Celebration of Costas Bachas

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Novel Modular Forms Arising in Correlators in N=4 SYM

Based on a number of papers with

Daniele Dorigoni, Congkao Wen, Fernando Alday, Shai Chester, Sylviu Pufu, Yifan Wang

- This talk will concern exact results in N=4 supersymmetric $SU(N)$ Yang-Mills

  with complex coupling constant $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{YM}} := \tau_1 + i\tau_2$

- Supersymmetric localisation determines certain **INTEGRATED CORRELATION FUNCTIONS**

  NON-LOCAL SUPERSYMMETRIC OBSERVABLES (cf. Wilson loops)

  Exact expressions for any N and all τ

  Holographic relationship with type IIB superstring amplitude on $AdS^5 \times S^5$

- **MODULAR INVARIANCE** $SL(2,\mathbb{Z})$ - manifest:

  MONTONEN-OLIVE DUALITY of gauge theory $\leftrightarrow$ **S-DUALITY** of Type IIB superstring

  AdS/CFT

$\mathcal{N}=4$ Supersymmetric Yang-Mills Integrated Correlators

- Consider the correlator of four superconformal primaries

factor out dependence on R-symmetry

cross ratios

$$\langle O_2(x_1)\, O_2(x_2)\, O_2(x_3)\, O_2(x_4)\rangle \sim \mathcal{I}_4(x_i;Y_i)\,\mathcal{T}_N(U,V,\tau,\bar\tau)$$

$$U=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}\,,\qquad V=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$$

Holographic dual of the four-graviton amplitude in $AdS_5\times S^5$

- Correlator is not supersymmetric but **integrated** correlator is **[Binder, Chester, Pufu, Wang** arXiv:1902.06263**]**

$$\mathcal{G}_N^i(\tau,\bar\tau)=\int dU\,dV\,\mu^i(U,V)\,\mathcal{T}_N(U,V,\tau,\bar\tau)$$

where the measure $\mu^i(U,V)$ is designed to preserve supersymmetry,

- Two examples of measures

$$\begin{cases}\mathcal{G}_N^1(\tau,\bar\tau)=-\dfrac{8}{\pi}\displaystyle\int_0^\infty dr\int_0^\pi d\theta\,\frac{r\sin^2\theta}{U^2}\,\mathcal{T}_N(U,V,\tau,\bar\tau)\\[2ex] \mathcal{G}_N^2(\tau,\bar\tau)=-\dfrac{96}{\pi}\displaystyle\int_0^\infty dr\int_0^\pi d\theta\,\frac{r\sin^2\theta}{U^2}\,\underbrace{\bar D_{1111}(U,V)}_{\text{box diagram}}\,\left(\mathcal{T}_N(U,V,\tau,\bar\tau)+\mathcal{T}_{free}(U,V)\right)\end{cases}$$

$$U=1+r^2-2r\cos\theta\,,\quad V=r^2$$

Supersymmetric Localisation

[Pestun arXiv:0712.2824]

- $\mathcal{N}=2^*$ supersymmetric YM $\to \mathcal{N}=4$ in the limit in which the mass of hypermultiplet in adjoint rep. vanishes

$$\mathcal{N}=2^* \underset{m\to 0}{\to} \mathcal{N}=4$$

- Localized partition function on S^4 of $\mathcal{N}=2^*$ is a $(N-1)$-dimensional integral over the Lie algebra $\mathfrak{su}(N)$ . . .
Integrate over VEV's of coulomb branch vector multiplet.

classical localized action

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2_{YM}} = \tau_1 + i\tau_2 \qquad Z_N(m,\tau,\bar\tau) = \int d^N a\,\delta(\sum_i a_i)\prod_{i<j}(a_i-a_j)^2\, e^{-\frac{8\pi^2}{g^2_{YM}}\sum_j a_j^2}\, \mathcal{Z}_{pert}(m,a_{ij})\,|\mathcal{Z}_{inst}(m,a_{ij},\tau)|^2 \qquad a_{ij}=a_i-a_j$$

mass parameter

Vandermonde determinant

perturbative factor

Nekrasov instanton partition function

$\mathcal{Z}_{pert}(m,a_i)$ is the one-loop determinant factor and is expressed in terms of a standard function (the **Barnes G-function**)

$\mathcal{Z}_{inst}(m,a_i)$ describes Coulomb branch instantons at the south pole and anti-instantons at the north pole of S^4. (Express as a sum of Young diagrams) [Nekrasov]

$\boxed{m\to 0 \text{ Limit}}$

- The partition function of $\mathcal{N}=4$ SYM $\qquad Z_N(0,\tau,\bar\tau).=1$
- But the $m=0$ limit of derivatives of $Z_N(m,\tau,\bar\tau)$ with respect to m may be nontrivial as we will see.

Relation to Localised $\mathcal{N}=2^*$ Partition Function

- Correlators are obtained by four derivatives acting on $Z_N(m,\tau,\bar\tau)$ the partition function of the $\mathcal{N}=2^*$ theory on S^4

$$\mathcal{G}^1_N(\tau,\bar\tau) = \frac{1}{4}\Delta_\tau\,\partial_m^2\,\log Z_N(m,\tau,\bar\tau)|_{m=0}$$ Considered in this talk $\Delta_\tau = 4\tau_2^2\partial_\tau\partial_{\bar\tau}$

$$\mathcal{G}^2_N(\tau,\bar\tau) = \partial_m^4\,\log Z_N(m,\tau,\bar\tau)|_{m=0}$$ Not considered in this talk

- Equality with integrated correlators on R^4 shown in [Binder, Chester, Pufu, Wang, arXiv:1902.06263]
  Uses supersymmetric Ward identities and accounts for operator mixing on S^4.

- Analysis of $\mathcal{G}_N \equiv \mathcal{G}^1_N$ is complicated.
  - Consider the exact perturbation expansion for many values of N.
  - Consider the exact 1-instanton contribution for many values of N.
  - Generalise to the k-instanton contribution.

Leads to a remarkably simple modular invariant expression for $\mathcal{G}_N(\tau,\bar\tau)$

2 DIM. LATTICE REPRESENTATION

THE MAIN FORMULA

$$\boxed{\mathcal{G}_N(\tau,\bar\tau) = \sum_{(m,n)\in\mathbb{Z}^2} \int_0^\infty e^{-\pi t \frac{|m+n\tau|^2}{\tau_2}} B_N(t)dt}$$

where $B_N(t) = \dfrac{\mathbb{Q}_N(t)}{(1+t)^{2N+1}}$ and $\mathbb{Q}_N(t)$ is a rational polynomial of order $(2N-1)$.

- $SL(2,\mathbb{Z})$ invariance is manifest: $\tau \to \dfrac{a\tau+b}{c\tau+d}$ $\quad a,b,c,d\in\mathbb{Z}$, $\; ad-bc=1$

  [Montonen-Olive]

Holographic image of S-duality in type IIB superstring.

- Note that $B_N(t) = \dfrac{1}{t} B_N(1/t)$, and $\displaystyle\int_0^\infty B_N(t)dt = \frac{N(N-1)}{4}$

.

e.g. $\quad SU(2)$: $\quad B_2(t) = \dfrac{9t^3-30t^2+9t}{(1+t)^5}$ $\qquad SU(3)$: $\quad B_3(t) = \dfrac{18t^5-99t^4+126t^3-99t^2+18t}{(1+t)^7}$

- General N:

$$\mathbb{Q}_N(t) = -\frac{1}{4}N(N-1)(1-t)^{N-1}(1+t)^{N+1}\left[(3+(8N+3t-6)t)\,P_N^{(1,-2)}(z) + \frac{3t^2-8Nt-3}{t+1}P_N^{(1,-1)}(z)\right]$$

where $z = \dfrac{1+t^2}{1-t^2}$ and $P_N^{(\alpha,\beta)}(z)$ is a JACOBI polynomial.

Perturbation Expansion

First non-planar contribution

't Hooft expansion $a = \frac{g_{YM}^2 N}{4\pi^2} = \frac{N}{\pi}\tau_2^{-1}$

$$\mathcal{G}_{N,}(\tau_2) = (N^2-1)\Big[\frac{3\,\zeta(3)a}{2} - \frac{75\,\zeta(5)a^2}{8} + \frac{735\,\zeta(7)a^3}{16} - \frac{6615\,\zeta(9)\left(1+\frac{2}{7}N^{-2}\right)a^4}{32}$$
$$+\frac{114345\,\zeta(11)\left(1+N^{-2}\right)a^5}{128} - \frac{3864861\,\zeta(13)\left(1+\frac{25}{11}N^{-2}+\frac{4}{11}N^{-4}\right)a^6}{1024}$$
$$+\frac{32207175\,\zeta(15)\left(1+\frac{55}{13}N^{-2}+\frac{332}{143}N^{-4}\right)a^7}{2048} + \mathcal{O}(a^8)\Big],$$

- Coefficients are RATIONAL MULTIPLES OF ODD ZETA VALUES.

- Recall the UNINTEGRATED CORRELATOR has very complicated dependence on cross ratios involving polylogs,

e.g. $L=1,\ 2$

$$f^{(L)}(z,\bar{z}) = \sum_{r=0}^{L} \frac{(-1)^r(2L-r)!}{r!(L-r)!L!} \log^r(z\,\bar{z})\left(\mathrm{Li}_{2L-r}(z) - \mathrm{Li}_{2L-r}(\bar{z})\right)$$

$z\bar{z} = U \qquad (1-z)(1-\bar{z}) = V$

- The **INTEGRATED CORRELATOR is much simpler.** The coefficients can be compared with calculations from Feynman diagrams. [Belokurov and Usyukina, 1983] [Usyukina, 1991] [Wen and Zhang 2022]

- **NON-PLANAR CORRECTIONS** BEGIN AT FOUR LOOPS – as is known from Feynman perturbation theory. Interesting pattern of non-planarity determined to arbitrary order.

[Eden, Heslop, Korchemsky, Sokatchev] [Boels, Kniehl, Tarasov, Yang] [Fleury and Pereira]

Laplace Difference Equation

- The integrated correlator satisfies a

Laplace difference equation:

$$\boxed{(\Delta_\tau - 2)\,\mathcal{G}_N(\tau,\bar\tau) = N^2\left[\mathcal{G}_{N+1}(\tau,\bar\tau) - 2\mathcal{G}_N(\tau,\bar\tau) + \mathcal{G}_{N-1}(\tau,\bar\tau)\right] - N\left[\mathcal{G}_{N+1}(\tau,\bar\tau) - \mathcal{G}_{N-1}(\tau,\bar\tau)\right]}$$

where $\Delta_\tau = \tau_2^2\left(\partial_{\tau_1}^2 + \partial_{\tau_2}^2\right)$ is the hyperbolic laplacian.

- Since $\mathcal{G}_1 = 0$ this equation determines $\mathcal{G}_N(\tau,\bar\tau)$ for all $N > 2$ In terms of $\mathcal{G}_2(\tau,\bar\tau)$.
- Solutions can be expressed in terms of **non-holomorphic Eisenstein series**

c.f. Non-Holomorphic Eisenstein Series

$$E(s,\tau,\bar\tau) = \frac{1}{\pi^s}\sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{|m+n\tau|^{2s}} = \sum_{(m,n)\neq(0,0)} \frac{1}{\Gamma(s)}\int_0^\infty e^{-t\pi\frac{|m+n\tau|^2}{\tau_2}} t^{s-1}dt \;=\sum_{k\in\mathbb{Z}} \mathcal{F}_k(s;\tau_2)\, e^{2\pi i k\tau_1} \qquad s\in\mathbb{C}$$

Fourier modes

Modular function $SL(2,\mathbb{Z}):\ \tau\to\frac{a\tau+b}{c\tau+d}$

$a,b,c,d\in\mathbb{Z} \quad ad-bc=1$

- Zero mode (perturbative)

$$\mathcal{F}_0(s;\tau_2) = \frac{2\zeta(2s)}{\pi^s}\tau_2^s + \frac{2\sqrt{\pi}\,\Gamma(s-\frac12)\zeta(2s-1)}{\pi^s\Gamma(s)}\,\tau_2^{1-s}$$

$(4\pi/g^2_{_{YM}})^s$ $\qquad$ $(g^2_{_{YM}}/4\pi)^{s-1}$

TWO POWER-BEHAVED TERMS

PERTURBATIVE (singular) $\qquad$ **PERTURBATIVE**

- Non-zero modes (instantons)

divisor sum $\qquad$ Bessel

$$\mathcal{F}_k(s;\tau_2) = \frac{4}{\Gamma(s)}\,|k|^{s-\frac12}\,\sigma_{1-2s}(|k|)\sqrt{\tau_2}\,K_{s-\frac12}(2\pi|k|\tau_2)\,, \quad k\neq 0 \qquad \sigma_r(k)=\sum_{d|k} d^r$$

$$\underset{\tau_2\to\infty}{\sim}(\dots)\,e^{-2\pi|k|\tau_2}$$

characteristic of INSTANTON or ANTI-INSTANTON

- **LAPLACE EIGENVALUE EQUATION** $(\Delta_\tau - s(s-1))\,E(s;\tau,\bar\tau)=0$

Expression for Integrated Correlator

Formal Infinite sum of Eisenstein series (with integer-index)

$$\boxed{\mathcal{G}_N(\tau,\bar\tau) = \frac{N(N-1)}{8} + \frac{1}{2}\sum_{s=2}^{\infty} c_N(s)\, E(s,\tau,\bar\tau)}$$

Integer index Eisenstein series.

where the coefficients are given by $B_N(t) = \sum_{s=2}^{\infty} c_N(s)\,\frac{t^{s-1}}{\Gamma(s)}$ with $B_N(t) = \frac{1}{t}\,B_N(1/t)$

e.g. for $SU(2)$ $c_2(s) = \frac{(-1)^s}{2}(s-1)(1-2s)^2\,\Gamma(s+1)$

- Perturbative terms.: Infinite sum of $\tau_2^s = (4\pi/g^2_{YM})^s$ terms $\underset{\text{after Borel resummation}}{=}$ infinite sum of $\tau_2^{1-s} = (g^2_{YM}/4\pi)^{s-1}$ terms!
- Instanton Contributions

e.g. $k=1$ in $SU(2)$

$$\mathcal{G}_{2,k=1}(\tau,\bar\tau) = e^{2\pi i\tau}\left[12y^2 - 3\sqrt{\pi}e^{4y}y^{3/2}(1+8y)\text{erfc}\,(2\sqrt{y})\right]$$

$$\underset{g^2_{YM}\to 0}{\sim} e^{2\pi i\tau}\left[-\frac{3}{8} + \frac{9}{32y} - \frac{135}{512y^2} + \frac{315}{1024y^3} + \cdots\right]$$

$y = \pi\tau_2 = \frac{4\pi^2}{g^2_{YM}}$

Large-N Expansion

't Hooft Expansion

Suppresses instantons so duality is not manifest

$$\mathcal{G}_N(\tau,\bar\tau) \sim \sum_{g=0}^{\infty} N^{2-2g}\,\mathcal{G}^{(g)}(\lambda)$$

$\lambda = g^2_{_{YM}}\,N = 4\pi\tau_2^{-1}\,N$
't Hooft coupling

1. Small-λ expansion

$$\mathcal{G}^{(0)}(\lambda) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\zeta(2n+1)\Gamma\left(n+\frac{3}{2}\right)^2}{\pi^{2n+1}\Gamma(n)\Gamma(n+3)}\lambda^n$$

Radius of convergence $|\lambda| \le \pi^2$

Proportional to N^2
Planar diagrams

Borel sum $$= \lambda \int_0^\infty dw\, w^3 \frac{{}_1F_2\left(\frac{5}{2};2,4;-\frac{w^2\lambda}{\pi^2}\right)}{4\pi^2\,\sinh^2(w)}\,.$$

2. Large-λ expansion

$$\mathcal{G}^{(0)}(\lambda) \sim \frac{1}{4} + \sum_{n=1}^{\infty} \frac{\Gamma\left(n-\frac{3}{2}\right)\Gamma\left(n+\frac{3}{2}\right)\Gamma(2n+1)\zeta(2n+1)}{2^{2n-2}\pi\,\Gamma(n)^2\lambda^{n+1/2}}$$

Divergent sum

- Asymptotic series that is **not Borel summable**. Requres non-perturbative completion (resurgence)

$$\Delta\mathcal{G}^{(0)}(\lambda) = i\left[8\mathrm{Li}_0(e^{-2\sqrt{\lambda}}) + \frac{18\mathrm{Li}_1(e^{-2\sqrt{\lambda}})}{\lambda^{1/2}} + \frac{117\mathrm{Li}_2(e^{-2\sqrt{\lambda}})}{4\lambda} + \frac{489\mathrm{Li}_3(e^{-2\sqrt{\lambda}})}{16\lambda^{3/2}} + \cdots\right]$$

- The behaviour $e^{-2\sqrt{\lambda}}$ is characteristic of a **world-sheet instanton** in string theory since $e^{-2\sqrt{\lambda}} = e^{-2L^2/\alpha'}$.

But since we know the exact function we can determine its large-N $SL(2,\mathbb{Z})$ completion analytically

MODULAR INVARIANT LARGE-N EXPANSION

Fixed - g^2_{YM} Expansion INSTANTONS NOT SUPPRESSED – S-duality is manifest.

- The $1/N$ expansion is holographically related to the low energy expansion of the dual IIB superstring amplitude in $AdS_5 \times S^5$.

$$N \to \frac{\tau_2 L^4}{\alpha'^2}\,, \qquad \tau_2 \to \frac{1}{g_s}$$

Supergravity

$$\mathcal{G}_N(\tau,\bar\tau) \sim \frac{N^2}{4} - \frac{3N^{\frac12}}{2^4}\underbrace{E(\tfrac32;\tau,\bar\tau)}_{R^4} + \frac{45}{2^8 N^{\frac12}}\underbrace{E(\tfrac52;\tau,\bar\tau)}_{d^4R^4}$$
$$+ \frac{3}{N^{\frac32}}\left[\frac{1575}{2^{15}}E(\tfrac72;\tau,\bar\tau) - \frac{13}{2^{13}}E(\tfrac32;\tau,\bar\tau)\right] + \frac{225}{N^{\frac52}}\left[\frac{441}{2^{18}}E(\tfrac92;\tau,\bar\tau) - \frac{5}{2^{16}}E(\tfrac52;\tau,\bar\tau)\right]$$
$$+ \frac{63}{N^{\frac72}}\left[\frac{3898125}{2^{27}}E(\tfrac{11}{2};\tau,\bar\tau) - \frac{44625}{2^{25}}E(\tfrac72;\tau,\bar\tau) + \frac{73}{2^{22}}E(\tfrac32;\tau,\bar\tau)\right] + O(N^{-\frac92})\,,$$

- Series of **½-integer** index Eisenstein series.

- Close connection to well-established BPS terms in low energy expansion of IIB superstring in the flat space limit.

- Note the absence of terms with integer powers of $1/N$, such as the term of order d^6R^4.
  Such terms arise in the $1/N$ expansion of $\mathcal{G}^2_N(\tau,\bar\tau) = \partial^4_m \log Z_N(m,\tau,\bar\tau)|_{m=0}$.

OTHER RESULTS

- **COMPLETION OF LARGE-N EXPANSION**: Non-convergent $1/N$ expansion completed by $e^{-(\dots)\sqrt{N}}$ term

Modular invariant completion $\displaystyle \sim \sum_{\ell=1}^{\infty} \sum_{\gcd(p,q)=1} \exp\left(-4\sqrt{N}\pi\ell\frac{|p+q\tau|}{\sqrt{\tau_2}}\right)\dots \sim \sum_{\ell=1}^{\infty} \sum_{\gcd(p,q)=1} \exp(-4\pi L^2 \ell\, T_{p,q})$

Tension of (p,q) strings

uses AdS/CFT dictionary $\sqrt{g_{_{YM}}^2 N} = L^2/\alpha'$

Sum over ℓ (p,q) string world-sheets wrapping an equatorial S^2 in S^5.

- **GENERAL CLASSICAL GROUPS** $SU(N),\ SO(N),\ USp(2N)$

Holographically related to type IIB in $AdS^5 \times (S^5/Z_2)$

- **THE SECOND CORRELATOR** $\mathcal{G}_N^2(\tau,\bar\tau) = \partial_m^4 \log Z_N(m,\tau,\bar\tau)|_{m=0}$

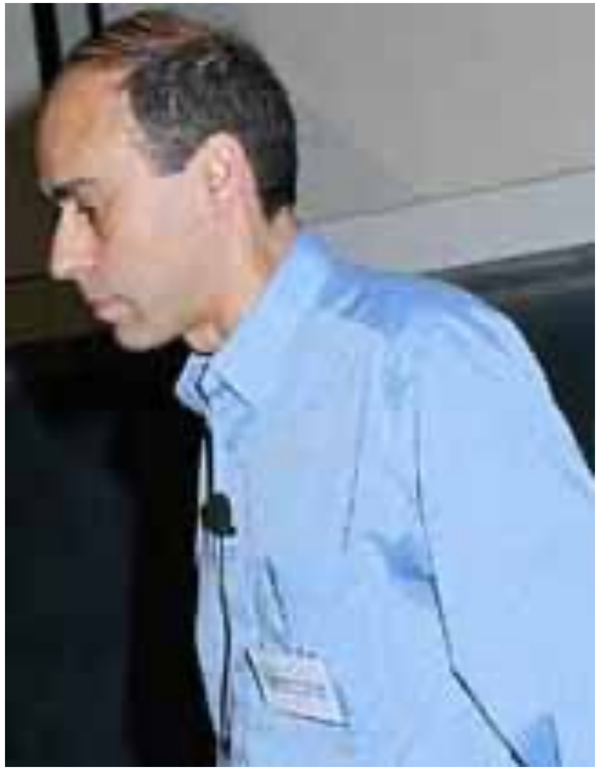
- Coefficients of the $1/N$ expansion: involve **GENERALISED EISENSTEIN SERIES**

Inhomogeneous Laplace eigenvalue equation

$$\boxed{(\Delta_\tau - s(s-1))\mathcal{E}(s,s_1,s_2) = E(s_1)\, E(s_2)}$$

Eisenstein series

intriguing number theoretic properties - lattice representation and holomorphic cusp forms.



It’s a very great pleasure to participate in this celebration of the contributions of Costas.

I have very happy memories of collaborating with Costas and absorbing his wisdom.

Less happy memories of struggling to complete forms for the EU network
organized by Lars Brink in which we were the ENS and Cambridge node leaders.

HAPPY RETIREMENT COSTAS !

Although it is difficult to imagine you genuinely retiring, .