Arithmetic Chaos inside Black Holes

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Prelude

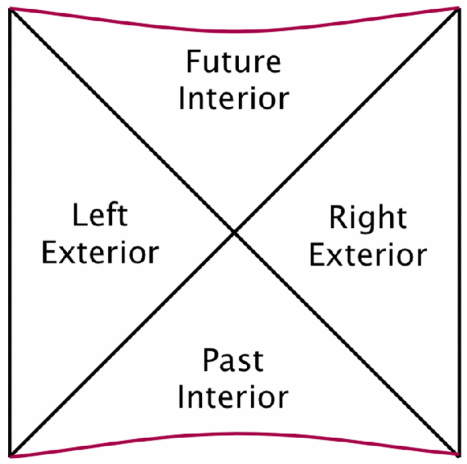
- This talk is about placing some old topics in a modern context. The old topics are Misner/BKL dynamics close to a singularity (1969) and Arithmetic Chaos (1990's). The modern context is holographic duality.

- The old results are well-known but perhaps not widely known, so I will spend some time discussing them too.

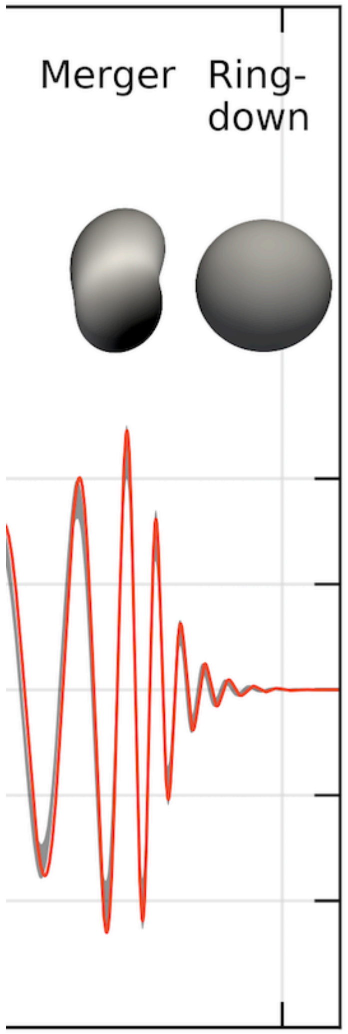
- More can found in the paper arXiv:2312.11622, w/ Marine De Clerck and Jorge Santos.

Black hole exteriors

In holography, eternal black holes are dual to the thermofield double state of a dual CFT. [Israel 76, Maldacena 01]



$$\Psi \sim \sum_n e^{-\frac{E_n}{2T}} |E_n\rangle_1 |E_n\rangle_2$$



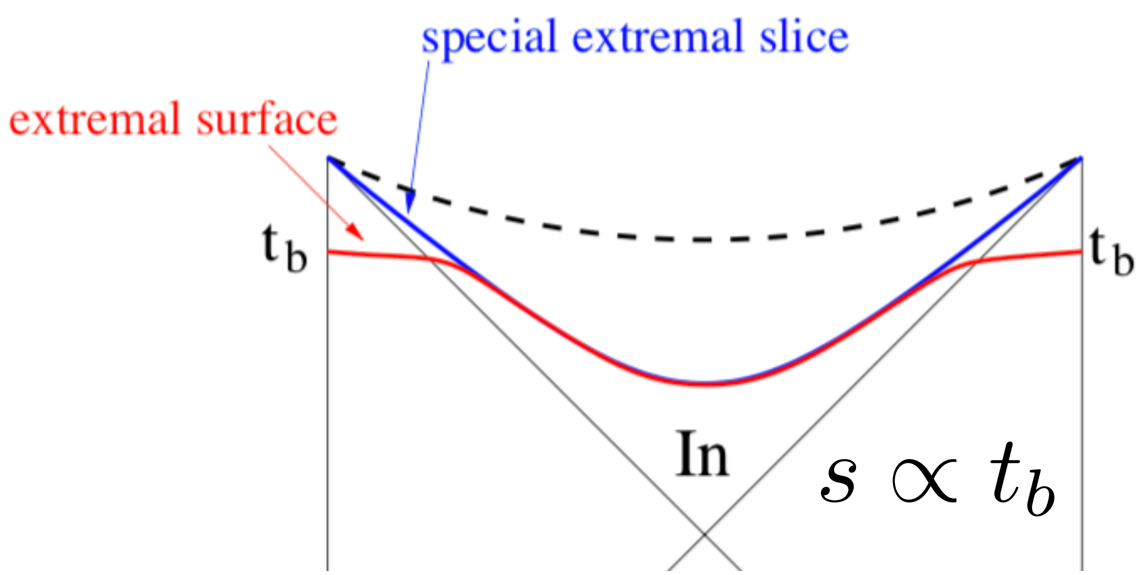
The dynamics of the exterior describes the approach to thermal equilibrium. E.g. quasinormal 'ringdown' followed by hydrodynamics. A rich source of inspiration for the dual dynamics of strongly quantum matter.

[e.g. SAH-Lucas-Sachdev 16]

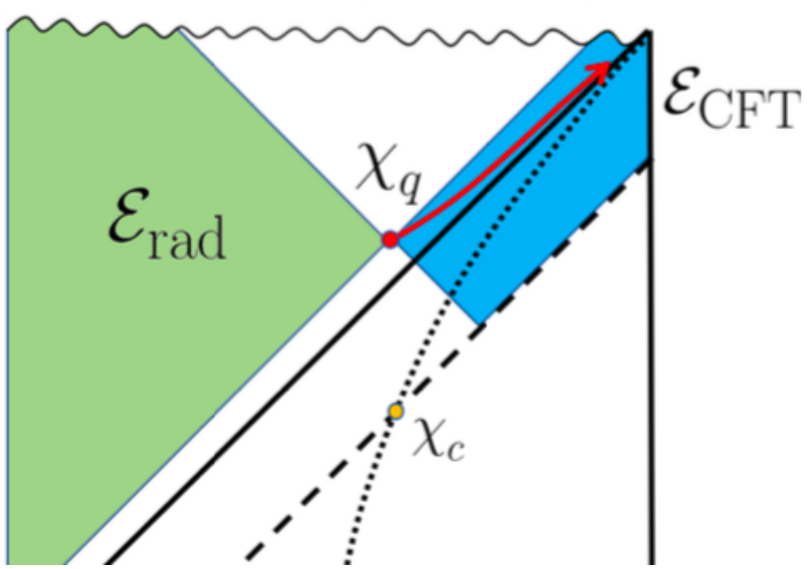
Black hole interiors

The black hole interior is tied up with deep questions involving infalling observers, the singularity and quantum cosmology.

Recent developments suggest geometric aspects of the black hole interior encode quantum-information theoretic facts about the dual state.



[Hartman-Maldacena 13]



[Penington; AEMM 19]

Which interior?

- The interior of cherished solutions such as Schwarzschild-AdS is unstable against small perturbations [cf. Fournodavlos-Sbierski 18] and should be physically irrelevant at late interior times.

- Interior dynamics widely studied by mathematicians. Holographic implications have not been explored. Are interior instabilities related to dual thermalization?

- Understanding actual, generic classical interiors may be necessary before addressing quantum gravity questions about e.g. the singularity.

Deformation of the CFT

- Deforming a CFT by a relevant operator triggers an RG flow. Does this more generic deformed QFT have a more generic singularity in the dual?

- Holographic renormalization group flow is described by radial evolution that breaks the scale invariance of AdS:

$$S = S_{\rm CFT} + \int dx \phi_{(0)} \mathcal{O} \quad \rightarrow \quad \phi(z)$$

- These are usually considered at zero temperature. At T>0, continue the RG flow past the horizon. In the interior the flow develops in time.

A first attempt

[2004.01192 w/ Alex Frenkel, Jorrit Kruthoff, Zhengyan Shi]

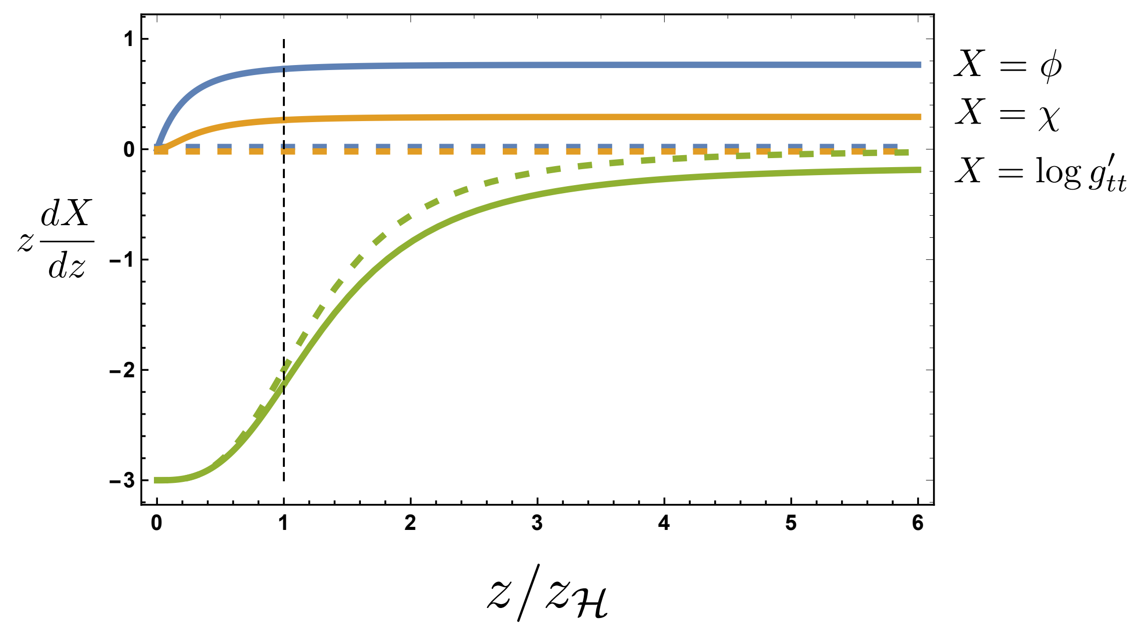
- Include scalar field dual to the deforming operator

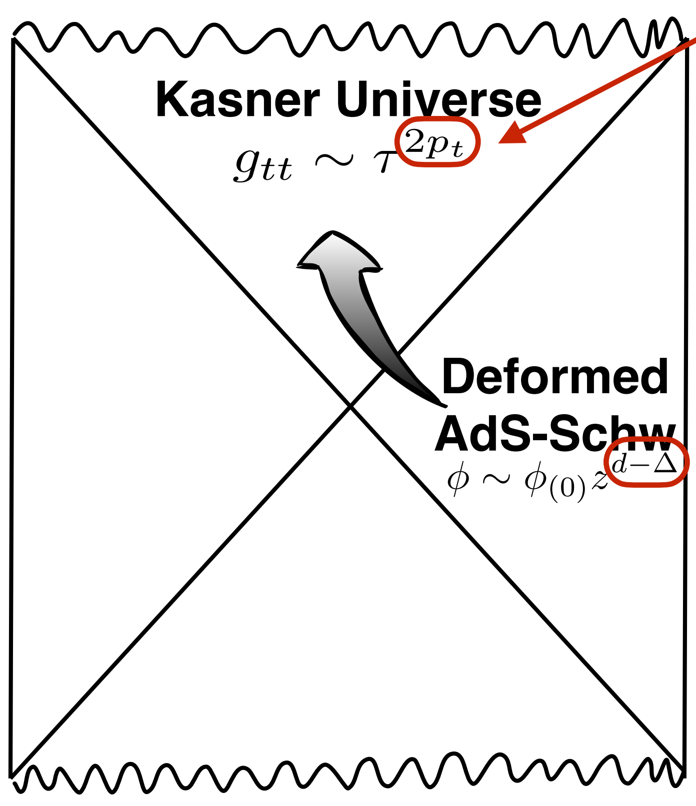
$$\mathcal{L} = R + 6 - g^{ab}\partial_a\phi\partial_b\phi - m^2\phi^2\,.$$

- Look for planar black hole solutions

Exponentiation of the logarithmic instability of Schwarzschild interior

$$ds^2 = \frac{1}{z^2}\left(-f(z)e^{-\chi(z)}dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2\right)$$





$$S = S_0 + \int dx\, \phi_{(0)} \mathcal{O}$$

The Kasner Universe

[Kasner 21; Belinski-Khalatnikov 73]

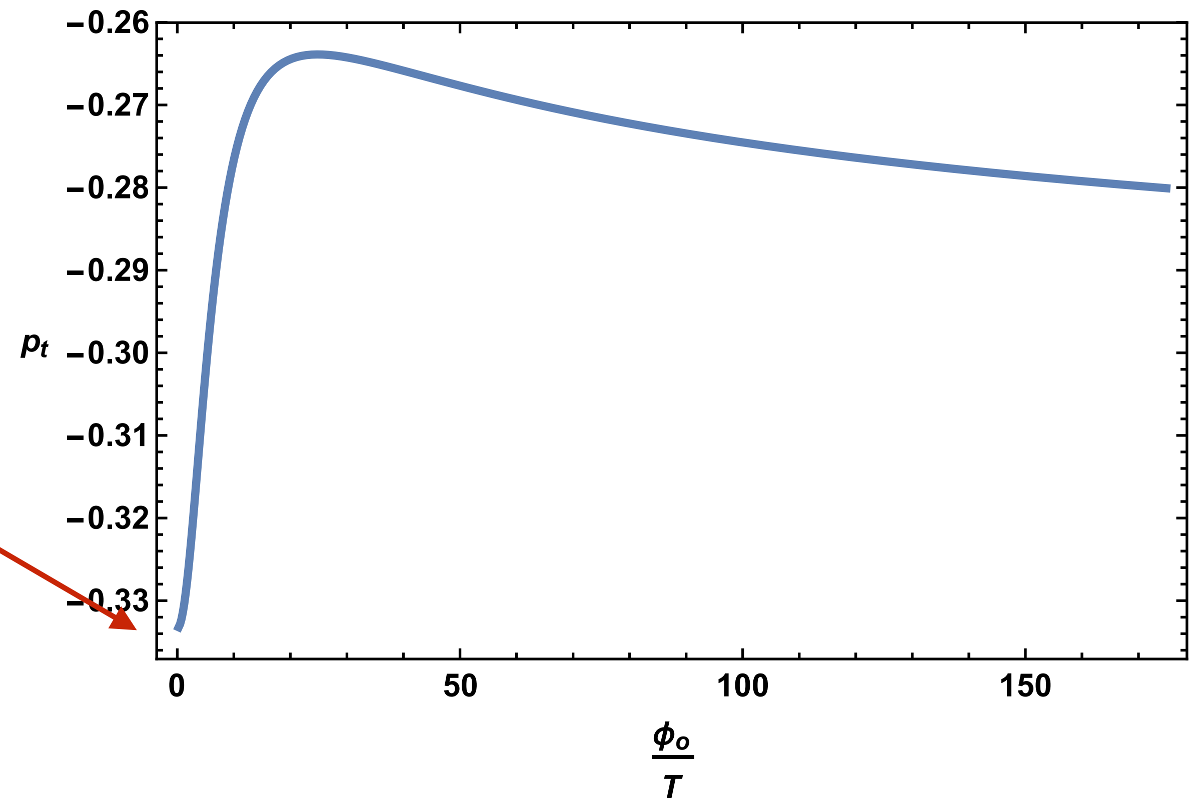
- The singularity is found to have scaling properties.
- In Kasner form ($\tau \to 0$ is proper time to singularity):

$$ds^2 \sim -d\tau^2 + \tau^{2p_t} dt^2 + \tau^{2p_x} \left(dx^2 + dy^2\right) , \qquad \phi(r) \sim -\sqrt{2} p_\phi \log \tau \, .$$

- In our model:

Schwarzschild

$p_t = -\frac{1}{3}$



$p_t < 0$

Einstein-Rosen bridge grows towards singularity

Interlude

- Marc Henneaux explained to us that the reason we had not landed on more interesting chaotic behaviour is that we did not have enough “walls”.

- Chaotic behaviour is (conjecturally) generic once all modes, including inhomogeneities, are present. [BKL 70] for pure gravity in D=4 [Damour-Henneaux-Nicolai 02] for $D \leq 11$ + SUSY.

- Objective: find a simple (non-generic) model in AdS with the expected chaotic interior behaviour.

Vector fields

[2312.11622 w/ Marine De Clerck and Jorge Santos]

- Three massive vector fields do the trick:

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[R + 6 - \sum_{i=1}^{3} \left(\frac{1}{4} F_i^2 + \frac{\mu_i^2}{2} A_i^2 \right) \right]$$

- With fields depending on a single ‘radial’ coordinate:

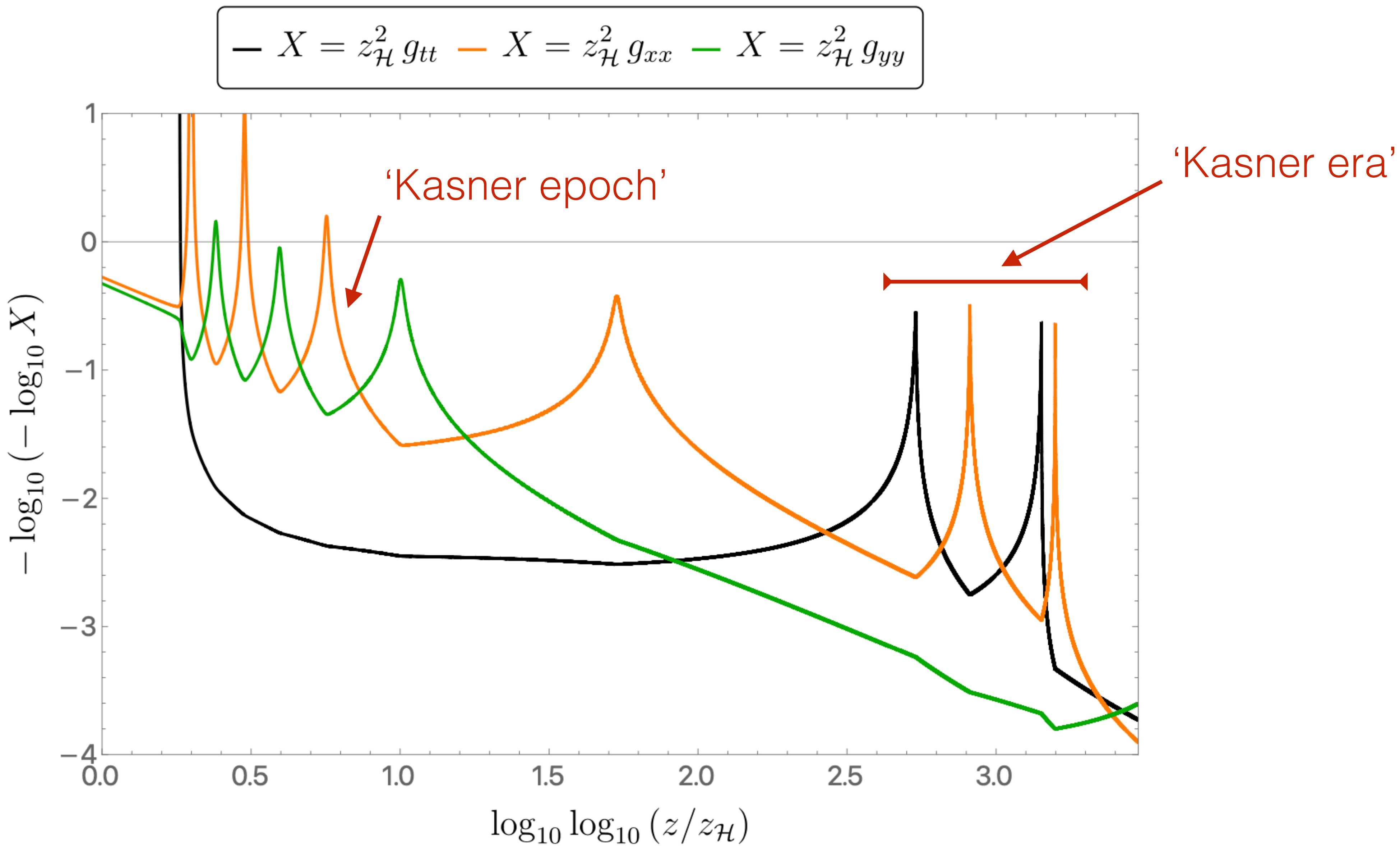
$$\mathrm{d}s^2 = \frac{1}{z^2} \left(-F e^{-2H} \mathrm{d}t^2 + \frac{\mathrm{d}z^2}{F} + e^{-2G} \mathrm{d}x^2 + e^{2G} \mathrm{d}y^2 \right) \quad A_1 = \phi_t \, \mathrm{d}t \, , \quad A_2 = \phi_x \, \mathrm{d}x \, , \quad A_3 = \phi_y \, \mathrm{d}y$$

- The mass (like the CC) drops out of the equations near the singularity, but is necessary to have a regular horizon in the presence of boundary sources.

- If μ_i^2 is small enough, the dual vector operator is relevant and preserves the AdS asymptotics.

Kasner epochs and eras

- As with the scalar case previously, the ODEs can be integrated from the boundary and through the horizon. The far interior dynamics is now very different:

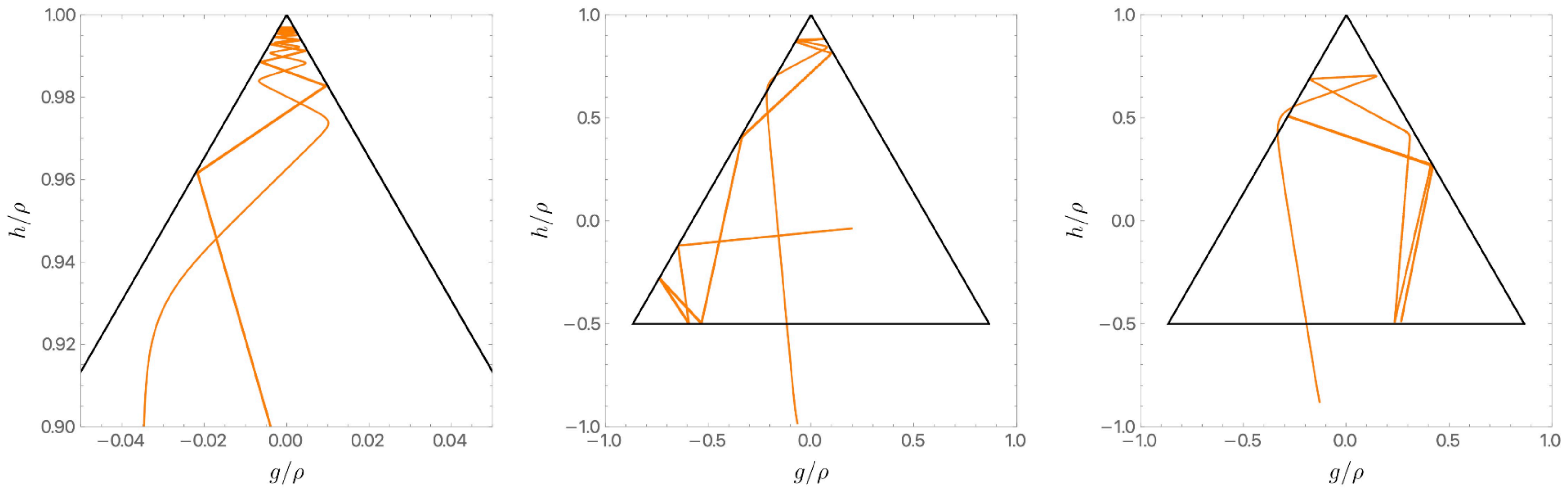


Walls

- The leading near-singularity equations in this model are identical to the Bianchi-IX 'mixmaster' universe studied by Misner and the Landau Institute school.

- Convenient near-singularity coordinates:

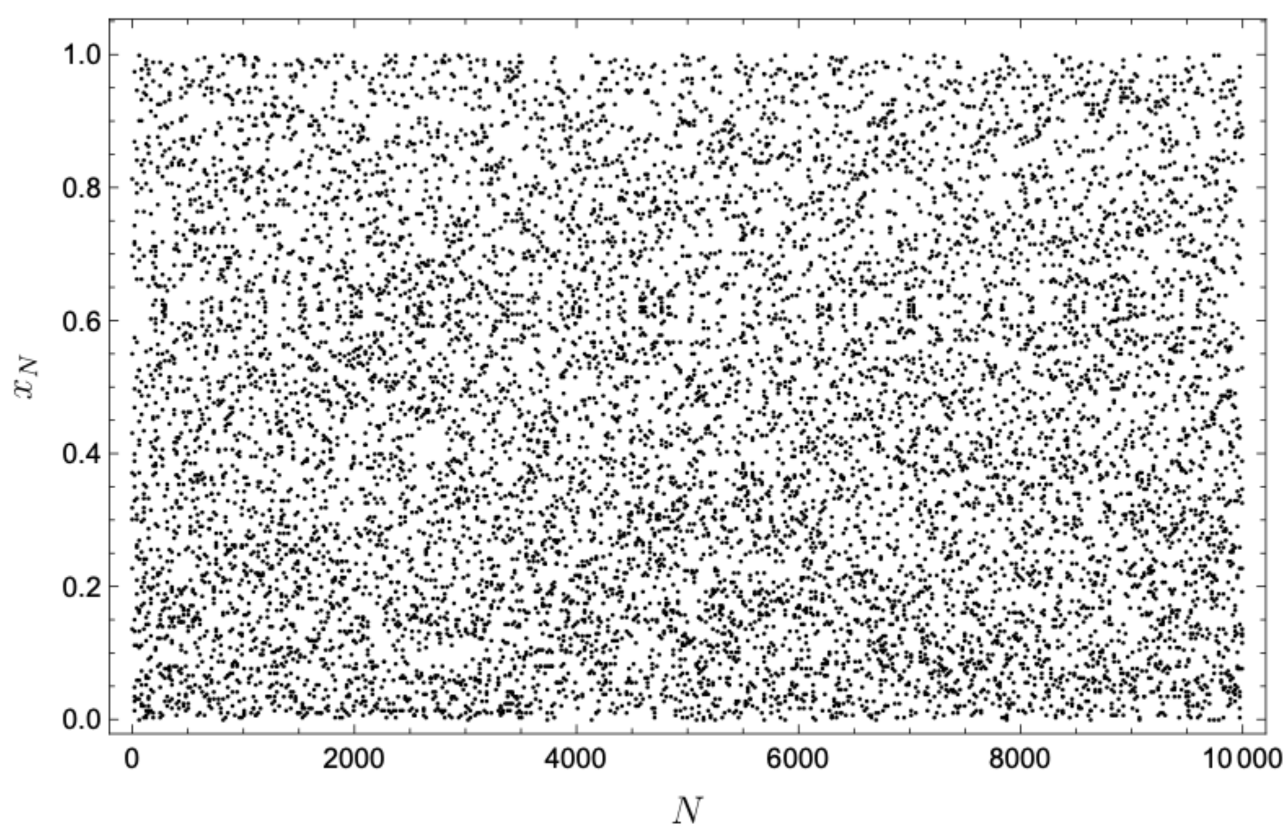
$$\mathrm{d}s^2 = e^{-\rho}e^{-2h}\mathrm{d}t^2 - n^2\mathrm{d}\rho^2 + e^{-\rho}e^{h}\left[e^{-\sqrt{3}g}\mathrm{d}x^2 + e^{\sqrt{3}g}\mathrm{d}y^2\right]$$

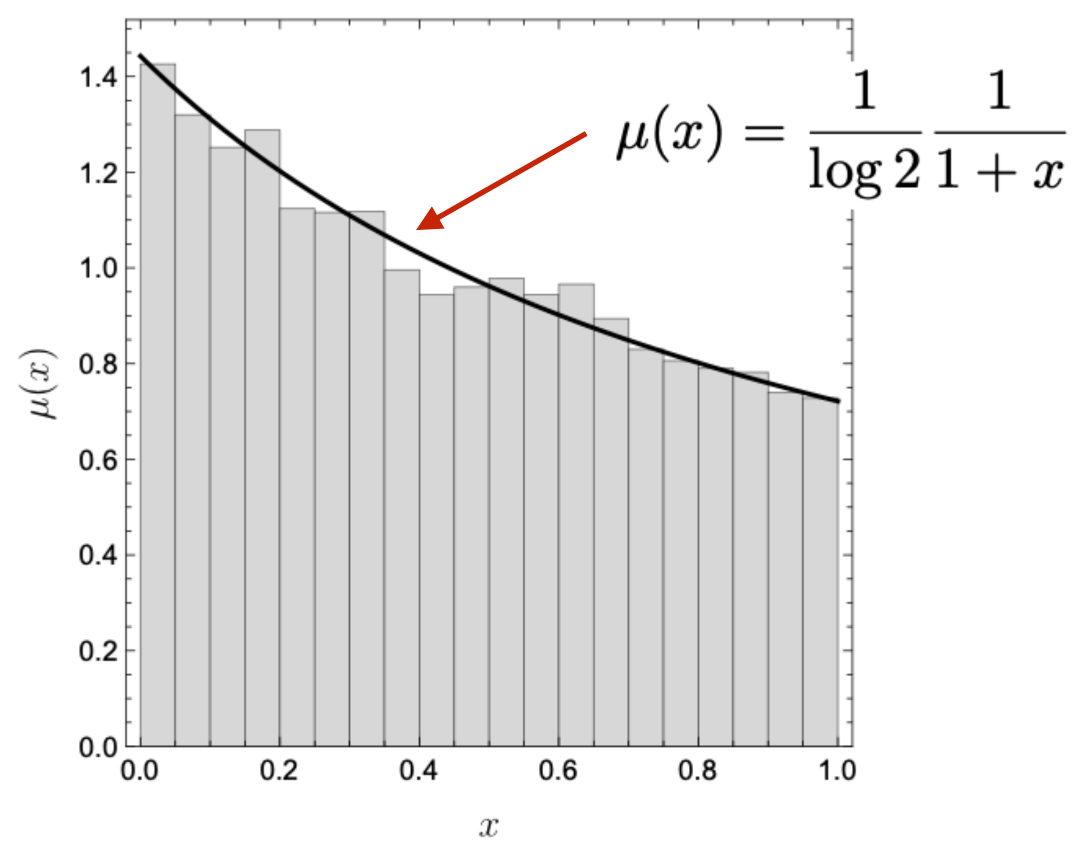


Stochastic properties

- BKL derived recursion relations for the Kasner exponents at the start of the N^{th} Kasner era.
- These reduce to the chaotic Gauss map for $x_N \in [0,1]$:

$$x_{N+1} = \frac{1}{x_N} - \left\lfloor \frac{1}{x_N} \right\rfloor$$



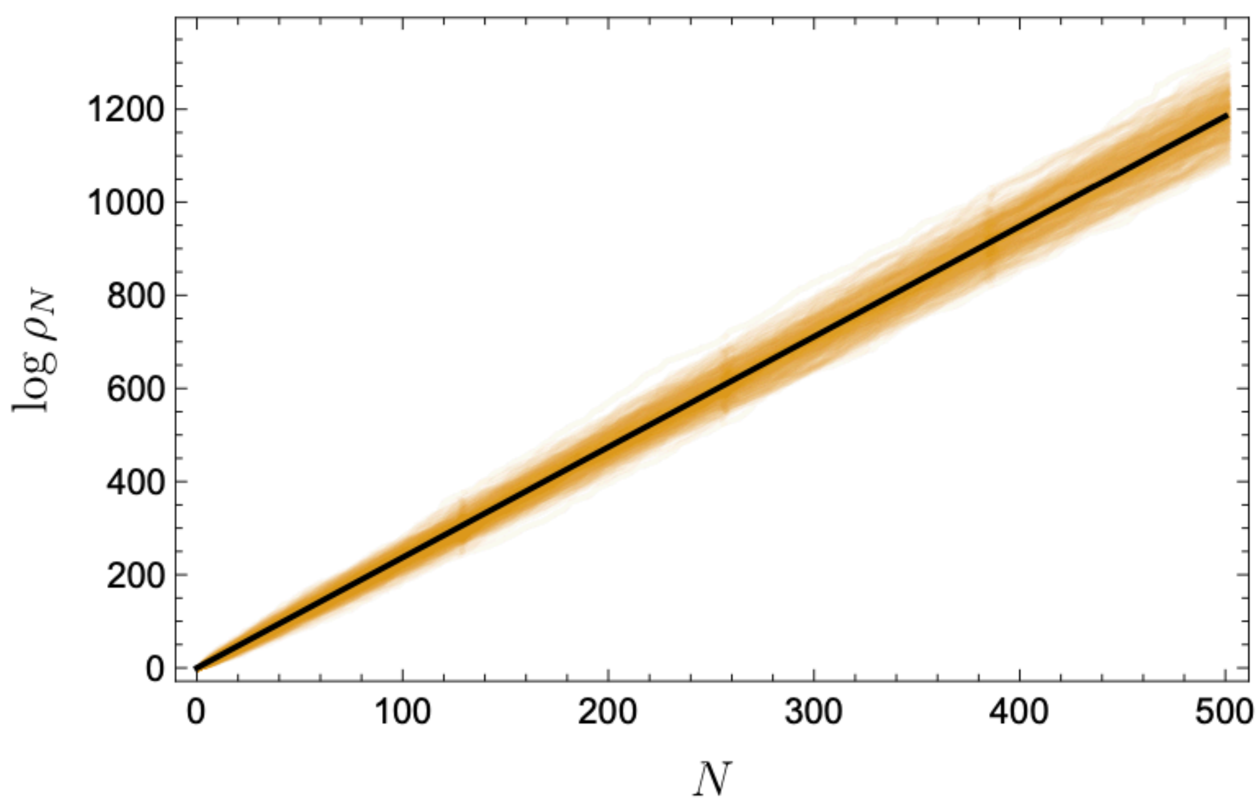


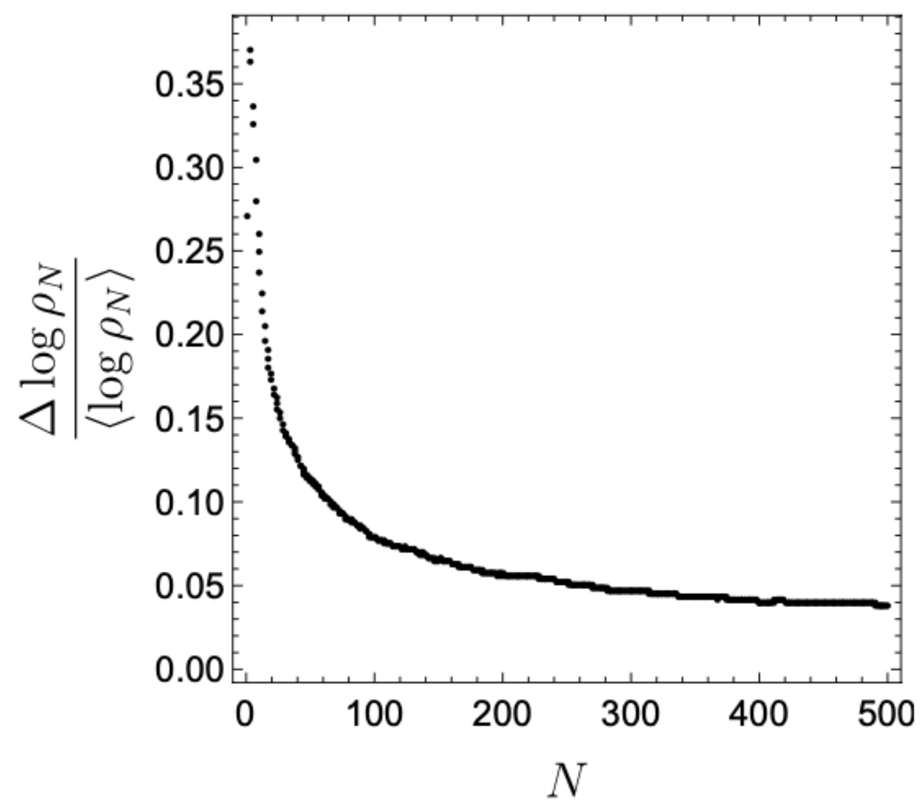
Stochastic properties

- The equilibrium distribution is reached very quickly and can be used to show that the spatial volume decreases doubly exponentially over each era:

$$\left\langle \log\left(-\frac{2}{3}\frac{\log V_N}{\rho_0}\right)\right\rangle = hN$$

h ≈ 2.37 is KS entropy





Interlude

- Known for ~ 50 years that the approach to cosmological singularities is governed by a chaotic dynamical system.
- We have given a simple, explicit embedding of this in an AdS black hole interior.
- A further aspect of the BKL story is that different points in space decouple. This is not addressed within our spatially homogeneous model.
- The true richness of the near-singularity behaviour is best appreciated from a Hamiltonian perspective …

Hamiltonian formalism

- The Hamiltonian constraint in our model is:

$$\mathcal{H} \equiv -\pi_\Omega^2 + \pi_g^2 + \pi_h^2 + 3e^{-\Omega}\left(e^{-2h}\pi_t^2 + e^{h-\sqrt{3}g}\pi_x^2 + e^{h+\sqrt{3}g}\pi_y^2\right) = 0$$

- In a relational description, one thinks of Ω as time. This constraint then determines time evolution.

- Chitre and Misner noticed that if one sets:

$$\Omega = e^\tau \cosh R\,, \qquad g = e^\tau \sinh R \cos\phi\,, \qquad h = e^\tau \sinh R \sin\phi$$

Then as $\tau \to \infty$ the exponential potential becomes an infinite barrier. One lands in a hyperbolic billiard.

Cosmological billiards

- The near-singularity ‘Hamiltonian’ generating evolution in τ is time-independent!

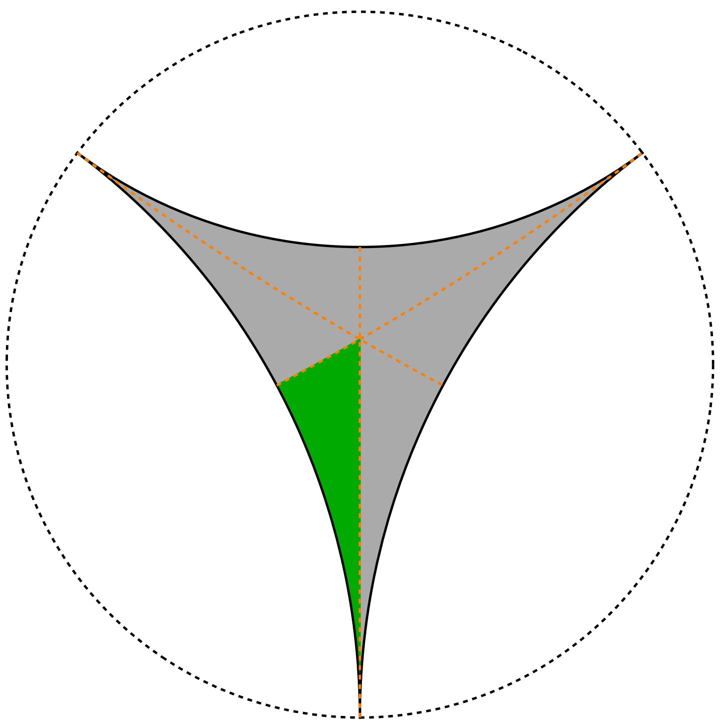
- Characterise the time-independent Hamiltonian via its spectrum — do a semi-classical canonical quantisation. (Essentially the same information is contained in the classical periodic orbits.)

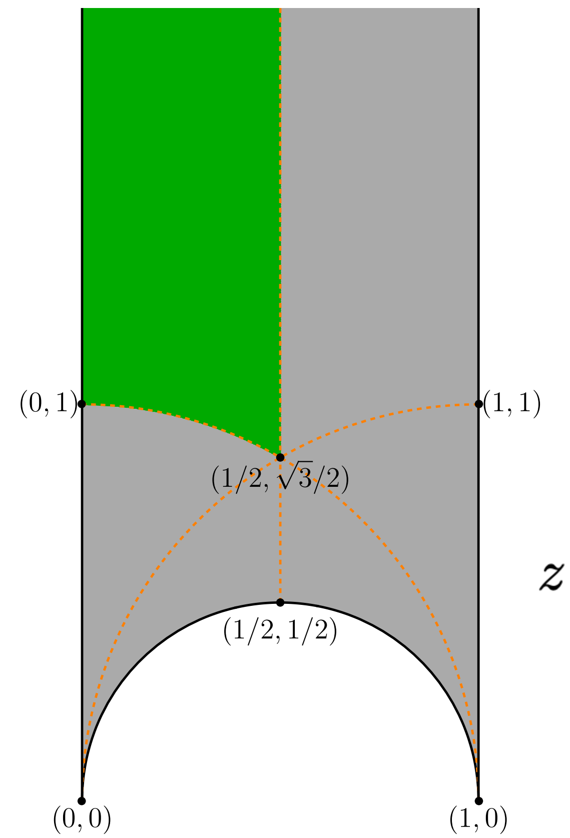
$$\Psi(\tau, R, \phi) = \sum_{n\pm} c_{n\pm} \Psi_n(R, \phi) e^{[-\frac{1}{2} \pm i\varepsilon_n]\tau}$$

$$-\nabla^2_{H^2} \Psi_n(R, \phi) = \left(\frac{1}{4} + \varepsilon_n^2\right) \Psi_n(R, \phi)$$

The billiard domain

- Map the Poincaré disc to the upper half plane, $z = x + iy$





- Boundaries are the fixed points of the reflections:

$$z \to -z^*\,, \qquad z \to 2 - z^*\,, \qquad z \to \frac{z^*}{2z^* - 1}$$

- Half the fundamental domain of $\Gamma(2)$, “the principal congruence subgroup of level 2” of SL(2,ℤ), with generators: $z \to z + 2\,, \qquad z \to \dfrac{z}{2z+1}$

The billiard domain

- The eigenfunctions we are after are the odd automorphic forms of $\Gamma(2)$:

$$\psi(\gamma z) = \psi(z), \qquad \forall \gamma \in \Gamma(2) \qquad \text{and} \qquad \psi(-z^*) = -\psi(z)$$

- The spectrum must be broken up with respect to the S_3 symmetry. Present results for the ‘sign’ representation, these are wave functions defined in the small green region and are precisely the odd automorphic forms of the modular group SL(2,ℤ).

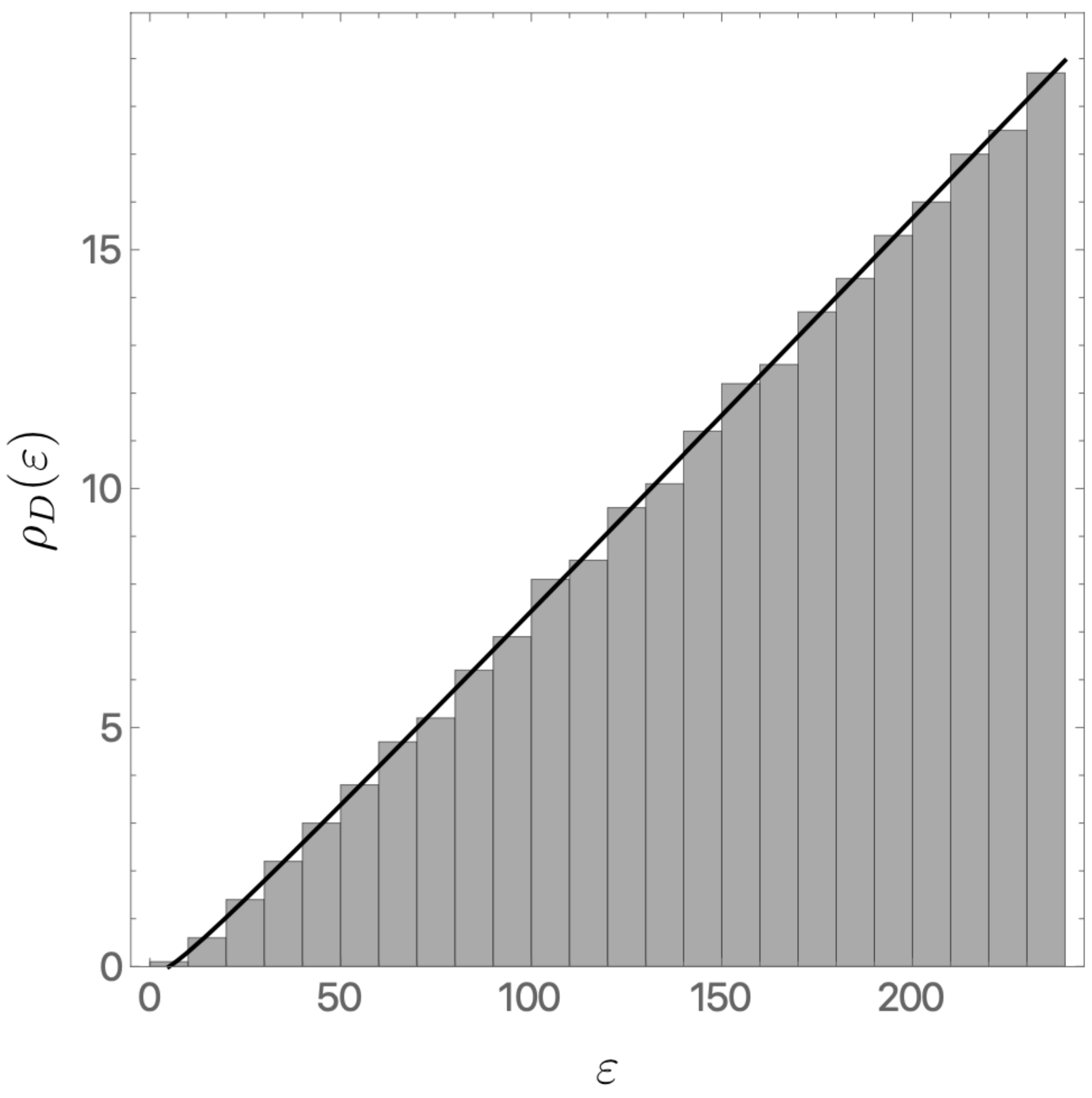
- Widely studied! We (re)computed the first 2250 eigenvalues. Other sectors less studied.

The Weyl law

- The density of states is known to behave asymptotically as:

$$\bar{\rho}_D(\varepsilon) \approx \frac{\varepsilon}{12} - \frac{1}{2\pi}\log\varepsilon - \frac{3}{4\pi}\log 2 + \mathcal{O}(\log\varepsilon/\varepsilon^2)$$

- The interesting features of quantum chaos have to do with the fluctuations of the density of states about this smooth asymptotic behaviour.

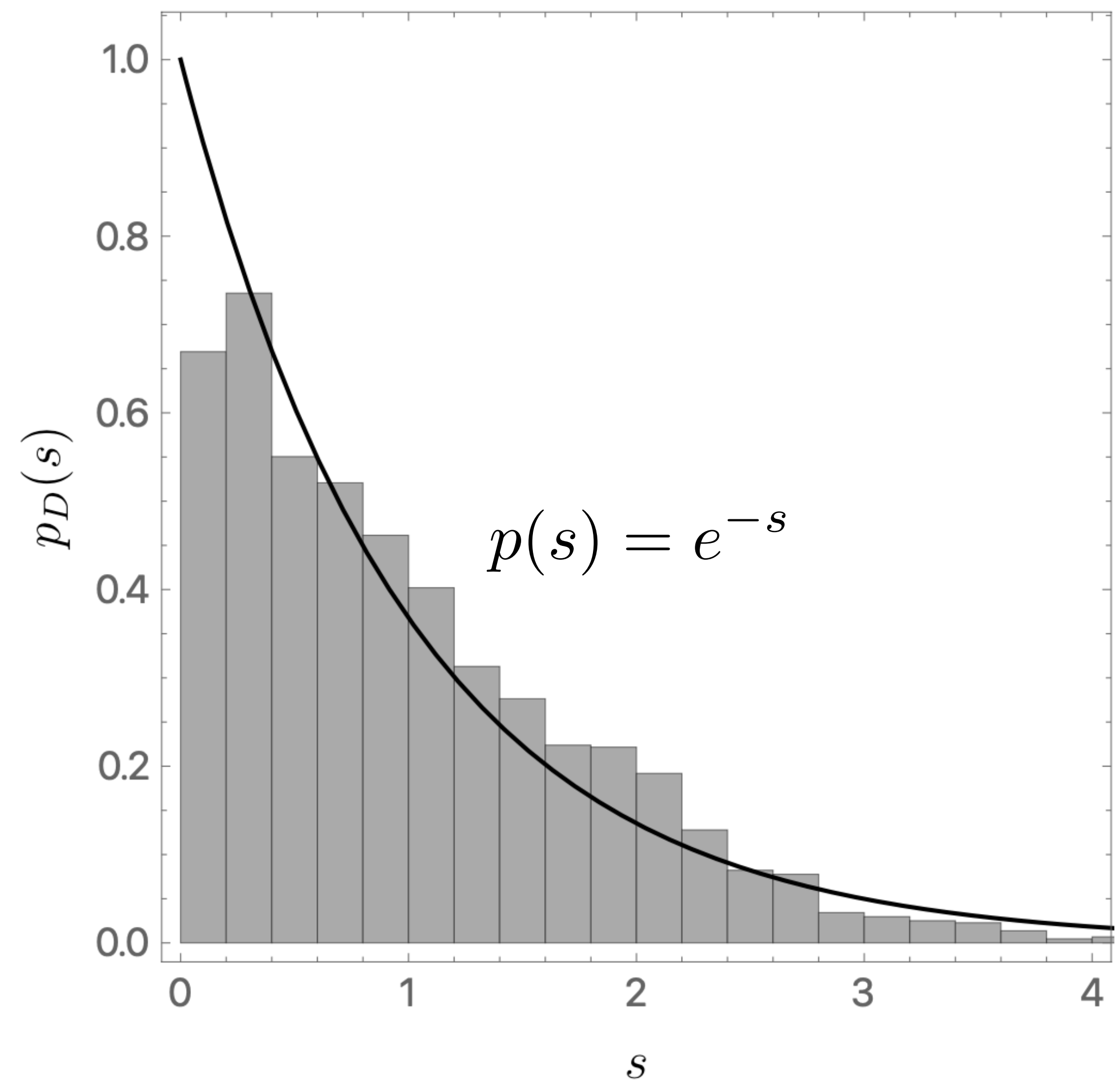


Nearest-neighbour spacing

- The 'unfolded' energy differences:

$$s_n = \bar{N}(\varepsilon_{n+1}) - \bar{N}(\varepsilon_n) \approx \bar{\rho}(\varepsilon_n)(\varepsilon_{n+1} - \varepsilon_n)$$

Capture universal aspects of late time behaviour.



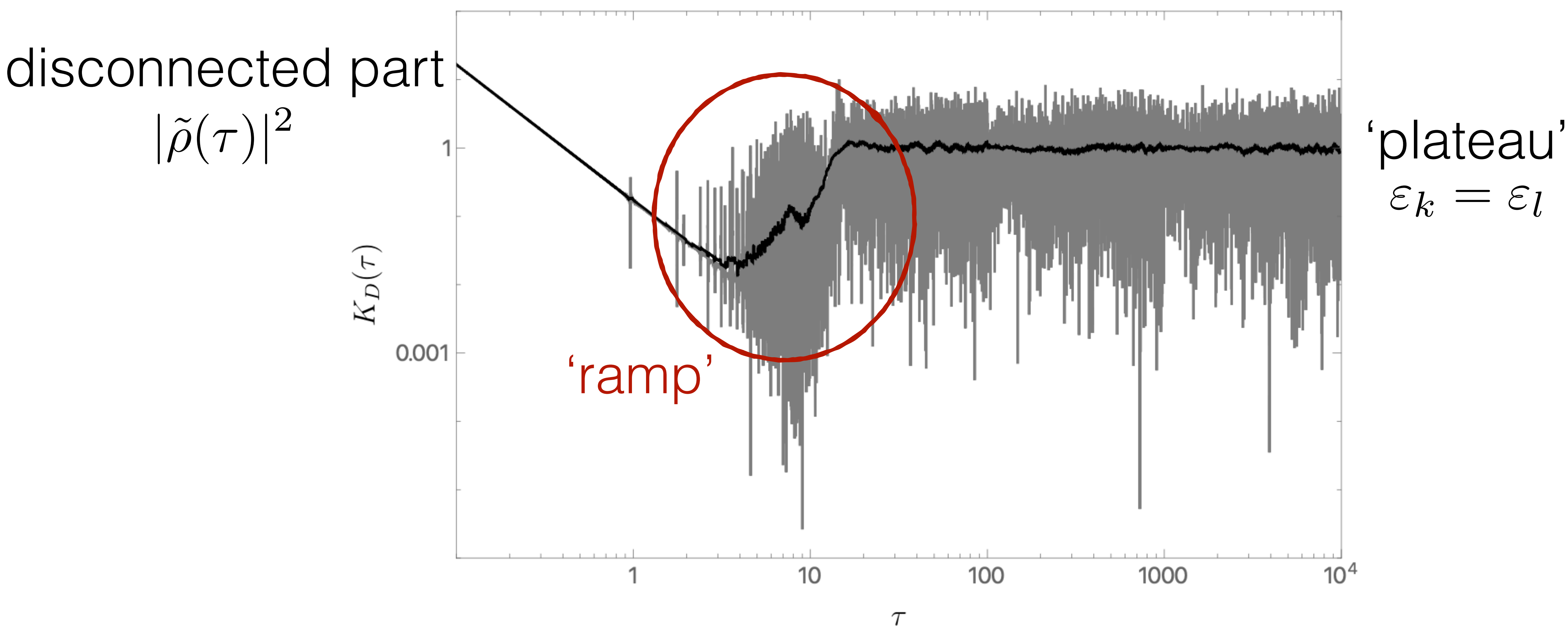
- Find (as is well-known) a Poisson rather than Wigner-Dyson distribution. No level repulsion.

- Usually a feature of integrable rather than chaotic systems.

The spectral form factor

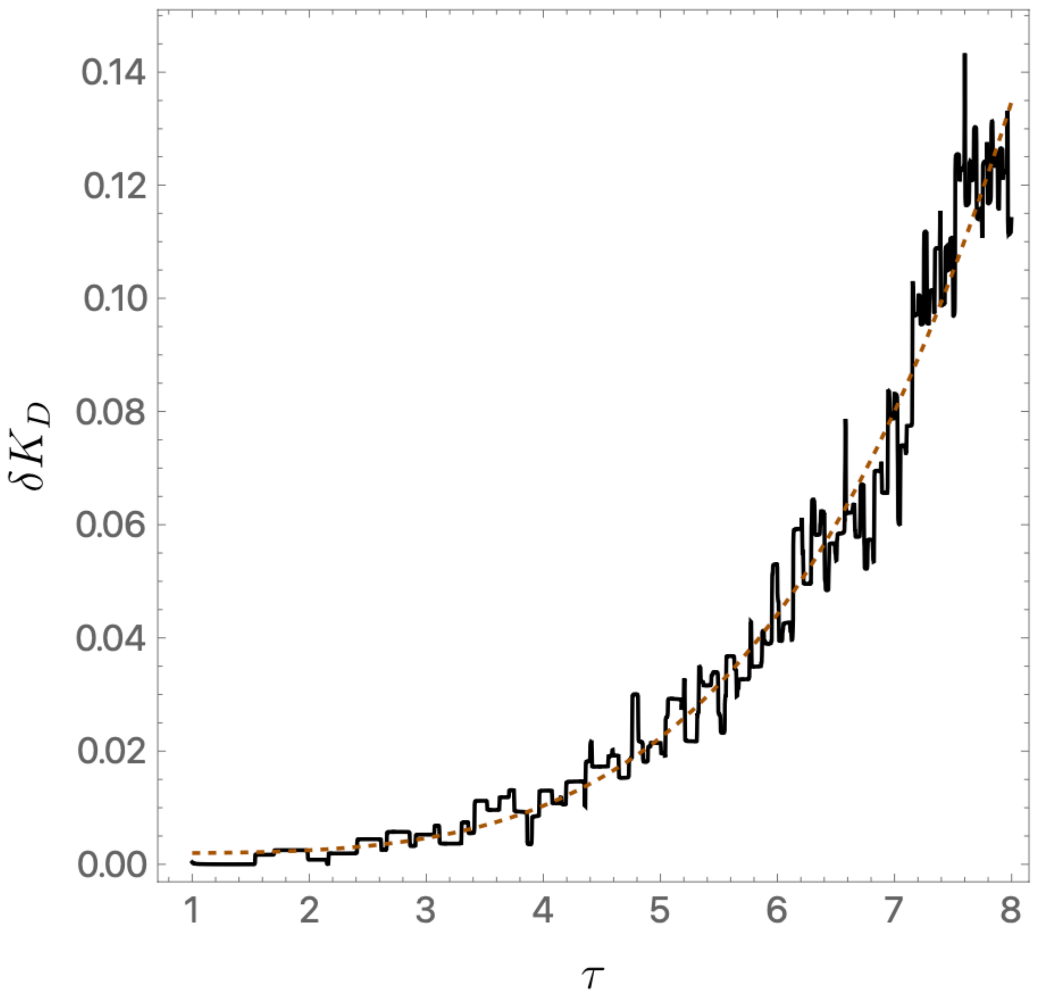
- A richer probe of the spectrum:

$$K(\tau) \equiv \frac{1}{\Lambda}\left|\sum_{k=1}^{\Lambda} e^{-i\,\varepsilon_k\,\tau}\right|^2 = \frac{1}{\Lambda}\sum_{k,l=1}^{\Lambda} e^{-i\,(\varepsilon_k-\varepsilon_l)\,\tau}$$



The spectral form factor

- Subtract off the disconnected part:



- Ramp is exponential rather than linear.
- Seen previously in arithmetic quantum chaotic system. Via periodic orbit theory, comes from an exponentially large degeneracy $e^{\frac{1}{2}L}$ of closed geodesics of length L.

- Also seen recently in integrable versions of the SYK model.

[Liao-Vikram-Galitski; Winer-Jian-Swingle 20]

Hecke relations

- The integrable-like features of arithmetic quantum chaotic systems are due to an infinite number of conserved ‘Hecke operators’.

- In the Fourier expansion

$$\Psi_n(x,y) = \sum_{m=1}^{+\infty} c_m^n \sqrt{y}\, K_{i\varepsilon_n}(2\pi m y) \sin(2\pi m x)$$

- One finds that all the non-prime coefficients are determined by the prime ones from Hecke relations (number theoretic voodoo):

$$c_{mp}^n = c_m^n c_p^n - c_{m/p}^n$$

Sato-Tate conjecture

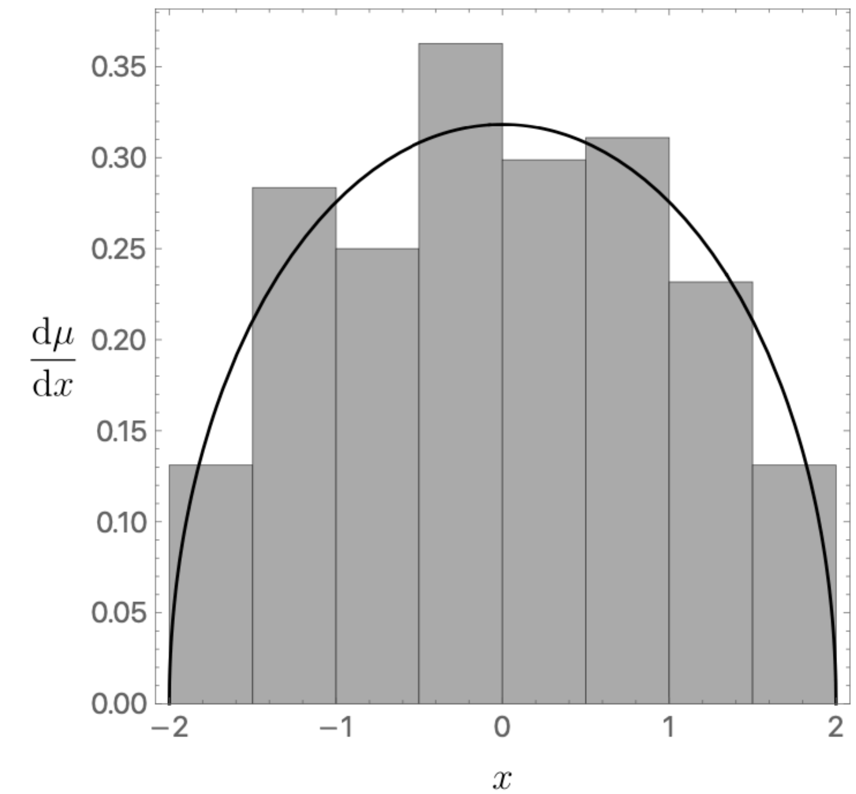
- The Hecke relations imply that an associated 'L-function' obeys an Euler product formula:

$$L_n(s) \equiv \sum_{m=1}^{\infty} \frac{c_m^n}{m^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - c_p^n p^{-s} + p^{-2s}}$$

- Generalisations of the Riemann zeta function.

- Prime c_p for fixed energy level conjectured to be distributed as Wigner semicircle (= the eigenvalues of a random matrix).

- Computed the first 656 c_p^1.



Work in Progress …

- General relativity near singularities follows chaotic dynamics with strong connections to number theory.

- Does a time-independent, arithmetically chaotic Hamiltonian near the singularity give a dual description of the interior. The wave functions share properties of CFT partition functions [cf Benjamin-Collier-Fitzpatrick-Maloney-Perlmutter ’21]

- Are the symmetries special to Einstein gravity or do they reflect a deeper principle? E.g. what happens with higher derivative terms?