Outline

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1 Lecture II : QCD

1.1 Global $SU(3)$ transformations

Lie group reminder I

Let us introduce very briefly some basic knowledges about groups. We will focus on Lie groups. In most of the cases we deal with, the action of the Lie group can be realised by matrices. More precisely, there are several kinds of Lie group realised by matrices:

1. The special linear groups over $\mathbb{R}$ or $\mathbb{C}$: SL$(n, \mathbb{R})$ or SL$(n, \mathbb{C})$ consisting of $n \times n$ matrices with determinant one with entries in $\mathbb{R}$ of $\mathbb{C}$.
2. The unitary groups and special unitary groups, U(n) and SU(n), consisting of $n \times n$ complex matrices satisfying $U^\dagger = U^{-1}$ (and also $\det(U) = 1$ in the case of SU(n))
3. The orthogonal groups and special orthogonal groups, O(n) and SO(n), consisting of $n \times n$ real matrices satisfying $R^{\mathrm{T}} = R^{-1}$ (and also $\det(R) = 1$ in the case of SO(n))

Lie group reminder : examples

A well known example is the rotations, say in 3-dimension, Since a rotation is linear : its action can be describe by a matrix; we also know that rotations preserve angles and lengths : $R^T R = 1$, and rotations preserve orientation (and area) : $\det R = 1$ thus the 3-dimensional rotations can be described by the Lie group $SO(3)$. In Quantum Field Theories, discrete symmetries ($\neq$ space–time symmetries) are described by SU(n) groups : gauge symmetries, classification of hadrons,...

Lie group reminder : Lie algebra

In addition to be a group, a Lie group is also a manifold : a topological space that locally resembles Euclidean space near each point, think about the area of sphere, to every points there is a plan tangent to the sphere at this point. Lie algebra Let us consider a Lie group, we find the identity of the group. Once we have done that, consider the tangent space at this identity, this flat space is the corresponding Lie algebra to the Lie group. The exponential map connects a point of the Lie algebra to a point of the Lie group. We want to reduce the Lie group to the Lie algebra But the Lie group is also a group! The group elements must satisfy some relations this implies the existence of Lie brackets : a specific operation that turns two tangent vectors into another tangent vector (way to construct the tangent vector of the composition of two elements of the Lie group $g \cdot h$ knowing the tangent vectors of g and h). This Lie brackets allow us to replicate the group multiplication entirely on the Lie algebra. So instead of working on curved space, as far as the group multiplication is concerned, we can work purely on a flat space. They must obey to certain relations because the group multiplication also obey to relations.

Let us construct a Lagrangian which is invariant under the $SU(3)$ gauge symmetry. Let us assume that there is a spin 1/2 field carrying a colour charge and this colour charge can take three values. Let us write the kinetic term for the spin 1/2 fields

Kinetic and mass terms for spin 1/2 fields

Introducing the **triplet** $\Psi(x)$ which contains the spin 1/2 field in three colour states

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \quad (1)$$

$$\mathcal{L} = \bar{\Psi}(x)\ (i\,\partial\!\!\!/ - m)\ \Psi(x) \quad (2)$$

To be more precise with the notation, since $\Psi(x)$ is a size 3 column matrix and $\bar{\Psi}(x)$ is a size 3 row matrix, the term in sandwich between $\bar{\Psi}(x)$ and $\Psi(x)$ must be a 3×3 matrix, thus we should have written

$$\mathcal{L} = \bar{\Psi}(x)\ (i\,\partial\!\!\!/ - m)\ \mathbb{1}_{3\times 3}\,\Psi(x) \tag{3}$$

Or write eq. (4) in terms of the matrix elements

$$\mathcal{L} = \sum_{i,j=1}^{3} \bar{\Psi}_j(x)\ (i\,\partial\!\!\!/ - m)\ \delta_{ij}\ \Psi_i(x) \tag{4}$$

The $SU(3)$ transformations

$\Psi(x)$ belongs to the **fundamental representation** of the $SU(3)$ Lie group, that is to say, if U is a group element, $\Psi(x)$ transforms as

$$\Psi(x) \to \Psi'(x) = U\,\Psi(x) \tag{5}$$

$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x)\,U^\dagger \tag{6}$$

Being an element of the $SU(3)$ Lie group, U is an unitary 3×3 matrix ($U^\dagger = U^{-1}$) having complex entries and with $\det(U) = 1$. Because of the exponential map, this matrix U can be parametrised as

$$U = \exp\left(i\,g\sum_{a=1}^{8}\alpha^a\,T^a\right) \quad \text{with} \quad \alpha^a \in \mathbb{R} \tag{7}$$

where the parameters α^a belongs to $\mathbb{R}$.

This said, it is easy to realise that the Lagrangian (4) is invariant under the transformations (5) and (6) because $\partial_\mu\,\Psi(x)$ transforms like $\Psi(x)$.

The generators of $SU(3)$

The 3×3 matrices T^a are the generators of the Lie algebra of $SU(3)$: the independent elementary transformations from which every transformations can be built. Indeed, in the Lie algebra, the generators can be thought as a "basis" for any elementary transformation.

The fact that a runs from 1 to 8 can be understood as follows: a $n\times n$ matrix V with complex entries has $2\,n^2$ unknowns. Since it is unitary $\sum_j V_{ij}\,V^\dagger_{jk} = \delta_{ik}$ gives n^2 constrains, in addition $\det(V) = 1$ gives 1 constrain more, thus the number of unknowns reduces to $n^2 - 1$.

Since the matrix U is unitary, this implies that the generators must be hermitian

$$U\,U^{\dagger} = 1 = \exp(i\,g\,\alpha^a\,T^a)\;\exp(-i\,g\,\alpha^b\,T^{b\,\dagger}) \to T^a = T^{a\,\dagger}$$

Notice that, from here, we will not write explicitly the sum over the colours but we will use the standard convention of repeated indices, that is to say $\alpha^a\,T^a \equiv \sum_{a=1}^{8} \alpha^a\,T^a$. In addition, using the well known relation

$$\det(\exp(V)) = \exp(\mathrm{Tr}[V])\,,$$

the fact that the determinant of U is equal to one implies that the generators T^a are traceless, indeed

$$\det(U) = 1 = \exp(i\,g\,\alpha^a\,\mathrm{Tr}[T^a]) \to \mathrm{Tr}[T^a] = 0$$

Furthermore, since the generators belongs to the Lie algebra, in order to be able to construct all the group transformations, they must obey to commutation relation (Lie brackets)

$$\left[T^a, T^b\right] = i\,f^{abc}\,T^c$$

where f^{abc} is the **structure constant** of the group. It is completely anti-symmetric under the permutation of any indices. It has real values.

1.2 Local $SU(3)$ transformations

Covariant derivative

Let us study now the case where the α^a are functions of the space time coordinates. It is easy to realise that the Lagrangian (4) is no more invariant under the following local transformations

$$\Psi(x) \to \Psi'(x) = U(x)\,\Psi(x) \tag{8}$$

$$\bar{\Psi}(x) \to \bar{\Psi}'(x) = \bar{\Psi}(x)\,U^{\dagger}(x) \tag{9}$$

Indeed, $\partial_\mu\,\Psi(x)$ does not transform any more like $\Psi(x)$ because the entries of the matrix U depends on x. To restore the invariance of the Lagrangian under local transformations, we have to introduce some new vector fields $A_\mu^b(x)$ whose role is to cancel the non wanted terms in the transformation of $\partial_\mu\,\Psi(x)$. In other words, we write a new Lagrangian

$$\mathcal{L} = \bar{\Psi}(x)\;(i\,\not{D} - m)\;\Psi(x) \tag{10}$$

with $D_\mu = \partial_\mu - i\, g\, A^b_\mu(x)\, T^b$, we then require that the transformation of the fields $A^b_\mu(x)$ are such that $D_\mu \Psi(x)$ transforms as $\Psi(x)$, that is to say

$$D_\mu \Psi(x) \to D'_\mu \Psi'(x) = U(x)\, D_\mu \Psi(x) \tag{11}$$

The transformation of the gauge fields

To lighten the formula, we introduce $\mathcal{A}_\mu(x) \equiv A^b_\mu(x)\, T^b$, the right-hand of eq. (11) can be written as

$$\begin{aligned} D'_\mu \Psi'(x) &= U(x)\, (\partial_\mu - i\, g\, \mathcal{A}_\mu(x))\, (U^{-1}(x)\, \Psi'(x)) \\ &= U(x)\, \partial_\mu\, (U^{-1}(x)\, \Psi'(x)) - i\, g\, U(x)\, \mathcal{A}_\mu(x)\, U^{-1}(x)\, \Psi'(x) \\ &= U(x)\, (\partial_\mu\, U^{-1}(x))\, \Psi'(x) + \partial_\mu\, \Psi'(x) - i\, g\, U(x)\, \mathcal{A}_\mu(x)\, U^{-1}(x)\, \Psi'(x) \\ &= \big(\partial_\mu - i\, \big[g\, U(x)\, \mathcal{A}_\mu(x)\, U^{-1}(x) + i\, U(x)\, (\partial_\mu\, U^{-1}(x))\big]\big)\; \Psi'(x) \\ &= \left(\partial_\mu - i\, g\, \left[U(x)\, \mathcal{A}_\mu(x)\, U^{-1}(x) + \frac{1}{i\, g}\, (\partial_\mu\, U(x))\, U^{-1}(x)\right]\right)\; \Psi'(x) \end{aligned} \tag{12}$$

But $D'_\mu = \partial_\mu - i\, g\, \mathcal{A}'_\mu(x)$, thus, to have the equality in eq. (12), the field $\mathcal{A}_\mu(x)$ must transform as

$$\mathcal{A}'_\mu(x) = U(x)\, \mathcal{A}_\mu(x)\, U^{-1}(x) + \frac{1}{i\, g}\, (\partial_\mu\, U(x))\, U^{-1}(x) \tag{13}$$

Note that the fields $A^b_\mu(x)$ belongs to the adjoint representation of $SU(3)$ (the octet representation).

Kinetic term for the gauge field I

Now, to consider the fields $A^b_\mu(x)$ as physical ones, we have to introduce a kinetic term which must be invariant under local transformations of $SU(3)$. In order to do that, we will start from a guess inspired by QED, let us form the tensor

$$F_{\mu\nu}(x) = \partial_\mu\, \mathcal{A}_\nu(x) - \partial_\nu\, \mathcal{A}_\mu(x)$$

and let us see how this tensor transforms under the local gauge transformations. We start from

$$
\begin{aligned}
F'_{\mu\nu}(x) &= \partial_\mu\, \mathcal{A}'_\nu(x) - \partial_\nu\, \mathcal{A}'_\mu(x) \\
&= \partial_\mu \left\{ \partial_\nu - i\, g \left[U(x)\, \mathcal{A}_\nu(x)\, U^{-1}(x) + \frac{1}{i\, g}\, (\partial_\nu\, U(x))\, U^{-1}(x) \right] \right\} \\
&\qquad - (\mu \leftrightarrow \nu) \\
&= (\partial_\mu\, U(x))\, \mathcal{A}_\nu(x)\, U^{-1}(x) + U(x)\, (\partial_\mu\, \mathcal{A}_\nu(x))\, U^{-1}(x) \\
&\quad + U(x)\, \mathcal{A}_\nu(x)\, (\partial_\mu\, U^{-1}(x)) + \frac{1}{i\, g}\, (\partial_\mu \partial_\nu\, U(x))\, U^{-1}(x) \\
&\quad + \frac{1}{i\, g}\, (\partial_\nu\, U(x))\, (\partial_\mu\, U^{-1}(x)) - (\mu \leftrightarrow \nu)
\end{aligned} \tag{14}
$$

Inserting $U^{-1}(x)\, U(x)$ in the first, the third and the fifth term and using that $U(x)\, (\partial_\mu\, U^{-1}(x)) = -(\partial_\mu\, U(x))\, U^{-1}(x)$ leads to

$$
\begin{aligned}
F'_{\mu\nu}(x) &= (\partial_\mu\, U(x))\, U^{-1}(x)\, U(x)\, \mathcal{A}_\nu(x)\, U^{-1}(x) + U(x)\, (\partial_\mu\, \mathcal{A}_\nu(x))\, U^{-1}(x) \\
&\quad - U(x)\, \mathcal{A}_\nu(x)\, U^{-1}(x)\, (\partial_\mu\, U(x))\, U^{-1}(x) + \frac{1}{i\, g}\, (\partial_\mu \partial_\nu\, U(x))\, U^{-1}(x) \\
&\quad - \frac{1}{i\, g}\, (\partial_\nu\, U(x))\, U^{-1}(x)\, (\partial_\mu\, U(x))\, U^{-1}(x) - (\mu \leftrightarrow \nu)
\end{aligned} \tag{15}
$$

The right hand side of the eq. (15) can be writing as

$$
\begin{aligned}
F'_{\mu\nu}(x) &= U(x)\, F_{\mu\nu}(x)\, U^{-1}(x) + ((\partial_\mu\, U(x))\, U^{-1}(x))\, (U(x)\, \mathcal{A}_\nu(x)\, U^{-1}(x)) \\
&\quad - ((\partial_\nu\, U(x))\, U^{-1}(x))\, (U(x)\, \mathcal{A}_\mu(x)\, U^{-1}(x)) \\
&\quad - (U(x)\, \mathcal{A}_\nu(x)\, U^{-1}(x))\, ((\partial_\mu\, U(x))\, U^{-1}(x)) \\
&\quad + (U(x)\, \mathcal{A}_\mu(x)\, U^{-1}(x))\, ((\partial_\nu\, U(x))\, U^{-1}(x)) \\
&\quad - \frac{1}{i\, g}\, ((\partial_\nu\, U(x))\, U^{-1}(x))\, ((\partial_\mu\, U(x))\, U^{-1}(x)) \\
&\quad + \frac{1}{i\, g}\, ((\partial_\mu\, U(x))\, U^{-1}(x))\, ((\partial_\nu\, U(x))\, U^{-1}(x))
\end{aligned} \tag{16}
$$

Now, it is easy to realise that the eq. (16) can be written in terms of commutators in the following form

$$
\begin{aligned}
F'_{\mu\nu}(x) = U(x)\,F_{\mu\nu}(x)\,U^{-1}(x) &+ \left[(\partial_\mu\,U(x))\,U^{-1}(x), U(x)\,\mathcal{A}_\nu(x)\,U^{-1}(x)\right] \\
&+ \left[U(x)\,\mathcal{A}_\mu(x)\,U^{-1}(x), (\partial_\nu\,U(x))\,U^{-1}(x)\right] \\
&+ \left[(\partial_\mu\,U(x))\,U^{-1}(x), (\partial_\nu\,U(x))\,U^{-1}(x)\right] \\
= U(x)\,F_{\mu\nu}(x)\,U^{-1}(x) &+ i\,g\,\left[\frac{1}{i\,g}\,(\partial_\mu\,U(x))\,U^{-1}(x), U(x)\,\mathcal{A}_\nu(x)\,U^{-1}(x)\right] \\
&+ i\,g\,\left[U(x)\,\mathcal{A}_\mu(x)\,U^{-1}(x), \frac{1}{i\,g}\,(\partial_\nu\,U(x))\,U^{-1}(x)\right] \\
&+ i\,g\,\left[\frac{1}{i\,g}\,(\partial_\mu\,U(x))\,U^{-1}(x), \frac{1}{i\,g}\,(\partial_\nu\,U(x))\,U^{-1}(x)\right]
\end{aligned} \tag{17}
$$

Using the linearity of the commutators yields

$$
\begin{aligned}
F'_{\mu\nu}(x) &= U(x)\,F_{\mu\nu}(x)\,U^{-1}(x) \\
&\quad + i\,g\,\left[U(x)\,\mathcal{A}_\mu(x)\,U^{-1}(x) + \frac{1}{i\,g}\,U(x)\,\mathcal{A}_\nu(x)\,U^{-1}(x)\right. \\
&\qquad \left., U(x)\,\mathcal{A}_\nu(x)\,U^{-1}(x) + \frac{1}{i\,g}\,U(x)\,\mathcal{A}_\mu(x)\,U^{-1}(x)\right] \\
&\quad - i\,g\,U(x)\,\left[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\right]\,U^{-1}(x) \\
&= U(x)\,\left(F_{\mu\nu}(x) - i\,g\,\left[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\right]\right)\,U^{-1}(x) + i\,g\,\left[\mathcal{A}'_\mu(x), \mathcal{A}'_\nu(x)\right]
\end{aligned} \tag{18}
$$

Equivalently, eq. (18) can be written as

$$
F'_{\mu\nu}(x) - i\,g\,\left[\mathcal{A}'_\mu(x), \mathcal{A}'_\nu(x)\right] = U(x)\,\left(F_{\mu\nu}(x) - i\,g\,\left[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\right]\right)\,U^{-1}(x) \tag{19}
$$

To sum up this long computation, we have shown that the tensor $\mathcal{G}_{\mu\nu}(x) \equiv F_{\mu\nu}(x) - i\,g\,\left[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\right]$ transforms under the local gauge transformations as

$$
\mathcal{G}'_{\mu\nu}(x) = U(x)\,\mathcal{G}_{\mu\nu}(x)\,U^{-1}(x) \tag{20}
$$

This tensor is said to be covariant because it transforms as D_μ, indeed,

$$
D'_\mu\,\Psi'(x) \to U(x)\,D_\mu\,\Psi(x)
$$

thus

$$
D'_\mu \to U(x)\,D_\mu\,U^{-1}(x)
$$

Kinetic term for the gauge field II

The sought kinetic term will be expressed in terms of the tensor $\mathcal{G}_{\mu\nu}(x)$

$$\mathcal{L}_c = -\frac{1}{2}\,\text{Tr}\,[\mathcal{G}_{\mu\nu}(x)\,\mathcal{G}^{\mu\nu}(x)] \tag{21}$$

Note that the appearance of the trace in eq. (21) comes from the fact that the kinetic terms must be a scalar with respect to the colour indices (a colour singlet), remember that $\mathcal{G}_{\mu\nu}$ is a 3×3 matrix. This kinetic term is invariant under the local gauge transformations, indeed

$$\begin{aligned}\text{Tr}\left[\mathcal{G}'_{\mu\nu}(x)\,\mathcal{G}'^{\,\mu\nu}(x)\right] &= \text{Tr}\left[U(x)\,\mathcal{G}_{\mu\nu}(x)\,U^{-1}(x)\,U(x)\,\mathcal{G}^{\mu\nu}(x)\,U^{-1}(x)\right]\\ &= \text{Tr}\left[U(x)\,\mathcal{G}_{\mu\nu}(x)\,\mathcal{G}^{\mu\nu}(x)\,U^{-1}(x)\right]\\ &= \text{Tr}\left[U^{-1}(x)\,U(x)\,\mathcal{G}_{\mu\nu}(x)\,\mathcal{G}^{\mu\nu}(x)\right]\\ &= \text{Tr}\,[\mathcal{G}_{\mu\nu}(x)\,\mathcal{G}^{\mu\nu}(x)]\end{aligned} \tag{22}$$

where we have used that the trace of product of matrices is cyclic to prove the gauge invariance. This property shows why the trace operation has been used to build a colour singlet term. Let us know come back to the notation where the colour indices appear explicitly

$$\begin{aligned}\mathcal{G}_{\mu\nu}(x) &= F_{\mu\nu}(x) - i\,g\,\left[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\right]\\ &= \partial_\mu\,\mathcal{A}_\nu(x) - \partial_\nu\,\mathcal{A}_\mu(x) - i\,g\,\left[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\right]\\ &= \partial_\mu\,A^a_\nu(x)\,T^a - \partial_\nu\,A^a_\mu(x)\,T^a - i\,g\,A^b_\mu(x)\,A^c_\nu(x)\,\left[T^b, T^c\right]\\ &= \left(\partial_\mu\,A^a_\nu(x) - \partial_\nu\,A^a_\mu(x)\right)\,T^a + g\,f^{bcd}\,A^b_\mu(x)\,A^c_\nu(x)\,T^d\\ &\equiv G^a_{\mu\nu}\,T^a\end{aligned} \tag{23}$$

where

$$G^a_{\mu\nu} = \partial_\mu\,A^a_\nu(x) - \partial_\nu\,A^a_\mu(x) + g\,f^{abc}\,A^b_\mu(x)\,A^c_\nu(x) \tag{24}$$

Kinetic term for the gauge field III

Thus, in this notation, the kinetic term reads

$$\mathcal{L}_c = -\frac{1}{2}\,G^a_{\mu\nu}(x)\,G^{b\,\mu\nu}(x)\,\text{Tr}\left[T^a\,T^b\right] \tag{25}$$

but it can be shown[1] that $\text{Tr}[T^a\,T^b] = 1/2\,\delta^{ab}$, thus eq. (26) becomes

$$\mathcal{L}_c = -\frac{1}{4}\,G^a_{\mu\nu}(x)\,G^{a\,\mu\nu}(x) \tag{26}$$

[1] See note on the colour matrices

Note that each $G^a_{\mu\nu}(x)$ by itself is not gauge invariant, it is the sum over the colour indices which is gauge invariant.

The total Lagrangian

We have shown that the following Lagrangian which includes kinetic terms for vector fields and spin 1/2 fields as well as interaction terms between both is invariant under local gauge transformations of the $SU(3)$ group.

$$\mathcal{L} = -\frac{1}{4}\, G^a_{\mu\nu}(x)\, G^{a\,\mu\nu}(x) + \bar{\Psi}_i(x)\, \left(i\, \not{D}_{ij} - m\, \delta_{ij}\right)\, \Psi_j(x) \tag{27}$$

To make connections with the literature where, very often, the gauge invariant Lagrangian is built starting from infinitesimal transformations, we will present them. It is easy to get them by considering that the parameters α^a are much smaller than one whatever the value of a. In other words, we consider transformations of the type

$$U(x) \simeq 1 + i\, g\, \alpha^a\, T^a \tag{28}$$

Under these kind of transformations, the variation of the different fields are given by

$$\delta\Psi_i(x) = i\, g\, \alpha^a(x)\, \left(T^a\right)_{ij}\, \Psi_j(x) \tag{29}$$

$$\delta\bar{\Psi}_i(x) = -i\, g\, \bar{\Psi}_j(x)\, \alpha^a(x)\, \left(T^a\right)_{ji} \tag{30}$$

$$\delta\left(D_\mu \Psi(x)\right)_i = i\, g\, \alpha^a(x)\, \left(T^a\right)_{ik}\, \left(D_\mu\right)_{kj}\, \Psi_j(x) \tag{31}$$

$$\delta A^a_\mu(x) = \left(D_\mu\right)^{ab}\, \alpha^b(x) \tag{32}$$

$$\delta G^a_{\mu\nu}(x) = g\, f^{abc}\, G^b_{\mu\nu}(x)\, \alpha^c(x) \tag{33}$$

The covariant derivative in the fundamental representation

where $\left(D_\mu\right)_{ij} = \partial_\mu\, \delta_{ij} - i\, g\, A^a_\mu(x)\, \left(T^a\right)_{ij}$ is the covariant derivative in the fundamental representation while $\left(D_\mu\right)^{ab} = \partial_\mu\, \delta^{ab} - g\, f^{abc}\, A^c_\mu(x)$ is the covariant derivative in the adjoint representation. To understand from where comes this last relation, we have to remind that the generators of the $SU(3)$ group in the adjoint representation have the following expression

$$\left(\mathcal{T}^c\right)_{ab} = -i\, f^{cab} = -i\, f^{abc} \tag{34}$$

This is a general result of the Lie group, obviously the generators must obey to the commutation rule whatever their representations are, that is to say

$$\left[\mathcal{T}^a, \mathcal{T}^b\right] = i\, f^{abc}\, \mathcal{T}^c \tag{35}$$

This relation is insured by the Jacobi identity which translates the associativity of the group multiplication

$$\left[T^a, \left[T^b, T^c\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] + \left[T^b, \left[T^c, T^a\right]\right] = 0 \tag{36}$$

or

$$f^{abd}\, f^{cde} + f^{bcd}\, f^{ade} + f^{cad}\, f^{bde} = 0 \tag{37}$$

The covariant derivative reads in the **adjoint** representation

$$\begin{aligned}(D_\mu)^{ab} &= \partial_\mu\, \delta^{ab} - i\, g\, \left(\mathcal{T}^c\right)_{ab}\, A^c_\mu(x) \\ &= \partial_\mu\, \delta^{ab} - g\, f^{abc}\, A^c_\mu(x)\end{aligned} \tag{38}$$

Quantum level

We built a Lagrangian which is invariant under local $SU(3)$ transformations, it describes the propagation of spin 1/2 fields as well as gauge fields and the interactions among them. At the classical level, everything is fine, but since the gauge fields are massless (we cannot built a mass term for the gauge field which is invariant under the local $SU(3)$ transformations), the quantification of this field is not straightforward because of the gauge constraint.

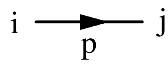
$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}\, G^a_{\mu\nu}(x)\, G^{a\,\mu\nu}(x) + \bar{\psi}_i(x)\, \left(i\, \not{D}_{ij} - m\, \delta_{ij}\right)\, \psi_j(x) \\ &-\frac{1}{2\,\xi}\, \left(\partial^\mu\, A^a_\mu(x)\right)^2 + \left(\partial^\mu \eta^{a\,\star}(x)\right)\, D^{ab}_\mu\, \eta^b(x)\end{aligned} \tag{39}$$

$-\frac{1}{2\,\xi}\, \left(\partial^\mu\, A^a_\mu(x)\right)^2$: **gauge fixing term**, whatever the way to quantify the field $A^a_\mu(x)$, some problems show up due to the gauge freedom. Need to fix the gauge : covariant gauge $\partial^\mu\, A^a_\mu(x) = 0$

$\left(\partial^\mu \eta^{a\,\star}(x)\right)\, D^{ab}_\mu\, \eta^b(x)$: **ghost term** the price to pay for using covariant gauge, the propagator propagates non physical polarisation states! Introduce "strange" fields $\eta(x)$ (ghost : scalar field which obeys to Fermi-Dirac statistic!) whose role is to cancel these spurious polarisation states.

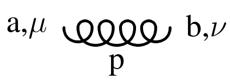
Feynman rules I

Quark propagator:



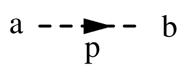
$$\frac{i\,\delta^{ij}\,(\not{p}+m)}{p^2-m^2+i\,\lambda}$$

Gluon propagator:



$$\frac{-i\,\delta^{ab}}{p^2+i\,\lambda}\left(g^{\mu\,\nu}-(1-\xi)\,\frac{p^{\mu}\,p^{\nu}}{p^2+i\,\lambda}\right)$$

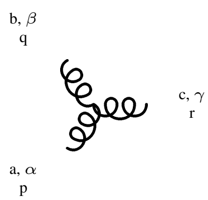
Ghost propagator:



$$\frac{i\,\delta^{ab}}{p^2+i\,\lambda}$$

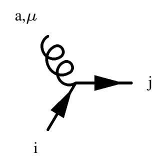
Feynman rules II

Vertex gluon-gluon-gluon (**all momentum are incoming**)



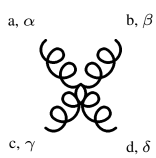
$$-g\,f^{abc}\left[g^{\alpha\,\beta}\,(p-q)^{\gamma} + g^{\beta\,\gamma}\,(q-r)^{\alpha} + g^{\gamma\,\alpha}\,(r-p)^{\beta}\right]$$

Vertex quark-quark-gluon



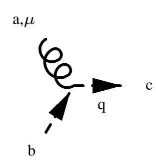
$$-i\,g\,(T^{a})_{ji}\,\gamma^{\mu}$$

Vertex gluon-gluon-gluon-gluon



$$\begin{aligned}&-ig^2\,f^{eac}\,f^{ebd}\,(g^{\alpha\,\beta}\,g^{\gamma\,\delta}-g^{\alpha\,\delta}\,g^{\beta\,\gamma})\\&-ig^2\,f^{ead}\,f^{ebc}\,(g^{\alpha\,\beta}\,g^{\gamma\,\delta}-g^{\alpha\,\gamma}\,g^{\beta\,\delta})\\&-ig^2\,f^{eab}\,f^{ecd}\,(g^{\alpha\,\gamma}\,g^{\beta\,\delta}-g^{\alpha\,\delta}\,g^{\beta\,\gamma})\end{aligned}$$

Vertex ghost-ghost-gluon



$$g\,f^{abc}\,q^{\mu}$$

What we learnt in lecture II

- A gauge theory can be built to describe the interactions between particles carrying colour charges : the gauge group is $SU(3)$
- The mediator of the strong interaction : the gluon carries a colour charge and thus there are different type of vertices : gluons among themselves, gluon – quark
- The gauge symmetry imposes that there is only one coupling constant

- The quantification breaks the classical gauge invariance, but it remains a quantum version of the gauge invariance : BRST symmetry