

The PARISI-SOURLAS UPLIFT

&

INFINITELY MANY SOLVABLE CFT<sub>4</sub>

19/06/2024  
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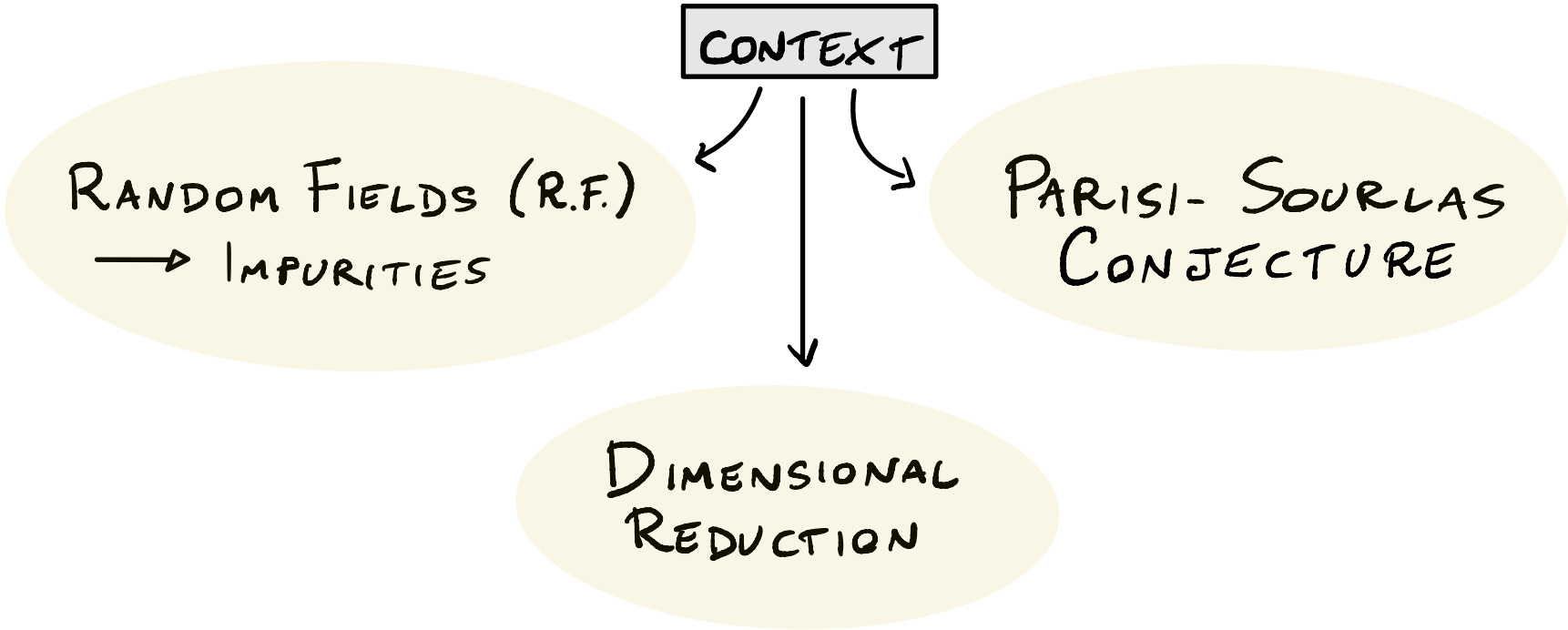
[1912.01617]  
[2009.10087]  
[2112.06942]  
[2203.12627]

with:

APRATIM KAVIRAJ  
SLAVA RYCHKOV

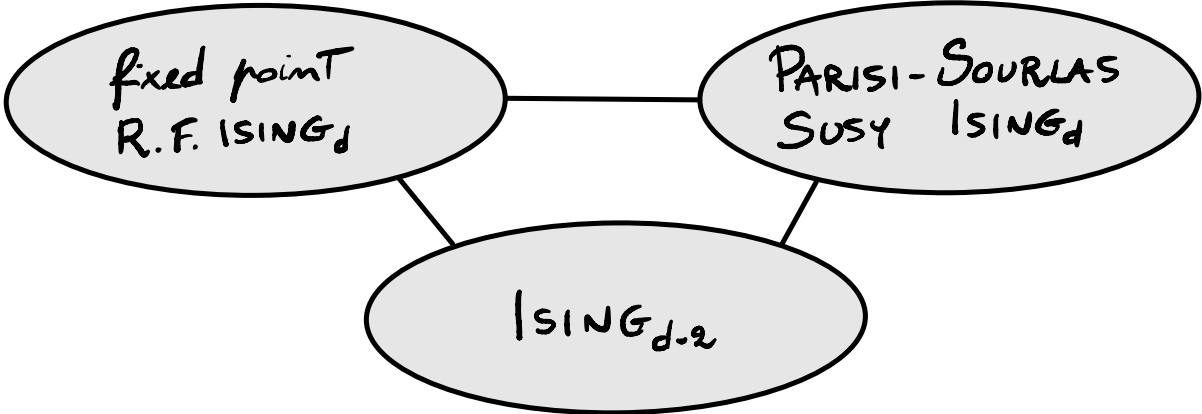
# MAIN IDEA

CFT<sub>d</sub>  $\xrightarrow{\text{UPLIFT}}$  CFT<sub>d+2</sub> with  
PARISI-SOURLAS (PS) SUSY



# CONTEXT

## P.S. CONJECTURE [PARISI, SOURLAS '79]



R.F.  $\phi^4$  (ISING)  $2 < d < 6$   
→  $d=3,4$  ✗  
→  $d=5$  ✓

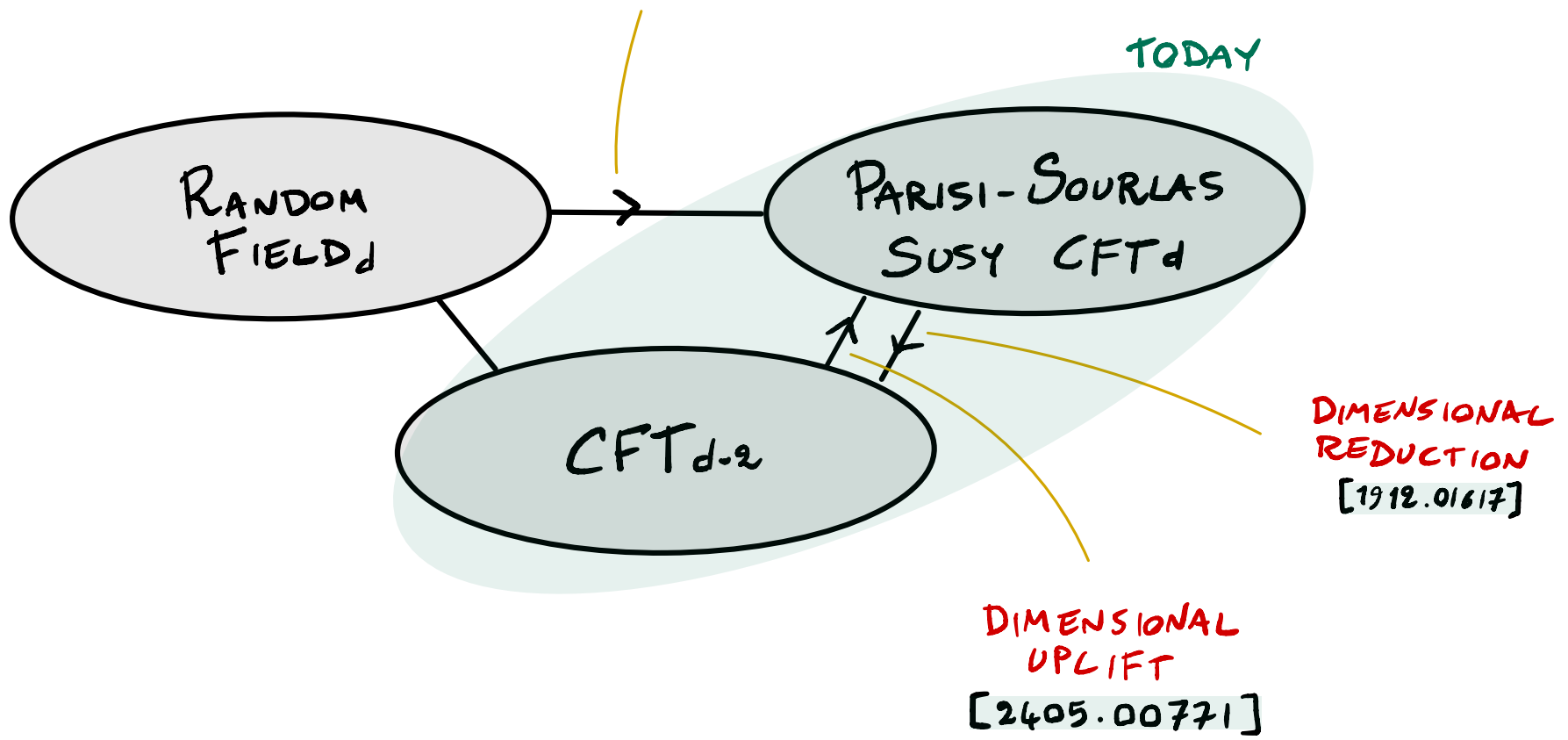
R.F.  $\phi^3$   $2 \leq d < 8$  →  $\forall d$  ✓

[FYTAS, MARTIN MAYOR, PARISI, PICCO, SOURLAS]

Q: when / why the conjecture works?

# UPSHOT of The RESULTS

[2009.10087]  
[2112.06942]  
[2203.12629] } EMERGENCE of SUSY  
PERTURB. R.G. }  
for  $RF\phi^4$ ,  $RF\phi^3$  } SUSY F.P.  
SOMETIMES UNSTABLE



# MOTIVATIONS for UPLIFT

- ITERATED UPLIFT  $CFT_d \rightarrow PS CFT_{d+2} \rightarrow PS^2 CFT_{d+4} \rightarrow \dots$
- defines SOLVABLE  $CFT_{d>2}$  by uplifting solvable  $CFT_2$   
[e.g. minimal models]
- UPLIFTED MODELS are PHYSICAL  
[e.g. UPLIFTED LEE-YANG MIN. MOD.  $\sim$  BRANCHED POLYMERS in  $d=4$ ]
- SUSY as a Tool  
[e.g. use SUSY of the  $PS CFT_{d+2}$  to solve a question of  $CFT_d$ ]
- KINEMATICAL RELATIONS  $CFT_d \leftrightarrow CFT_{d+2}$   
[e.g. relations on CONFORM. BLOCKS]

# OUTLINE

UPLIFT

PS CFT

PS REDUCT./UPLIFT

STRUCTURAL CHECK of UPLIFT

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APPLICATIONS

BOOTSTRAP GENER. FREE FIELD (GFF) THEORY

DEFINE SOLVABLE CFT<sub>d>2</sub>

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PS

CFT

# PS CFT<sub>d</sub>

SYMMETRIES:

$P^a$   
TRANSL.

$L^{ab}$   
ROTATION

$D$   
DILATION

$K^a$   
SPECIAL  
CONF. TR.

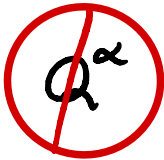
USUAL CFT<sub>d</sub>:  $a = 1, \dots, d \rightarrow SO(d+1, 1)$

P.S. CFT<sub>d</sub>:  $a = 1, \dots, d, \theta, \bar{\theta} \rightarrow OSP(d+1, 1|2) \subset SO(d+1, 1) \times \underbrace{Sp(2)}_{R\text{-SYM}}$

e.g.  $\left\{ \begin{aligned} P^\theta &= \partial^\theta \\ L^{\theta m} &= \theta \partial^m - x^m \partial^\theta \end{aligned} \right.$

$\theta, \bar{\theta}$ : ANTICOMMUTING  
SCALARS

NO SPINORIAL  
CHARGES



→ VIOLATES  
SPIN-STATISTICS





# OPERATORS

$$\mathcal{U}(y)$$

$$y^a = (x^r, \theta, \bar{\theta}) \in \mathbb{R}^{d|2}$$

SUPER PRIMARY  
LABELLED BY

$$\left\{ \begin{array}{l} \Delta \rightarrow D \\ \text{osp}(d|2)_{\text{SPIN}} \rightarrow L^{a\bar{b}} \end{array} \right.$$

COMPONENTS

$$\mathcal{U}^{a_1 \dots a_\ell}(y) = \mathcal{U}_0^{a_1 \dots a_\ell}(x) + \theta \mathcal{U}_{\bar{\theta}}^{a_1 \dots a_\ell}(x) + \bar{\theta} \mathcal{U}_{\theta}^{a_1 \dots a_\ell}(x) + \theta \bar{\theta} \mathcal{U}_{\theta \bar{\theta}}^{a_1 \dots a_\ell}(x)$$

$\Delta$                        $\Delta$                        $\Delta+1$                        $\Delta+1$                        $\Delta+2$

$\text{osp}(d|2)$  irreps.  $\longrightarrow$   $\text{SO}(d)$  irreps.

e.g.  $V^a = V^r \oplus V^\theta \oplus V^{\bar{\theta}}$

$\text{osp}(d|2)$  VECTOR =  $\text{SO}(d)$  VECTOR  $\oplus$  2  $\text{SO}(d)$  SCALARS

# CORRELATORS in PS CFT

$\gamma_{ij}^2 = (x_i - x_j)^2 - 2(\theta_i - \theta_j)(\bar{\theta}_i - \bar{\theta}_j)$ 
SUPERSPACE DISTANCE

$$\left\{ \begin{aligned}
 \langle \mathcal{V}(y_1) \mathcal{V}(y_2) \rangle &= \frac{1}{|\gamma_{12}|^{2\Delta}} \\
 \langle \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \rangle &= \frac{\lambda_{123}}{|\gamma_{12}|^{\Delta_1 + \Delta_2 - \Delta_3} |\gamma_{13}|^{\Delta_1 + \Delta_3 - \Delta_2} |\gamma_{23}|^{\Delta_2 + \Delta_3 - \Delta_1}} \\
 \langle \mathcal{V}(y_1) \mathcal{V}(y_2) \mathcal{V}(y_3) \mathcal{V}(y_4) \rangle &= \frac{1}{|\gamma_{12}|^{2\Delta} |\gamma_{34}|^{2\Delta}} f(u, v)
 \end{aligned} \right.$$

SUPER CROSS-RATIOS

$$\left\{ \begin{aligned}
 u &= \frac{\gamma_{12}^2 \gamma_{34}^2}{\gamma_{13}^2 \gamma_{24}^2} \\
 v &= \frac{\gamma_{14}^2 \gamma_{23}^2}{\gamma_{13}^2 \gamma_{24}^2}
 \end{aligned} \right.$$

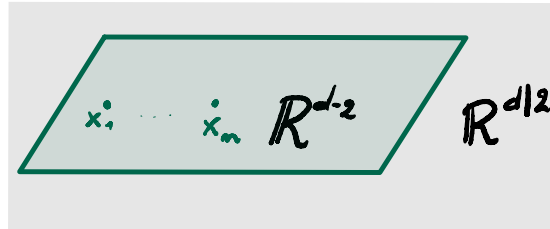
same as usual CFT with  $\gamma_{ij}^2 \rightarrow \gamma_{ij}^2$

PS

REDUCTION/UPLIFT

# DIMENSIONAL REDUCTION

PRESCRIPTION:



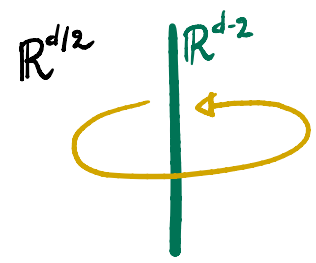
$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{\text{P.S. CFT}_d} = \langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle_{\text{CFT}_{d-2}}$$

Decoupling of  
 $\infty$  many ops

# DECOUPLING ARGUMENT

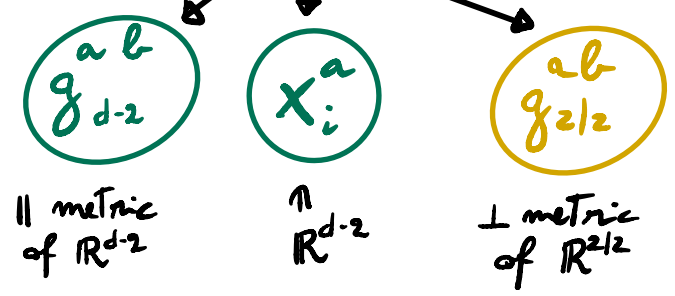
TRIVIAL DEFECT

$$\begin{array}{ccc}
 \text{Osp}(d+1, 1|2) & \supset & \text{SO}(d-1, 1) \times \text{Osp}(2|2) \\
 \text{PS CFT}_d & & \text{CFT}_{d-2} \quad \text{global sym}
 \end{array}$$



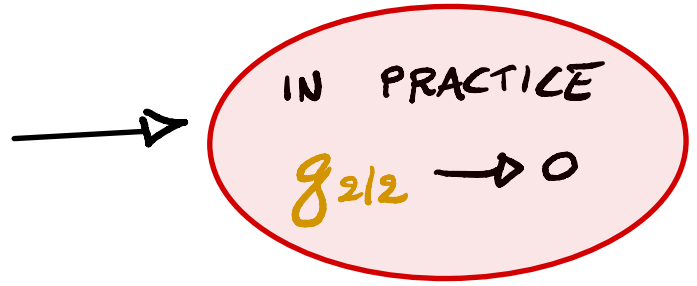
DIMENS. REDUCTION: CORRELATOR of  $\mathcal{O}_i$ :  $\text{Osp}(2|2)$ -SINGLET

$$\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_n(x_n) \rangle = f(\text{INVARIANTS BUILT OUT OF})$$



$\mathcal{I}$  invariant built out of  $g_{2|2}$

$$\begin{cases} g^{ab} & x_a y_b = 0 \\ \text{Tr} & g_{2|2} = 0 \end{cases}$$



# CONSEQUENCES of DECOUPLING

○ MAP  $\mathcal{O} \leftrightarrow \mathcal{O}$       e.g.  $\mathcal{O} \begin{cases} \mathcal{O}(x) \in \text{CFT}_{d-2} \\ (\lambda_{\perp}^2)^m \mathcal{O}(x) \text{ DECOUPLE} \end{cases}$

○  $\exists \mathbb{T}^{ab}$  in PS  $\text{CFT}_d \longrightarrow \exists \mathbb{T}^{m\nu}$  in  $\text{CFT}_{d-2}$

○  $\mathcal{O}$  OPE EXCHANGE  $\longleftrightarrow$   $\mathcal{O}$  OPE EXCHANGE

○  $G_{\Delta, \ell}^{(d)}$  =  $g_{\Delta, \ell}^{(d-2)}$       CONF. BLOCK

|  
SUPERCONF. BLOCK

# UPLIFT CFT

given  $CFT_{d-2} \longrightarrow$  Fully reconstruct PS  $CFT_d$ ? HARD ▽!

$\exists U$  s.t.  
 $U \longrightarrow 0$   
 NOT REALLY 1-1  
 $PS CFT_d \supset CFT_{d-2}$

however:  $CFT_{d-2} \longrightarrow$  a lot of exact info about PS  $CFT_d$  EASY

e.g. 4PT of SCALARS  $U_i$   $d \geq 4$

$$\langle U_1 U_2 U_3 U_4 \rangle_{d-2} = \langle U_1 U_2 U_3 U_4 \rangle_{d/2} \Big|_{\mathbb{R}^{d-2}} \xrightarrow{\text{SYMMETRY}} \langle U_1 U_2 U_3 U_4 \rangle_{d/2}$$

FULL EXACT RESULT

PRESCRIPTION FOR THE UPLIFT :

$x_{ij}^2 \longleftrightarrow y_{ij}^2$

A STRUCTURAL CHECK  
of PS UPLIFT



# STRUCTURAL CHECK of The UPLIFT

if UPLIFT WORKS  $\rightarrow$   $\left\{ \begin{array}{l} \text{all COMPONENTS must be} \\ \text{well defined CFT}_d \text{ observables} \end{array} \right.$

$$\langle UUUU \rangle_{d-2} \rightarrow \langle UUUU \rangle_{d/2} = \begin{array}{l} \text{COMPONENTS} \\ \langle U_0 U_0 U_0 U_0 \rangle_d \quad 1 \\ \langle U_0 U_{\bar{0}} U_0 U_0 \rangle_d \quad 2 \\ \vdots \\ \langle U_{\bar{0}\bar{0}} U_{\bar{0}\bar{0}} U_{\bar{0}\bar{0}} U_{\bar{0}\bar{0}} \rangle_d \quad 43 \end{array}$$

$$f(u, v)$$

$$f(u, v)$$

$$f_s(u, v) = D_s f(u, v)$$

○  $D_s$  : COMPUTED  $\forall s$  ✓ [e.g.  $D_1 = 1, D_2 = -(\Delta_1 + \Delta_2) + 2u\partial_u$ ]

○ CHECK  $f_s$  :  $\left\{ \begin{array}{l} \text{CROSSING} \\ \text{CONF. BLOCK DECOMP. in } d\text{-dimens.} \end{array} \right.$  ✓ ✓

# CONFORMAL BLOCKS RELATIONS

43 relations between  $\mathcal{G}^{(d-2)} \leftrightarrow \mathcal{G}^{(d)}$

$$\Rightarrow \mathcal{D}_S \mathcal{G}_{\Delta, \ell}^{(d-2)} = \sum_{\substack{i, j \\ \leq 5 \text{ TERMS}}} C_{i, j}^{(S)} \mathcal{G}_{\Delta+i, \ell+j}^{(d)} \quad S=1, \dots, 43$$

e.g. from  $\langle U_+ U_+ U_- U_- \rangle \Rightarrow \mathcal{D} \mathcal{G}_{\Delta, \ell}^{(d-2)} = 2(\Delta-1)\ell \mathcal{G}_{\Delta+1, \ell-1}^{(d)}$

[Dolan Osborn '11]

SUSY as a Tool

# SUSY as a TOOL $\leadsto$ BOOTSTRAPPING GFF

GENER. FREE FIELD (GFF): secretly constrained by PS SUSY!

$$\begin{array}{ccc} \text{GFF} & & \text{PS GFF} \\ \langle \phi \cdots \phi \rangle = \sum \langle \phi\phi \rangle \cdots \langle \phi\phi \rangle & \longrightarrow & \langle \Phi \cdots \Phi \rangle = \sum \langle \Phi\Phi \rangle \cdots \langle \Phi\Phi \rangle \end{array}$$

e.g.

$$\langle \phi\phi\phi\phi \rangle = f(u, v)$$

$$0 = \langle \Phi_{\theta\bar{\theta}} \Phi_0 \Phi_0 \Phi_0 \rangle = D_s f(u, v)$$

$$\langle \Phi_0 \Phi_{\theta\bar{\theta}} \rangle = 0$$

$$\Rightarrow \underline{f(u, v) \text{ satisfies } D_s f = 0}$$

# SOLVE The OPE COEFF.

$$0 = D_S f(u, v)$$

$$= D_S \sum_{\Delta e} a_{\Delta e} g_{\Delta e}^{(d-2)}(u, v)$$

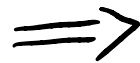
USE RELATIONS:

$$D_S g_{\Delta e}^{(d-2)}(u, v) = \sum_{ij} C_{ij}^{(s)} g_{\Delta_i, \ell+j}^{(d)}$$

$$= \sum_{\Delta e} a_{\Delta e} \sum_{ij} C_{ij}^{(s)} g_{\Delta_i, \ell+j}^{(d)}$$

$$= \sum_{\Delta' e'} [\sum_{ij} a_{c}] g_{\Delta' e'}^{(d)}$$

0



RECURRENCE  
RELATION for  $a_{\Delta e}$

SOLVE RECURRENCE

for  $\langle \phi^{\Delta_1} \phi^{\Delta_2} \phi^{\Delta_3} \rangle$

$$a_{n, \ell}^{n_2, n_3, n_3} = a_0^{n_2, n_3} \frac{\left(\frac{\beta_{12}-1}{2}\right)_\ell \left(\frac{\beta_{12}}{2}\right)_\ell \left(\frac{\Delta_\phi}{2}\right)_\ell \left(\frac{\Delta_\phi+1}{2}\right)_\ell}{\ell! \left(\frac{\beta_{12}-1}{4}\right)_\ell \left(\frac{\beta_{12}+1}{4}\right)_\ell} \times$$

$$\times \frac{(-d+\beta_{12}+3)_n \left(-\frac{d}{2}+\Delta_\phi+2\right)_n \left(\frac{\ell+\beta_{12}-1}{2}\right)_n \left(\frac{\ell+\beta_{12}}{2}\right)_n (\ell+\Delta_\phi)_n \left(-\frac{d}{2}+\ell+\beta_{12}+1\right)_n}{2^{4n} n! \left(\frac{d}{2}+\ell-1\right)_n \left(-\frac{d+\beta_{12}+3}{2}\right)_n \left(\ell+\frac{\beta_{12}-1}{2}\right)_n \left(\frac{-d+2\ell+2\beta_{12}+2}{4}\right)_n \left(\frac{-d+2\ell+2\beta_{12}+4}{4}\right)_n}$$

$$\beta_{12} = \Delta_1 + \Delta_2$$

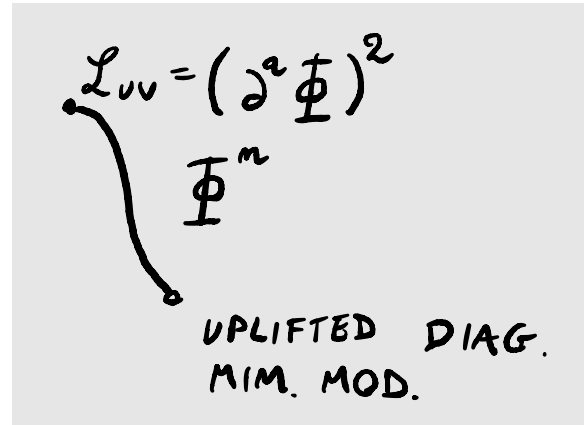
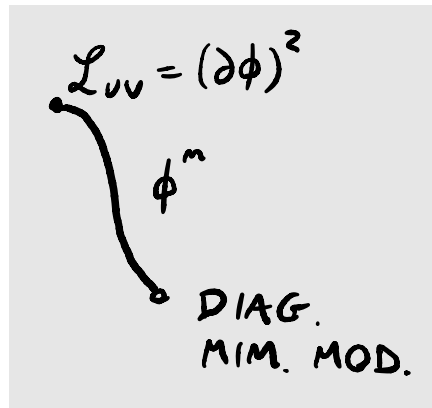
$$a_0^{m_2, m_3} = m_3! \binom{m_2+1}{\frac{m_2+1}{2}} \binom{1-m_2+2m_3}{2} \frac{m_2+1}{2}$$

UPLIFT MINIMAL MODELS

# UPLIFT MINIMAL MODELS

○ UPLIFT of DIAGONAL MIN. MODELS EXISTS

↳  $\mathcal{L}$  argument: all scalar  $\mathcal{L}$  can be uplifted



CAN UPLIFT The whole RG FLOW  
→ ALSO The IR FIXED POINT

# EXAMPLE: UPLIFT 2d ISING

ISING:  $\sigma, \epsilon, \mathbb{1}$  Virasoro primaries

4PT KNOWN

$$\langle \epsilon \epsilon \epsilon \epsilon \rangle = f_{\epsilon \epsilon \epsilon \epsilon}(u, v)$$

$$\langle \sigma \sigma \sigma \sigma \rangle = f_{\sigma \sigma \sigma \sigma}(u, v)$$

$$\langle \sigma \epsilon \epsilon \sigma \rangle = f_{\sigma \epsilon \epsilon \sigma}(u, v)$$

$$\langle \sigma \sigma \epsilon \epsilon \rangle = f_{\sigma \sigma \epsilon \epsilon}(u, v)$$



UPLIFTED 4PT  
KNOW EXACTLY



$$D_S f_{\epsilon \epsilon \epsilon \epsilon}$$

$$D_S f_{\sigma \sigma \sigma \sigma}$$

$$D_S f_{\sigma \epsilon \epsilon \sigma}$$

$$D_S f_{\sigma \sigma \epsilon \epsilon}$$

DECOMPOSE IN  
CONF. BLOCKS





$[\vec{S}]$	$D_{[\vec{S}]} f_{\ell\ell\ell\ell}$
[ ]	$v^{-1}(u^2 - uv - u + v^2 - v + 1)$
[1]	$4v^{-1}(u^2(v+1) - u^3 + u(v^2+1) - (v-1)^2(v+1))$
[2]	$4v^{-2}(u^2(v+1) - u^3 + u(v^2+1) - (v-1)^2(v+1))$
[3]	$4v^{-2}(u^2(v+1) - u^3 + u(v^2+1) - (v-1)^2(v+1))$
[4]	$4v^{-1}(u^2(v+1) - u^3 + u(v^2+1) - (v-1)^2(v+1))$
[12]	$2v^{-1}(u^2 - v^2 + v - 1)$
[34]	$2v^{-1}(u^2 - v^2 + v - 1)$
[1234]	$-4v^{-2}(u-v)((u-1)u + (v-1)v + 1)$
[1243]	$-4v^{-1}(-1+u)(1+u^2 - (1+u)v + v^2)$
[134]	$-8v^{-1}(-u^2(v+1) + u^3 + u(v^2+1) - (v-1)^2(v+1))$
[234]	$-8v^{-2}(-u^2(v+1) + u^3 + u(v^2+1) - (v-1)^2(v+1))$
[312]	$-8v^{-2}(-u^2(v+1) + u^3 + u(v^2+1) - (v-1)^2(v+1))$
[412]	$-8v^{-1}(-u^2(v+1) + u^3 + u(v^2+1) - (v-1)^2(v+1))$
[12]	$16v^{-2}(u^2(v^2+1) - u^3(v+1) + u^4 + u(-3v^3 + v^2 + v - 3) + 2(v-1)^2(v^2+1))$
[13]	$16v^{-2}(u^2(4v^2 + v + 1) - u^3(4v + 3) + 2u^4 + u(-4v^3 + v^2 - 1) + (v-1)^2(2v^2 + v + 1))$
[14]	$16v^{-1}((u^2+1)v^2 + (-3u^3 + u^2 + u - 3)v + 2(u-1)^2(u^2+1) - (u+1)v^3 + v^4)$
[23]	$16v^{-3}((u^2+1)v^2 + (-3u^3 + u^2 + u - 3)v + 2(u-1)^2(u^2+1) - (u+1)v^3 + v^4)$
[24]	$16v^{-2}(u^2(4v^2 + v + 1) - u^3(4v + 3) + 2u^4 + u(-4v^3 + v^2 - 1) + (v-1)^2(2v^2 + v + 1))$
[34]	$16v^{-2}(u^2(v^2+1) - u^3(v+1) + u^4 + u(-3v^3 + v^2 + v - 3) + 2(v-1)^2(v^2+1))$
[1234]	$-32v^{-2}(u^2(v^2+1) + u^3(v+1) - u^4 + u(-3v^3 + v^2 + v - 3) + 2(v-1)^2(v^2+1))$
[1234]	$-32v^{-2}(u^2(v^2+1) + u^3(v+1) - u^4 + u(-3v^3 + v^2 + v - 3) + 2(v-1)^2(v^2+1))$
[123]	$-256v^{-3}(u^2 - u(v+1) + (v-1)v + 1)(-u^2(v+1) + u^3 - u(v^2+1) + (v-1)^2(v+1))$
[124]	$-256v^{-2}(u^2 - u(v+1) + (v-1)v + 1)(-u^2(v+1) + u^3 - u(v^2+1) + (v-1)^2(v+1))$
[134]	$-256v^{-2}(u^2 - u(v+1) + (v-1)v + 1)(-u^2(v+1) + u^3 - u(v^2+1) + (v-1)^2(v+1))$
[234]	$-256v^{-3}(u^2 - u(v+1) + (v-1)v + 1)(-u^2(v+1) + u^3 - u(v^2+1) + (v-1)^2(v+1))$
⋮	⋮

43 CASES

$$\rightarrow = \sum_{\Delta \ell} a_{\Delta \ell} g_{\Delta \ell}^{(d=4)}(u, v)$$

$(\Delta, \ell)$	(0,0)	(2,0)	(2,2)	(4,0)	(4,2)	(4,4)	(6,0)
$a_{\Delta \ell}$	1	-1	4	1	$-\frac{2}{3}$	$\frac{8}{5}$	$-\frac{1}{6}$
$(\Delta, \ell)$	(6,2)	(6,4)	(6,6)	(8,0)	(8,2)	(8,4)	(8,6)
$a_{\Delta \ell}$	$\frac{2}{5}$	$-\frac{8}{35}$	$\frac{32}{63}$	$\frac{1}{60}$	$-\frac{2}{35}$	$\frac{8}{63}$	$-\frac{16}{231}$
$(\Delta, \ell)$	(8,8)	(10,0)	(10,2)	(10,4)	(10,6)	(10,8)	(10,10)
$a_{\Delta \ell}$	$\frac{64}{429}$	$-\frac{1}{700}$	$\frac{1}{189}$	$-\frac{4}{231}$	$\frac{16}{429}$	$-\frac{128}{6435}$	$\frac{512}{12155}$

...

### FEATURES:

- $a_{\Delta \ell}$  CAN BE NEGATIVE
- $\Delta, \ell$  BELOW UNITARITY BOUND  $\Delta \geq \ell + d - 2$   
 $\Delta \geq \frac{d}{2} - 1$
- $\infty$  - MANY CONSERVED HIGHER-SPIN CURRENTS  
 $\Delta = \ell + 2$

# SYMMETRIES

2d ISING  
GLOBAL CURRENTS

$$L_{-k}^{m_k} \dots L_{-2}^{m_2} \mathbb{1}$$



UPLIFTED  
GLOBAL CURRENTS

$$\mathbb{J}_{\{m_2, \dots, m_k\}}^{a_1, \dots, a_\ell}$$

# grows as  $e^{\sqrt{\ell}}$

each

$$\mathbb{J}_{\{m_2, \dots, m_k\}}^{a_1, \dots, a_\ell}$$



TOPOLOGICAL  
OPERATORS



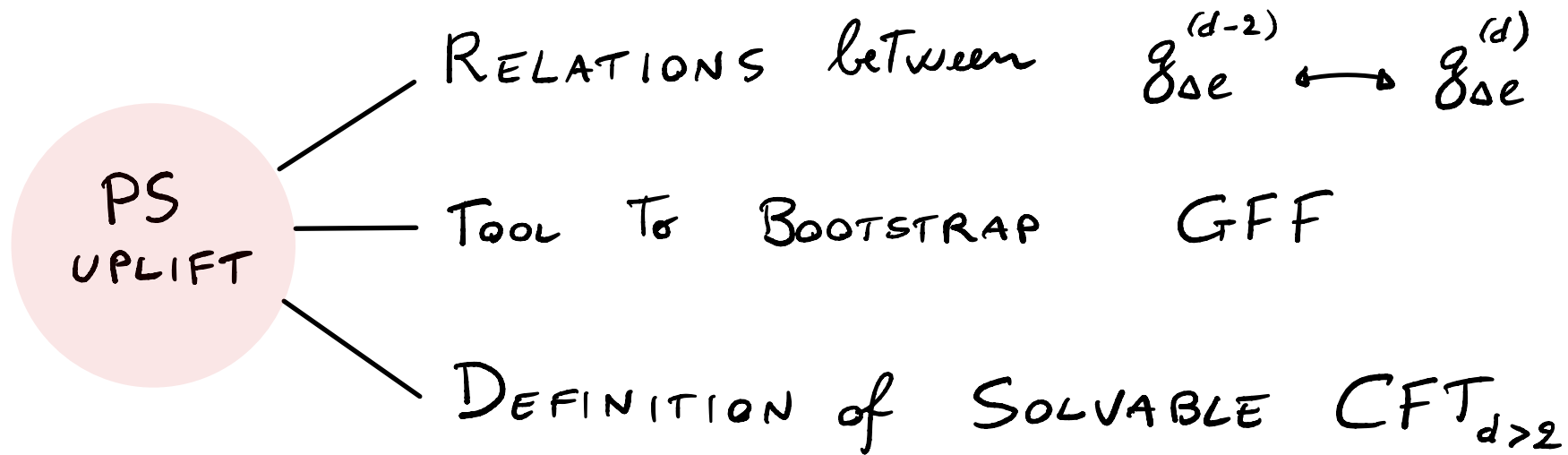
CHARGES

⇒ EXTENSION of VIRASORO  $\rightsquigarrow$   $T_0$   $d_0$

e.g.  $L_{-1}, \bar{L}_{-1}$  v.s.  $P^a = 1, \dots, 4, 0, \bar{0}$

# SUMMARY

EXPLAIN/CHECK PS REDUCTION/UPLIFT



# OUTLOOK

- EXTEND TO SPINNING CORRELATORS



- FULLY SOLVE UPLIFTED MIN. MODELS

- SUSY DEFORMATIONS of UPLIFTED CFT<sub>d</sub>

- PROVE/DISPROVE CONJECTURE:

GIVEN CFT<sub>d-2</sub>,  $\exists$  UNIQUE PS UPLIFT

Thanks