Efficient solver for relativistic hydrodynamics with implicit Runge-Kutta method

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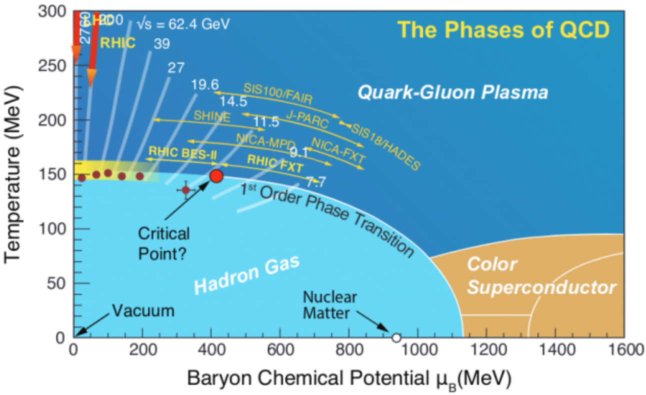
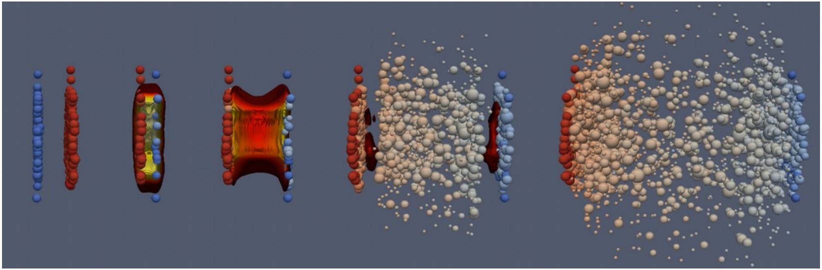
QCD

QCD : Quantum chromo-dynamics



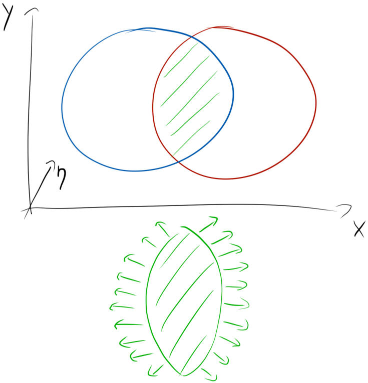
QCD phase diagram

Heavy ion collisions evolution

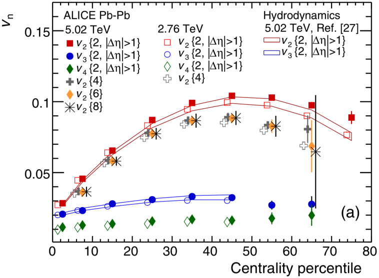


- Collision of the 2 Lorentz boosted nucleus
- interaction and creation of the QGP → fluid dynamics
- chemical and kinetic freeze-out → particlization and free streaming

Why a fluid?



Almond shape → elliptic flow



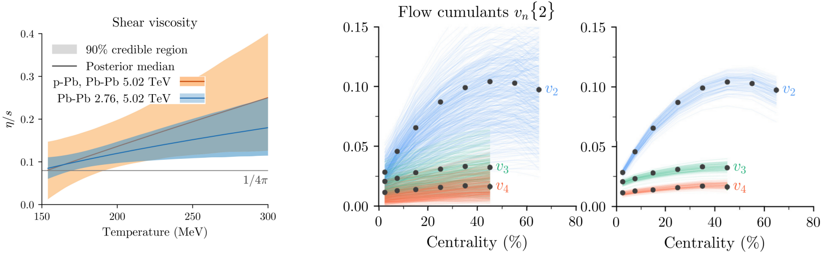
J. Adam et al, PRL 116, 132302 (2016)

Simulation framework

Heavy-ion collision stages:
Initial collision → Pre-hydrodynamics → Hydrodynamics → Freeze-out → Cascade/Kinetic

Numerical framework:
Trento → skipped → ImplHydro → frzout → UrQMD

To fit all the involved parameters, Bayesian analysis are performed. For instance: PRC 101, 024911 (2020)



Relativistic hydrodynamics

Energy–momentum tensor conservation:

$$\partial_{;\,\mu} T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^{\mu}{}_{\alpha\mu} T^{\alpha\nu} + \Gamma^{\nu}{}_{\alpha\mu} T^{\mu\alpha} = 0$$

$$T^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\Delta_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu, \qquad u^\mu u_\mu = 1,$$

Israel-Stewart equations:

$$u^\lambda \partial_{;\lambda}\Pi = -\frac{\Pi - \Pi_{NS}}{\tau_\Pi} - \frac{4}{3}\Pi\partial_{;\lambda}u^\lambda$$

$$\left\langle u^\lambda \partial_{;\lambda}\pi^{\mu\nu}\right\rangle = -\frac{\pi - \pi^{\mu\nu}_{NS}}{\tau_\pi} - \frac{4}{3}\pi\partial_{;\lambda}u^\lambda$$

$$\Pi_{NS} = -\zeta\partial_{;\lambda}u^\lambda,$$

$$\pi^{\mu\nu}_{NS} = \eta\left(\Delta^{\mu\lambda}\partial_{;\lambda}u^\nu + \Delta^{\nu\lambda}\partial_{;\lambda}u^\nu\right) - \frac{2}{3}\eta\Delta^{\mu\nu}\partial_{;\lambda}u^\lambda,$$

$$\tau_\pi = \tau_\Pi = \frac{5\eta}{\epsilon + P}$$

Metric

Cartesion coordinates:

(t,x,y,z)

$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

Milne coordinates:

(τ, x, y, η)

$\tau = \sqrt{t^2 - z^2}$, $\eta = \tanh^{-1} \frac{z}{t}$

$t = \tau \cosh \eta$, $z = \tau \sinh \eta$

$g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -\tau^2)$

$\Gamma^{\eta}{}_{\tau\eta} = \Gamma^{\eta}{}_{\eta\tau} = 1/\tau$, $\Gamma^{\tau}{}_{\eta\eta} = \tau$.

Relativistic Hydrodynamics

Rewriting the equations:

$$\partial_\tau \left(\tau T^{\tau\nu}_{\mathrm{id}}\right) = -\partial_x \left(\tau T^{x\nu}_{\mathrm{id}}\right) - \partial_y \left(\tau T^{y\nu}_{\mathrm{id}}\right) - \partial_\eta \left(T^{\eta\nu}_{\mathrm{id}}\right) + A(\bullet, \partial_\tau \bullet)$$
$$\partial_\tau \left(u^\tau \pi^{\mu\nu}\right) = -\partial_x \left(u^x \pi^{\mu\nu}\right) - \partial_y \left(u^y \pi^{\mu\nu}\right) - \partial_\eta \left(u^\eta \pi^{\mu\nu}\right) + B(\bullet, \partial_\tau \bullet)$$
$$\partial_\tau \left(u^\tau \Pi\right) = -\partial_x \left(u^x \Pi\right) - \partial_y \left(u^y \Pi\right) - \partial_\eta \left(u^\eta \Pi\right) + C(\bullet, \partial_\tau \bullet)$$

where $\bullet = \epsilon, u^\mu, \pi^{\mu\nu}, \Pi$.

Right-hand-side time derivatives partial solving

Extraction of the necessary time derivatives:

$$
\begin{aligned}
\partial_\tau\left(\tau T_{\mathrm{id}}^{\tau\nu}\right) &= \tau\partial_\tau T_{\mathrm{id}}^{\tau\nu} + T_{\mathrm{id}}^{\tau\nu} \\
\partial_\tau\left(u^\tau \pi^{\mu\nu}\right) &= u^\tau \partial_\tau \pi^{\mu\nu} + \pi^{\mu\nu}\partial_\tau u^\tau \\
\partial_\tau\left(u^\tau \Pi\right) &= u^\tau \partial_\tau \Pi + \Pi\partial_\tau u^\tau \\
\partial_\tau\left(u^\mu u_\mu\right) &= 0 = u^\tau\partial_\tau u^\tau + u^x\partial_\tau u^x + u^y\partial_\tau u^y + u^\eta\partial_\tau u^\eta \\
V &= \left(\partial_\tau\epsilon, \partial_\tau u^x, \partial_\tau u^y, \partial_\tau u^\eta\right) \\
\partial_\tau T_{\mathrm{id}}^{\tau\nu} &= \frac{\partial\left(\partial_\tau T_{\mathrm{id}}^{\tau\nu}\right)}{\partial V} V = mV \Rightarrow V = m^{-1}\partial_\tau T_{\mathrm{id}}^{\tau\nu}
\end{aligned}
$$

Numerical scheme

Hydro equations with the dynamical variables $y = (\tau T_{\rm id}^{\tau\nu}, u^\tau \pi^{\mu\nu}, u^\tau \Pi)$:

$$\partial_t y = f(t, y)$$

Kurganov-Tadmor space discretization: $f(t,y) = f_{KT}(t,y) + O(\Delta x^2)$

- **Independent of time discretization**
- Numerical diffusion $\propto \Delta x^2$
- Flux limiter $\rightarrow$ avoid numerical oscillations

Runge-Kutta time discretization

$$y(t+\Delta t) = y(t) + \Delta t \sum_j b_j k_j$$

$$k_i = f(t, y(t) + \Delta t \sum_j a_{ij} k_j) \qquad \text{``}\vec{K} = \vec{F}(\vec{K})\text{''}$$

Choice of Runge-Kutta coefficients

Accuracy order:

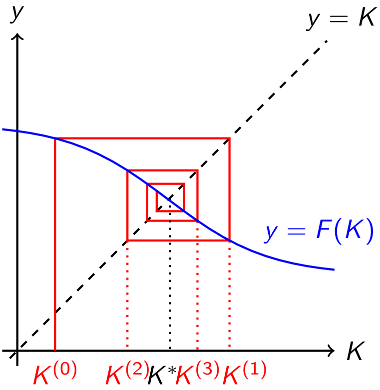
$$||y^*(t) - y(t)|| < C\Delta t^p$$

Second order choice:

	Heun	Gauss-Legendre 1 (GL1)
Type	Explicit	Implicit
Stage S	2	1
Order p	2	2
$\begin{array}{c\|c} c_n & a_{nm} \\ \hline & b_m \end{array}$	$\begin{array}{c\|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 0.5 & 0.5 \end{array}$	$\begin{array}{c\|c} 0.5 & 0.5 \\ \hline & 1 \end{array}$

Implicit solver

Solve implicit equation "$\vec{K} = \vec{F}(\vec{K})$" by the fixed-point solver



$$\vec{K}^{(l+1)} = \vec{F}(\vec{K}^{(l)})$$

$$\vec{K}^{(0)} = \vec{0} \text{ or last time step solution}$$

Solve iteratively

$$\vec{K}^{(0)} \to \vec{K}^{(1)} \to \vec{K}^{(2)} \to \ldots \to \vec{K}^{(l+1)}$$

Local optimization

1. Update all cells once to obtain $\vec{K}^{(1)}$
2. check the convergence in every cell $[\bullet]_j$:

$$\left\|[\vec{F}(\vec{K}^{(l)})]_j - [\vec{K}^{(l)}]_j\right\| < e\frac{\langle T^{\tau\tau}\rangle}{\Delta\tau}\left(\frac{\Delta\tau}{\Delta x}\right)^{(p+1)}$$

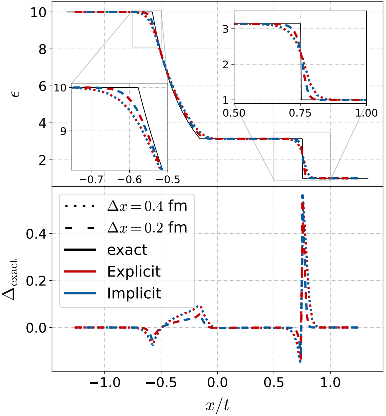
3. To obtain $\vec{K}^{(l+1)}$ $(l+1 \geq 2)$, only update a cell if itself or any surrounding cells does not satisfy the threshold
4. Repeat (2) and (3) until all cells satisfy the threshold

This dramatically reduces the computation cost.

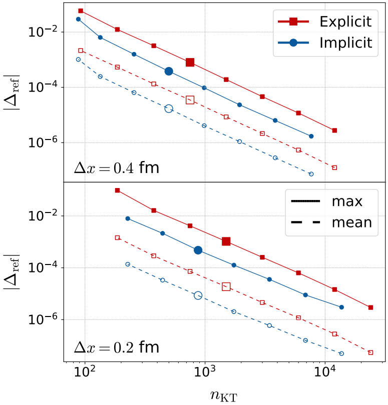
Disclaimer: This partial update breaks conservation and leads to inconsistencies, but this is controlled within the error threshold.

1D ideal Riemann problem

Vanishing Δt comparison

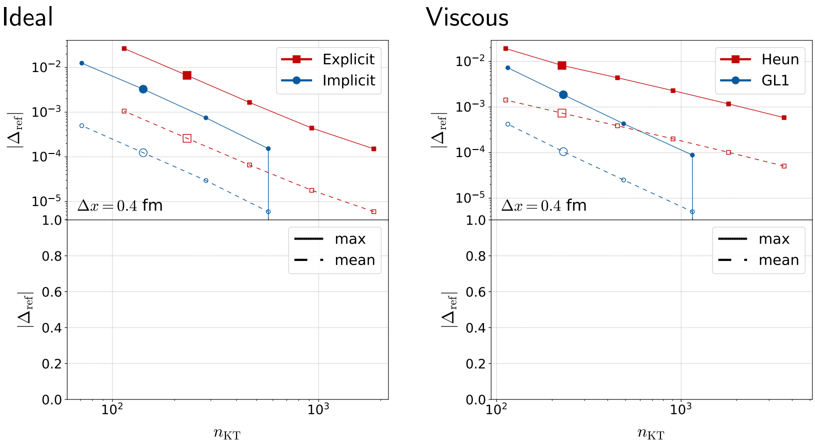


Error-cost comparison



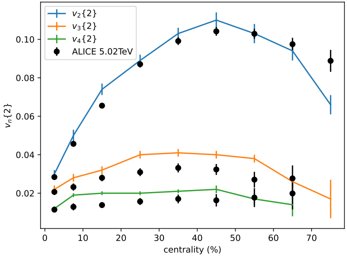
- At vanishing Δt, both explicit and implicit agree
- As function of cost, implicit is more efficient than explicit

3D Trento initial conditions: ideal vs viscous (Preliminary)

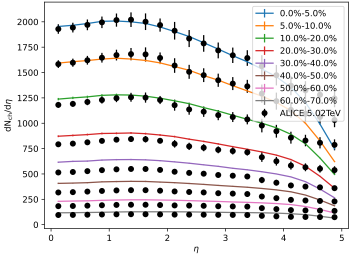


Physical results

Elliptic flow



Particle spectra



Conclusion

Achievement

- Implicit method can be more efficient and stable than explicit
- First true 2^{nd} order time integrator for viscous hydrodynamics

Outlook

- Including other charges (baryon number, electric charge, ...)
- Including fluctuations

Thank you for your attention

KT 2nd order algorithm

$$KT[f,\rho(f),y,\Delta x]_i = \frac{H_{i+1/2}-H_{i-1/2}}{\Delta x}$$

$$H_{i+1/2} = \frac{f(y^+_{j+1/2}) + f(y^-_{j+1/2})}{2} + \frac{a_{i+1/2}}{2}\left(y^+_{j+1/2} - y^-_{i+1/2}\right)$$

$$a_{i+1/2} = max\left\{\rho\left(\frac{\partial f}{\partial y}\left(y^+_{i+1/2}\right)\right), \rho\left(\frac{\partial f}{\partial y}\left(y^-_{i+1/2}\right)\right)\right\}$$

$$y^+_{i+1/2} = y_{i+1} - \frac{\Delta x}{2}(\partial_x y)_{i+1}$$

$$y^-_{i+1/2} = y_i + \frac{\Delta x}{2}(\partial_x y)_i$$

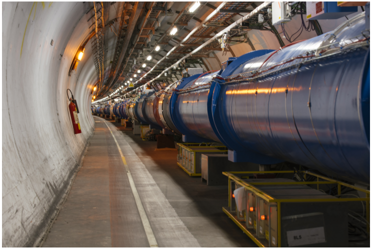
$$(\partial_x y)_i = \text{minmod}\left(\theta\frac{y_i - y_{i-1}}{\Delta x}, \frac{y_{i+1}-y_{i-1}}{2\Delta x}, \theta\frac{y_{i+1}-y_i}{\Delta x}\right)$$

$$1 \leq \theta \leq 2$$

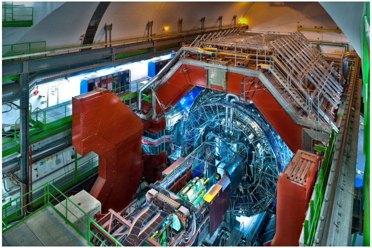
needs $y_{i-2}, y_{i-1}, y_i, y_{i+1}, y_{i+2}$.

Heavy ion collisions

Experimental study of the QGP through heavy ion collisions



LHC accelerator



Alice particle detector

$\Rightarrow$ Give enough energy to the colliding nucleus to rich the QGP state.