Warning

- QCD is a vast subject which cannot be covered in four lectures! I had to select some topics whose the main line is "QCD for LHC"
- The global outline will
  - Lecture I : Pre QCD, quark model, parton model
  - Lecture II : QCD as gauge theory
  - Lecture III : Renormalisation
  - Lecture IV : Soft/Collinear divergences, the QCD improved parton model
- I will not address the non perturbative regime of QCD, the low energy one as well as many other subjects....
- For each lecture, there will be some slides as well as some more detailed notes. They can be downloaded from this URL : `https://mydrive.lapth.cnrs.fr/s/rK9Qb6Nggfo78aT` They are labelled, for lecture X : note_cX.pdf and slide_cX.pdf where $X \in [I, II, III, IV]$

Outline

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1 Lecture I : The naive parton model

1.1 The hadrons are not elementary particles

Status of strong interaction

Hadrons particles sensitive to the strong interaction, ex. proton, neutron, $\cdots$
mesons : hadrons having integer spin baryons : hadrons with half-integer spin

Particle accelerators Before the 50', people thought that hadrons were elementary particles

With the coming of accelerators, hundred of hadrons have been discovered! (cf. Particle Data Book) Clearly not elementary...

1.2 The quark model

Isospin symmetry

A first attempt to classify The proton and the neutron undergo the same strong interaction, their masses are similar, only the electric charge distinguishes between them.

Isospin symmetry

$$\begin{array}{lll} p \equiv (I = 1/2\,;\; I_3 = +1/2) & \text{isospin "up",} & \text{proton} \\ n \equiv (I = 1/2\,;\; I_3 = -1/2) & \text{isospin "down",} & \text{neutron} \end{array}$$

Only for the couple proton–neutron!

The idea of the quark model is to classify the hadrons.
Among all the hadrons, some of them have "strange" properties: they have an "extremely" long live time. For example,

$$\begin{array}{ll} \Sigma^- \to n\,\pi^- & \tau \simeq 10^{-10}s \\ \Delta^- \to n\,\pi^- & \tau \simeq 10^{-23}s \end{array}$$

with $m_\Sigma \simeq m_\Delta$ or

$$\begin{array}{rl} K(m_K \simeq 500\text{MeV}) & \tau \simeq 10^{-8}s \\ \rho(m_\rho \simeq 770\text{MeV}) & \tau \simeq 10^{-23}s \end{array}$$

Those particles were produced by pair:

$$\sigma(\pi\ p \to K\ \Sigma) \simeq \sigma(\pi\ p \to \rho\ \Delta)$$

A new additive quantum number is introduced: strangeness This quantum number is conserved by the strong interaction as well as the electro-magnetic one. As an example:

$$\begin{array}{cc} s=0 & \pi, \rho, \cdots, \Delta \\ s=1 & K^+, K^0 \\ s=-1 & \Lambda, \Sigma^-, \Sigma^0 \end{array}$$

Note that an anti-particle has a strangeness of opposite sign with respect to the particle.

The quark model of Gell-Mann and Zweig

Three quarks are introduced with the following quantum numbers

quark	saveur	spin	I	I_3	S	B	Y	Q
u	up	$1/2$	$1/2$	$1/2$	0	$1/3$	$1/3$	$2/3$
d	down	$1/2$	$1/2$	$-1/2$	0	$1/3$	$1/3$	$-1/3$
s	étrange	$1/2$	0	0	-1	$1/3$	$-2/3$	$-1/3$

The baryonic quantum number B is additive and can take the values

- $B = 0$ for mesons (hadrons with integer spin),
- $B = 1$ for the baryons (hadrons with half integer spin $\frac{1}{2}$, $\frac{3}{2}$, ...)
- $B = -1$ for the antibaryons. It has been introduced to "explain" why the nucleon or more generally baryons do not decay in pions. There is also the **strangeness** S and the **hypercharge**, Y, which is not independent because it satisfies

$$Y = B + S$$

in such a way that with the choice $B = \frac{1}{3}$ for the quarks, the sum of the hypercharge of the quark triplet members vanishes. Finally, the electric charge is related to the other quantum numbers by

$$Q = I_3 + \frac{Y}{2}.$$

The Hadrons

They are made of **quarks** and **antiquarks** in such a way that their charge and their baryonic numbers have integer values.

The **mesons** which have a baryonic number equal to zero $(B = 0)$ are bound states quark–antiquark

$$M = (q_i\,\bar{q}_j) \qquad i, j = u, d, s...$$

The **baryons**, which have a baryonic quantum number equal to 1, are made of three quarks

$$B = (q_i q_j q_k) \qquad i, j, k = u, d, s...$$

All the knowns hadrons (at that time!) were arranged in the irreducible **representations** of $SU(3)_{\text{flavour}}$

Weaknesses

Despite its success : prediction of a new resonance $\Omega = (sss)$, the quark model has some weaknesses

- The symmetry described by the Lie group $SU(3)_{\text{flavour}}$ is not exact : $m_u \simeq m_d \neq m_s$
- Other quarks have been discovered : c, b and t. The symmetry group has to be extended to $SU(6)$ but huge mass difference $m_t/m_u \sim 10^4$
- No information on the dynamic! How the quarks interact between themselves?

Colour

they are several indications that the quarks have to carry a new quantum number : the colour. The first indication is the following. Let us consider, for example, the hadron$\Delta^{++} = (uuu)$ and more precisely the Δ^{++} in a spin state $s_z = \frac{3}{2}$ (each quarks has its spin up) this can be written in the quark model

$$\Delta^{++}(s_z = \frac{3}{2}) = (u^{\uparrow} u^{\uparrow} u^{\uparrow})$$

and the Δ^{++} wave function is **symmetric** when exchanging two quarks in contraction with the Fermi-Dirac statistic which requires that the wave function is

antisymmetric! To solve this problem, a new quantum number is introduced : the colour. Each quark occurs with three type of colour $i = R, G, B$, in such way that

$$u = \begin{pmatrix} u_R \\ u_G \\ u_B \end{pmatrix} \quad d = \begin{pmatrix} d_R \\ d_G \\ d_B \end{pmatrix} \quad s = \begin{pmatrix} s_R \\ s_G \\ s_B \end{pmatrix}$$

Colour

To this new quantum number is associated a **colour symmetry group** $\mathrm{SU}(3)$ (to be distinguished from $\mathrm{SU}(3)_{\text{flavour}}$). Each quark is a colour triplet and the hadrons are colour singlets (their wave functions are invariant under this group of transformation). In this way, the Δ^{++} wave function is

$$\begin{aligned} \Delta^{++} &= \frac{1}{\sqrt{6}}\,\epsilon_{ijk}\,u_i^\uparrow u_j^\uparrow u_k^\uparrow \\ &= \frac{1}{\sqrt{6}}\left(u_R^\uparrow u_G^\uparrow u_B^\uparrow - u_R^\uparrow u_B^\uparrow u_G^\uparrow + u_B^\uparrow u_R^\uparrow u_G^\uparrow - u_B^\uparrow u_G^\uparrow u_R^\uparrow + u_G^\uparrow u_B^\uparrow u_R^\uparrow - u_G^\uparrow u_R^\uparrow u_B^\uparrow\right) \end{aligned}$$

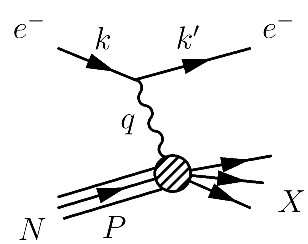
which is antisymmetric under the permutation of two elements. Note that the baryon wave function is constructed in such a way that it is totally antisymmetric in the colour space but totally symmetric with respect to the orbital momentum $\otimes$ spin $\otimes$ flavour.

The mesons are also colour singlet and their wave function, concerning the colour, is written

$$M = \frac{1}{\sqrt{3}}\sum_i^3 \bar{q}_i q'_i = \frac{1}{\sqrt{3}}\left(\bar{q}_R q'_R + \bar{q}_G q'_G + \bar{q}_B q'_B\right)$$

1.3 The parton model

The electron–nucleus scattering



In the laboratory frame (the frame in which the nucleon is at rest), the different external 4-momenta can be parametrised as follows

$$\begin{aligned} P &= (M, \vec{0}) \\ k &= (\omega, 0, 0, \omega) \\ k' &= (\omega', \omega' \sin\theta, 0, \omega' \cos\theta) \end{aligned}$$

we assume that the mass of the lepton is negligible with respect to the energy of the initial lepton ($\omega \gg m$).

By energy-momentum conservation, $q = k - k'$, its components are

$$q = (\omega - \omega', -\omega' \sin\theta, 0, \omega - \omega' \cos\theta)$$

The virtuality of the exchanged photon Q^2 is given by

$$\begin{aligned} Q^2 &= -q^2 \\ &= -((\omega - \omega')^2 - \omega'^2 \sin^2\theta - (\omega - \omega' \cos\theta)^2) \\ &= 4\,\omega\,\omega' \sin^2\frac{\theta}{2} > 0 \end{aligned}$$

The different scalar products are

$$\begin{aligned} P \cdot q &= M\,(\omega - \omega'), \\ P \cdot k &= M\,\omega \end{aligned}$$

Let us introduce some new variables :

$$\begin{aligned} \nu &\equiv \omega - \omega' \\ y &\equiv \frac{2\,P \cdot q}{2\,P \cdot k} = \frac{\nu}{\omega} \\ x &\equiv \frac{Q^2}{2\,P \cdot q} = \frac{Q^2}{2\,M\,\nu} \end{aligned}$$

The invariant mass of the hadronic final state $M_X^2 \equiv (P+q)^2$ is given

$$M_X^2 = M^2 + Q^2\,\frac{1-x}{x}$$

But $M_X^2 \geq M^2$, thus $0 \leq x \leq 1$. In addition, the variable y also belongs to $[0, 1]$.

The photon–nucleon coupling

The problem here is that the coupling of the photon to the nucleon is unknown. Indeed, the QED Feynman rules inform us about the coupling of a photon to a point like fermion and the nucleon is not point like, it has a size! Thus, the photon – nucleon coupling will have to be parametrised, but how? Let us start to split the squared matrix element into a leptonic tensor $L^{\mu\nu}$ and an hadronic one $W_{\mu\nu}$

$$\sum_{\text{spin}} |M|^2 = \frac{e^4}{Q^2}\, L^{\mu\nu}\, W_{\mu\nu}$$

The most general parametrisation in terms of the two independent 4-momenta P and q

$$\begin{aligned} W_{\mu\nu} = {} & V_1\, g_{\mu\nu} + V_2\, P_\mu P_\nu + V_3\, (q_\mu P_\nu + q_\nu P_\mu) + V_4\, (q_\mu P_\nu - q_\nu P_\mu) \\ & + V_5\, q_\mu q_\nu + V_6\, \epsilon_{\mu\nu\rho\sigma} P^\rho q^\sigma \end{aligned}$$

where the parameters V_i, $i = 1, \cdots, 6$ are some functions of Q^2, x and M^2.

Constraints on the V_i parameters

But this a **QED interaction** thus the hadronic tensor is expected to be **transverse**

$$q^\mu\, W_{\mu\nu} = q^\nu\, W_{\mu\nu} = 0$$

These two constrains give rise to a set of four equations

$$\begin{aligned} V_1 + P \cdot q\, (V_3 + V_4) + q^2\, V_5 &= 0 \\ V_2\, P \cdot q + q^2\, (V_3 - V_4) &= 0 \\ V_1 + P \cdot q\, (V_3 - V_4) + q^2\, V_5 &= 0 \\ V_2\, P \cdot q + q^2\, (V_3 + V_4) &= 0 \end{aligned}$$

Solving these equations leads to

$$\begin{aligned} V_4 &= 0 \\ V_5 &= -\frac{1}{q^2}\left(V_1 - V_2\, \frac{(P\cdot q)^2}{q^2}\right) \\ V_3 &= -V_2\, \frac{P \cdot q}{q^2} \end{aligned}$$

Note that the parameter V_6 is not constraint by the fact that the hadronic tensor is transverse because the coefficient in front of this parameter is always zero. But the contraction of the tensor in front of V_6 which is antisymmetric in $\mu\,\nu$ with the leptonic tensor which is symmetric in these indices is always zero! So there is no need to take care of it.

Thus the hadronic tensor can be expressed in terms of **two parameters** only : V_1 et V_2

$$W_{\mu\nu} = V_1\,\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + V_2\,\left(P_\mu - q_\mu\,\frac{P\cdot q}{q^2}\right)\,\left(P_\nu - q_\nu\,\frac{P\cdot q}{q^2}\right)$$

It is more convenient to introduce **two other parameters** W_1 and W_2 such that $W_1 = -V_2/(2\,M)$ and $W_2 = M/2\,V_2$.

The amplitude squared

The **leptonic tensor** can be easily obtained using the standard QED Feynman rules yielding

$$L^{\mu\nu} = 2\,(k^\mu k'^{\,\nu} + k^\nu k'^{\,\mu} - k\cdot k'\,g^{\mu\nu})$$

The contraction of the hadronic tensor with the leptonic one leads to

$$\begin{aligned} L^{\mu\nu}\,W_{\mu\nu} = 2\,\Bigg[2\,M\,W_1\,\left(2\,\frac{q\cdot k\,q\cdot k'}{q^2} + k\cdot k'\right) + 2\,\frac{W_2}{M}\,\Bigg(2\,k\cdot P\,k'\cdot P \\ -\,2\,k\cdot P\,k'\cdot q\,\frac{P\cdot q}{q^2} - 2\,k\cdot q\,k'\cdot P\,\frac{P\cdot q}{q^2} + 2\,k\cdot q\,k'\cdot q\,\left(\frac{P\cdot q}{q^2}\right)^2 \\ -k\cdot k'\,\left(P^2 - \frac{(P\cdot q)^2}{q^2}\right)\Bigg)\Bigg] \end{aligned}$$

This formula can be simplified using that

$$\begin{aligned} k\cdot k' &= \frac{Q^2}{2} \\ q\cdot k &= -\frac{Q^2}{2} \\ q\cdot k' &= \frac{Q^2}{2} \end{aligned}$$

this yields

The **contraction** of the leptonic and the hadronic tensors gives

$$L^{\mu\nu}\, W_{\mu\nu} = 2\, M\, \left[2\, Q^2\, W_1 + W_2\, \left(4\, \frac{k\cdot P\, k'\cdot P}{M^2} - Q^2 \right) \right]$$

In the **laboratory frame** :

$$L^{\mu\nu}\, W_{\mu\nu} = 8\, M\, \omega\, \omega'\, \left[2\, W_1\, \sin^2 \frac{\theta}{2} + W_2\, \cos^2 \frac{\theta}{2} \right]$$

The cross section

The cross section is given by

$$\begin{aligned} \sigma &= \frac{1}{4P\cdot k} \int \frac{d^3k'}{(2\,\pi)^3\, 2\,\omega'}\, \frac{d^4P_X}{(2\,\pi)^3}\, (2\pi)^4\, \delta^4(k+P-k'-P_X)\, \overline{\sum} |M|^2 \\ &= \frac{1}{4\,M\,\omega} \int \frac{d^3k'}{(2\,\pi)^3\, 2\,\omega'}\, 2\,\pi\, \frac{e^4}{Q^4}\, L^{\mu\nu}\, W_{\mu\nu} \\ &= 4\,\pi\, e^4 \int \frac{d^3k'}{(2\,\pi)^3\, 2\,\omega'}\, \frac{\omega'}{Q^4}\, \left[2\, W_1\, \sin^2 \frac{\theta}{2} + W_2\, \cos^2 \frac{\theta}{2} \right] \end{aligned}$$

Note that since the invariant mass M_X is not fixed, the measure d^4P_X must be used instead the traditional one $d^3P_X/(2\,P_X^0)$. Furthermore, using spherical coordinates $d^3k' = \omega'^2\, d\omega'\, dcos\theta\, d\phi$ but $L^{\mu\nu}\, W_{\mu\nu}$ does not depend on ϕ, thus the integration on this variable gives a factor $2\,\pi$.

The **differential cross section** is given by

$$\frac{d\sigma}{d\omega'\, d\cos\theta} = \frac{\pi\,\alpha^2}{2\,\omega^2\, sin^4\frac{\theta}{2}}\, \left[2\, W_1\, \sin^2 \frac{\theta}{2} + W_2\, \cos^2 \frac{\theta}{2} \right]$$

One can use also the variables Q^2 and ν instead of ω and ω', the new differential cross section becomes

$$\frac{d\sigma}{dQ^2\, d\nu} = \frac{4\,\pi\,\alpha^2}{Q^4}\, \frac{\omega-\nu}{\omega}\, \left[2\, W_1\, \sin^2 \frac{\theta}{2} + W_2\, \cos^2 \frac{\theta}{2} \right]$$

Remarks

- The **dynamic** of the interaction $\gamma^* N$ is encoded inside the functions W_1, W_2.

- The cross section **decreases** when ω or θ **increases**, or equivalently, when $Q,\ \nu$ increases. Since the cross section to be measured is very small, it requires high luminosity lepton beams to get some data at high $\omega,\ \theta \Leftrightarrow$ high $Q^2,\ \nu$

- at fixed initial energy (ω), modifying ω', θ, the variables x and Q^2 varies and $W_1(x, Q^2, M^2),\ W_2(x, Q^2, M^2)$ can be extracted from experiment.

- the functions W_1, W_2 have the dimension of the inverse of an energy (in GeV^{-1} for example), if the symbol $[\ldots]$ denotes the dimension in unit of energy of a quantity, we have that

$$\left[\frac{d\sigma}{dQ^2 d\nu}\right] = \left[\frac{1}{M^5}\right] = \left[\frac{W_1}{M^4}\right] = \left[\frac{W_2}{M^4}\right].$$

  MW_1, MW_2, νW_1 νW_2 are then dimensionless. Considering only the functions MW_1 and νW_2, which will play a role after, they can be expressed as functions of dimensionless variables, thus we may write

$$MW_1\left(x, \frac{M^2}{Q^2}\right) = \mathcal{F}_1(x, \frac{M^2}{Q^2}), \quad \nu W_2\left(x, \frac{M^2}{Q^2}\right) = \mathcal{F}_2(x, \frac{M^2}{Q^2}).$$

  where MW_1 and νW_2 are the historical notations and $\mathcal{F}_1$ and $\mathcal{F}_2$ the modern ones.

Results of the experiment

The experiment reveals **two important facts** :

1. $$\nu W_2\left(x, \frac{M^2}{Q^2}\right) \equiv \nu W_2(x)$$

   that it is to say, there is no explicit dependence on Q^2, inside the experimental error bars (see fig. 1). This is the property of **scale invariance**. This property is verified for small value of Q as small as the proton mass ($\sim$ 1GeV. If we have a model taking into account the radius R of the proton, we should expect a dependence of the type $\exp(-R^2Q^2)$ which is not the case. Everything happens as if the virtual photon is insensible to the proton size, in other words, the virtual photon which has a resolution power which is better than the size of the proton, couple to point like proton constituents, instead of the proton itself.

2. The relation

$$2MW_1(x) \equiv \frac{\nu W_2(x)}{x} = \frac{P.q\ W_2(x)}{Mx}$$

is satisfied experimentally (**Callan-Gross relation**). The last expression is simply the invariant form of $\nu W_2(x)/x$.

The SLAC experiment

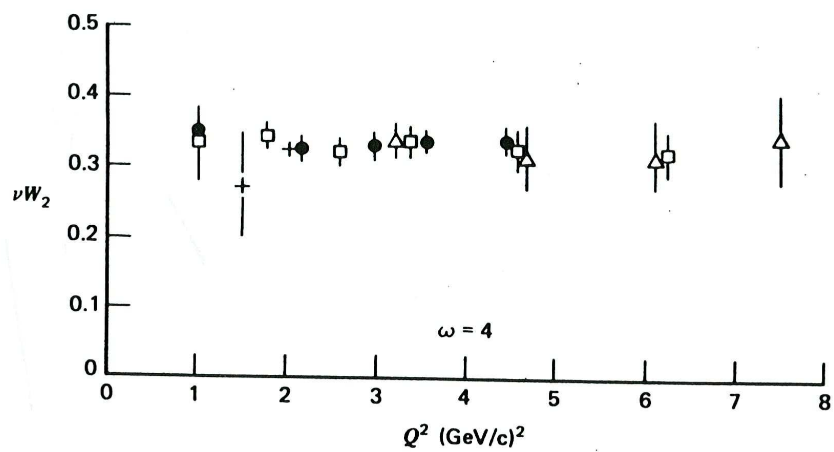


Figure 2.1: *Résultats expérimentaux de SLAC montrant l'invariance d'échelle de la fonction νW_2.*

Figure 1: *Results of the SLAC experiment in 1968, showing the scale invariance of the function νW_2 at the value of $x = 1/\omega = 0,25$.*

Lorentz invariant form

It is useful to study the differential cross section in a form which is explicitly Lorentz invariant. From preceding results, the cross section can be written as

$$\sigma = \frac{1}{4P.k}\frac{e^4}{(2\pi)^2}\int \frac{d^3k'}{2\omega'}\frac{2M}{(Q^2)^2}\left(2W_1Q^2 + W_2\left(\frac{4(k.P)(k'.P)}{M^2} - Q^2\right)\right).$$

Let us introduce "Mandelstam variables" $s,\ t,\ u$ of the electron proton scattering

$$\begin{array}{rclcc}
(P+k)^2 & = & s & 2k.P = s - M^2 & 2k.q = -Q^2 \\
(k-k')^2 & = & t = -Q^2 & & \\
(P-k')^2 & = & u & 2k'.P = M^2 - u & 2k'.q = \ Q^2,
\end{array} \tag{1}$$

where the mass of the lepton has been systematically neglected, the coefficient of W_2 becomes

$$-\left(\frac{(s-M^2)(u-M^2)}{M^2} + Q^2\right)$$

and, using the relation $s+t+u = M_X^2 + M^2$, $\Rightarrow s+u = Q^2/x + M^2 \sim Q^2/x$ when $Q^2 \to \infty$, we find

$$\frac{\omega' d\sigma}{d^3k'} = \frac{\alpha^2}{s}\frac{2}{Q^4}\left\{Q^2\,(2\,M\,W_1 - \frac{W_2}{M}\,\frac{P.q}{x}) + \frac{W_2}{2M}\,(s^2+u^2)\right\}. \tag{2}$$

Let us remark that the coefficient of the term in Q^2 inside the curly brackets is nothing else than the invariant form of $2\,M\,W_1 - \nu\,W_2/x$ which is zero (an experimental fact), we thus get

$$\frac{\omega' d\sigma^{\text{exp}}}{d^3k'} = \frac{\alpha^2}{s}\,\frac{(s^2+u^2)}{Q^4}\,\frac{W_2}{M}$$

which is the invariant cross section $e^-N \to e^-X$ ou $\mu^{\pm}N \to \mu^{\pm}X$ taking into account the experimental constraint (2).

Note: The invariant form is necessary because the parton model is not formulated in the laboratory frame but in a frame where $P^0 = E_{\text{nucléon}} \to \infty$.

1.4 The parton model in the deep inelastic

The parton model

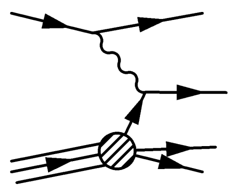
Feynman has proposed to consider the proton (or nucleon) as made of partons which are point like objects whose quantum numbers are a priori unknown (charge, spin, etc) but which have to form an object with a definite spin, charge, etc. Let us assume that the proton is made of parton of type i carrying a 4-momentum p_i.

$$p_i = y_i P$$

with $\sum_i y_i = 1$ and $P = (E,0,0,E) =$ the proton 4-momentum.

All the momentum of the proton is carried by the partons. We stand in a frame where the components of $P \to \infty$ and we will neglect the mass of the proton and the partons.

The fundamental postulate consists in describing the γ^*-hadron interaction in terms of γ^*-parton interaction since the resolving power of the virtual photon is large, it can "feel" constituents inside the proton. This can be symbolised by the following diagram where the virtual photon is absorbed by the parton of 4-momentum p_i



A very intuitive reasoning

We can use a very intuitive reasoning to compare the time of the electromagnetic interaction with the one characteristic of the interaction which binds the parton inside the proton.

- The "life time of the virtual photon" in its rest frame is $1/\sqrt{Q^2}$ and in the centre of mass frame photon-proton, it can be estimated to ($q_0/\sqrt{Q^2}$ being the boost factor to go from the virtual photon rest frame to centre of mass frame)

$$\Delta\tau_{\text{em}} \sim \frac{1}{\sqrt{Q^2}} \frac{q_0}{\sqrt{Q^2}} \sim \frac{1}{\sqrt{Q^2}},$$

  where a multiplicative factor depending on x of order 1 has been neglected $\Delta\tau_{\text{em}}$ can be considered as the time that the electromagnetic interaction lasts in the centre of mass frame γ^*-proton and $\Delta\tau_{\text{em}} \to 0$, when $\sqrt{Q}^2 \to \infty$;

- The characteristic time of the strong interaction which binds the partons inside the proton in the rest frame of the proton is $1/M$ (M, the proton mass is the only energy scale!); in the frame γ^*-proton, it is given by

$$\Delta\tau_{\text{strong int.}} \sim \frac{1}{M}\frac{E}{M} \sim \frac{\sqrt{Q^2}}{M^2}$$

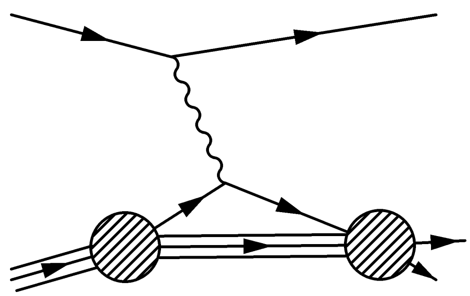
  .

If the characteristic times are compared, we get that

$$\Delta\tau_{\text{em}} \sim \frac{1}{\sqrt{Q^2}} \ll \Delta\tau_{\text{strong int.}} \sim \frac{\sqrt{Q^2}}{M^2}.$$

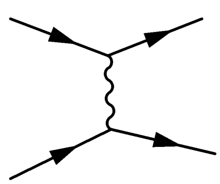
A very intuitive reasoning

One can thus assume that during the time $\Delta\tau_{\text{em}}$ that the $\gamma^* p_i$ interaction lasts, one can neglect the hadronic interaction which lasts on a much larger time scale : the partons seem free and independent. Well after the electromagnetic interaction the partons combine themselves to from a hadron with a probability one since no partons are observed in X system as shown by the following diagram



A very intuitive reasoning

The **confinement interactions** do not affect the **interaction** γ^***–parton**, we thus have to compute



and add in a **incoherent manner** the cross sections electron–parton to form the cross section electron–proton.

1.4.1 The electron–parton cross section

Squared amplitude γ^*–parton

The amplitude will be decomposed in the following way

$$| \mathcal{M} |^2_{ep_i} = \frac{q_i^2\, e^4}{Q^4}\, L^{\mu\nu} \underbrace{\widehat{W}_{\mu\nu}}_{\gamma^*\text{-parton int.}} .$$

where q_i is the parton charge in unit of proton one e. Let assume that the spin of the partons is $1/2$ (some partons must have some half integer spin because the proton has a spin $1/2$). Let us also assume that the interaction photon–parton takes the form $q_i e \gamma_\mu$, this leads to

$$\widehat{W}_{\mu\nu} = 2\,(p_{i\mu}\, p'_{i\nu} + p_{i\nu}\, p'_{i\mu} - p_i.p'_i\, g_{\mu\nu}),$$

where the final parton 4-momentum is $p'_i = p_i + q$. Neglecting the parton mass, we get

$$| \mathcal{M} |^2_{ep_i} = 8\, \frac{e^4\, q_i^2}{Q^4} \left((p_i.k)^2 + (p_i.k')^2 \right) = 2\, \frac{e^4\, q_i^2}{Q^4} \left(\hat{s}^2 + \hat{u}^2 \right),$$

with the partonic invariants $\hat{s} = (p_i + k)^2$ et $\hat{u} = (p_i - k')^2$.

The cross section γ^*–parton

The cross section will be

$$\begin{aligned} \hat{\sigma} &= \frac{1}{2\hat{s}} \frac{1}{(2\pi)^2} \int \frac{d^3k'}{2\omega'} \frac{d^3p'_i}{2p_i'^0}\, \delta^{(4)}(k + p_i - k' - p'_i)\ | \mathcal{M} |^2_{ep_i} && (3) \\ &= 2\, \frac{\alpha^2 q_i^2}{Q^4} \int \frac{d^3k'}{\omega'}\, \delta\left(2p_i.q - Q^2\right)\ \frac{\hat{s}^2 + \hat{u}^2}{\hat{s}}. && (4) \end{aligned}$$

At the **partonic level**, the differential cross section will have the following form

$$\frac{\omega' d\hat{\sigma}}{d^3k'} = 2\, \frac{\alpha^2 q_i^2}{Q^4}\, \frac{\hat{s}^2 + \hat{u}^2}{\hat{s}}\, \delta(2p_i.q - Q^2). \tag{5}$$

1.4.2 The electron–proton cross section

The cross section γ^*–proton

To get the hadronic cross section, the partonic cross sections will be summed incoherently

$$\frac{\omega' d\sigma}{d^3k'} = \sum_i \int_0^1 dy\, F_i(y)\, \frac{\omega' d\hat{\sigma}}{d^3k'}\bigg|_{p_i=yP}, \tag{6}$$

where the hadronic invariants are expressed in terms of the partonic invariants and the fraction y of 4-momentum of the parton in the proton

$\hat{s} = ys$ et $\hat{u} = yu$, $2p_i \cdot q = y\, 2P \cdot q$. The quantities $F_i(y)$ are the **number of partons** of type i carrying a **4-momentum fraction** y of the proton one. In our model, the hadronic cross section is given by

$$\begin{aligned}
\frac{\omega' d\sigma}{d^3k'} &= 2\, \frac{\alpha^2}{Q^4} \sum_i e_i^2 \int_0^1 \frac{dy}{y} F_i(y)\, y^2 \frac{s^2+u^2}{s}\, \delta(2yP.q - Q^2) \\
&= \underbrace{2\, \frac{\alpha^2}{s} \frac{s^2+u^2}{Q^4}}_{indep.\ y_i} \sum_i e_i^2 \int_0^1 dy\ F_i(y)\ y\ \delta(2yP.q - Q^2) \\
&= \frac{\alpha^2}{s} \frac{s^2+u^2}{Q^4} \sum_i q_i^2 \frac{x}{P.q} F_i(x)
\end{aligned} \tag{7}$$

with $y = Q^2/2Pq = x$.

Comparison

Comparing with the formula given the differential cross section electron–proton, one can identify

$$\frac{W_2}{M}(x, \frac{M^2}{Q^2}) = \sum_i q_i^2\, \frac{x}{P.q} F_i(x) \tag{8}$$

which is equivalent to (in the laboratory frame $P.q = M\nu$)

$$\frac{1}{x}\ \nu W_2(x, \frac{M^2}{Q^2}) = \sum_i e_i^2\ F_i(x) \tag{9}$$

and since there is no term in Q^2 in $\frac{\omega' d\sigma}{d^3k'}$ (see eq. (2)), we also recover

$$2\, M\, W_1(x, \frac{M^2}{Q^2}) = \frac{1}{x}\ \nu W_2(x). \tag{10}$$

Remarks

- The parton model well reproduces the "scale invariance", that is to say $W_2(x, M^2/Q^2) = W_2(x)$
- The variable $x = Q^2/2P.q$ has the following physical meaning : this is the normalised parton 4-momentum in the proton which scatters with the virtual photon
- $\nu\, W_2/x$ is the sum weighted by the squared charge q_i^2, of the probabilities of finding a parton of type i being scattered by the photon with a x fixed.
- The relation $2\, M\, W_1(x) = \nu\, W_2(x)/x$ is a direct consequence of the fact that the partons interacting with the virtual photon has a spin $1/2$.

Exercice : Show that for spin 0 partons (coupling to the γ given by $q_i\,(p_i + p_i')^\mu$) one has $W_1 \equiv 0$.

1.4.3 Partons $\equiv$ quarks + ...

Feynman partons I

It is tempting to identify the Feynman partons with the Gell-Mann and Zweig quarks and to assume that the proton and the neutron, in the deep inelastic scattering can be expressed as in the quark model by

$$\text{proton} = (uud)$$
$$\text{neutron} = (udd).$$

This is the "valence" quarks and we coined $u_v(x)$ and $d_v(x)$ the u and d quarks in the proton. The quantum number of the nucleon are carried by the "valence" quarks. Using isospin symmetry, it exists some relations between partonic densities in the proton (p) and the neutron (n)

$$\begin{aligned} F_u^p(x) &= F_d^n(x) = u_v(x) \\ F_d^p(x) &= F_u^n(x) = d_v(x) \end{aligned}$$

But the experimental results show that the proton and the neutron are more complex than this "3 quarks" model, they contain also antiquarks. They are called "sea" quarks whose distribution is $u_m(x) = \bar{u}_m(x),\; d_m(x) = \bar{d}_m(x)$.

The sum of the quantum number carried by this quarks is zero! Let us define

Feynman partons II

$$\begin{aligned} u(x) &= u_v(x) + u_m(x) \\ d(x) &= d_v(x) + d_m(x). \end{aligned}$$

Neglecting the role of s, c and b quarks, we can write following the parton model

$$\begin{aligned} \frac{1}{x}\nu W_2^{ep} &= \frac{4}{9}(u(x) + \bar{u}(x)) + \frac{1}{9}(d(x) + \bar{d}(x)) \\ \frac{1}{x}\nu W_2^{en} &= \frac{1}{9}(u(x) + \bar{u}(x)) + \frac{4}{9}(d(x) + \bar{d}(x)). \end{aligned} \tag{11}$$

where $u_m = \bar{u}_m = \bar{u}$, $d_m = \bar{d}_m = \bar{d}$.

Feynman partons III

Moreover, it is possible to measure experimentally using the deep inelastic scattering on an isoscalar target of deuterium $= p + n$

$$\frac{1}{x}\nu W_2^{ep+en} = \frac{5}{9}\left(u(x) + \bar{u}(x) + d(x) + \bar{d}(x)\right)$$

and thus to compute the integral

$$\frac{9}{5}\int_0^1 dx\ \nu W_2^{ep+en} = \int_0^1 dx\ x\ \left(u(x) + \bar{u}(x) + d(x) + \bar{d}(x)\right)$$

which measures the total momentum, normalised to the proton one, carried by all the quarks u, d, $\bar{u}$ and $\bar{d}$ in the nucleon. If the proton and the neutron were made of quarks and antiquarks only, then the right hand member should be equal to 1, but the result of this experiment is

$$\langle x \rangle_{q+\bar{q}} \simeq 0.45 \neq 1. \tag{12}$$

This means that the quarks carry half of the proton momentum, the other half is carried by neutral partons.

The parton model: general formulation I

• Under certain conditions that will precise below, we consider that a hadron is made of partons. We "work" in the infinite momentum frame. We have

$H = \{p_i\} \quad i = 1, \infty$
$P = \sum_i p_i$ where P, p_i are resp. the hadron and partons 4-momenta.

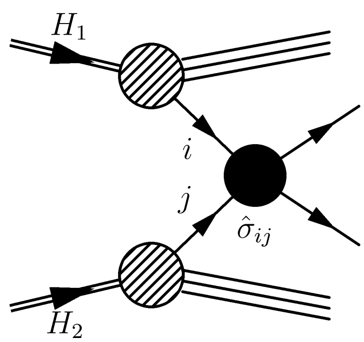
All the masses (hadron and partons) are neglected, so we can write

$$p_i = x_i P \text{ avec } \sum_i x_i = 1.$$

Partons are **point like** and their interactions is **ignored** inside the hadron

The parton model: general formulation II

• **Interactions** between hadrons reduce to interactions between partons following the diagram



$\hat{\sigma_{ij}}$ is the **"hard" cross section** describing the interactions between partons. The hadronic cross section is a **incoherent superposition** of partonic cross sections, the probabilities are added, not the amplitudes!

The parton model: general formulation III

We can the write

$$\sigma^{H_1 H_2} = \sum_{i,j} \int dx_1 dx_2 \, F_i^{H_1}(x_1) \, F_j^{H_2}(x_2) \, \alpha_s^p \, \hat{\sigma}_{ij}(x_1, x_2, s).$$

The function $F_i^H(x)$ is the the partonic density, that is to say it is proportional to the probability of finding in the hadron H a parton of type i carrying the fraction x of the 4-momentum of the hadron. This function is "scale invariant", that is to say independent of dimensional variables s, t, u. It contains the "long distance" effects (confinement) and its x dependence is not predict by the perturbative theory. The "short distance" effects are contained in the "hard" cross section $\hat{\sigma}_{ij}$. This factorisation between "long" and "short" distance is similar to the one realised in the more rigorous, but less general, approach of the operator product expansion.

• The parton model is a valid postulate when all the dimensional variables s, t, u are **large** compared to the proton mass ($\sim 1\,\mathrm{GeV}^2$).

What we learnt in lecture I

- The hadrons are not elementary particles. They can be classified following the representations of SU$(n)_{flavour}$ (n = number of flavours = number of quarks (spin 1/2 particles))
- These quarks have to carry a new quantum number : colour
- To describe the scale invariance (experimental fact), hadrons made of partons such that a probe (γ^*), with a large momentum transfer, interacts with them as if they were free
- partons $\in$ [quarks,...]
- No information on the dynamics between partons