Two Tales about 2d CFTs

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Permeable conformal walls and holography

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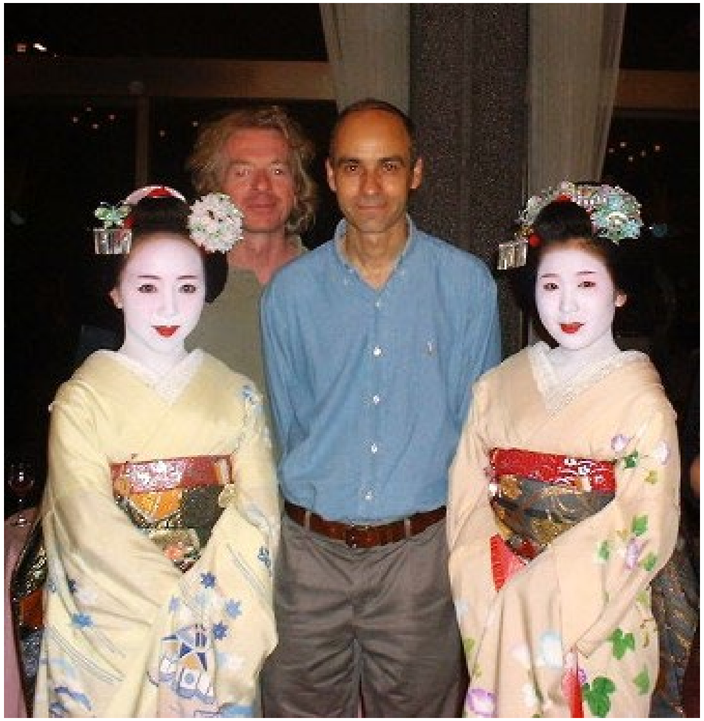
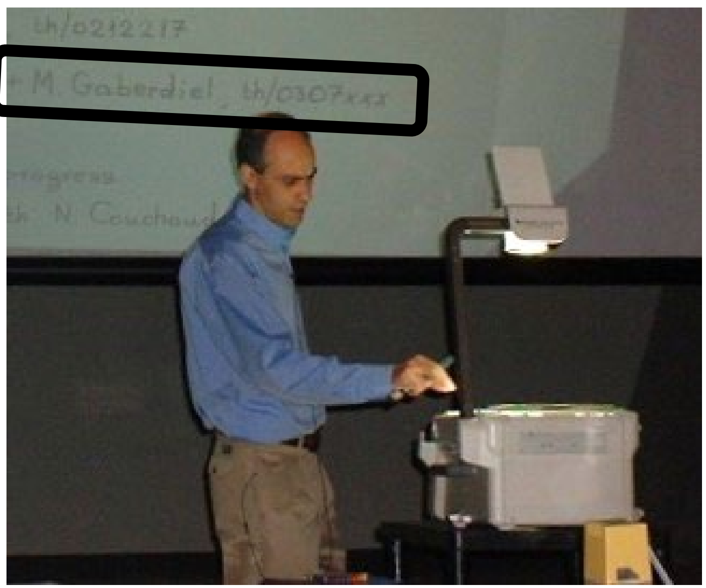
Abstract: We study conformal field theories in two dimensions separated by domain walls, which preserve at least one Virasoro algebra. We develop tools to study such domain walls, extending and clarifying the concept of 'folding' discussed in the condensed-matter literature. We analyze the conditions for unbroken supersymmetry, and discuss the holographic duals in AdS3 when they exist. One of the interesting observables is the Casimir energy between a wall and an anti-wall. When these separate free scalar field theories with different target-space radii, the Casimir energy is given by the dilogarithm function of the reflection probability. The walls with holographic duals in AdS3 separate two sigma models, whose target spaces are moduli spaces of Yang-Mills instantons on T4 or K3. In the supergravity limit, the Casimir energy is computable as classical energy of a brane that connects the walls through AdS3. We compare this result with expectations from the sigma-model point of view.



Strings 2003 in Kyoto



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Two tales about 2d CFTs:

1. $$\Delta = \exp(-\,\alpha\, t + O(1)),$$

$$\frac{1}{\sqrt{c}} \;\le\; \alpha \;\le\; 1\,.$$

2. $$c_{LR} \le c_{\text{eff}} \le \min(c_L, c_R).$$

1. Universal Bounds on CFT Distance Conjecture

Wang + H.O.: 2405.00674

For any unitary 2d CFT, if there is a primary operator whose conformal dimension Δ vanishes in some limit on the conformal manifold,

- **The Zamolodchikov distance t to the limit is infinite.**
- **The approach to this limit is exponential $\Delta = \exp\left(-\alpha\, t + O(1)\right)$.**
- **The decay rate obeys the universal bounds $c^{-1/2} \leq \alpha \leq 1$.**

In the limit, an infinite tower of primary operators emerges without a gap above the vacuum and that the conformal field theory becomes locally a tensor product of a sigma-model in the large radius limit and a compact theory.

This work was motivated by the Distance Conjecture

Vafa + H.O.: 0605264

Conjecture 0: Every parameter in quantum gravity is an expectation value of a dynamical field and can be varied by changing its expectation value.

Conjecture 1: Choose any point p_0 in the moduli space $\mathcal{M}$. For any positive t, there is another point $p \in \mathcal{M}$ such that $d(p, p_0) > t$.

Conjecture 2: Compared to the theory at $p_0 \in \mathcal{M}$, the theory at p with $d(p, p_0) > t$ has an infinite tower of light particles starting with mass of the order of $e^{-\alpha t}$ for some $\alpha > 0$.

Examples

Sigma model on T^2

- Complexified Kähler moduli ρ
- Complex structure moduli τ

Zamolodchikov metric: $ds^2 = \frac{d\rho d\bar{\rho}}{\rho_2^2} + \frac{d\tau d\bar{\tau}}{\tau_2^2}$

- $\mathbb{Z}_3$ orbifold point at **finite distance**

$$\Delta_{\text{gap}} = \frac{2}{3}$$ is saturated by $SU(3)_1$ primary fields

- Large volume limit at **infinite distance** $\rho_2 \to \infty$

$$\Delta_{\text{gap}} = \frac{1}{2\rho_2\tau_2} \sim e^{-t} \to 0$$ and $\alpha = 1$.

$\mathcal{N} = (2,2)$ sigma-model on the quintic Calabi-Yau manifold

- $\mathbb{Z}_5$ orbifold point at **finite distance**

  It is a Gepner point described by $(SU(2)_3/U(1))^{\otimes 5}/{\mathbb{Z}_5}^{\otimes 3}$.

  $\Delta_{\text{gap}} = \frac{2}{5}$ is saturated by a non-BPS primary with zero $U(1)_R$ charge.

- Conifold point at **finite distance**

  Continuous spectrum above $\Delta_{\text{gap}} = \frac{1}{2}$ described by $SL(2)_1/U(1)$.

- Large volume limit at **infinite distance** $\rho_2 \to \infty$

  $ds^2 = \frac{6}{\rho_2^2} d\rho d\bar{\rho}$ and $\alpha = \frac{1}{\sqrt{6}} < 1$ ⇐ $^{\exists}$Marginal operators that are **not exactly marginal**.

Proof of:

$$\Delta = \exp(-\alpha\, t + O(1))$$

$$\alpha \leq 1$$

Start with the simple case when there is only **one marginal operator $\boldsymbol{M}$** and when it is exact.

Suppose there is a primary field $\mathcal{O}$, whose conformal dimension Δ vanishes at some point on the conformal manifold. Choose a geodesic coordinate t so that $\Delta(t)$ monotonically decreases toward the point.

$$\frac{d\Delta(t)}{dt} = -C_{\mathcal{O}\mathcal{O}\boldsymbol{M}}$$

The distance t diverges if $C_{\mathcal{O}\mathcal{O}M}$ vanishes at least linearly in Δ.

We can show the stronger statement $C_{\mathcal{O}\mathcal{O}M} = \Delta\,(1 + O(\Delta))$.
Therefore, $\Delta(t) = \exp\big(-t + O(1)\big)$ with $\alpha = 1$.

In view of time, I will present a **simple but not rigorous proof**.

Since $[L_1, L_{-1}] = 2L_0$, there is an operator J of weights (Δ/2+1, Δ/2) such that $\partial\mathcal{O} = i\sqrt{\Delta}\, J$.

Therefore, $C_{\mathcal{O}\mathcal{O}M} = \Delta\, C_{J\bar{J}M}$. Need to show $C_{J\bar{J}M} = 1 + O(\Delta)$.

From $\langle\mathcal{O}(z)\mathcal{O}(w)\rangle = \frac{1}{|z-w|^{2\Delta}}$,

$$\langle J(z)J(w)\rangle = \frac{1}{(z-w)^2} + O(\Delta), \quad \langle\bar{J}(\bar{z})\bar{J}(\bar{w})\rangle = \frac{1}{(\bar{z}-\bar{w})^2} + O(\Delta),$$

$$\langle J(z)\bar{J}(\bar{w})\rangle = \frac{\Delta}{|z-w|^2} + O(\Delta^2).$$

$$\langle J(z)J(w)\rangle = \frac{1}{(z-w)^2} + O(\Delta),\quad \langle \bar{J}(\bar{z})\bar{J}(\bar{w})\rangle = \frac{1}{(\bar{z}-\bar{w})^2} + O(\Delta),$$

$$\langle J(z)\bar{J}(\bar{w})\rangle = \frac{\Delta}{|z-w|^2} + O(\Delta^2).$$

$$\langle \bar{J}(w)\, J(z)\, \bar{J}(u)\, J(v)\, \rangle$$

$$= \frac{\mathbf{1}}{(z-v)^2(\bar{w}-\bar{u})^2} + \cdots + O(\Delta) \ \text{ in the } t\text{-channel}$$

$$= \frac{\Delta^2}{|z-w|^2|u-v|^2} + \frac{(C_{J\bar{J}M})^2}{(z-v)^2(\bar{w}-\bar{u})^2} + \cdots + O(\Delta) \ \text{ in the } s\text{-channel}$$

⇧ $\mathbf{1}$ exchange

⇧ $\boldsymbol{M}$ exchange

⇧ exchange of other operators

By the crossing symmetry, $C_{J\bar{J}M} = \mathbf{1} + O(\Delta)$.

Therefore, $\Delta(t) = \exp\big(-t + O(1)\big)$ with $\boldsymbol{\alpha} = \mathbf{1}$.

With several marginal operators M_i, the crossing symmetry gives

$$G^{ij} C_{J\bar{J}M_i} C_{J\bar{J}M_j} = 1 + O(\Delta).$$

For exactly marginal operators M_a, define $\alpha_a = \lim_{t\to\infty} C_{J\bar{J}M_a}$.

- $\Delta(t) = \exp(-\alpha_a t^a + O(1))$.
- $\|\alpha\| = \sqrt{G^{ab}\alpha_a\alpha_b} \leq 1$.
- $\|\alpha\|$ = 1 if and only if $C_{J\bar{J}M_i} = 0$ for all non-exact operators.

Parametrizing $t^a = e^a\, t$ by the geodesic distance t and a unit vector $e^a = \cos\theta\ G^{ab}\alpha_b + \sin\theta\, e_\perp^a$, where $0 \leq \theta \leq \pi/2$ and $e_\perp^a$ is a unit vector satisfying $e_\perp^a c_a = 0$, Δ in the limit is,

$\Delta(t^a = e^a t) = \exp(-\alpha\, t + O(1))$ with $\alpha = \cos\theta\, \|\alpha\| \leq 1$.

Proof of:

$$\Delta = \exp(-\alpha\, t + O(1))$$

$$\frac{1}{\sqrt{c}} \leq \alpha$$

$\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3, \cdots$: primary fields whose conformal weights vanish $\Delta_n \to 0$ toward $t \to \infty$.

Define J_n by $\partial \mathcal{O}_n = i\sqrt{\Delta_n}\, J_n$.

J_n's may not be linearly independent in the limit.

Consider operator product expansion:

$$\mathcal{O}_n(z)\mathcal{O}_m(w) = \sum_k C_{nm}^k |z-w|^{\Delta_k - \Delta_n - \Delta_m} \mathcal{O}_k(w) + O\left(|z-w|^{\Delta_{\mathrm{finite}}}\right)$$

$O\left(|z-w|^{\Delta_{\mathrm{finite}}}\right)$ represents contributions of operators whose conformal weights remain above Δ_{finite}.

$$\mathcal{O}_n(z)\mathcal{O}_m(w) = \sum_k C_{nm}^k |z-w|^{\Delta_k - \Delta_n - \Delta_m} \mathcal{O}_k(w) + \cdots$$

- Acting $(\boldsymbol{\partial_z} + \boldsymbol{\partial_w})$ on both sides and using $\partial\mathcal{O}_n = i\sqrt{\Delta_n}\, J_n$
  ⇒ **Linear relations among $\boldsymbol{J_n}$'s** in the limit.

$$\sqrt{\Delta_n}\, J_n + \sqrt{\Delta_m}\, J_m = \sum_k C_{nm}^k \sqrt{\Delta_k}\, J_k$$

- Acting $(\boldsymbol{\partial_z} \times \boldsymbol{\partial_w})$ on both sides and using $\partial\mathcal{O}_n = i\sqrt{\Delta_n}\, J_n$
  ⇒ **Quadratic relations among $\boldsymbol{J_n}$'s** in the limit.

$$J_n(z)\, J_m(w) = \sum_k C_{nm}^k \frac{\Delta_k - \Delta_n - \Delta_m}{\sqrt{\Delta_n \Delta_m}} \left(\frac{1}{(z-w)^2} - \frac{i\sqrt{\Delta_k}}{z-w} J_k(w) \right) + \cdots$$

In linearly independent basis:

$$\mathcal{J}^{\mu}(z)\,\mathcal{J}^{\mu}(w) = \frac{\delta^{\mu\nu}}{(z-w)^2} + O(1).$$

Bosonization: $\mathcal{J}^{\mu}(z) = i\partial X^{\mu}$

Since $\partial\mathcal{O}(z) \propto \mathcal{J}(z)$, $\mathcal{O}(z) = e^{ip_{\mu}X^{\mu}(z)}$.

p_{μ} becomes continuous the $t \to \infty$ limit.

- CFT in the limit contains a subalgebra of local operators described by the sigma-model on $\mathbb{R}^N$.
- $N \leq c$: the central charge of the original CFT.

The limiting CFT is **locally** the $\mathbb{R}^N$ sigma-model $\otimes$ compact CFT.

Examples with nontrivial global structures

- $S^1_R/\mathbb{Z}_2$: Consider $\mathcal{O}_n = \sqrt{2}\cos(nX/R)$.

  $J_n = i\partial X \cdot \tilde{\mathcal{O}}_n$ with $\tilde{\mathcal{O}}_n = \sqrt{2}\sin(nX/R)$.

  In the $R \to \infty$ limit, $\tilde{\mathcal{O}}_n$ becomes a topological operator at the end-point of the topological defect line that implements the quantum $\mathbb{Z}_2$ symmetry of the orbifold.

- The $k \to \infty$ limit of the A_k-type Virasoro minimal model is the $c = 1$ sigma-model with a pair of walls infinitely distant from each other.

Runkel, Watts: 0107118
Mazel, Sandor, Wang, Yin: 2403.14544

Marginal operators that couple of the light operators $\mathcal{O}(z) = e^{ip_\mu X^\mu}$ in the $t \to \infty$ limit are of the form $\partial X^\mu \bar{\partial} X^\nu$.

Since the light sector in the limit is parity invariant, the perturbation $t \int \kappa_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu$ should be positive and parity preserving. Therefore, $\kappa_{\nu\mu}$ is symmetric with non-negative eigenvalues.

For $\mathcal{O} = e^{ip_\mu X^\mu}$, $\Delta = e^{-\alpha t + O(1)}$ with $\alpha = \frac{\sum \kappa_{\mu\nu} p_\mu p_\nu}{\sum (p_\mu)^2}$.

We can choose p_μ so that α is the largest eigenvalue of $\kappa_{\mu\nu}$, which is bounded below by $N^{-1/2}$. Thus, there is always a light operator for which $c^{-1/2} \leq N^{-1/2} \leq \alpha$.

To summarize:

Δ can vanish only in infinite distant limits on the conformal manifold, where

$$\Delta_{\text{gap}} = \exp(-\,\alpha\, t + O(1)) \quad \text{and} \quad \frac{1}{\sqrt{c}} \leq \alpha \leq 1 .$$

$$\left(\sqrt{\frac{3}{2c}} \leq \alpha \leq 1 \ \text{ with superconformal symmetry.} \right)$$

The large volume limit of the quintic Calabi-Yau saturates the lower bound at $\alpha = 1/\sqrt{6}$.

If CFT_2 has a holographic dual in AdS_3

$$\Delta = \exp(-\,\alpha_{\mathrm{AdS}}\,\phi + O(1))$$

$$\left(\frac{2}{3}\,L_{\mathrm{Planck}}\right)^{1/2} \leq \alpha_{\mathrm{AdS}} \leq (8\pi L_{\mathrm{AdS}})^{1/2}$$

where $L_{\mathrm{Planck}} = 8\pi G_N$.

If CFT_2 has a holographic dual in AdS_3

$$\Delta = \exp(-\alpha_{\mathrm{AdS}}\,\phi + O(1))$$

$$\left(\frac{2}{3}L_{\mathrm{Planck}}\right)^{1/2} \leq \alpha_{\mathrm{AdS}} \leq (8\pi L_{\mathrm{AdS}})^{1/2}$$

- The tower of light particles **must emerge** when $\phi \geq \left(\frac{2}{3}L_{\mathrm{Planck}}\right)^{-1/2}$.
- The tower of light particles **can emerge** when $\phi \geq (8\pi L_{\mathrm{AdS}})^{-1/2}$.

With supersymmetry

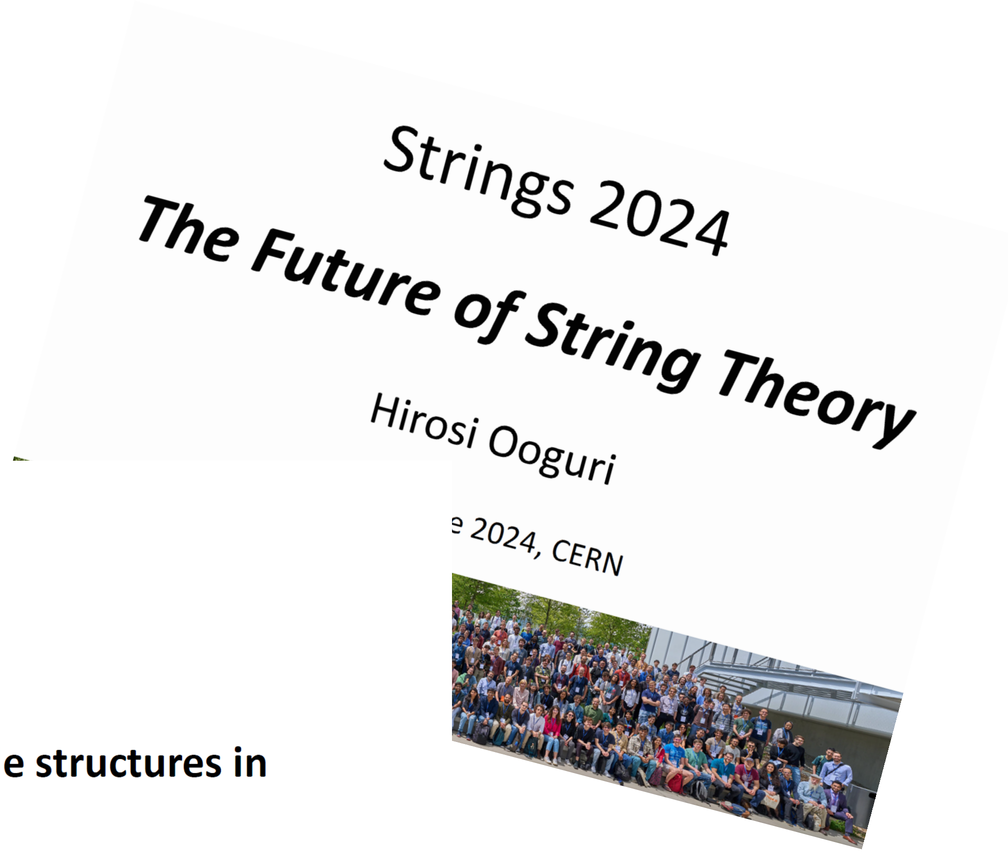
$$\Delta = \exp(-\,\alpha_{\mathrm{AdS}}\,\phi + O(1))$$

$$(L_{\mathrm{Planck}})^{1/2} \leq \alpha_{\mathrm{AdS}} \leq (8\pi L_{\mathrm{AdS}})^{1/2}$$

The lower bound agrees with
the Sharpened Distance Conjecture.

Lee, Lerche, Weigand: 1910.01135
Etheredge, Heidenreich, Kaya, Qiu, Rudelius: 2206.04063

From my Strings 2024 closing remarks



Discrete Families of CFTs

Eric Perlmutter

66. Can we quantitatively describe large scale structures in the space of unitary, generic CFTs?

For example, how CFTs are distributed as a function of c.

H.O.

56. Can we define a distance between any pair of conformal field theories that are not necessarily related by marginal perturbations?

Hint: Can we use a domain wall between such a pair?

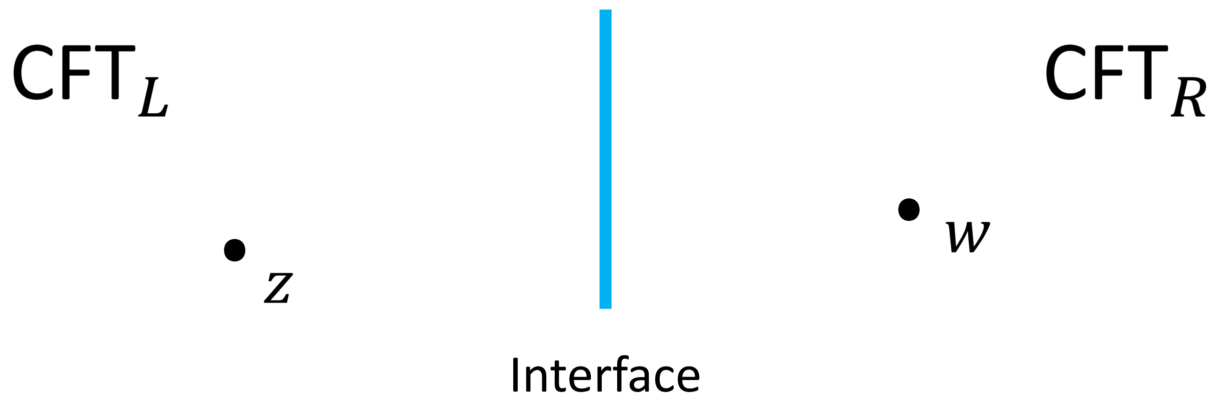
2. Universal Bound on Effective Central Charge

Karch, Kusuki, Sun, Wang + H.O.: 2308.05436 and 2404.01515

The effective central charge c_{eff} measures the entanglement across a CFT interface, while the transmission coefficient encoded in c_{LR} measures the energy transmission through the interface.

We prove the upper bound on c_{eff}
and give evidence for the lower bound.

$$c_{LR} \leq c_{\text{eff}} \leq \min(c_L, c_R)$$



$$\langle T_L(z) T_R(w) \rangle = \frac{c_{LR}}{2(z-w)^4}$$

Energy transmission coefficients: $\mathcal{T}_L = \dfrac{c_{LR}}{c_L}$, $\mathcal{T}_R = \dfrac{c_{LR}}{c_R}$

This requires $c_{LR} \leq \min(c_L, c_R)$.

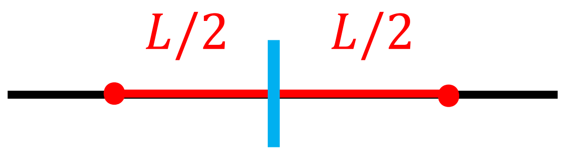
Quella, Runkel, Watts: 0611296
Meineri, Penedones, Rousset: 1904.10974

The effective central charge is defined in terms of the entanglement entropy.

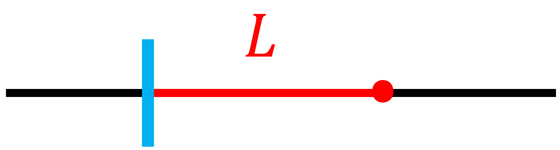
Without an interface:

$$S = \frac{c}{3}\log\left(\frac{L}{\pi\varepsilon}\right)$$

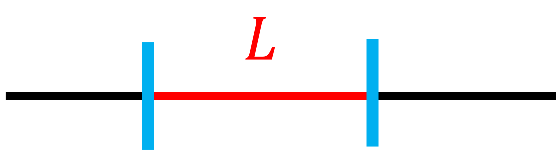
With interfaces:



$$S = \frac{c_L + c_R}{6}\log\left(\frac{L}{\pi\varepsilon}\right)$$



$$S = \frac{c_{\text{eff}} + c_R}{6}\log\left(\frac{L}{\pi\varepsilon}\right)$$



$$S = \frac{c_{\text{eff}}}{3}\log\left(\frac{L}{\pi\varepsilon}\right)$$

We proved that c_{eff} defined by the last two equations are the same.

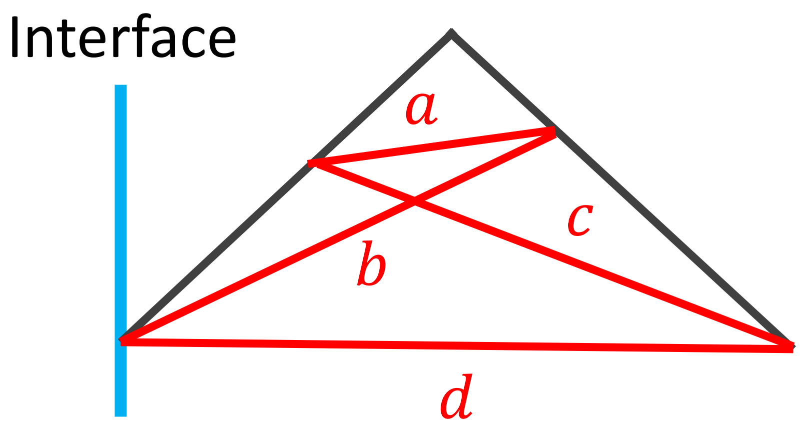
Karch, Kusuki, Sun, Wang + H.O.: 2308.05436

Proof of $c_{\text{eff}} \leq \min(c_L, c_R)$

Karch, Kusuki, Sun, Wang + H.O.: 2308.05436

Add the interface to the proof of the c theorem

by Casini, Huerta: 0610375.



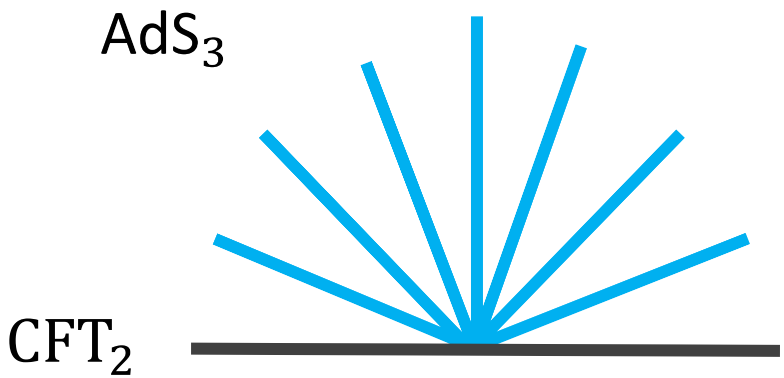
$$|a||d| = |b||c|$$

The strong subadditivity $S(b) + S(c) \geq S(a) + S(d)$ implies

$$\frac{c_R - c_{\text{eff}}}{6} \log\left(\frac{|b|}{|a|}\right) \geq 0 \; .$$

Evidence for $c_{LR} \leq c_{\text{eff}}$

Karch, Kusuki, Sun, Wang + H.O.: 2404.01515



$$ds^2 = a^2(\theta)\left(\frac{-dt^2 + dx^2}{x^2} + d\theta^2\right)$$

$$c_{LR} = \frac{4}{L_L + L_R}\left(\frac{1}{L_L} + \frac{1}{L_R} + 8\pi G_N \sigma\right)^{-1}$$

Bachas, Chapman, Ge, Policastro: 2006.11333

$$\leq \frac{3\min[a(\theta)]}{2G_N} = c_{\text{eff}}$$

The inequality also holds in free theories and in the defect perturbation theory.

$$c_{LR} \leq c_{\text{eff}} \leq \min(c_L, c_R)$$

- The upper bound is proven.
- The lower bound holds in holographic CFTs, free CFTs, and the defect perturbation theory.
- The lower bound means that the amount of energy transmitted across the interface cannot exceed the amount of information transmitted.
- The inequalities are sharp and can be saturated.
- $c_{LR} = c_{\text{eff}}$ only if $c_{\text{eff}} = 0$ or $\min(c_L, c_R)$, *i.e.*, the interface is either a boundary or topological.

Two tales about 2d CFTs:

1. $$\Delta = \exp(-\alpha\, t + O(1)),$$

$$\frac{1}{\sqrt{c}} \leq \alpha \leq 1\,.$$

2. $$c_{LR} \leq \min(c_L, c_R)$$

$$\Rightarrow\ c_{LR} \leq c_{\text{eff}} \leq \min(c_L, c_R).$$

Congratulations, Costas!

$$\frac{1}{\sqrt{c}} \leq \alpha \leq 1$$

$$c_{LR} \leq c_{\mathrm{eff}} \leq \min(c_L, c_R)$$