AdS3/CFT2 @ the free point and beyond

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Based on work with Rajesh Gopakumar and Beat Nairz

An exact AdS/CFT duality

Tensionless string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ with 1 unit of NS-NS flux is exactly dual to symmetric orbifold:

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 = \mathrm{Sym}_N(\mathbb{T}^4)$$

1 unit of NS-NS flux

solvable worldsheet theory (almost free)

almost free 2d CFT

In particular, the **spectrum** agrees precisely, and the structure of the symmetric orbifold **correlators** is reproduced from the worldsheet.

[Eberhardt, MRG, Gopakumar, '19 & '20]

AdS3

This is the description with pure NS-NS flux, but in order to generalise this to N=4 SYM it is important to understand how R-R flux affects this picture.

In the context of the symmetric orbifold, switching on R-R flux corresponds to perturbing the theory by the 2-cycle twisted sector operator.

→ can study and understand this in quite some detail….

cf. [Gava, Narain, '02], [Gomis, Motl, Strominger, '02], [David, Sahoo, '08], [Pakman, Rastelli, Razamat, '09]

Symmetric orbifold basics

Recall basic structure of symmetric orbifold

$$(\mathbb{T}^4)^N / S_N$$

untwisted sector: permutation invariant combinations

twisted sectors: associated to conjugacy classes of S_N

labelled by cycle shapes, i.e. partitions of N

concentrate on **single cycle sectors** $\longleftrightarrow$ analogue of single trace

Magnons

In each w-cycle twisted sector consider reference BPS state $|\mathrm{BPS}\rangle_w$ with

$$h = \tilde{h} = j = \tilde{j} = \frac{w+1}{2} \ .$$

The single-particle states of the twisted sector are generated by the action of the magnon excitations

$$\begin{array}{lll} \psi^{-}_{-\frac{1}{2}+\frac{n}{w}} \ , & \psi^{+}_{-\frac{3}{2}+\frac{n}{w}} \ , & \alpha^{i}_{-1+\frac{n}{w}} \\ \tilde{\psi}^{-}_{-\frac{1}{2}+\frac{n}{w}} \ , & \tilde{\psi}^{+}_{-\frac{3}{2}+\frac{n}{w}} \ , & \tilde{\alpha}^{i}_{-1+\frac{n}{w}} \end{array} \qquad \text{(i=1,2) + c.c.}$$

subject to the orbifold invariance (momentum conservation).

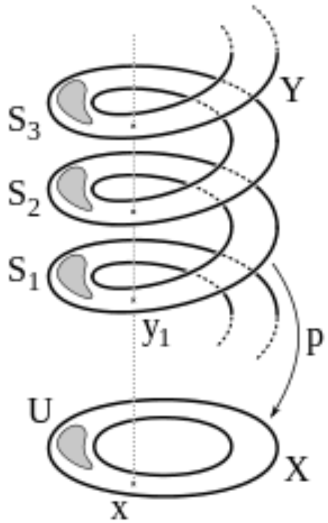
cf. [Gava, Narain, '02]

Correlators in sym orbifold

Symmetric orbifold correlators can be calculated by lifting to covering surface

[Lunin, Mathur ’00]
[Pakman, Rastelli, Razamat ’09]

z



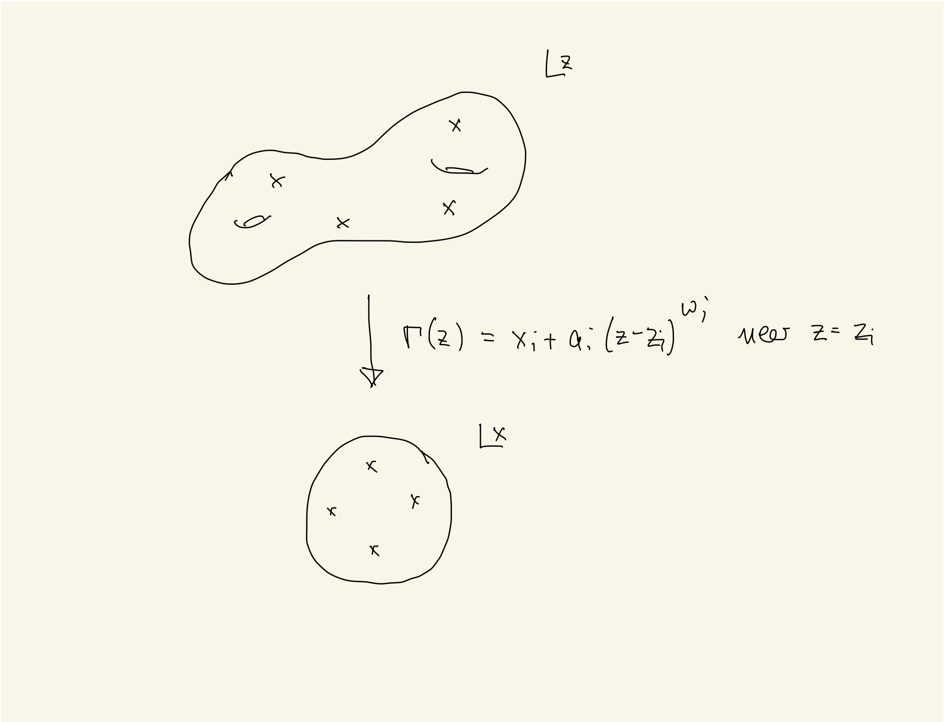
$x = \Gamma(z)$

Locally, the effect of a w-cycle twist field is to introduce a w-fold covering.

$$z \mapsto \Gamma(z) = x_0 + a(z - z_0)^w + \cdots$$

Correlators in sym orbifold

To describe the full correlator combine these local coverings into a global covering surface.



The covering surface has, in general, a non-trivial genus. The genus captures the 1/N corrections of the symmetric orbifold correlators.

[Lunin, Mathur '00]
[Pakman, Rastelli, Razamat '09]

Correlators in sym orbifold

Since string coupling is related to 1/N this suggests that **covering surface = world-sheet**.

[Pakman, Rastelli, Razamat '09]

Realised explicitly by our worldsheet theory:

$$\langle \prod_{i=1}^{n} \mathcal{O}_{h_i}^{w_i}(x_i) \rangle = \int_{\mathcal{M}} d\mu(z_i) \, \langle \prod_{i=1}^{n} V_{h_i}^{w_i}(x_i; z_i) \rangle$$

↑ symmetric orbifold correlator on sphere

↑ integral over worldsheet moduli

↑ corresponding worldsheet correlator

[Eberhardt, MRG, Gopakumar, '19]
[Dei, MRG, Gopakumar, Knighton, '20]
[Eberhardt, '20], [Knighton, '21]

Perturbation

From perspective of symmetric orbifold, R-R deformation comes from 2-cycle twisted sector perturbation.

The perturbing field is an su(2) singlet, but since it comes from the 2-cycle twisted sector, it changes the twisted sectors

$$\mathcal{Z}_{\pm} |\mathrm{BPS}\rangle_w = |\mathrm{BPS}\rangle_{w \pm 1}$$

Study effect of perturbation by analysing the modified action of supercharges on the magnon descendants.

cf. [Beisert '05]

Perturbation

To simplify the exposition, let us concentrate on the subalgebra

$$Q_1 \equiv G^-_{+\frac{1}{2}} \;, \qquad S_1 \equiv G^+_{-\frac{1}{2}} \;, \qquad \tilde{Q}_2 \equiv \tilde{G}'^-_{+\frac{1}{2}} \;, \qquad \tilde{S}_2 \equiv \tilde{G}'^+_{-\frac{1}{2}}$$

that annihilate the BPS states and satisfy

$$\begin{aligned}
\{Q_1, S_1\} &\equiv \mathcal{C} = (L_0 - K_0^3) \;, & \{\tilde{Q}_2, \tilde{S}_2\} &\equiv \tilde{\mathcal{C}} = (\tilde{L}_0 - \tilde{K}_0^3) \;, \\
\{Q_1, Q_1\} &= \{S_1, S_1\} = 0 \;, & \{\tilde{Q}_2, \tilde{Q}_2\} &= \{\tilde{S}_2, \tilde{S}_2\} = 0 \;, \\
\{Q_1, \tilde{Q}_2\} &= \{Q_1, \tilde{S}_2\} = 0 \;, & \{S_1, \tilde{Q}_2\} &= \{S_1, \tilde{S}_2\} = 0 \;.
\end{aligned}$$

[David, Sahoo, ’08], [Hoare, Tseytlin, ’13]
[Borsato, Ohlsson Sax, Sfondrini, Stefanski, ’14]

Then can also restrict to magnons associated to

$$\psi^- \;, \quad \alpha^2 \;, \quad \tilde{\psi}^- \;, \quad \tilde{\alpha}^1 \;.$$

Perturbation

In the free theory, the single magnon excitations transform under these generators as

$$Q_1\, \alpha^2_{-1+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = (1 - \tfrac{n}{w})\, \psi^-_{-\frac{1}{2}+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w\,, \qquad \tilde{Q}_2\, \alpha^2_{-1+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = 0$$

$$Q_1\, \psi^-_{-\frac{1}{2}+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = 0\,, \qquad \tilde{Q}_2\, \psi^-_{-\frac{1}{2}+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = 0$$

$$S_1\, \alpha^2_{-1+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = 0\,, \qquad \tilde{S}_2\, \alpha^2_{-1+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = 0$$

$$S_1\, \psi^-_{-\frac{1}{2}+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = \alpha^2_{-1+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w\,, \qquad \tilde{S}_2\, \psi^-_{-\frac{1}{2}+\frac{n}{w}}\, |\mathrm{BPS}\rangle_w = 0\,.$$

What happens under the perturbation?

Can use su(2) + conformal symmetry to make the most general ansatz….

Perturbation

[MRG, Gopakumar, Nairz, '23]

Under the perturbation we expect

$$
\begin{aligned}
&Q_1\,\alpha^2_{-1+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = a\,\psi^-_{-\frac{1}{2}+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w\,, &&\tilde{Q}_2\,\alpha^2_{-1+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = 0\\
&Q_1\,\psi^-_{-\frac{1}{2}+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = 0\,, &&\tilde{Q}_2\,\psi^-_{-\frac{1}{2}+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = b_{m,n}\,\alpha^2_{-1+\frac{m}{w-1}}\,\mathcal{Z}_-\,|\mathrm{BPS}\rangle_w\\
&S_1\,\alpha^2_{-1+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = 0\,, &&\tilde{S}_2\,\alpha^2_{-1+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = c_{m,n}\,\psi^-_{-\frac{1}{2}+\frac{m}{w+1}}\,\mathcal{Z}_+\,|\mathrm{BPS}\rangle_w\\
&S_1\,\psi^-_{-\frac{1}{2}+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = d\,\alpha^2_{-1+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w\,, &&\tilde{S}_2\,\psi^-_{-\frac{1}{2}+\frac{n}{w}}\,|\mathrm{BPS}\rangle_w = 0\;.
\end{aligned}
$$

E.g. $\tilde{S}_2 = \tilde{G}'^{+}_{-\frac{1}{2}}\;:\quad \delta K^3_0 = \delta L_0 = 0\,,\quad \underline{\delta\tilde{K}^3_0 = \delta\tilde{L}_0 = +\tfrac{1}{2}}$

$\mathcal{Z}_+\;:\quad \delta K^3_0 = \delta L_0 = \underline{\delta\tilde{K}^3_0 = \delta\tilde{L}_0 = +\tfrac{1}{2}}$

cf. [Gava, Narain, '02], [David, Sahoo, '08]

Perturbation

Can actually calculate the coefficient explicitly to first order in the deformation [MRG, Gopakumar, Nairz, '23]

$$c_{m,n} \sim \int d^2 z \; {}_{w+1}\langle \mathrm{BPS}| \, \bar{\psi}^{+}_{\frac{1}{2}-\frac{m}{w+1}} \, \tilde{G}'^{+}_{-\frac{1}{2}} \; \Phi_2(z,\bar{z}) \; \alpha^2_{-1+\frac{n}{w}} |\mathrm{BPS}\rangle_w$$

where perturbing field is $\mathbf{\Phi_2} = G^{-}_{-\frac{1}{2}} \, \tilde{G}'^{-}_{-\frac{1}{2}} \, \sigma^{(2)}_{+\frac{1}{2},+\frac{1}{2}}$

$$G^{-} \sim \bar{\alpha}^1 \psi^{-}$$

Perturbation

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where perturbing field is

$$\boldsymbol{\Phi_2} = G^{-}_{-\frac{1}{2}} \tilde{G}'^{-}_{-\frac{1}{2}} \sigma^{(2)}_{+\frac{1}{2},+\frac{1}{2}}$$

$$G^{-} \sim \bar{\alpha}^{1} \psi^{-}$$

This is most easily done upon lifting to covering surface, i.e. effectively on the worldsheet…

Finite w

[MRG, Gopakumar, Nairz, '23]

The exact finite w answer turns out to be

$$c_{m,n} = \left(\frac{w}{w+1}\right)^{m-n+1} \frac{(w+1)^{\frac{n}{w}}}{w^{\frac{m}{w+1}}} \frac{\Gamma(w - m\frac{w}{w+1})\,\Gamma(w+1-n\frac{w+1}{w})}{\Gamma(w-m)\,\Gamma(w+1-n)}$$

$$\times \boxed{\frac{\sqrt{1-\frac{n}{w}}}{1-\frac{m}{w}}\,\Gamma(1-\tfrac{m}{w+1})\Gamma(\tfrac{n}{w})\,\sin\left(\pi\tfrac{n}{w}\right)\,\sin\left(\pi\tfrac{m}{w+1}\right)}$$

$$\times\ \delta(\text{momentum conservation from integral over } \Phi(z,\bar{z}))$$

In the large w limit this becomes

$$\boxed{c_{q,p} \cong \frac{\sin(\pi p)}{\sqrt{1-p}} \times \delta(p-q)} \qquad \left(p = \tfrac{n}{w}\ ,\ q = \tfrac{m}{w+1}\right)$$

Multi-magnons

[MRG, Gopakumar, Nairz, '23]

To generalise the calculation to multi-magnon states need essentially only one additional ingredient:

$$d_{m,n} = {}_{w+1}\langle \mathrm{BPS}|\, \bar{\alpha}^{1}_{1-\frac{m}{w+1}}\, \sigma^{(2)}(1,1)\, \alpha^{2}_{-1+\frac{n}{w}}\, |\mathrm{BPS}\rangle_{w}$$
$$= \left(\tfrac{w}{w+1}\right)^{m-n-1} \frac{(w+1)^{\frac{n}{w}}}{w^{\frac{m}{w+1}}}\, \frac{\Gamma\left(w-\frac{n(w+1)}{w}\right)\Gamma\left(w-\frac{mw}{w+1}\right)}{\Gamma(w-n)\,\Gamma(w-m)}$$
$$\times \boxed{(-1)^{m-n}\frac{1}{\left(\frac{m}{w+1}-\frac{n}{w}\right)}\sqrt{\frac{1-\frac{n}{w}}{1-\frac{m}{w+1}}}\,\sin\left(\pi\tfrac{mw}{w+1}\right)\sin\left(\pi\tfrac{n(w+1)}{w}\right)\Gamma\left(\tfrac{n}{w}\right)\Gamma\left(1-\tfrac{m}{w+1}\right)}\;.$$

and similar formula for fermions $\hat{d}_{m,n}$. In the large w limit this leads to

$$\boxed{d_{q,p} = \hat{d}_{q,p} = e^{\pm i\pi p}\,\delta(p-q)} \qquad \left(p = \frac{n}{w}\;,\; q = \frac{m}{w+1}\right)$$

General action

[MRG, Gopakumar, Nairz, '23]

The action of the generators can then be described in terms of successive commutators

$$
\begin{aligned}
[Q_1, \alpha^2(p)] &= a(p)\,\psi^-(p) \ , & [\tilde{Q}_2, \alpha^2(p)] &= 0 \ , \\
\{Q_1, \psi^-(p)\} &= 0 \ , & \{\tilde{Q}_2, \psi^-(p)\} &= b(p)\,\alpha^2(p)\,\mathcal{Z}_- \ , \\
[S_1, \alpha^2(p)] &= 0 \ , & [\tilde{S}_2, \alpha^2(p)] &= c(p)\,\psi^-(p)\,\mathcal{Z}_+ \ , \\
\{S_1, \psi^-(p)\} &= d(p)\,\alpha^2(p) \ , & \{\tilde{S}_2, \psi^-(p)\} &= 0 \ ,
\end{aligned}
$$

where

$$a(p)b(p) = c(p)d(p) = g\,\sin(\pi p) + \mathcal{O}(g^2)$$

$$\left[\begin{array}{ll} \alpha^2(p) & = \frac{1}{\sqrt{1-p}}\alpha^2_{-1+p} \\ \chi^-(p) & = \chi^-_{-\frac{1}{2}+p} \end{array}\right]$$

Central extension

On physical (orbifold invariant) states, the left-right anti-commutator must vanish:

$$\{S_1, \tilde{S}_2\} = 0 = \{Q_1, \tilde{Q}_2\} \ .$$

For finite w, this is automatically the case, but in the large w limit, it only remains true provided we choose the correct phases in

$$d_{q,p} = \hat{d}_{q,p} = e^{\pm i\pi p}\, \delta(p-q) \qquad \left(p = \tfrac{n}{w} \ , \ q = \tfrac{m}{w+1}\right)$$

This requires that

$$\mathcal{Z}_{\pm} \mathcal{O}(p) = e^{\mp 2\pi i p} \mathcal{O}(p)\, \mathcal{Z}_{\pm}$$

cf. [Beisert '05]

Anomalous dimension

Can then use the usual integrability arguments to conclude that each magnon contributes

$$\mathcal{C}(p) + \tilde{\mathcal{C}}(p) = \sqrt{(1-p)^2 + 4g^2 \sin^2(\pi p)}$$

to anomalous conformal dimension.

cf. [Berenstein, Maldacena, Nastase, ’02], [Hoare, Stepanchuk, Tseytlin, ’14]

Furthermore, the problem is integrable: can read off S-matrix from above action — satisfies the Yang-Baxter equation.

cf. [Babichenko, Stefanski, Zarembo, ’09], [Lloyd, Ohlson Sax, Sfondrini, Stefanski, ’13]

Conclusions and Outlook

Studied the perturbation of duality involving tensionless string theory on $\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4$ and symmetric orbifold

$$\mathrm{AdS}_3 \times \mathrm{S}^3 \times \mathbb{T}^4 \quad = \quad \mathrm{Sym}_N(\mathbb{T}^4)$$

1 unit of NS-NS flux

almost free 2d CFT

under the perturbation that switches on RR flux, thereby also giving tension to the string.

Conclusions and Outlook

Performed the perturbation analysis in symmetric orbifold, but given that the calculation was done on covering surface = worldsheet calculation.

We have solved the perturbed symmetric orbifold spectrum, using effectively a dynamical spin chain approach —- very similar to N=4 SYM.

cf. [Minahan, Zarembo, '02]
[Beisert '05]

However, unlike the case for N=4, we can actually determine the structure constants at finite w, and obtain asymptotic answer by taking large w limit.

Conclusions and Outlook

The deformed theory possesses an S-matrix, satisfying the Yang-Baxter-equation $\longrightarrow$ **integrable.**

One feature of the deformation is that the left- and right-moving degrees of freedom start to mix once the R-R flux is switched on.

cf. [Gava, Narain, '02]

This is likely to be an important ingredient in further understanding our proposal for the duality relating free N=4 SYM in D=4 to the tensionless limit of strings on $\mathrm{AdS}_5 \times \mathrm{S}^5$.

[MRG, Gopakumar, '21]

Costas @ETH

During 2003-2006 Costas taught courses at ETH each summer.

Much appreciated help in getting the group at ETH off the ground.

Thank you very much, Costas

— and all the very best for the years ahead!