Exponential S-Matrix for Classical Observables

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Based on work with

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A motivation from Costas

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Spécialité :

Physique Théorique

par

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Au bout de la corde... la théorie M

soutenue le 17 avril 1998 devant le jury composé de :

C. Bachas	Directeur de thèse
É. Brézin	Président
J.P. Derendinger	Rapporteur
E. Kiritsis	
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V. Rivasseau	
E. Verlinde	
P. Windey	Rapporteur

A motivation from Costas

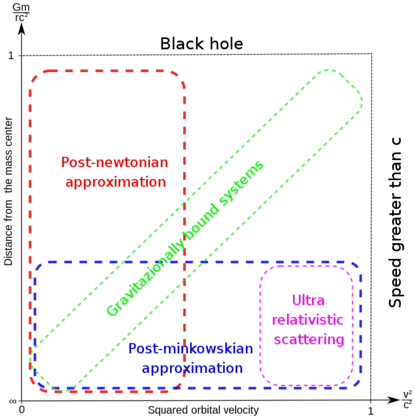
Early on my phd period Costas made me familiar with the paper famous Amati, Ciafaloni, Veneziano (ACV) papers in which classical gravity effects emerge from the scattering of two strings strings at trans-planckian energies in a flat Minkowski background. They analyzed how the effective spacetime experienced by the two colliding strings is modified.

I left my phd without a satisfactory understanding of the consequences of their results. Still this was a motivation behind futher work on the eikonal resommation in classical and quantum gravity.

«Το μυαλό δεν είναι ένα δοχείο που πρέπει να γεμίσει αλλά μια φωτιά που πρέπει ν' ανάψει». Πλούταρχος
The mind is not a vessel to be filled but a fire to be lit. Plutarch

Actually switching from UV to IR question and looking at the long range gravitational interactions between two massive bodies as led to interesting reformulation of the S-matrix and the calculation of the scattering angle in classical and quantum gravity.

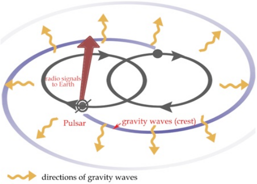
Two-body gravitational interactions



One can connect the scattering angle in the ultrarelativistic regime computed by ACV to the ultra-high energy limit of classical post-Minkowskian two-body scattering [Damour; Heissenberg, di Vecchia, Veneziano; Bjerrum-Bohr, Damgaard, Planté, Vanhove]

Post-Minkowskian expansion for the binary system

We want to compute gravitational interaction of two massive objects.



A relativistic Hamiltonian for the two body dynamics in centre-of-mass

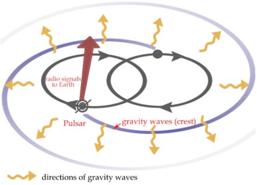
$$\mathcal{H}_{\rm PM}(\gamma, r) = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} + \sum_{L\geq 0} \underbrace{\mathcal{V}_{L+1}(\gamma, r)}_{\propto G_N^{L+1}/r^{L+1}}$$

with a relativistic potential organised in a series of Newton's constant G_N which is the general relativity correction to Newton's potential $L=0$

$$\mathcal{V}_1(\gamma, r) = -\frac{G_N}{E_1E_2}\frac{m_1^2m_2^2}{r}(2\gamma^2-1) \qquad \gamma = \frac{p_1\cdot p_2}{m_1m_2} = \frac{1}{\sqrt{1-\frac{\vec v^2}{c^2}}} \geq 1$$

Post-Minkowskian expansion for the binary system

We want to compute gravitational interaction of two massive objects.



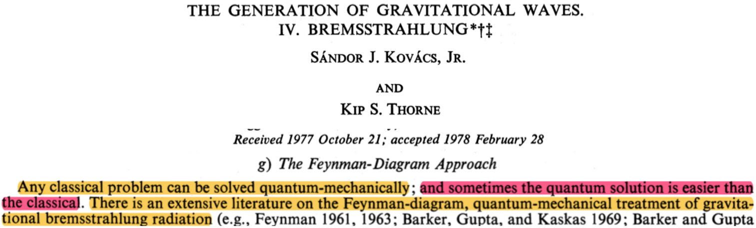
The $L+1$PM potential has polynomial mass dependence

$$\mathcal{V}_{L+1}(\gamma, r) = \frac{G_N^{L+1} m_1^2 m_2^2}{r^{L+1}} \sum_{r_1+r_2=L} v_{r_1,r_2}(\gamma) m_1^{r_1} m_2^{r_2}$$

Consider its Fourier transform to momentum space

$$\mathcal{M}_{L+1}(\gamma, \underline{q}^2) \propto \int e^{-i\underline{q}\cdot\vec{r}} \mathcal{V}_{L+1}(\gamma, r) d^3\vec{r} \propto \frac{G_N^{L+1} m_1^2 m_2^2}{\underline{q}^{2-L}} \sum_{r_1+r_2=L} v_{r_1,r_2}(\gamma) m_1^{r_1} m_2^{r_2}$$

Classical physics from quantum loops



THE GENERATION OF GRAVITATIONAL WAVES.
IV. BREMSSTRAHLUNG*†‡

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AND

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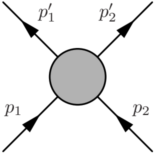
g) *The Feynman-Diagram Approach*

Any classical problem can be solved quantum-mechanically; and sometimes the quantum solution is easier than the classical. There is an extensive literature on the Feynman-diagram, quantum-mechanical treatment of gravitational bremsstrahlung radiation (e.g., Feynman 1961, 1963; Barker, Gupta, and Kaskas 1969; Barker and Gupta

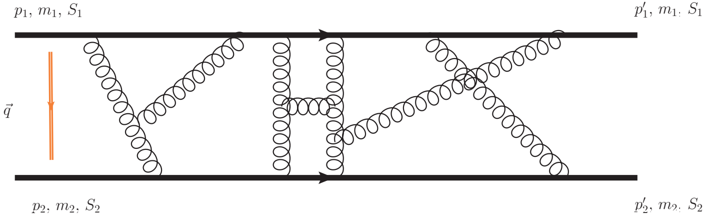
We seek quantum gravity formalism where the classical limit $\hbar \to 0$ gives the general relativity potential

$\lim_{\hbar\to 0}$



$$\to \mathcal{M}_L(\gamma, \underline{q}^2) \to \mathcal{V}_{L+1}(p, r)$$

Perturbative gravity



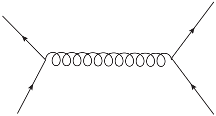
We will be considering the pure gravitational interaction between massive and massless matter of various spin $g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{G_N} h_{\mu\nu}$

$$\mathcal{S}_{\rm EH} = \int d^4x \sqrt{-g} \left(-\frac{\mathcal{R}}{16\pi G_N} + \frac{1}{2} \sum_{a=1}^{2} \left(g^{\mu\nu} \partial_\mu \phi_a \partial_\nu \phi_a - m_1^2 \phi_a^2 \right) \right)$$

Evaluating the quantum scattering S-matrix

$$\mathfrak{M}^{\rm GR}(p_1 \cdot p_2, \underline{q}, \hbar) = \sum_{L \geq 0} G_N^{L+1} \mathfrak{M}_L(p_1 \cdot p_2, \underline{q}, \hbar)$$

One graviton exchange : tree-level amplitude



$$\mathfrak{M}_0 = -16\pi G_N \hbar \frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2 - |\hbar\vec{\underline{q}}|^2 (p_1\cdot p_2)]}{|\hbar\vec{\underline{q}}|^2}$$

The $\hbar$ expansion of the tree-level amplitude

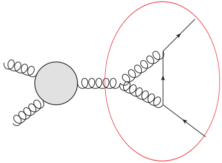
$$\mathfrak{M}_0 = \frac{\mathcal{M}_1^{(-1)}(\gamma)}{\hbar|\underline{q}|^2} + \hbar 4\pi G_N p_1\cdot p_2$$

The relativistic classical Newtonian potential is obtained by taking the Fourier transform

$$V_1(\gamma, r) = \int \frac{d^3\vec{\underline{q}}}{(2\pi)^3} \frac{1}{4E_1E_2} \mathcal{M}_1^{(-1)}(\gamma)\, e^{i\vec{\underline{q}}\cdot\vec{r}} = -\frac{G_N m_1 m_2^2}{E_1E_2} \frac{2\gamma^2-1}{r}$$

The higher order in $\underline{q}^2$ is the quantum contact interaction of order $\hbar$

Classical physics from loops : the one-loop triangle



The Klein-Gordon equation reads

$$(\Box - \frac{m^2c^2}{\hbar^2})\phi = 0$$

The triangle with a massive leg $p_1^2 = p_2^2 = m^2$ reads

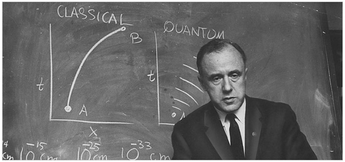
$$\int \frac{\kappa^2 \, d^4\ell}{(\ell+p_1)^2(\ell^2 - \frac{m^2c^2}{\hbar^2})(\ell-p_2)^2}\Bigg|_{\text{finite part}}$$

$$\sim \frac{\kappa^2}{m^2}\left(\log(q^2) + \frac{\pi^2 mc}{\hbar\sqrt{\underline{q}^2}}\right)$$

This one-loop amplitude contains [Iwasaki, Holstein, Donoghue, Bjerrum-Bohr, Vanhove]

- The classical 2nd post-Minkowskian correction G_N^2/r^2 to Newton's potential of order $1/\hbar$
- An infrared quantum correction of order $\hbar^0$

Classical physics from loops : $\hbar$ counting



The classical limit $\hbar \to 0$ fixed $\underline{q} \ll m_1, m_2$ of the amplitude [Bjerrum-Bohr, Damgaard, Vanhove, Planté]

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma)}{\hbar^{L+1}|\underline{q}|^{2+\frac{L(4-D)}{2}}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma)}{\hbar|\underline{q}|^{2-L+\frac{L(4-D)}{2}}} + O(\hbar^0)$$

- A classical contribution of order $1/\hbar$ from all loop orders
- The dimensional regularisation scheme gives a control of the IR divergences from radiation
- The computation is explicitly relativistic invariant

Classical physics from loops : $\hbar$ counting

The connection between quantum scattering and classical gravitational physics has forced to rethink the S matrix for dealing with the $\hbar$ expansion

[Damgaard, Planté, Vanhove]

$$\hat{S} = \mathbb{I} + \frac{i}{\hbar}\hat{T} =: \exp\left(\frac{i\hat{N}}{\hbar}\right)$$

doing the Dyson expansion with the conservative and radiation part

$$\hat{T} = G_N \sum_{L\geq 0} G_N^L \hat{T}_L + G_N^{\frac{1}{2}} \sum_{L\geq 0} G_N^L \hat{T}_L^{\rm rad}, \hat{N} = G_N \sum_{L\geq 0} G_N^L \hat{N}_L + G_N^{\frac{1}{2}} \sum_{L\geq 0} G_N^L \hat{N}_L^{\rm rad}$$

The higher powers of $1/\hbar$ are more singular than the classical contribution, but **are needed for the consistency of the full quantum amplitude and the correct exponentiation of the amplitude**

$$\mathfrak{M}_L(\gamma, \underline{q}^2, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma)}{\hbar^{L+1}|\underline{q}|^{2+\frac{L(4-D)}{2}}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma)}{\hbar|\underline{q}|^{2-L+\frac{L(4-D)}{2}}} + O(\hbar^0)$$

Classical observables

The change in an observable $\hat{O}$ is given by the [Kosower, Maybee, O'Connel] expression
$\langle \Delta\hat{O}\rangle := \langle \text{out}|\hat{O}|\text{out}\rangle - \langle \text{in}|\hat{O}|\text{in}\rangle$

$$\langle \Delta\hat{O}\rangle(p_1,p_2,r) = \int \frac{d^D(\hbar \underline{q})}{(2\pi)^{D-2}} \delta(2\hbar p_1\cdot \underline{q} - \hbar^2 \underline{q}^2)\delta(2\hbar p_2\cdot \underline{q} + \hbar^2 \underline{q}^2) e^{ir\cdot \underline{q}} \langle p_1', p_2'|\hat{O}|p_1,p_2\rangle$$

which can be expanded using the $\hat{N}$-operator

$$\langle \Delta\hat{O}\rangle = \langle \text{in}|\hat{S}^\dagger \hat{O}\hat{S} - \hat{O}|\text{in}\rangle = \sum_{n\geq 1} \frac{(-i)^n}{\hbar^n n!} \langle \text{in}|\underbrace{[\hat{N},[\hat{N},\dots,[\hat{N},\hat{O}],\dots,]]}_{n \text{ times}}|\text{in}\rangle$$

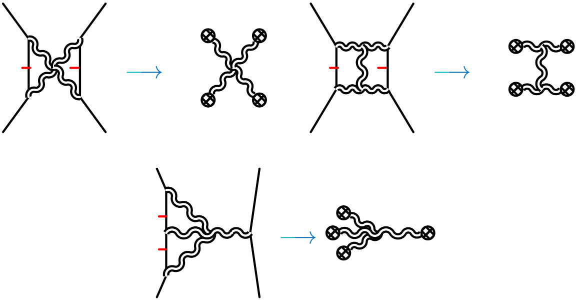
The $\hbar \to 0$ limit gives directly the classical answer [Damgaard, Planté, Vanhove]

$$\lim_{\hbar\to 0} \langle \Delta\hat{O}\rangle = \Delta O^{\text{classical}}(p_1,p_2,r) + O(\hbar)$$

with the exponential representation all superclassical pieces cancel automatically

Classical observables

There is a direct equivalence with the word-line formalism



The completeness relation induces velocity cuts delta-functions on the massive line

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^D k_1}{(2\pi\hbar)^{D-1}} \delta^+(k_1^2 - m_1^2) \frac{d^D k_2}{(2\pi\hbar)^{D-1}} \delta^+(k_2^2 - m_2^2)$$
$$\times \frac{d^{D-1}\ell_1}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_1}} \cdots \frac{d^{D-1}\ell_n}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_n}} \times |k_1, k_2; \ell_1, \dots, \ell_n\rangle\langle \ell_1, \dots, \ell_n; k_1, k_2|$$

The $\hat{N}$ operator upto 1PM and 2PM

$$\hat{S} = \exp\left(\frac{i\hat{N}}{\hbar}\right), \qquad N(\gamma, \underline{q}^2) := \langle p_1, p_2|\hat{N}|p_1', p_2'\rangle$$

$$\tilde{N}(\gamma, J) := \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{N(\gamma, \underline{q}^2)}{4m_1 m_2\sqrt{\gamma^2-1}} e^{i\frac{J\cdot \underline{q}}{p_\infty}}; \qquad p_\infty = \frac{m_1 m_2\sqrt{\gamma^2-1}}{\sqrt{m_1^2+m_2^2+2m_1m_2\gamma}}$$

with the result evaluated at tree-level and one-loop using dimensional regularisation $D = 4-2\epsilon$

$$\tilde{N}^{1PM}(\gamma, J) = \frac{G_N m_1 m_2(2\gamma^2-1)}{\sqrt{\gamma^2-1}}\Gamma(-\epsilon)J^{2\epsilon}$$

$$\tilde{N}^{2PM}(\gamma, J) = \frac{3\pi G_N^2 m_1^2 m_2^2(m_1+m_2)(5\gamma^2-1)}{4\sqrt{(p_1+p_2)^2}}\frac{1}{J}$$

The 1PM (tree-level) and 2PM (one-loop) contributions are the same as for a test mass in the Schwarzschild black hole of mass $M = m_1 + m_2$.

The $\hat{N}$ operator at 3PM

$$\tilde{N}^{3PM}(\gamma,J) = \frac{G_N^3 m_1^3 m_2^3 \sqrt{\gamma^2-1}}{sJ^2} \times \left(\frac{s(64\gamma^6-120\gamma^4+60\gamma^2-5)}{3(\gamma^2-1)^2} - \frac{4m_1m_2(14\gamma^2+25)}{3} + \frac{4m_1m_2(3+12\gamma^2-4\gamma^4)\operatorname{arccosh}(\gamma)}{\sqrt{\gamma^2-1}} + \frac{2m_1m_2(2\gamma^2-1)^2}{\sqrt{\gamma^2-1}}\left(\frac{8-5\gamma^2}{3(\gamma^2-1)} + \frac{\gamma(2\gamma^2-1)\operatorname{arccosh}(\gamma)}{(\gamma^2-1)^{\frac{3}{2}}}\right)\right)$$

At 3PM (two-loop) new phenomena arise

- The conservative part deviates from Schwarzschild as we have contributions which depends (linearly) on the relative mass $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$
- And the important Radiation-reaction terms for the correct high-energy behaviour [Bjerrum-Bohr et al., Para-martinez et al.; Damour; Veneziano et al.]

The radial action

Applying the previous formalism to the momentum kick $\hat{O}^\mu = \hat{P}_1^\mu$ gives in the conservative sector [Bjerrum-Bohr, Damgaard, Planté, Vanhove]

$$\Delta \tilde{P}_1^\mu(\gamma, r)|_{\text{cons}} = -\frac{p_\infty r^\mu}{|r|} \sin\left(-\frac{\partial \tilde{\mathrm{N}}(\gamma, J)}{\partial J}\right) + p_\infty^2 L^\mu \left(\cos\left(-\frac{\partial \tilde{\mathrm{N}}(\gamma, J)}{\partial J}\right) - 1\right)$$

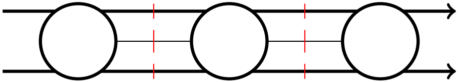
with the angular momentum

$$u_1^\mu \equiv p_\infty \frac{m_1 \gamma p_2^\mu - m_2 p_1^\mu}{m_1^2 m_2(\gamma^2 - 1)}, \qquad u_2^\mu \equiv p_\infty \frac{m_2 \gamma p_1^\mu - m_1 p_2^\mu}{m_1 m_2^2(\gamma^2 - 1)}, L^\mu := \frac{u_2^\mu - u_1^\mu}{p}$$

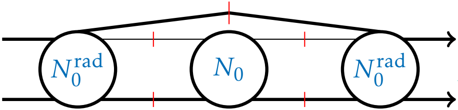
this shows that in the conservative sector the $\tilde{N}(\gamma, J)$ is the radial action used by [Landau, Lifshitz; Damour] for computing the scattering angle in classical GR

$$\chi = -\frac{\partial \tilde{\mathrm{N}}(\gamma, J)}{\partial J} = \sum_{n=0}^{\infty} G_N^{n+1} \chi_{\text{cons}}^{(n)}$$

The momentum kick dissipation



which has two pieces contributing to radiation at 4PM order:



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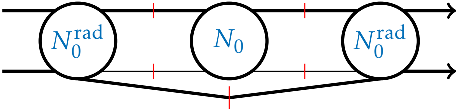
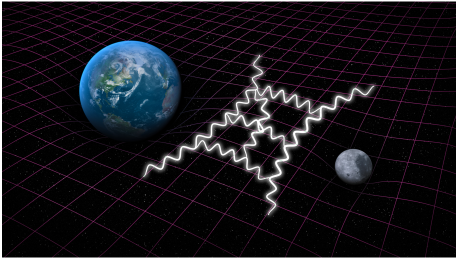


$$\Delta\tilde P_1^{\nu,4PM} = G_N^4 \begin{pmatrix} u_1^\mu & u_2^\mu & \frac{b^\mu}{|b|} \end{pmatrix} \begin{pmatrix} p_\infty\left(-\frac{(\chi_{\rm cons}^{(0)})^4}{24} + \frac{(\chi_{\rm cons}^{(1)})^2}{2} + \chi_{\rm cons}^{(0)}\chi_{\rm cons}^{(2)}\right) \\ p_\infty\left(\frac{(\chi_{\rm cons}^{(0)})^4}{24} - \frac{(\chi_{\rm cons}^{(1)})^2}{2} - \chi_{\rm cons}^{(0)}\chi_{\rm cons}^{(2)}\right) - \frac{\tilde P_{1,1}^{u_2,(3)}}{2} + \frac{i}{6}\tilde P_{1,2}^{u_2,(3)} \\ p_\infty\left(\frac{(\chi_{\rm cons}^{(0)})^2\chi_{\rm cons}^{(1)}}{2} - \chi_{\rm cons}^{(3)} - \frac{\partial \tilde L_2(\gamma,J)}{\partial J}\right) + \frac{\mathcal{E}_1\chi_{\rm cons}^{(0)}\tilde P_{1,1}^{u_2,(2)}}{2} - \frac{\tilde P_{1,1}^{b,(3)}}{2} + \frac{i}{6}\tilde P_{1,2}^{b,(3)} \end{pmatrix}$$

Recycling of lower-order terms a feature that generalizes to higher orders.

Outlook: beyond Einstein gravity

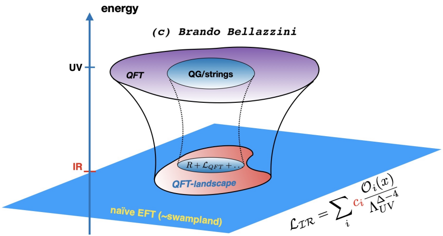
The exponential formalism gives the right eikonal framework when including as well radiation or quantum corrections



- Quantum gravity correction to the star light bending [Bjerrum-Bohr, Donoghue, Holstein, Vanhove, Planté]
- Quantum gravity corrections effects to the metric of black hole solutions
- Quantum contributions to the causal cone [Bellazzini, Isabella, Riva; Madalcena, Zhiboedov; Caron-Huot, Para-Martinez, ...]

Outlook: beyond Einstein gravity

This provides way of constraining possible corrections to Einstein's gravity



- How does the quantum corrections affect the classical picture from Einstein's gravity in particular the nature of Black holes?
- Causality constraint on possible extensions of Einstein gravity : what are the physically acceptable corrections from high energy completion?