Symplectic classical mechanics

Making life harder to make it easier

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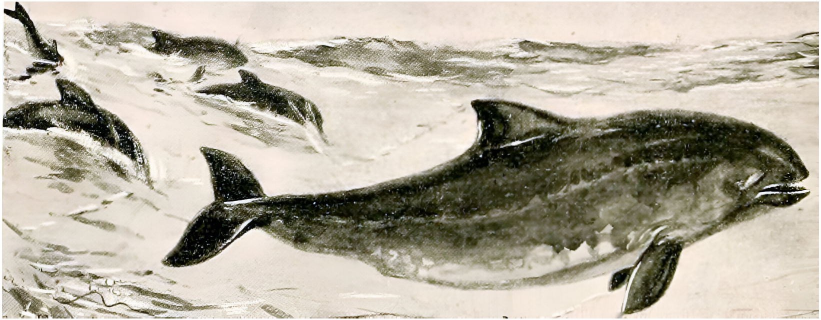


The Lagrangian: who needs arrows?

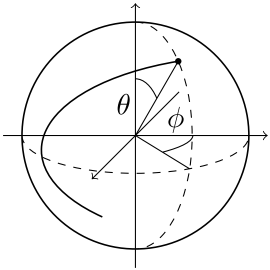
Lagrangian of a physical system, coordinates q, kinetic energy K, potential energy V:

$$\mathcal{L} = K(\dot{q}) - V(q)$$

Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\dot{q}} - \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}q} = 0$$

Solutions are $(q, \dot{q}, t)$ paths of the system!



The Hamiltonian: who needs time derivatives?

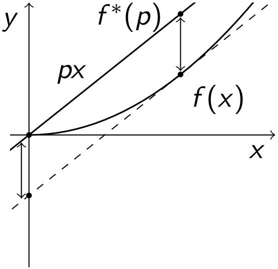
Legendre-transforming $\mathcal{L}$ from velocities $\dot{q}$ to momenta $p = \frac{\partial \mathcal{L}}{\partial \dot{q}}$ gives the **Hamiltonian**

$$\mathcal{H}(q,p) = p \cdot \dot{q}(p) - \mathcal{L}\big(q, \dot{q}(p), t\big)$$

and paths obey the **canonical equations**

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{q} \qquad \frac{\partial \mathcal{H}}{\partial q} = -\dot{p}$$

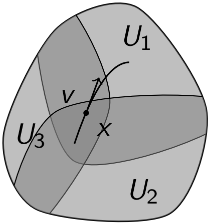
(q,p) are coordinates of $2n$-D **phase space**.



Manifolds: many definitions

Smooth **manifold**: **set** X and **atlas** (U_i, ϕ_i).
Charts ϕ_i let us locally treat manifolds as $\mathbb{R}^n$.

- **Tangent vectors** v: paths through x.
- **Vector fields**: smooth maps $x \mapsto v$.
- **Cotangent vectors** α: duals of paths.
- **1-forms**: smooth maps $x \mapsto \alpha$.
- **Differential**: $\mathrm{d}H : v \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} H(x + tv)$.
- **2-form**: signed area between 1-forms.
- **Exterior derivative** $\mathrm{d}\alpha$: 1-form $\mapsto$ 2-form.



Cotangent bundle $\{(x, \alpha)\}$ is phase space!

Symplectic mechanics: who needs coordinates?

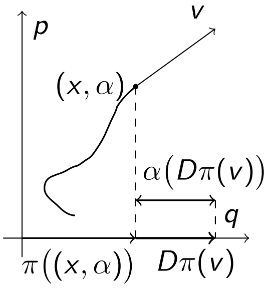
Symplectic form: 2-form ω which is **closed** and **non-degenerate**.

For cotangent bundles: **canonical form** is $\omega = -\mathrm{d}\lambda$, where λ is α applied to the first n tangent coordinates to the bundle.

Hamiltonian vector field: V_H such that

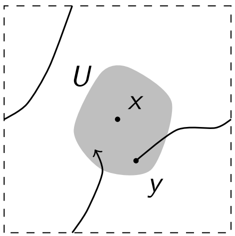
$$\omega(V_H, \cdot) = \mathrm{d}H$$

Solutions to $\frac{\mathrm{d}}{\mathrm{d}t}\big((x,\alpha)\big) = V_H\big((x,\alpha)\big)$ are phase space paths of the system!



Some applications: why do all this?

- Coordinate-independent formulation
  - Liouville's theorem
  - Poincaré recurrence
- Non-cotangent bundle manifolds: symplectic reduction of symmetries
- Lie algebra and geometric quantization
  - Lagrangian submanifolds and states



Thank you!

Now for question time :)