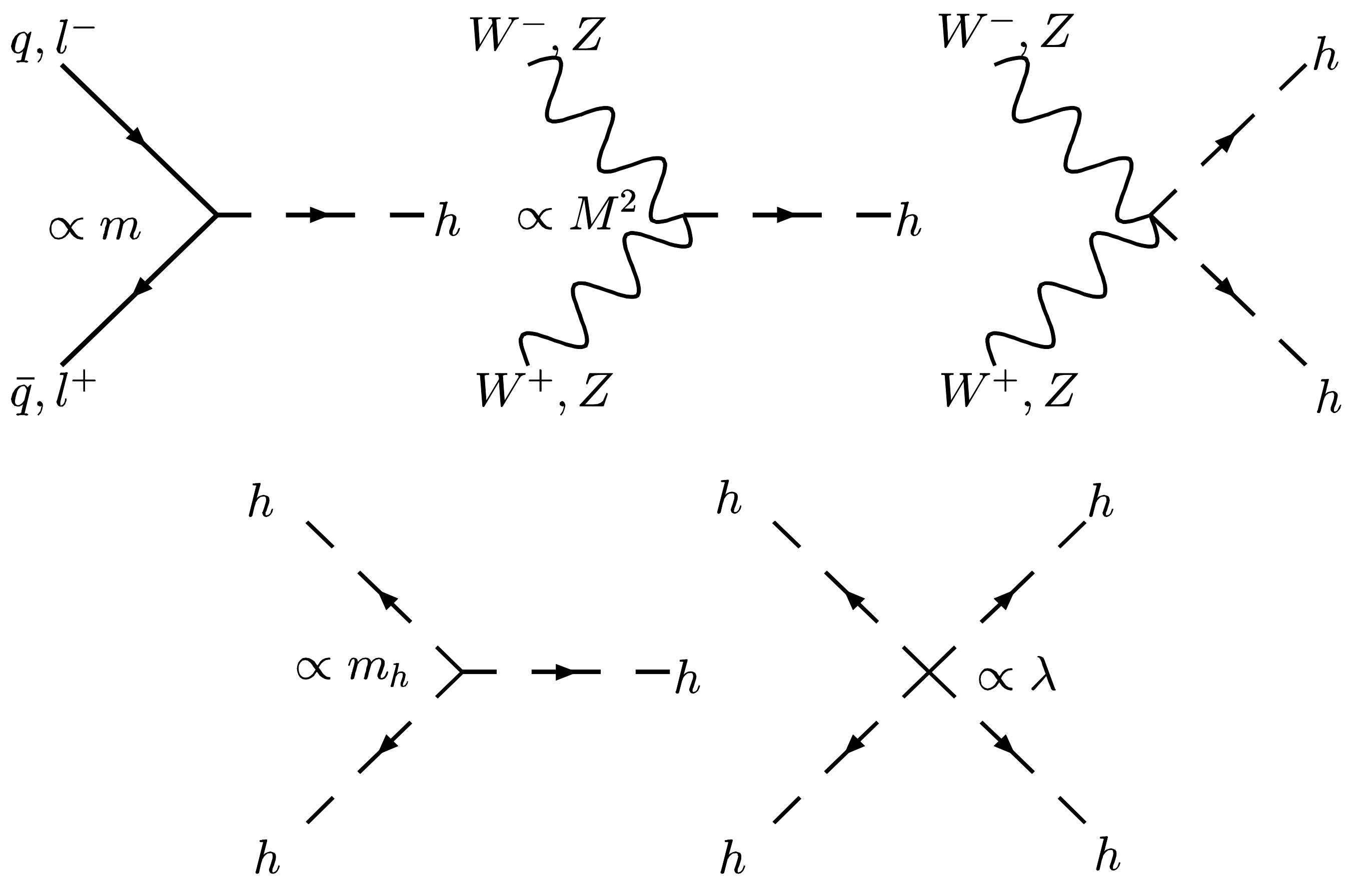
$W^{\pm}$ and Z boson masses arise from coupling to the v.e.v
$\boldsymbol{hBB}$ **coupling** $\propto \boldsymbol{M_B^2}$

Since SU(2) interaction violates parity, it can only works for massless fermions

Coupling of fermions to the Higgs field also gives **fermion mass terms**

$$\mathcal{L}_{yukawa} = -\frac{\lambda}{2}\varphi\bar{\psi}\psi = -\frac{\lambda v}{2}\bar{\psi}\psi - \frac{\lambda}{2}h\bar{\psi}\psi$$

$\boldsymbol{hf\bar{f}}$ **coupling** $\propto \boldsymbol{m_f}$



Limitations of the SM

(and what might lie beyond)

Free parameters of the Standard Model

Free parameters
Values that are not predicted by the model : they must be measured !

Gauge coupling (3)
Each gauge symmetry introduce a coupling constant :
$\alpha_{EM}, \alpha_W, \alpha_S$

Fermion masses (9+3)
Yukawa coupling to the Higgs fields :
$m_u, m_d, m_s, m_c, m_b, m_t, m_e, m_\mu, m_\tau, (m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau})$

Higgs Field (2)
Parameters of the Higgs potential : its mass and self-coupling
$m_h = \sqrt{-2\mu^2} \qquad \lambda$

Flavour Mixing (4+4/6)
CKM matrix can be parametrized by
3 angles and 1 phase
Mixing also exists in the neutrino sector :
3 angles and 1 or 3 phases

CP violation in strong interaction (1)
It is possible to have parity violating terms in the strong interaction but it is not observed
One angle $\theta_{strong} (= 0\ ?)$

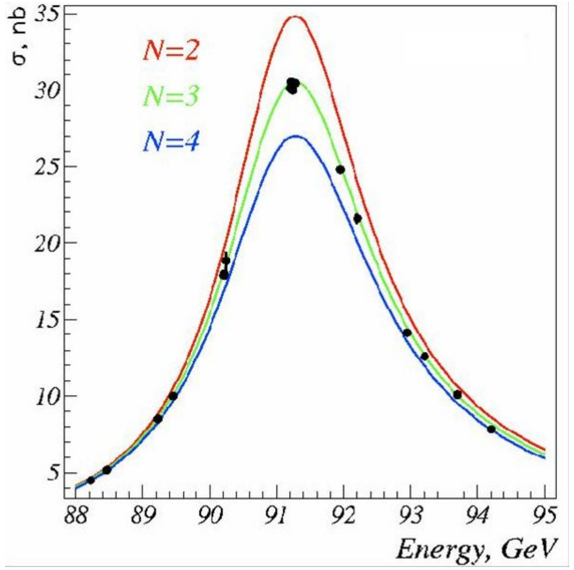
Some limitations

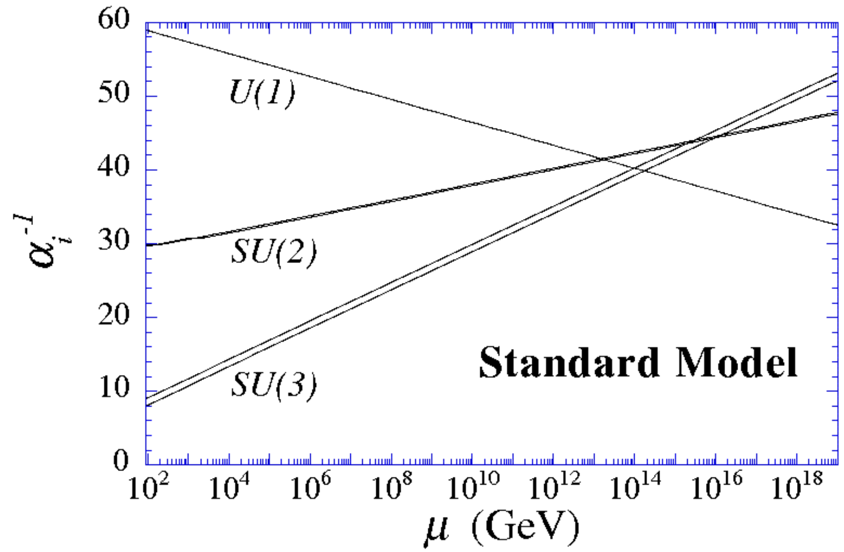
Theoretical consistency

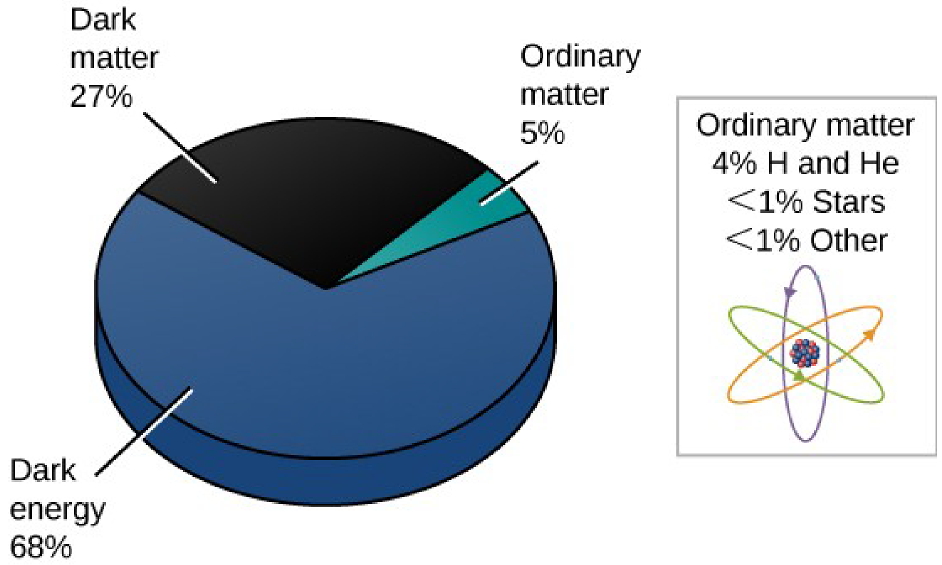
Why 3 families ?
Quadratic divergences in radiative corrections to the Higgs field (hierachy problem), fine tuning
Suppression of the strong CP violation

Astrophysics and cosmology

Matter/antimater assymetry in the universe
Nature of the dark matter (27% of the universe)
Nature of the dark energy (68% of the universe)







Esthetics

19 to 26 free parameters : it's a lot
No unification of couplings at high energy

Missing stuff

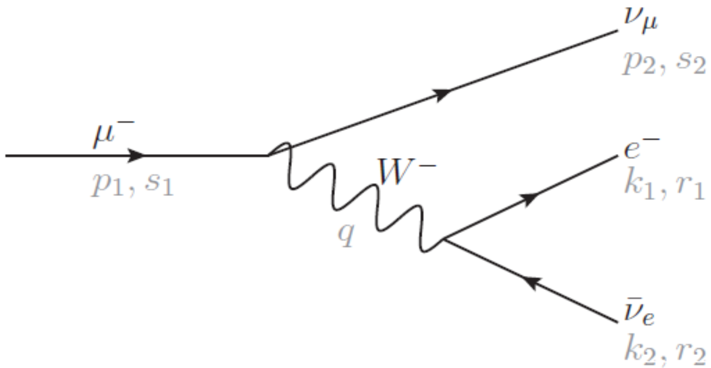
Neutrino masses and mixing
Gravitation

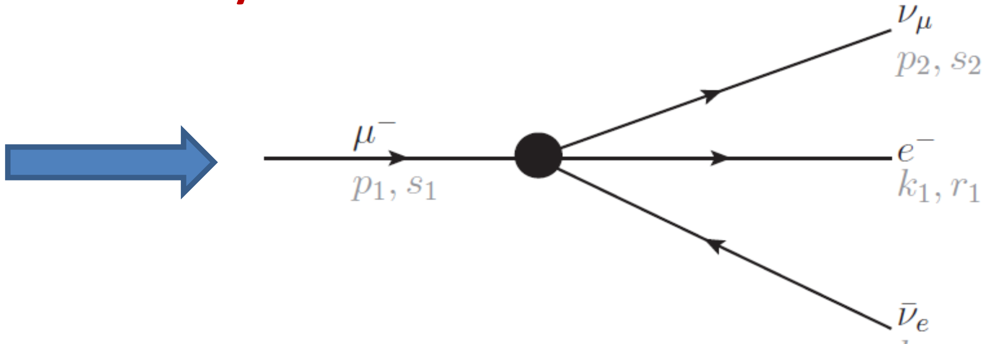
Going Beyond

Add **new fields and symmetries** to the SM and derive the Lagrangian or use **effective field theory**

Effective field theory

Muon decay





Effective 4-fermion Lagrangian

W propagator $D_W^{\mu\nu} = \dfrac{-i\left(g^{\mu\nu}+\frac{q^\mu q^\nu}{M_W^2}\right)}{q^2-M_W^2} \xrightarrow[q^2\sim m_\mu^2 \ll M_W^2]{} \dfrac{ig^{\mu\nu}}{M_W^2}$

$$\mathcal{L} = \frac{g^2}{M_W^2}\bar{\psi}\psi\bar{\psi}\psi$$

Dimensional analysis

Spinors $[\psi]\sim E^{\frac{3}{2}}$,
Scalar, vectors $[\phi]\sim[A_\mu]\sim E$
All interaction Lagrangians $\mathcal{L}\sim E^4$

Only **dimensionless constants** or the theory is non-renormisable

Effective field theory

Add higher order operators that encapsulate the low energy effect of high energy new interactions

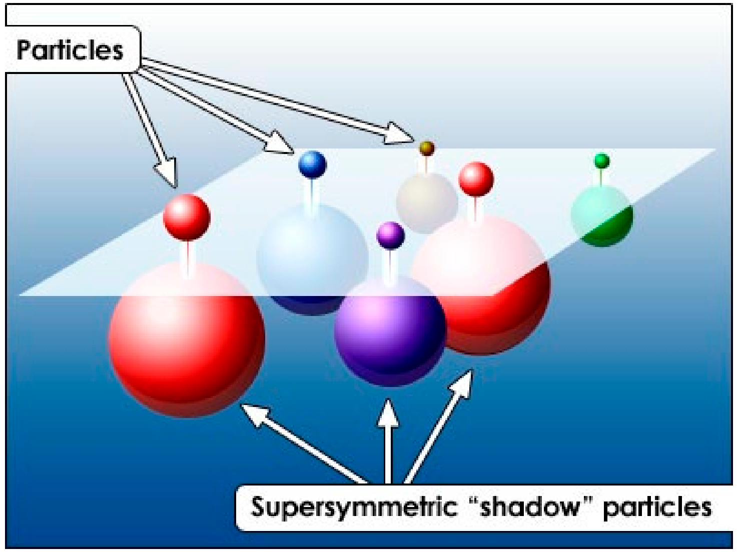
$$\mathcal{L} = \sum_i \frac{c_i}{\Lambda} O_5^i + \sum_j \frac{c_j}{\Lambda^2} O_6^j + \cdots$$

Λ : energy scale of new physics
c_i: Wilson coefficient

Supersymmetry

New kind of symmetry

Symmetry between Boson ⟷ Fermion
≠ gauge symmetry



Each SM particles have a superpartner

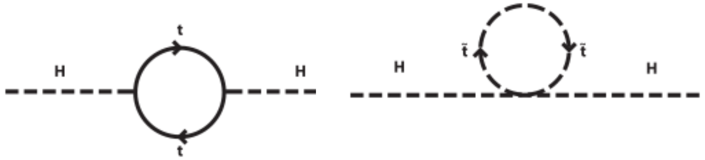
Quarks, leptons : fermions (s=½) → **Squarks, Slepton** : bosons (s=0)
Bosons de Jauges : bosons (s=1) → **Jauginos** : fermions (s=½)
Bosons de Higgs : bosons (s=0) → **Higgsinos** : fermions (s=½)

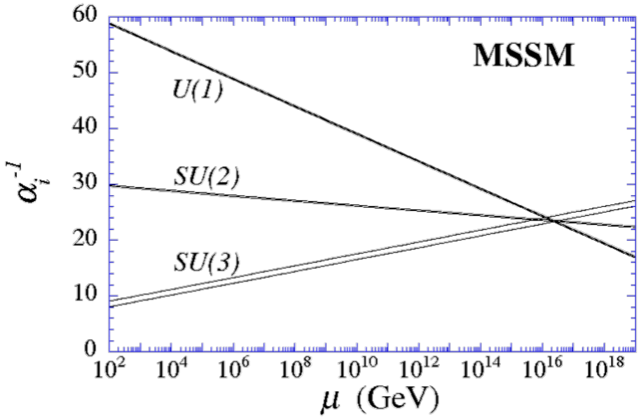
Pros

Suppression of quadratic divergence

Coupling unification at high energy

Candidate for dark matter : if lightest SUSY particle is neutral and stable





Extended Higgs sector

5 Higgs bosons : h, H, A, H^+, H^-
Higgsinos and EW Gauginos are mixing
→ 4 charginos $\tilde{\chi}^{\pm}_{1,2}$
→ 4 neutralinos : $\tilde{\chi}^{0}_{1,2,3,4}$

Cons

More than 120 parameters
Broken symmetry : what’s the breaking mechanism
Not observed yet…

Other ideas…

Grand Unification Theory

Embbed the SM gauge group in a larger one :

$SU(2)_L \times U(1)_Y \times SU(3)_C \subset G_{GUT}$

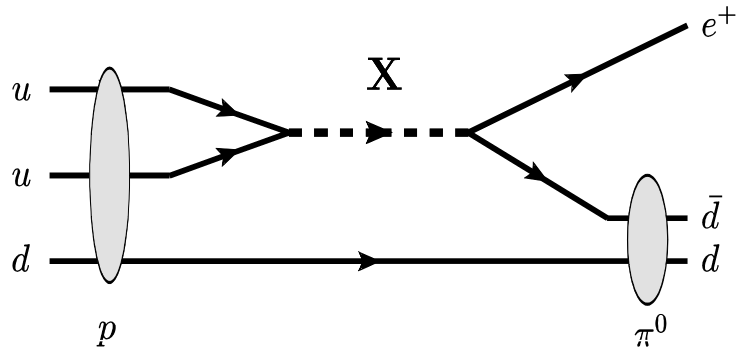
$$G_{GUT} = SU(5), SO(10), E_8 \ldots$$

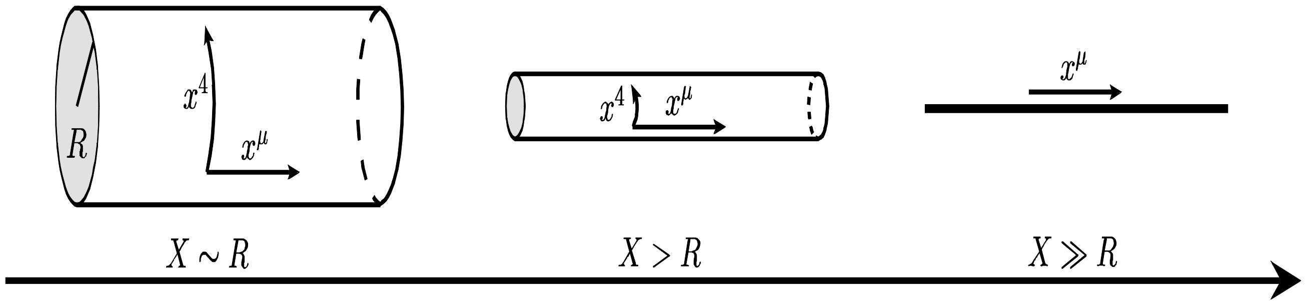
Only one coupling constant

New couplings between quarks and leptons : makes proton unstable

Broken symmetry

- new « Higgs »-like bosons
- new heavy gauge bosons : Z', W'
- new very heavy gauge bosons of a new type : Leptoquark

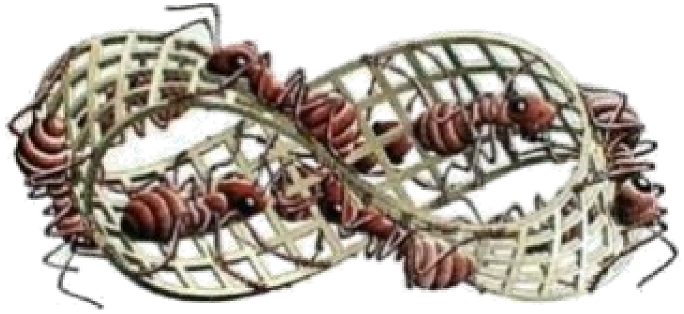




Extra spatial dimensions

Compactified extra dimensions: no large scale effects

Only some particles can propagate in these dimensions



More…

Axions and axion-like particles

Beyond QFT : Ads/CFT, Supercords, Loop quantum gravity, …