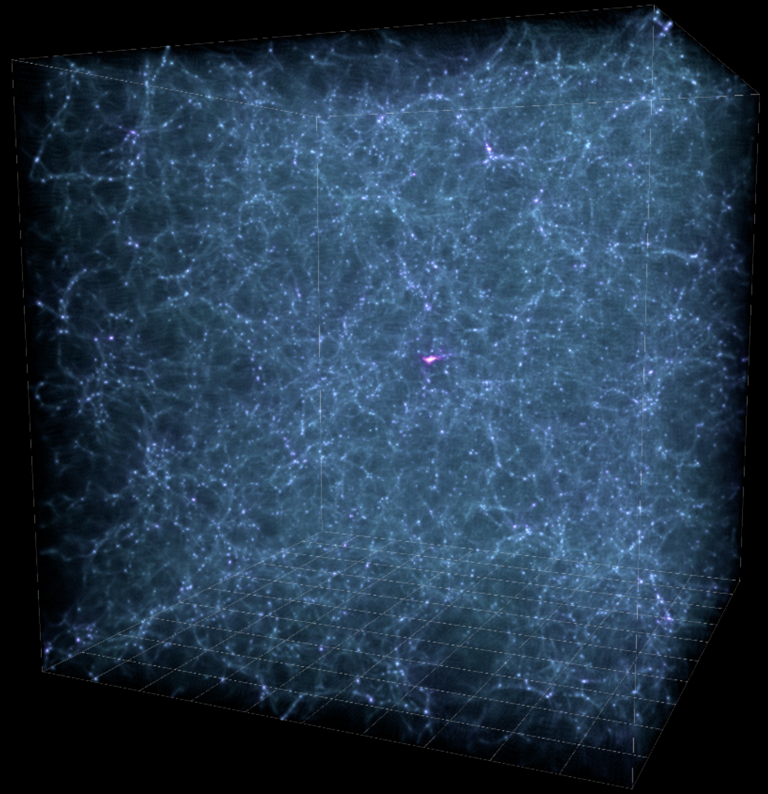
Towards Fully General Relativistic Cosmology (1/N)

(a.k.a. What the hell am I doing?)

(a.k.a. The Master Plan)



LSST Group Meeting – January 30th, 2024 – LAPP, Annecy

Vincent Reverdy

Goal

Unit systems

Profunctor optics

Proxy-types
in qualified languages

Implicit trees

Abstract
iteration

$\Rightarrow$ Higher-order
HyperGraphs

Group symmetries
of types

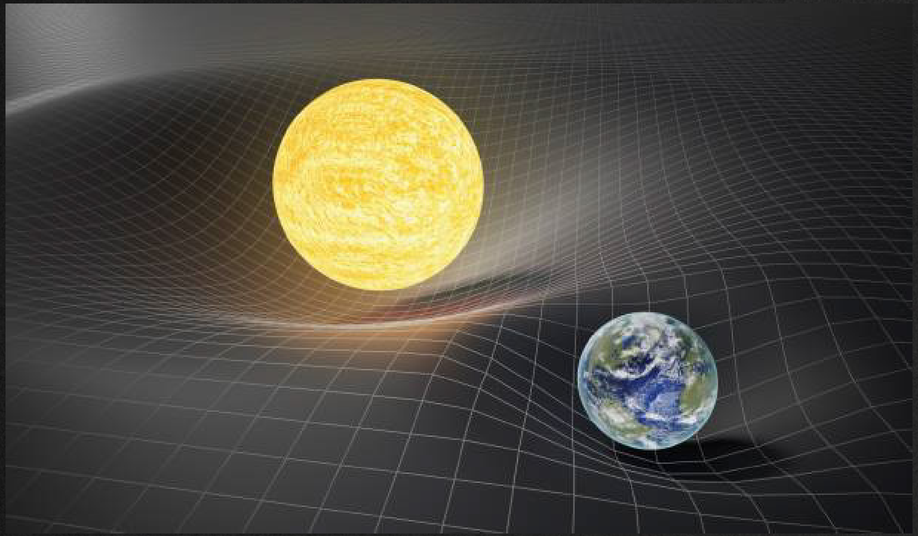
Category theory

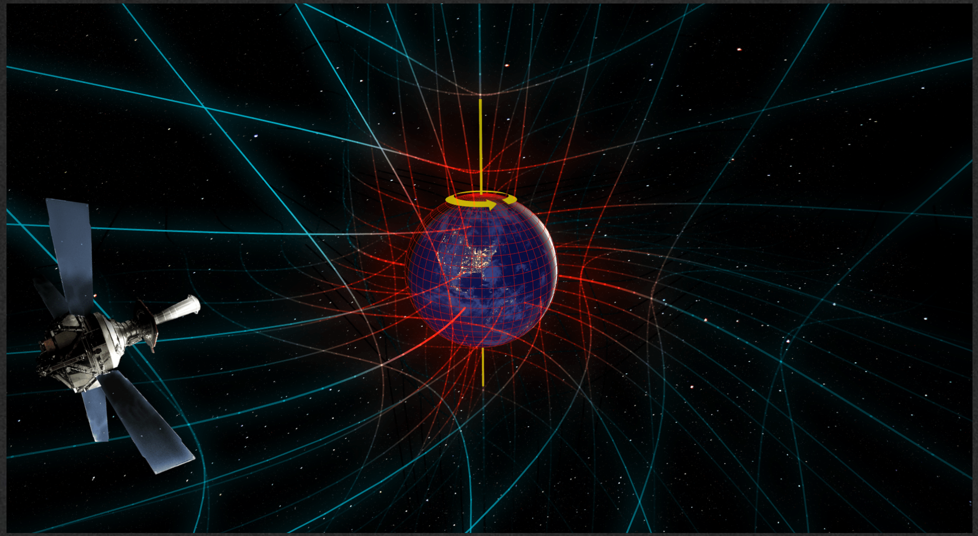
1. Equivalence principle $\Rightarrow$ Spacetime is a manifold

2. Vanishing torsion $\nabla_X Y - \nabla_Y X = [X, Y]$ $\Rightarrow$ Symmetric Christoffel symbols (geodesic equations from a variational principle)

3. Poisson equation $\Rightarrow$ Fix the constants to be compatible with Newton

General relativity





$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Einstein tensor
Spacetime geometry
($g_{\mu\nu}$ metric tensor)

Stress energy tensor
Spacetime contents

Metric theories

- Scalar field theories → (Nordström)
- Quasilinear theories → (Whitehead)
- Tensor theories → (General Relativity)
- Scalar-tensor theories → (Brans-Dicke)
- Vector-tensor theories → (Will–Nordtvedt)
- Bimetric theories → (Lightman-Lee)

Non-metric theories

Einstein Field Equations (EFE)

$$\begin{pmatrix} G_{tt} & G_{tx} & G_{ty} & G_{tz} \\ G_{xt} & G_{xx} & G_{xy} & G_{xz} \\ G_{yt} & G_{yx} & G_{yy} & G_{yz} \\ G_{zt} & G_{zx} & G_{zy} & G_{zz} \end{pmatrix} = \frac{8\pi G}{c^4} \begin{pmatrix} T_{tt} & T_{tx} & T_{ty} & T_{tz} \\ T_{xt} & T_{xx} & T_{xy} & T_{xz} \\ T_{yt} & T_{yx} & T_{yy} & T_{yz} \\ T_{zt} & T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

⬇

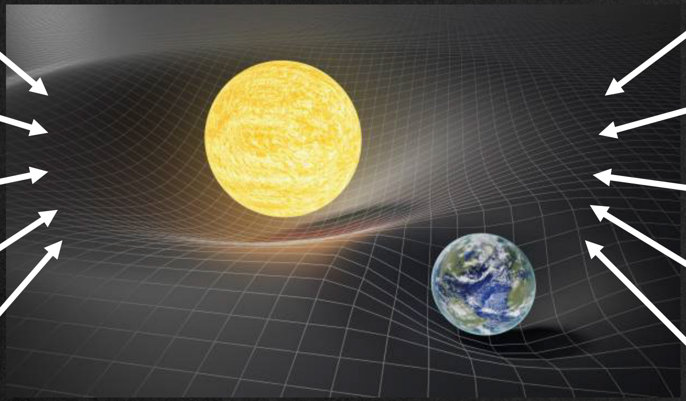
Symmetric tensors

⬇

10 non-linear coupled differential equations

...at each point of the 4D-spacetime (tensor field)

$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ (×5)



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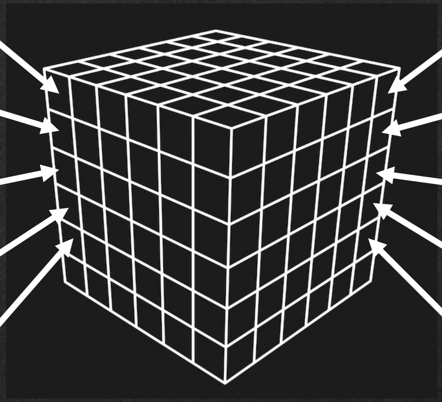
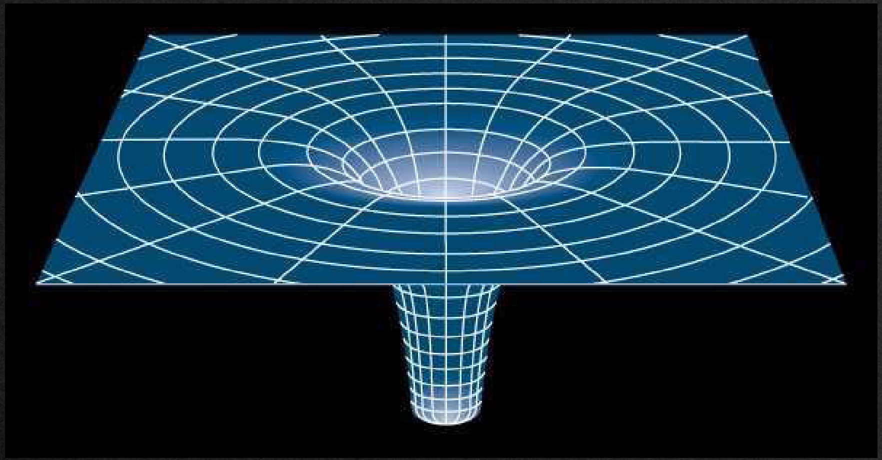


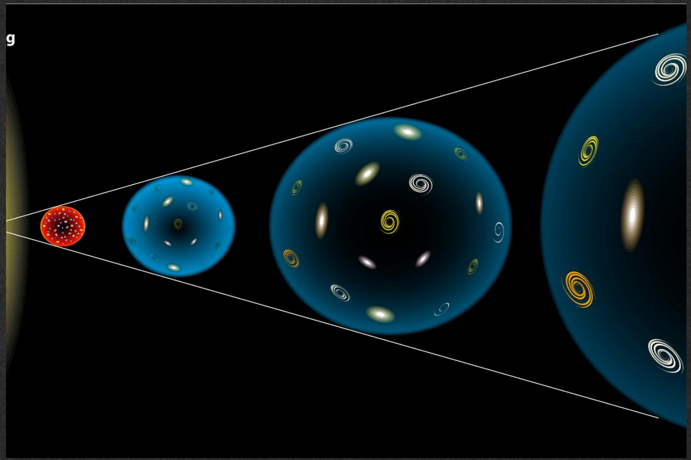
$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ (×5)

Too complicated

1. Analytical formulas $\Rightarrow$ 10 differential equations that applies everywhere
2. Diagonal terms only $\Rightarrow$ 4 differential equations
3. Spatial symmetries $\Rightarrow$ 2 differential equations



Schwarzschild metric
(black hole)



Friedmann-Lemaître-Robertson-Walker metric
(FLRW) (expansion of the Universe)

10 equations: too complicated

⬇

The Cosmological Principle
The most reasonable first thing to do to compute a model of the Universe by hand

1915: General relativity (Einstein)

1922-1924: First derivation of FLRW (Friedmann)

1924: M331, M32, M31 [Andromeda] are outside of the Milky Way (Hubble)

Cosmological Principle

At a sufficiently large scale, the Universe is homogeneous and isotropic

No cosmological principle ⇝ no calculation by hand

$$ds^2 = -c^2 dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j$$

FLRW metric

$$T_{\mu\nu} = (\rho c^2 + P) u_\mu u_\nu + P g_{\mu\nu}$$

Perfect fluid

$$\begin{pmatrix} G_{00} & 0 & 0 & 0 \\ 0 & G_{ii} & 0 & 0 \\ 0 & 0 & G_{ii} & 0 \\ 0 & 0 & 0 & G_{ii} \end{pmatrix} = \frac{8\pi G}{c^4} \begin{pmatrix} T_{00} & 0 & 0 & 0 \\ 0 & T_{ii} & 0 & 0 \\ 0 & 0 & T_{ii} & 0 \\ 0 & 0 & 0 & T_{ii} \end{pmatrix}$$

$$\begin{pmatrix} G_{00} & 0 & 0 & 0 \\ 0 & G_{ii} & 0 & 0 \\ 0 & 0 & G_{ii} & 0 \\ 0 & 0 & 0 & G_{ii} \end{pmatrix} = \frac{8\pi G}{c^4} \begin{pmatrix} T_{00} & 0 & 0 & 0 \\ 0 & T_{ii} & 0 & 0 \\ 0 & 0 & T_{ii} & 0 \\ 0 & 0 & 0 & T_{ii} \end{pmatrix}$$

$$\begin{cases} G_{00} = T_{00} \\ G_{ii} = T_{ii} \end{cases} \Rightarrow \begin{cases} 3H^2 + 3\dfrac{kc^2}{a^2} = 8\pi G\rho \\ -3\dfrac{H^2}{c^2} - 2\dfrac{\dot{H}}{c^2} - \dfrac{k}{a^2} = \dfrac{8\pi G}{c^4} P \end{cases} \Rightarrow \boxed{\begin{cases} \left(\dfrac{\dot{a}}{a}\right)^2 = \dfrac{8\pi G}{3}\rho - \dfrac{kc^2}{a^2} \\ \dfrac{\ddot{a}}{a} = -\dfrac{4\pi G}{3}\left(\rho + \dfrac{3P}{c^2}\right) \end{cases}}$$

Cosmological Principle $\Leftrightarrow$ 2 differential equations on $a(t)$

Mix of non-interacting perfect fluids

$$\boxed{P_X = w_X \rho_X c^2}$$

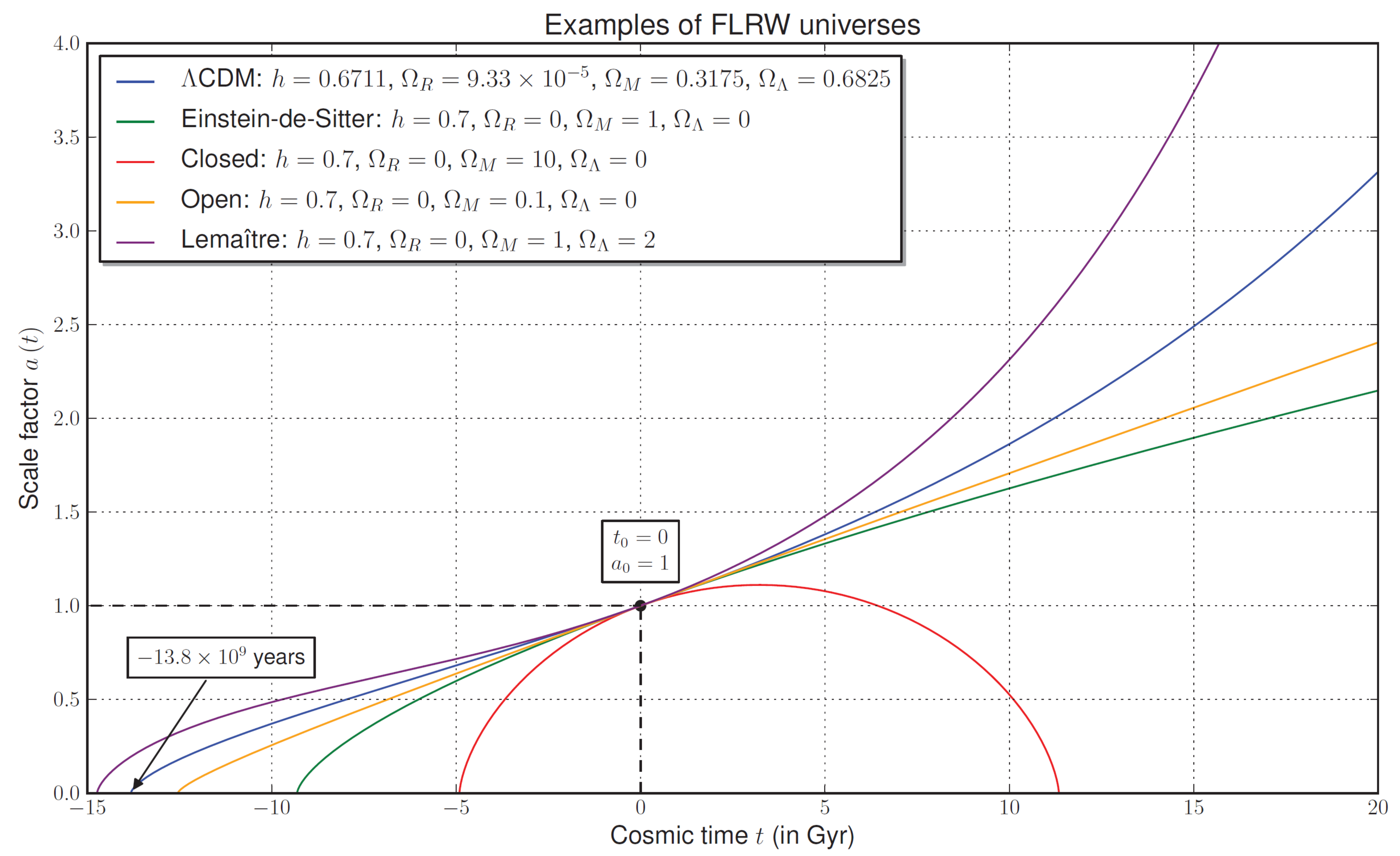
$$w_X = \begin{cases} 1/3 \text{ for radiation } R \\ 0 \text{ for matter } M \\ -1 \text{ for dark energy } \Lambda \end{cases}$$

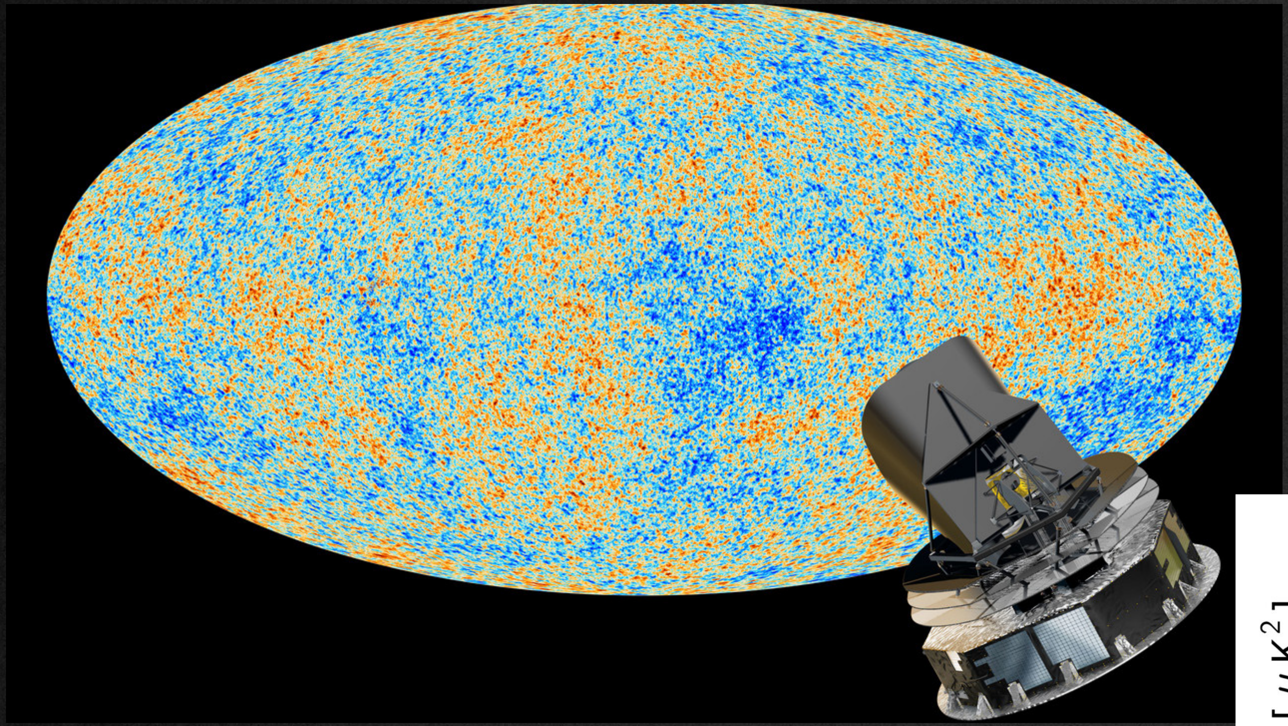
$$\Omega_X = \frac{\rho_X}{\rho_C} = \frac{8\pi G}{3H^2}\rho_X$$

$$\boxed{\frac{H(t)^2}{H_0^2} = \Omega_{R_0} a(t)^{-4} + \Omega_{M_0} a(t)^{-3} + \Omega_{k_0} a(t)^{-2} + \Omega_{\Lambda_0}}$$

Final equation under its canonical form

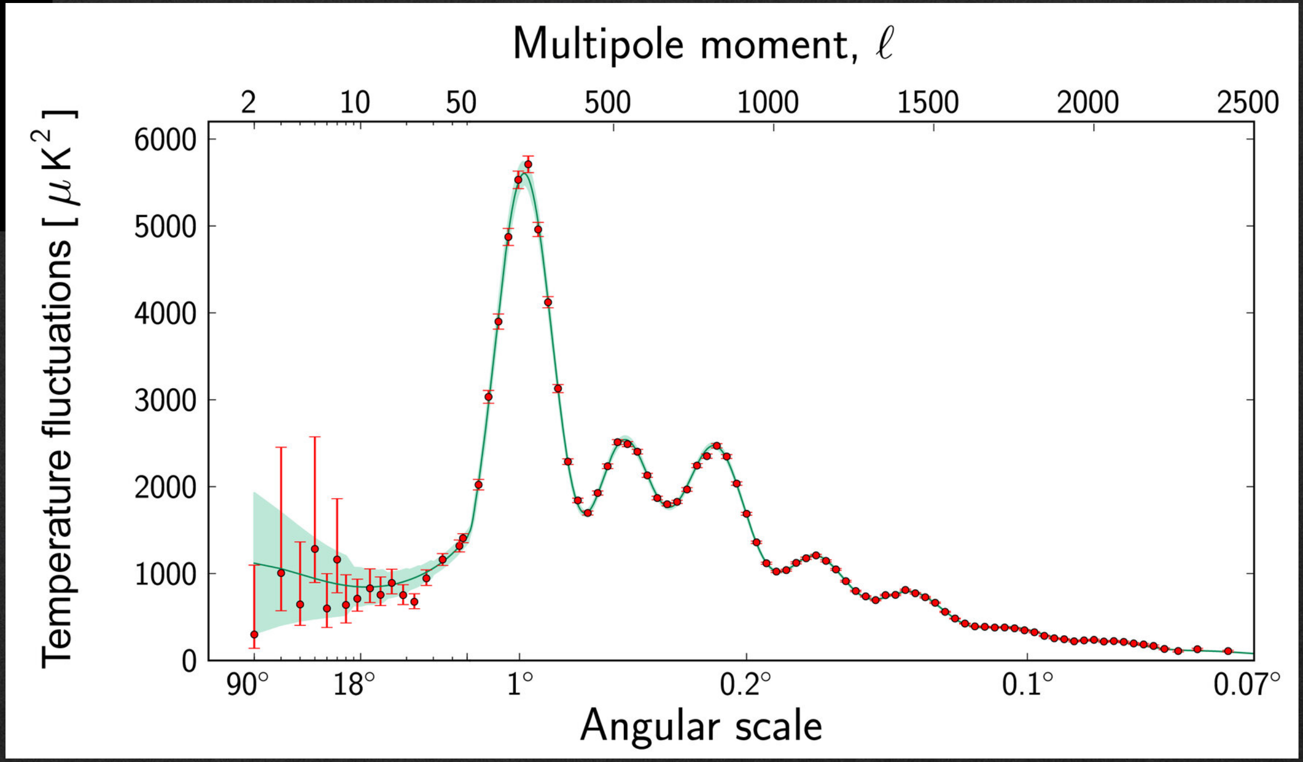
Universe evolution

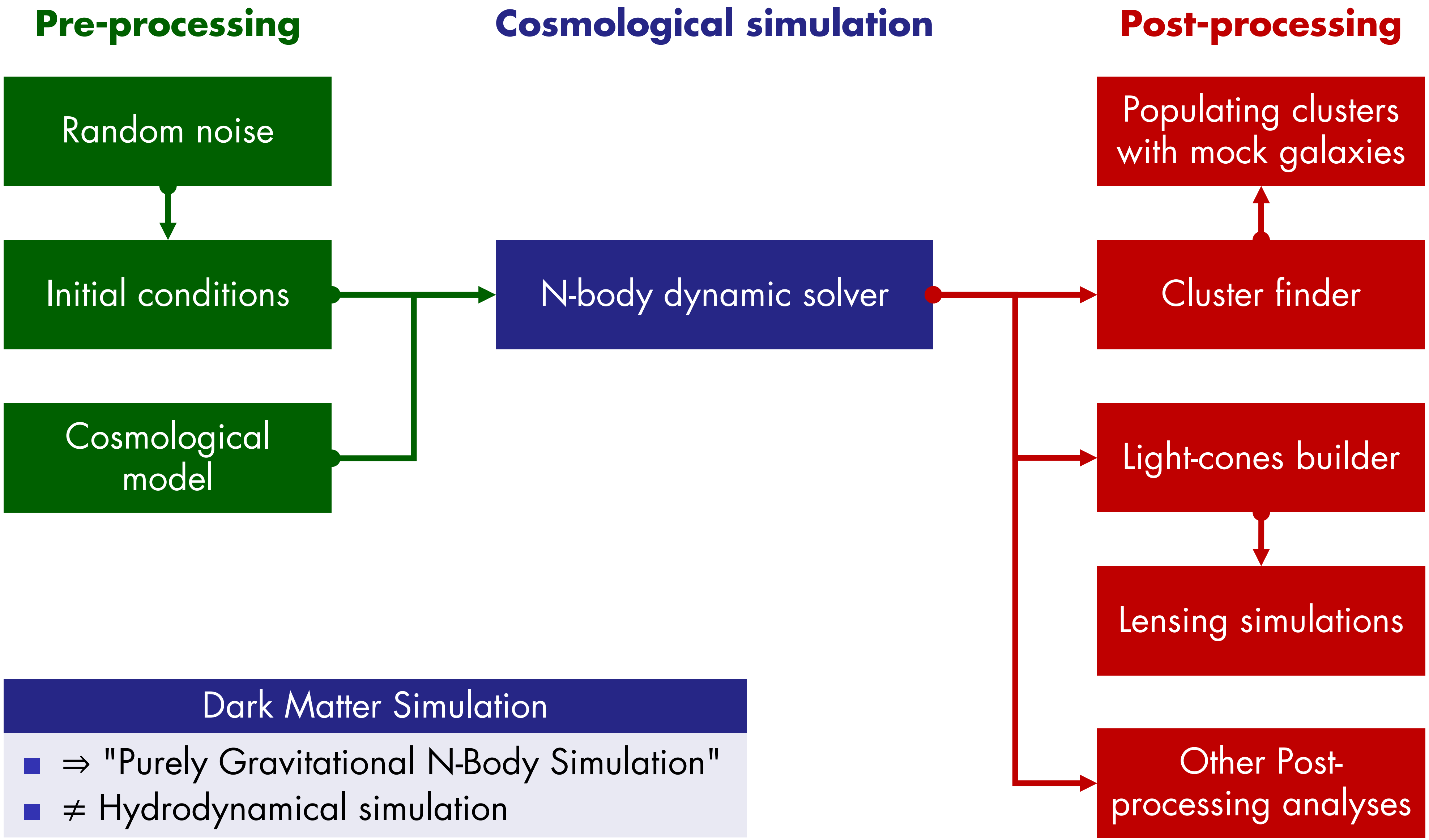




Power-spectrum only captures Gaussian processes

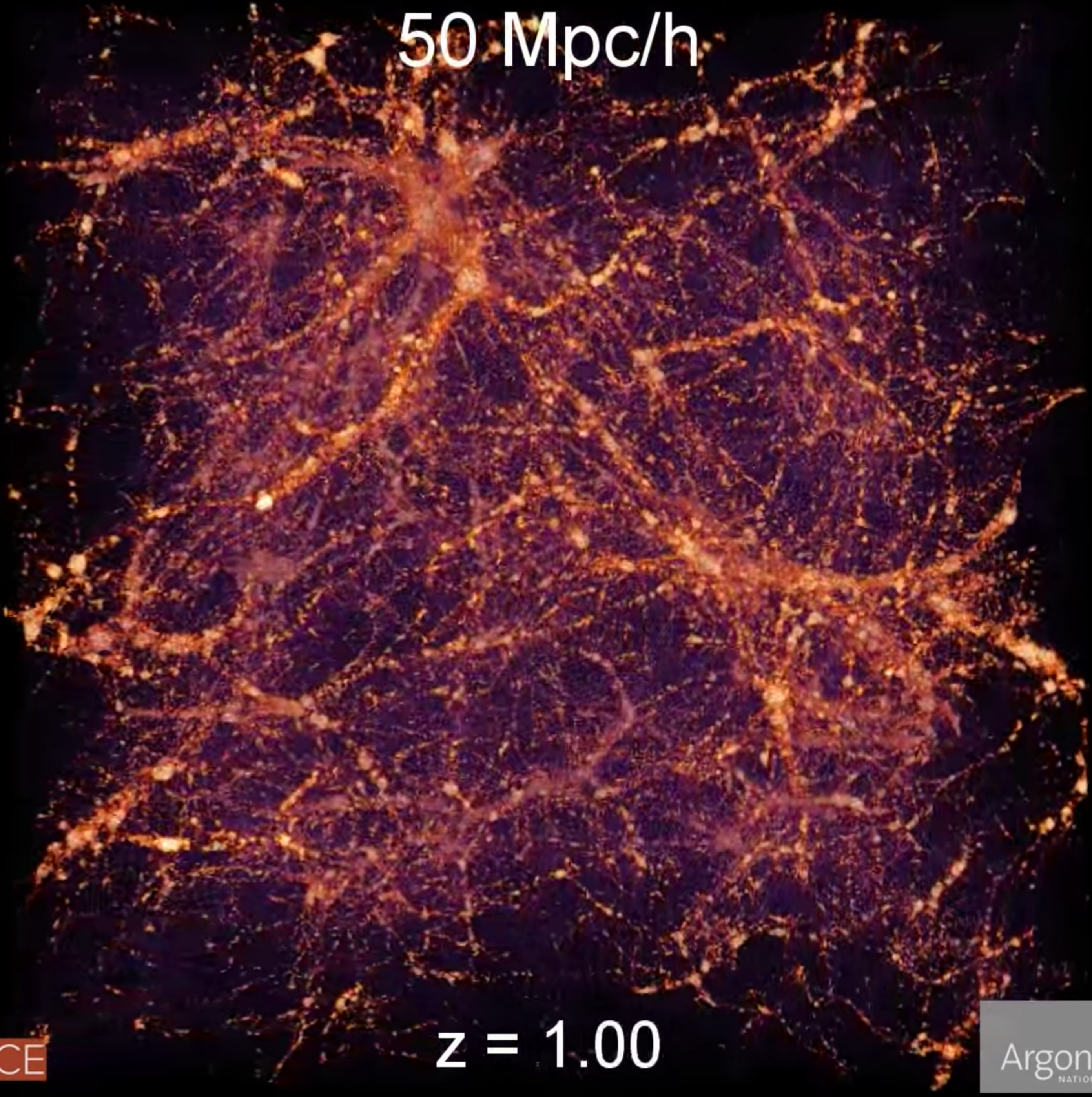
Need bispectrum, trispectrum… to capture primordial non-gaussianities





Dark Matter Simulation

- ⇒ "Purely Gravitational N-Body Simulation"
- ≠ Hydrodynamical simulation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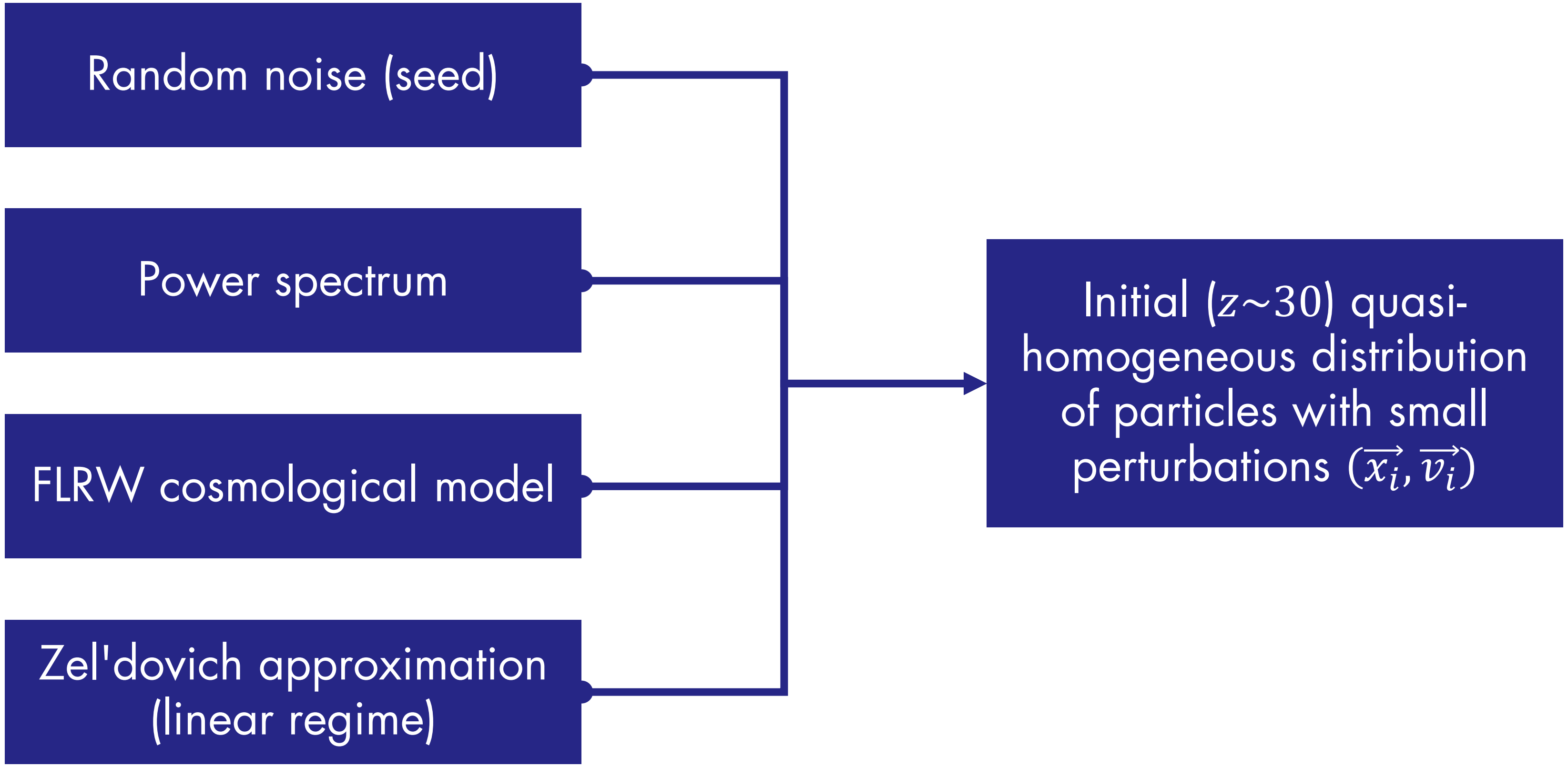
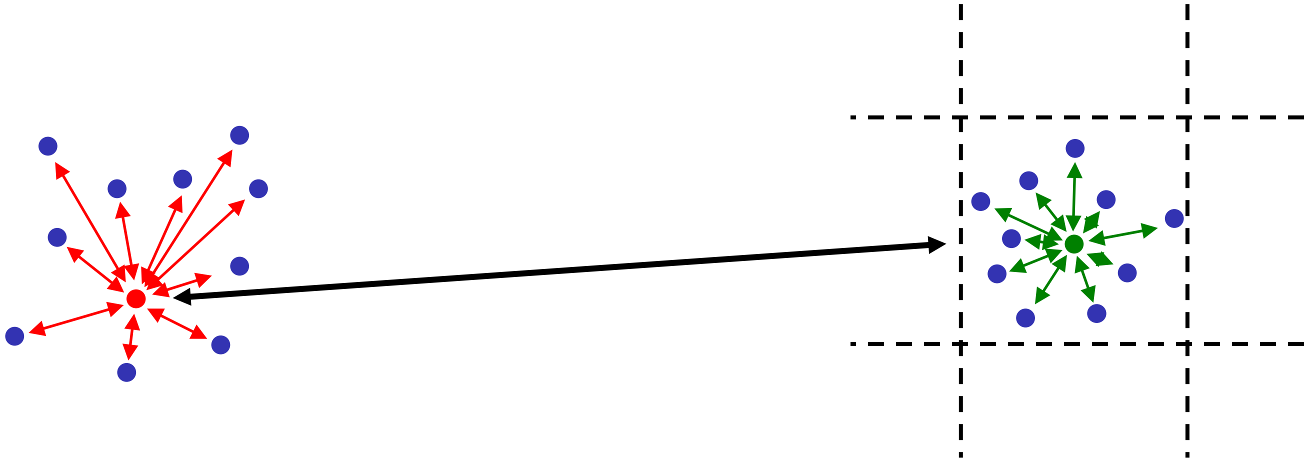
ALCF: EARLY SCIENCE
Argonne NATIONAL LABORATORY
Los Alamos NATIONAL LABORATORY
BERKELEY LAB

Initial conditions



N-body gravitational solvers: particles and meshes

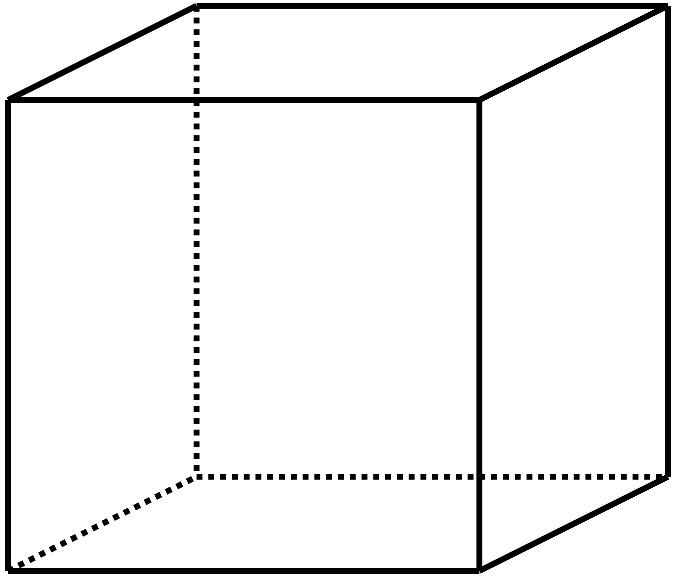


PP: Particle-Particle
▪ $\mathcal{O}(N^2)$
▪ Short distance: Particle-Particle
▪ Long distance: Particle-Particle

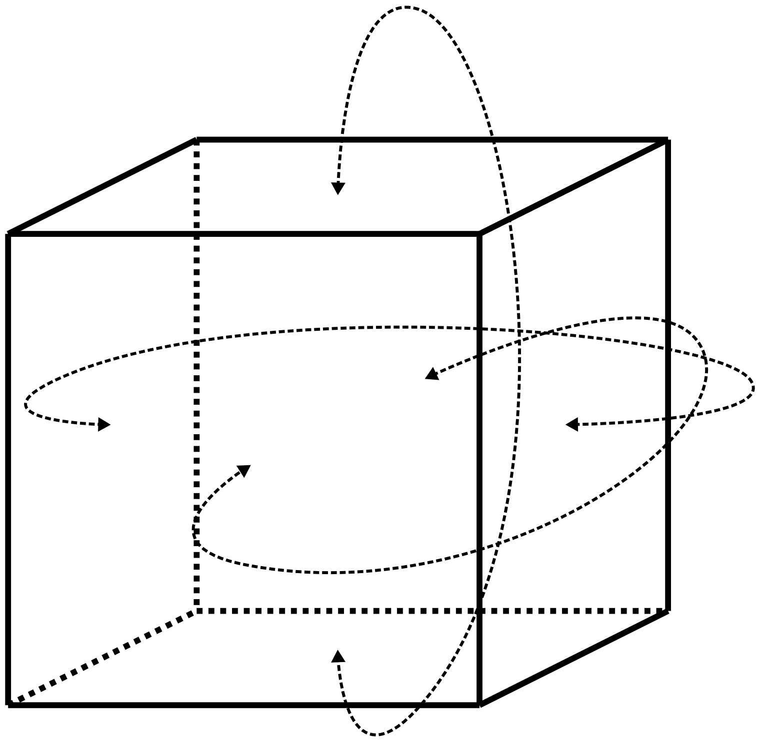
PM: Particle-Mesh
▪ $\mathcal{O}(N \log N)$
▪ Short distance: Particle-Mesh
▪ Long distance: Particle-Mesh

P^3M: Particle-Particle Particle-Mesh
▪ $\mathcal{O}(N \log N)/\mathcal{O}(N)$
▪ Short distance: Particle-Particle
▪ Long distance: Particle-Mesh

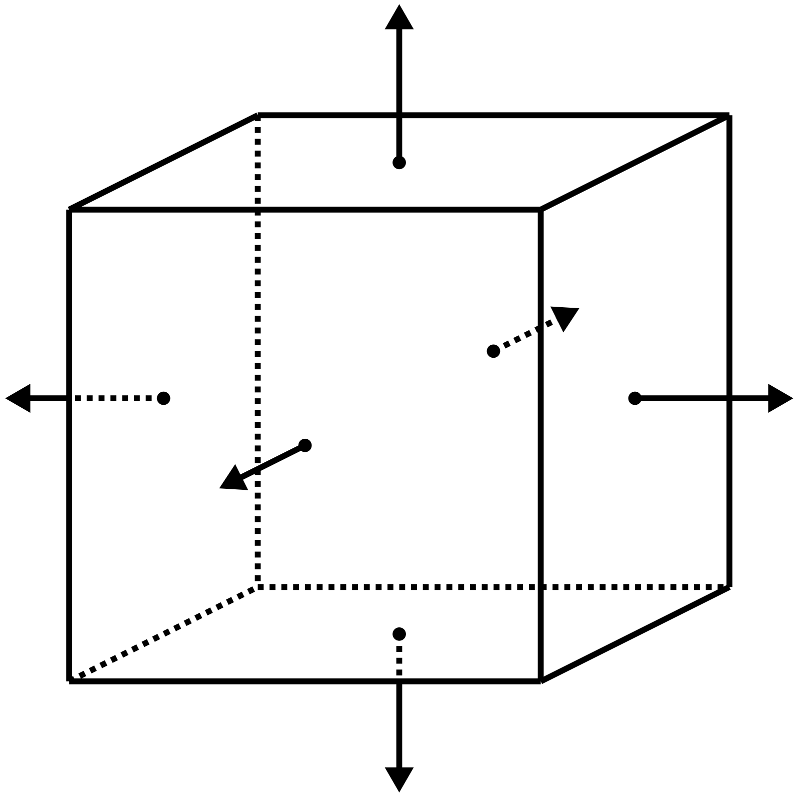
Simulation box: periodic and expanding



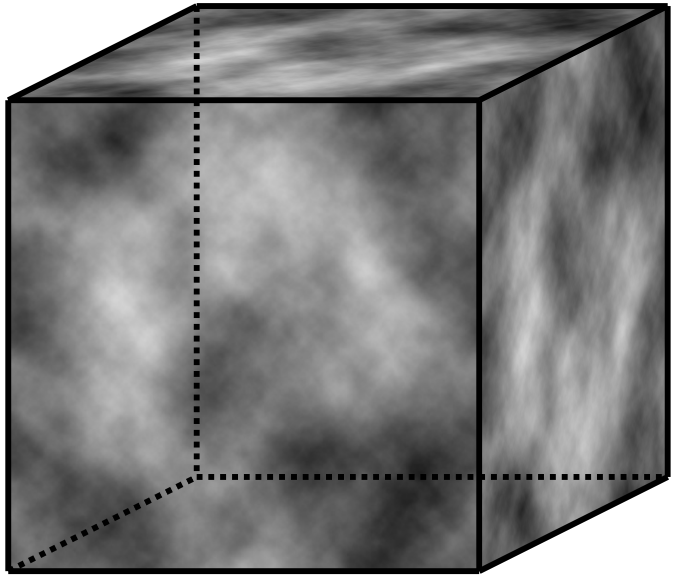
Simulation box



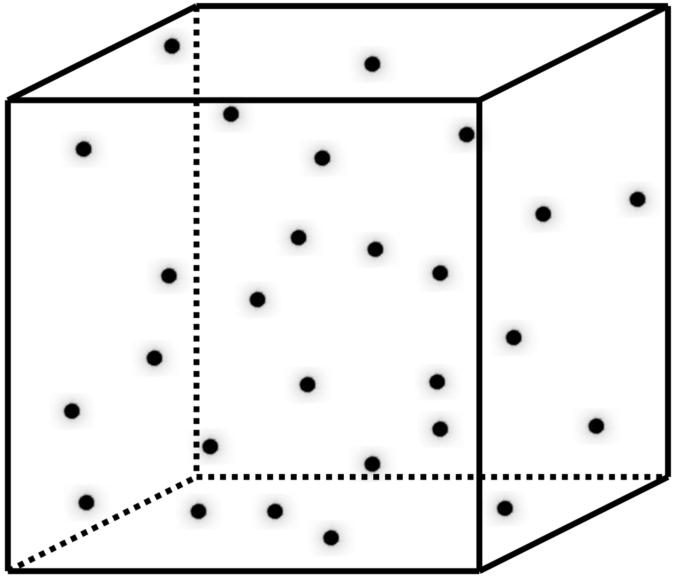
Periodic boundary conditions



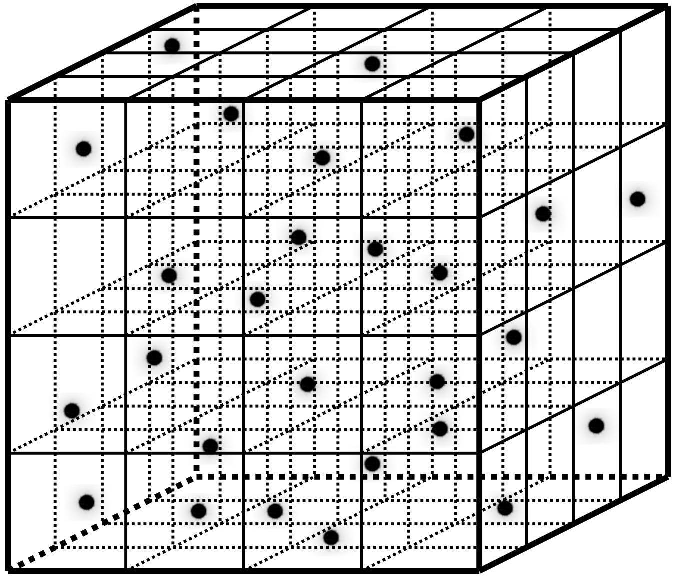
Coordinate system to take expansion into account



Initial density distribution



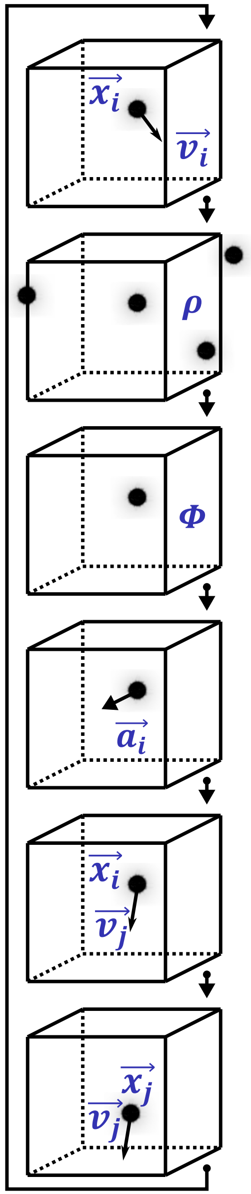
Matter particles



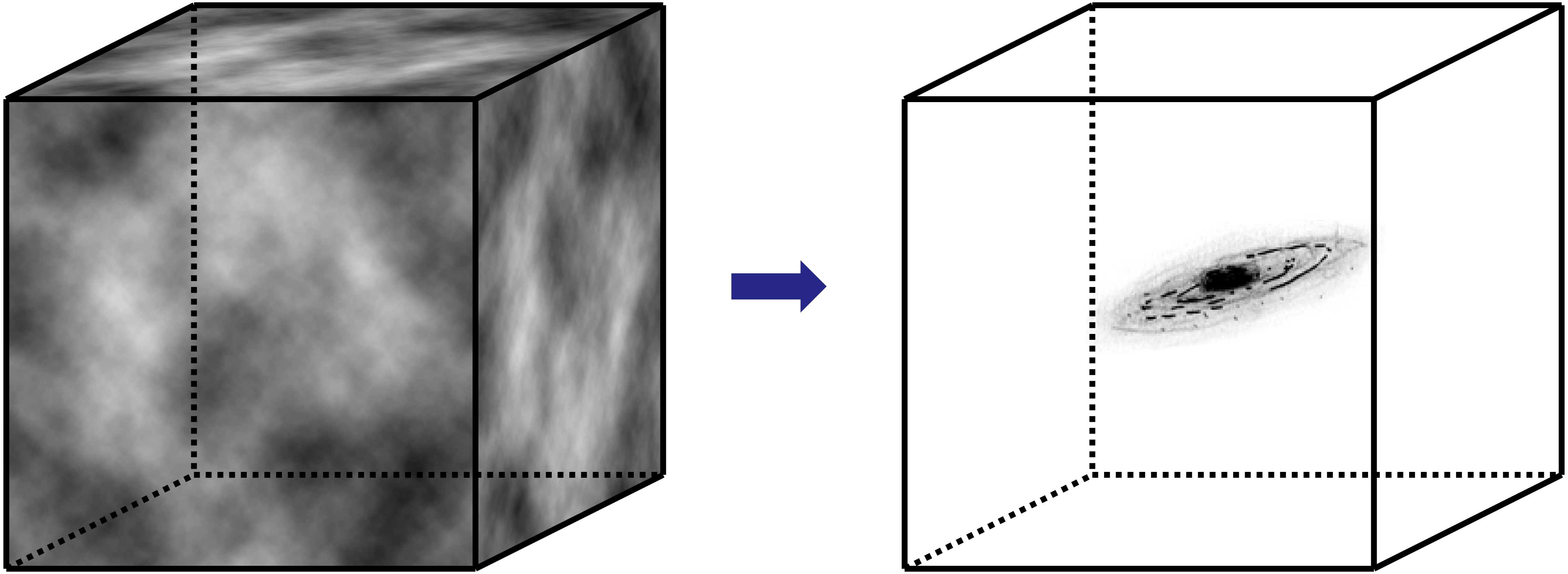
Cubic mesh: Regular or Adaptive Mesh Refinement

Core of the iterative process for each particle

- 1) For each cell $\boldsymbol{c}$ containing particles with position $\overrightarrow{x_i}$ and velocity $\overrightarrow{v_i}$
- 2) Interpolate density $\boldsymbol{\rho}$ in cell $\boldsymbol{c}$ depending on surrounding particles
- 3) From $\boldsymbol{\rho}$ compute the gravitational potential $\boldsymbol{\Phi}$
- 4) From $\boldsymbol{\Phi}$ interpolate back the acceleration $\vec{a}$ at position $\overrightarrow{x_i}$
- 5) From $\vec{a}$ compute the new speed $\overrightarrow{v_j}$ of each particle
- 6) From $\overrightarrow{v_j}$ compute the new position $\overrightarrow{x_j}$ of each particle
- 7) Restart at 1) with uploaded position $\vec{x}$ and speed $\vec{v}$



And that's how large scale cosmic structure formation is simulated

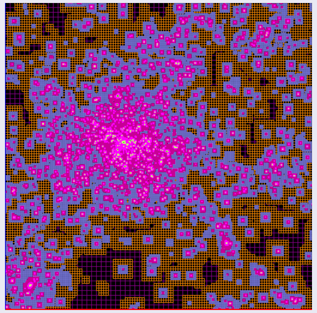


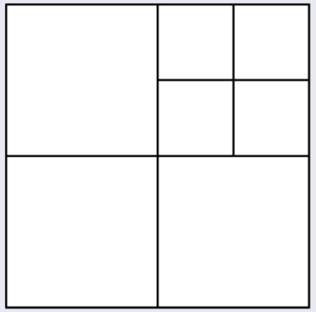
Initial conditions of the simulation (~homogeneous)

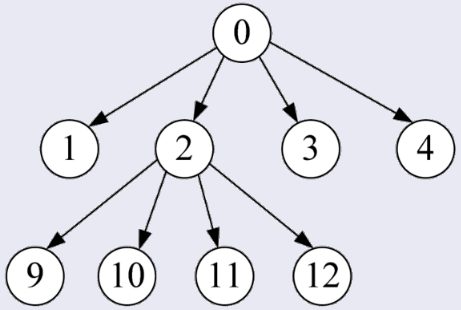
Gravitational collapse and structure formation

PM-AMR: Adaptive Mesh Refinement

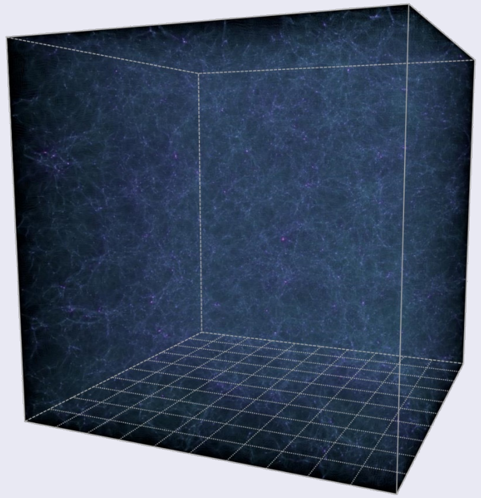
Quadtrees in 2D

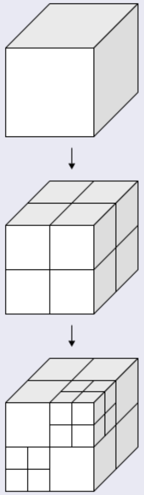


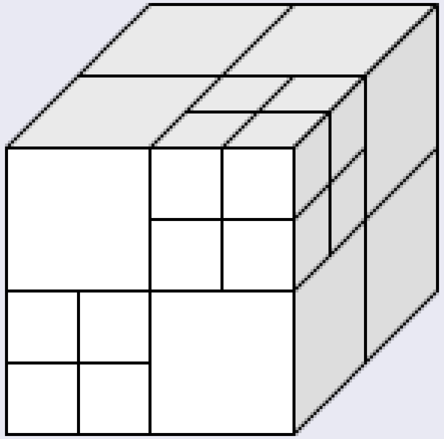


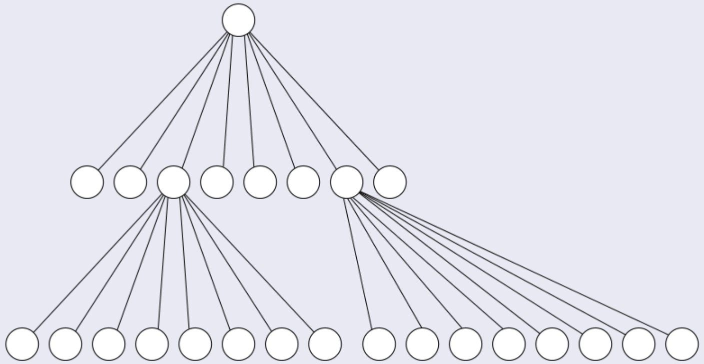


Octrees in 3D



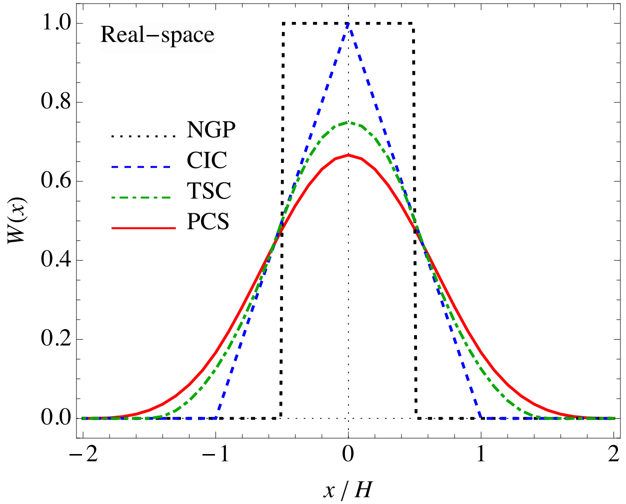






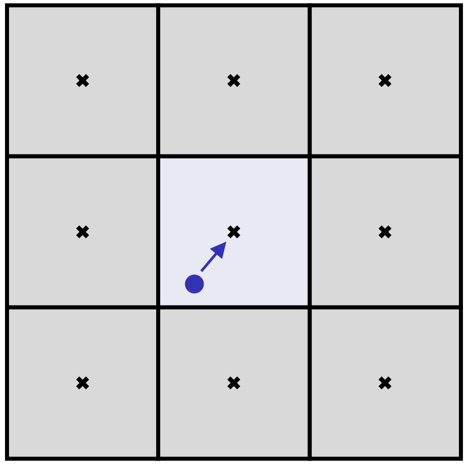
Interpolation schemes: NGP, CIC, TSC...

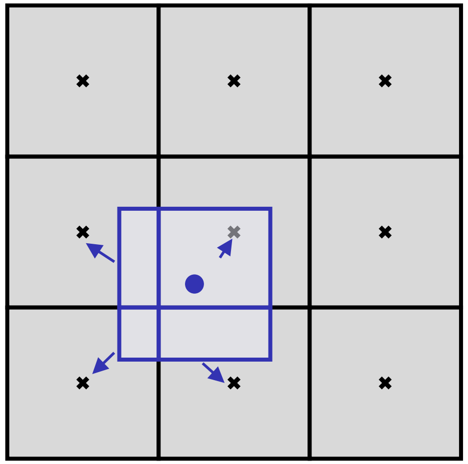
1D



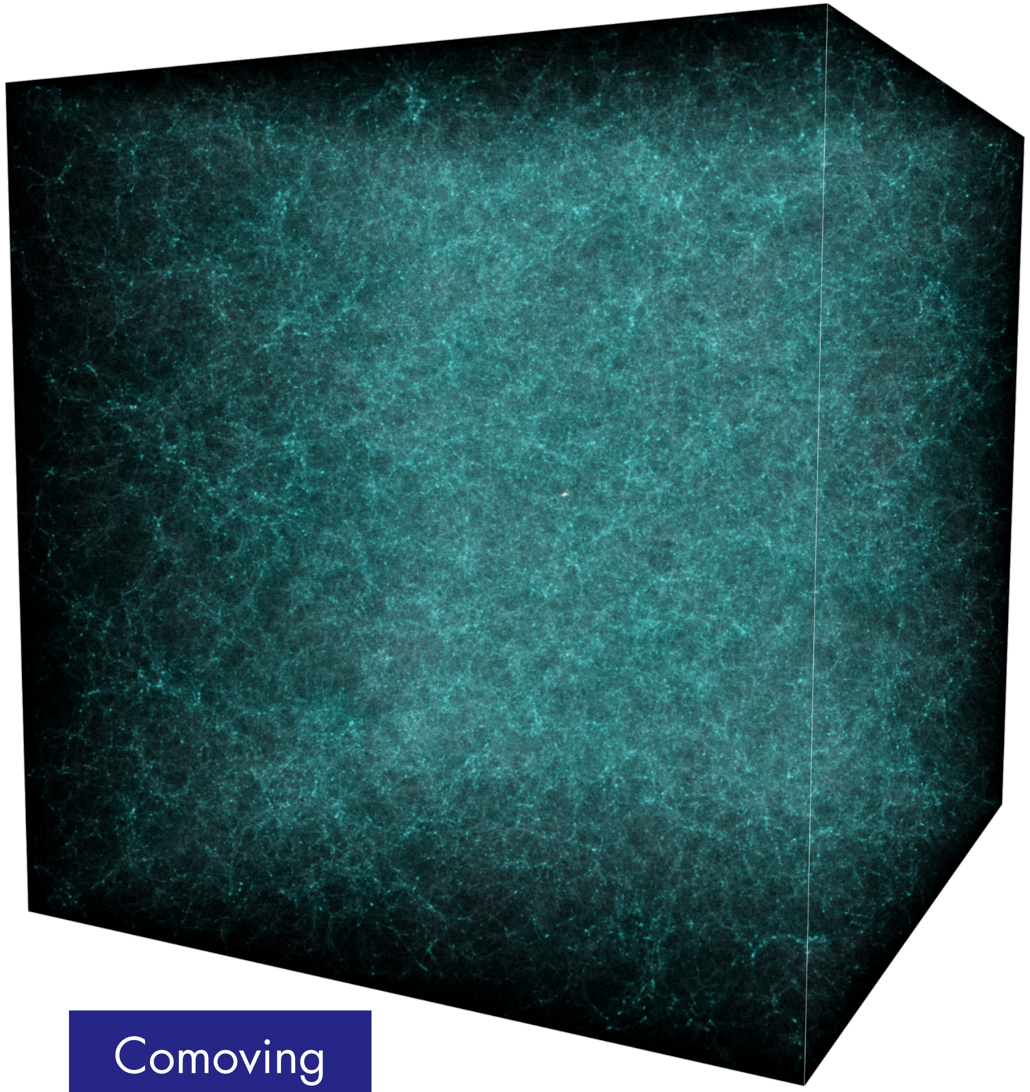
NGP	CIC	TSC
Nearest Grid Point	Cloud-in-Cell	Triangular Shaped Cloud

2D

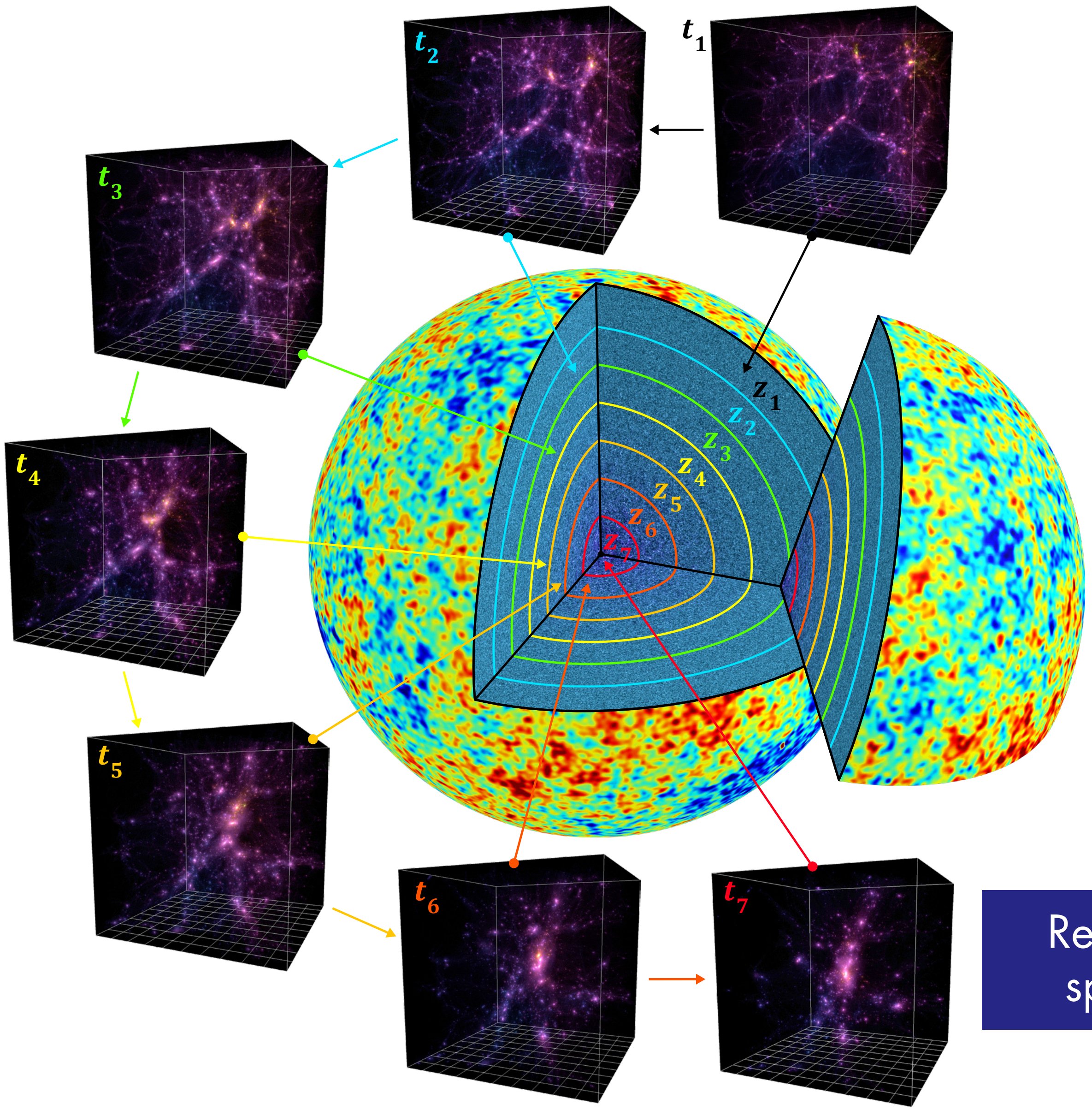




Cells	1D	2D	3D
NGP	1	1	1
CIC	2	4	8
TSC	3	9	27

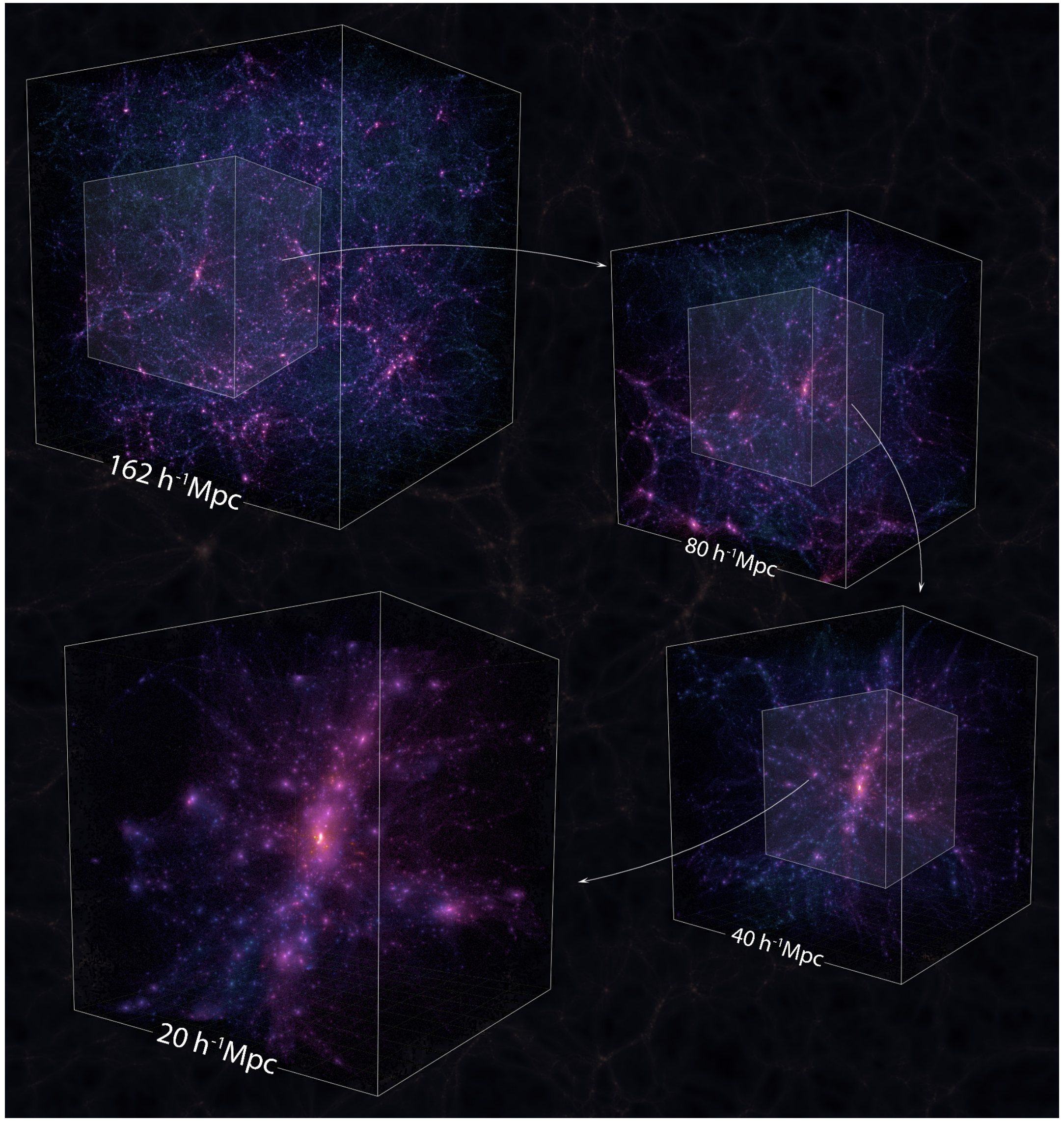


Comoving space



Redshift space

Properties of pure dark matter simulations



Main parameters

- Initial positions and speed of particles
- Cosmological model
- Box size
- Number of particles
- Resolution in mass (particle mass)
- Resolution in size (minimum cell size)
- Resolution in time (time step)

Solver parameters (examples)

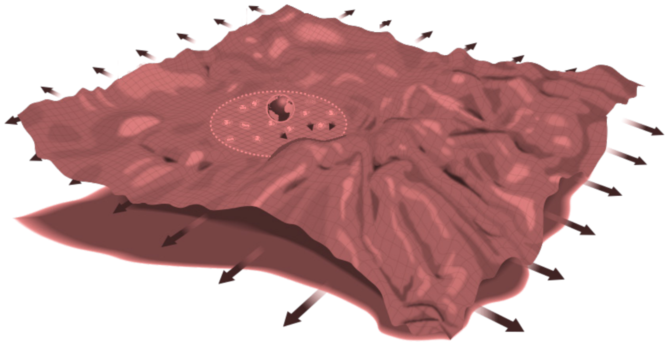
- Algorithm
- Discretization strategy
- Refinement strategy
- Floating-point precision
- Interpolation scheme
- Parallelization strategy

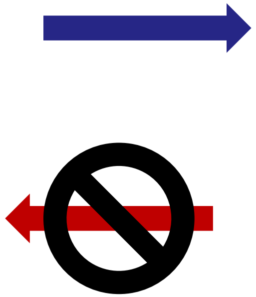
What is NOT done in cosmological simulations

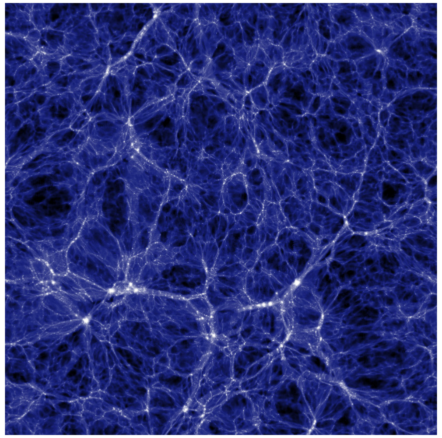
- Cosmological simulations are NOT solving general relativity

What is done in cosmological simulation

- Solve newtonian gravity in a homogeneous expanding background
- Expansion is pre-computed (FLRW solver)
- Instantaneous propagation of gravity
- ⇒ *see debates on the Backreaction Conjecture*







Precomputed FLRW metric

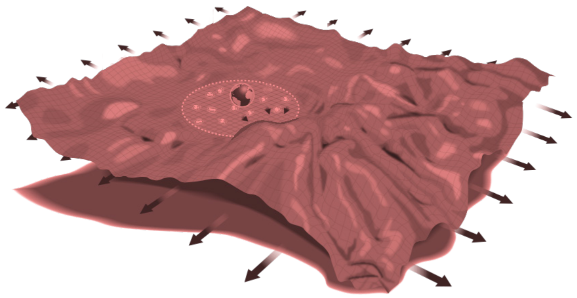
$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

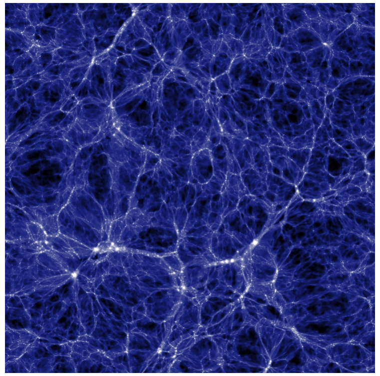
Matter density dynamics

Averaging does not commute

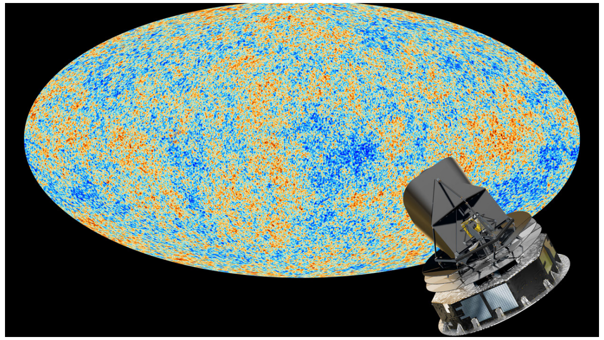
$$\Omega_M, \Omega_R, \Omega_\Lambda, \ldots$$

Averaging ↗ **Evolution** ↖





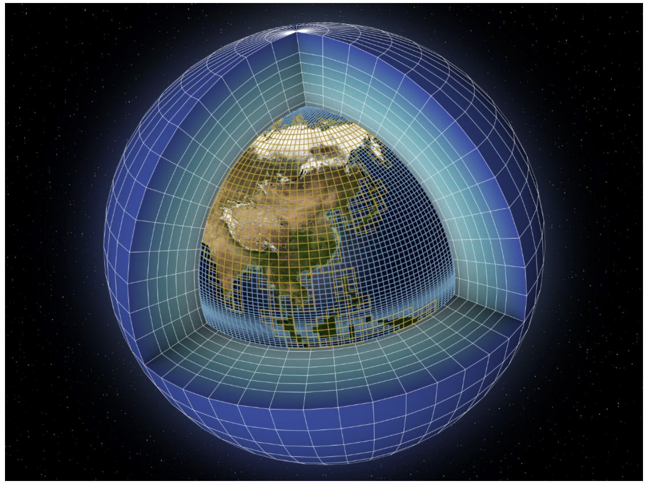
Evolution ↖ **Averaging** ↗

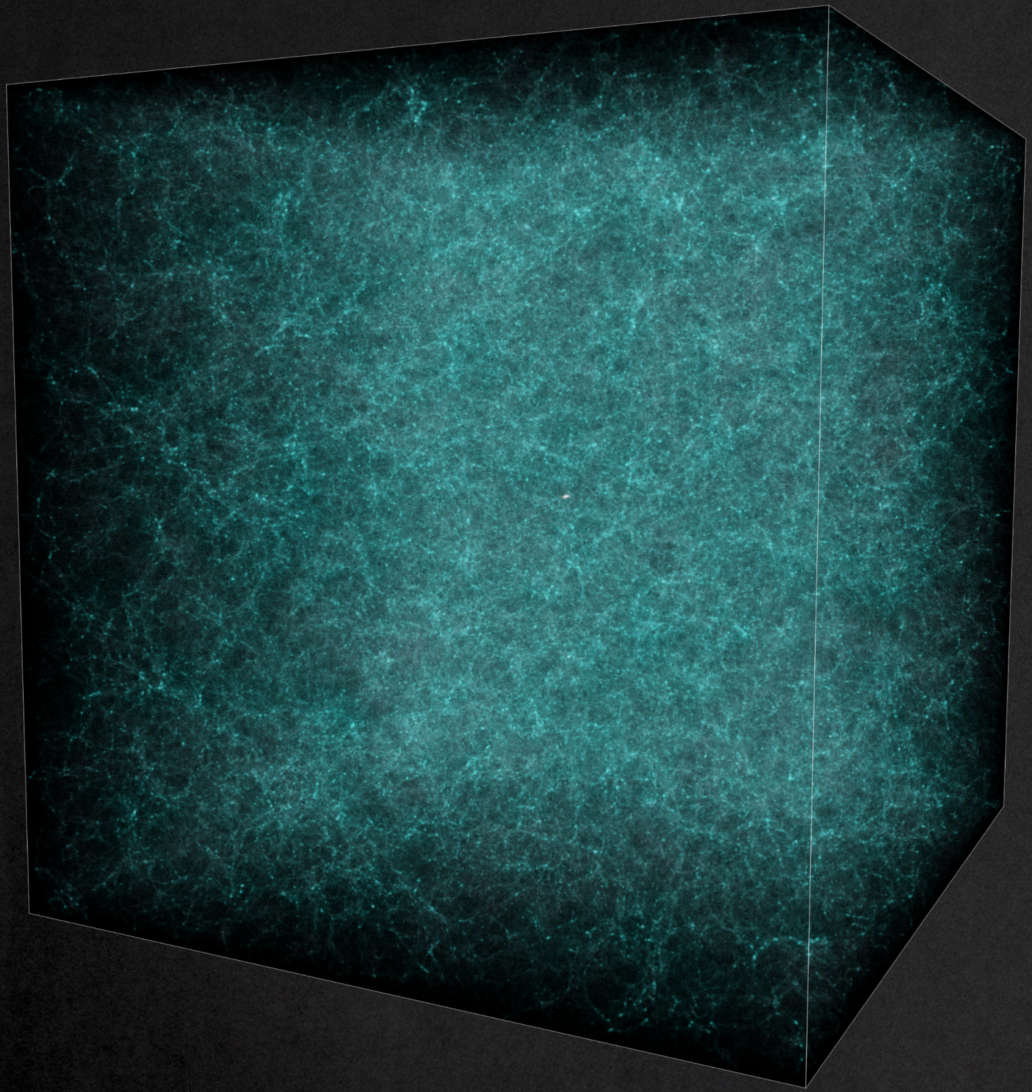


$$\text{averaging}(\text{evolution}(X)) \neq \text{evolution}(\text{averaging}(X))$$

Analogy

Temperature field in climate simulations





General relativity is a purely local theory

Comoving coordinates is an approximation

Each observer has a different notion of space and time

We aim at doing *precision* cosmology with petabytes of observational data using a simplifying 100-years old model back when we didn't know if there was anything else than the Milky Way.

Next time

How can we do better?