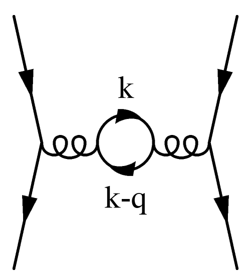
Outline

Contents

1 Lecture III : Renormalisation

The problem

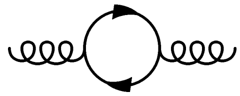
In this section, we will scout out superficially the quantum aspect of Quantum field theory. To start with, let us imagine that we want to compute the second order in perturbation of a QCD process, let us say the scattering of quarks of different flavour $q_i\,\bar{q}_i \to q_k\,\bar{q}_k$. Among all the diagrams, let us pick one, the following



What is new, in this diagram is the loop inside the gluon propagator. In the rest of this section, we will focus on this part, disregarding the physical initial and final states.

UV divergences

Let us try to compute this part of the diagram. From the left to the right, a virtual incoming gluon with a colour a, a Lorentz index μ and a 4-momentum q ($q^2 \neq 0$) splits into a quark – anti-quark pair which annihilates itself into a outgoing virtual gluon with colour b, Lorentz index ν and 4-momentum q (because of the global energy-momentum conservation).



Note that the energy-momentum conservation at each vertex does not fix the four-momentum of the quark and the anti-quark propagators. We thus have to sum over all the possibilities, i. e. integrate over the four-momentum k.

$$\mathcal{P}^{(1)}_{\mu\nu}(q) \simeq \int \frac{d^4k}{(2\pi)^n} Tr\left[\gamma_\mu \frac{\not{k}+m}{(k^2-m^2+i\lambda)}\gamma_\nu \frac{(\not{k}-\not{q})+m}{((k-q)^2-m^2+i\lambda)}\right]$$

There is a problem with this integration over k, indeed, we have to integrate up to infinity every component of the four-momentum k, let us study the behaviour of the integrand when $k \to \infty$. Here, we have to be careful because k lies on a Minkowski space, that is to say $k^2 = k_0^2 - |\vec{k}|^2$ and it is not guarantee that if k_0 and $|\vec{k}|$ go to infinity, k^2 goes to infinity too!. To do the things correctly, we have to apply a "Wick rotation" to go from a Minkowski space (k) to an Euclidean one ($\bar{k}$) (change k_0 into $i\,k_0$)

$$\int d^4k \, \frac{k_\mu\, k_\nu}{k^4} \sim \int_0^\infty d|\bar{k}|\, |\bar{k}| \to \infty \quad \textit{UV divergence}$$

The behaviour of the integrand is such that the integral over k diverges when the components of k go to infinity! This is an example of a Ultra-Violet divergence : there is a divergence when the frequency associated to the particles running into the loop goes to infinity.

Origin

What does that mean ? The mathematical origin of this divergence is that

distributions have been handled without precautions (treated like functions). But above that, there is also a physical origin which is the following.

We are getting a contribution from intermediate states involving $q\,\bar{q}$ pairs but the energy of these intermediate states is **arbitrarily high**! We have no idea what the interaction of gluons with arbitrarily high momentum quarks is.

We made the assumption, at the very beginning, that the $q-g$ interaction is **point-like** ($-i\,g\,T^a\,\gamma^\mu$). But we cannot test at such high energies that the interaction $q-g$ is like that

Regularisation

What can we do ?

Firstly, give a **meaning** to the expression of $\mathcal{P}^{(1)}_{\mu\nu}(q)$ by regularising the integral.

- **cut-off** $\int_0^\Lambda d\bar{k}\, f(\bar{k}) = F(\Lambda) \rightarrow$ breaks some symmetries of the Lagrangian, **not a good idea!**
- **dimensional regularisation** $d^4k \rightarrow d^nk$, preserves Lorentz and gauge symmetry

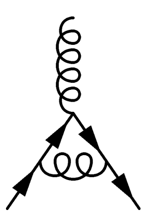
Note that the dimensional regularisation also breaks some symmetries of the Lagrangian, scale invariance for example but they are minor ones

Back to our example:

$$\int_0^\infty |\bar{k}|^{n-1}\, d|\bar{k}|\, |\bar{k}|^{-2} = \left[\frac{|\bar{k}|^{n-2}}{n-2}\right]_0^\infty \qquad \text{convergent for } n < 2$$

Regularisation

The question one can ask is: Is it an isolated case? The answer is no, other Green functions have UV divergences at one loop, for example:



$$\int d^4k\, \frac{k_\mu\, k_\nu}{k^6} \rightarrow \int_0^\infty \frac{d|\bar{k}|}{|\bar{k}|} \quad \text{logarithmic UV divergence}$$

$$\int_0^\infty d|\bar{k}|\, |\bar{k}|^{n-5} \quad \text{converge for } n < 4$$

But how many Green functions diverge?

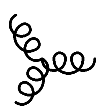
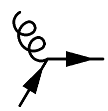
1.1 Superficial degree of divergence

A Simple tool

To answer this last question, we can build a simple tool based on the power counting of k, let us define

$\omega(G)$: **superficial degree of divergence** of a Feynman diagram G. For an arbitrary diagram :

- E_F external fermions
- E_B external bosons
- I_F fermion propagators
- I_B boson propagators
- n_i number of vertices of type i, $N = \sum_i n_i$ the total number of vertices. Some vertices may be derivative coupling d_i power of k coming from the vertex i, for instance

 $d_i = 1$  $d_i = 0$

- L the number of independent four-momenta (number of loops), each term corresponds to k^4 (d^4k)

Thus

A Simple tool

$$\omega(G) = 4\,L - I_F - 2\,I_B + \sum_i n_i\,d_i \tag{1}$$

But this formula is not very handy since it depends on the number of internal lines!

But we have the following relations:

- $L = I_B + I_F - (N-1)$ because the number of independent four-momenta L is equal to the number of momenta, one per internal lines ($I_F + I_B$) minus the number of constrains coming from the energy-momentum conservation ($N-1$) (it is $N-1$ and not N because of the global energy-momentum conservation)

- $E_F + 2\,I_F \;=\; \sum_i n_i\, f_i$ where f_i is the number of fermions attached to the vertex of type i
- $E_B + 2\,I_B \;=\; \sum_i n_i\, b_i$ where b_i is the number of bosons attached to the vertex of type i

Thus, using these extra relations, the equation (1) becomes

$$\omega(G) = 4 - E_B - \frac{3}{2}\,E_F + \sum_i n_i\,\left(b_i + d_i + \frac{3}{2}\,f_i - 4\right) \tag{2}$$

A Simple tool

But if the vertex of type i originates from a term in the Lagrangian of the type

$$g_i\,\underbrace{\psi\cdots\psi}_{f_i}\,\underbrace{A\cdots A}_{b_i}\,\underbrace{\partial\cdots\partial}_{d_i}$$

This term must have a dimension 4 as any element of the Lagrangian, introducing $[g_i]$ the dimension of the coupling constant g_i we have then that

$$[g_i] + b_i + d_i + \frac{3}{2}\,f_i = 4$$

because the A field has dimension 1 as the derivative ∂_μ and the dimension of the ψ field is 3/2. Thus, the superficial degree of divergence can be written

$$\omega(G) = 4 - E_B - \frac{3}{2}\,E_F - \sum_i n_i\,[g_i] \tag{3}$$

Reminder about dimensional analysis. The system of unit used in high energy is the system where $\hbar = c = 1$. In this unit system, every quantity can be expressed in unit of energy : the energy has a dimension 1 $[E] = 1$ and a length has a dimension -1 $[L] = -1$ because of the relation $\Delta p\,\Delta x \sim \hbar$. The action $S = \int d^4x\,\mathcal{L}(x)$ has no dimension $[S] = 0$, thus each term of the Lagrangian must have a dimension 4. The kinetic term for the fermions must have the dimension 4 :

$$[\bar\psi\,\partial_\mu\,\psi] = 4 \to 2\,[\psi] + [\partial_\mu] = 4 \to 2\,[\psi] + 1 = 4 \to [\psi] = \frac{3}{2}$$

the kinetic term for the gauge field have also a dimension 4 :

$$[\partial_\mu\,A_\nu\,\partial^\mu\,A^\nu] = 4 \to 2\,[\partial_\mu] + 2\,[A] = 4 \to [A] = 1$$

For the QED like interaction term

$$[g\,\bar\psi\,\not{A}\,\psi] = 4 \to [g] + 2\,[\psi] + [A] = 4 \to [g] = 0$$

thus a general interaction term with a coupling constant g_i generates the following relation

$$\begin{aligned}
\Big[g_i\,\underbrace{\psi\cdots\psi}_{f_i}\,\underbrace{A\cdots A}_{b_i}\,\underbrace{\partial\cdots\partial}_{d_i}\Big] &= 4\\
\to [g_i] + f_i\,[\psi] + b_i\,[A] + d_i\,[\partial_\mu] &= 4\\
\to [g_i] + \frac{3}{2}\,f_i + b_i + d_i &= 4
\end{aligned}$$

Be careful that these dimensions depend on space-time dimension. In a space-time of dimension n, the action given by $S = \int d^n x\,\mathcal{L}(x)$ is still dimensionless, thus:

$$\begin{aligned}
[\bar\psi\,\partial_\mu\,\psi] = n \to [\psi] &= \frac{n-1}{2}\\
[\partial_\mu\,A_\nu\,\partial^\mu\,A^\nu] = n \to [A] &= \frac{n-2}{2}\\
[g\,\bar\psi\,\not{A}\,\psi] = n \to [g] &= \frac{4-n}{2}
\end{aligned}$$

Exercise

1)Rederive the formula for the superficial degree of divergence in a space-time of dimensions n

2) Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\,\Phi(x))\,(\partial^\mu\,\Phi(x)) - \frac{m^2}{2}\,\Phi^2(x) - \frac{\lambda}{4!}\,\Phi^4(x)$$

where $\Phi(x)$ is a scalar field. Determine for which value of n, this theory is super renormalisable, renormalisable, non renormalisable.

QCD case

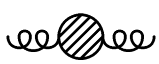
In the case of QCD, there is only one type of coupling constant whose dimension is zero! But the ghosts must be included, thus

$$\omega(G) = 4 - (E_B + E_G) - \frac{3}{2} E_F \tag{4}$$

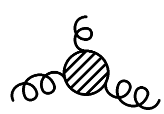
Note that eq. (4) depends uniquely on the number of external particles! It is rather remarkable and made this tool very handy

- $w(G) < 0$ convergent Green function
- $w(G) = 0$ Green function with a logarithmic divergence
- $w(G) = 1$ Green function with a linear divergence
- $w(G) = 2$ Green function with a quadratic divergence
- $\vdots$

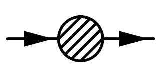
2-points, 3-points



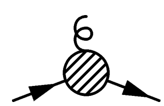
ω(G) = 2



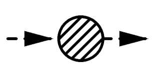
ω(G) = 1



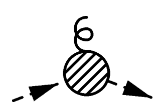
ω(G) = 1



ω(G) = 0

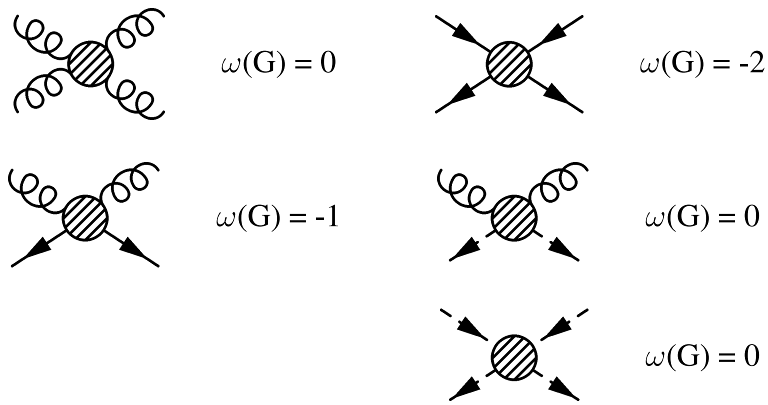


ω(G) = 2



ω(G) = 1

4-points



If the number of external legs (whatever its type) is greater than or equal to 5, the superficial degree of divergence becomes negative.

Renormalisation

There is a finite number of Green functions which diverge (9). Note that due to the symmetries of the Lagrangian (Lorentz symmetry, gauge symmetry), the real degree of divergence is less severe, it is in fact 0 for any divergent Green functions : all the divergences are of logarithmic types. Due to the residual gauge symmetry at quantum level all these nine divergent Green functions are not independent, there are relations among them : this is the Slavnov-Taylor identities (generalisation of Ward identities in QED). Since the number of divergent Green functions is finite, we can expect to absorb theses divergences into a redefinition of the parameters of the Lagrangian.

The Lagrangian, at quantum level, is expressed in terms of the bare quantities $\mathcal{L}(\psi_B, A_B, \eta_B, m_B, g_B)$. All these bare parameters are not physical (they are infinite!). By a multiplicative renormalisation, the Lagrangian can be expressed in terms of the renormalised parameters (the renormalised parameters differ from the physical one by finite transformations).

$$\psi_B(x) = Z_2^{1/2}\,\psi(x), \quad A^a_{B\,\mu}(x) = Z_3^{1/2}\,A^a_\mu(x), \quad \eta^a_B(x) = \tilde{Z}_3^{1/2}\,\eta^a(x),$$
$$m_B = \frac{Z_0}{Z_2}\,m, \quad g_B = \frac{Z_{1\,F}}{Z_2\,Z_3^{1/2}}\,g' = \frac{Z_1}{Z_3^{3/2}}\,g' = \frac{\tilde{Z}_1}{\tilde{Z}_3\,Z_3^{1/2}}\,g', \quad \xi_B = Z_3\,\xi$$

Note that, because of the Slavnov-Taylor identities, only 7 renormalised constant are required It will clear later why I named the renormalised coupling by g'. Thus the Lagrangian can be expressed in terms of the renormalised quantities

$$\begin{aligned}\mathcal{L}(\psi, A, \eta, m, g') = &-\frac{1}{4}\, Z_3\, \left(\partial_\mu\, A^a_\nu(x) - \partial_\nu\, A^a_\mu(x) + \frac{Z_1}{Z_3}\, g'\, f^{abc}\, A^b_\mu(x)\, A^c_\nu(x)\right)^2 \\ &+ \tilde{Z}_3\, (\partial_\mu\, \eta^{\star\, a}(x))\, \left(\delta^{ab}\, \partial_\mu - \frac{\tilde{Z}_1}{\tilde{Z}_3}\, g'\, f^{abc}\, A^c_\mu(x)\right)\, \eta^b(x) \\ &+ Z_2\, \bar{\psi}_j(x)\left[i\, \left(\partial_\mu - i\, \frac{Z_{1\,F}}{Z_2}\, g'\, A^b_\mu(x)\, T^b_{ji}\right)\gamma^\mu - m\right]\, \psi_i(x) \\ &- Z_0\, m\, \bar{\psi}_i(x)\, \psi_i(x) - \frac{1}{2\,\xi}\, \left(\partial^\mu\, A^a_\mu(x)\right)^2\end{aligned}$$

But remember that all the couplings must be equal even the renormalised ones, this is a consequence of the gauge invariance, thus

$$\frac{Z_{1\,F}}{Z_2} = \frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3}$$

This is what the Taylor-Slavnov identities tell us! For matter of convenience, we introduce

$$Z_i = 1 + \delta Z_i, \quad \tilde{Z}_i = 1 + \delta\, \tilde{Z}_i$$

The Lagrangian can be expanded in terms of the $\delta\, Z_i$ and $\delta\tilde{Z}_i$ under the following form

$$\begin{aligned}\mathcal{L}(\psi, A, \eta, m, g') = &-\frac{1}{4}\, F^a_{\mu\nu}(x)\, F^{a\,\mu\nu}(x) + (\partial^\mu\, \eta^{\star\, a}(x))\, D^{ab}_\mu\, \eta^b(x) - \frac{1}{2\,\xi}\, \left(\partial^\mu\, A^a_\mu(x)\right)^2 \\ &+ \bar{\psi}_j(x)\, (i\not{D}_{ji} - m)\, \psi_i(x) - \frac{1}{4}\, \delta Z_3\, \left(\partial_\mu\, A^a_\nu(x) - \partial_\nu\, A^a_\mu(x)\right)^2 \\ &+ \delta\tilde{Z}_3\, (\partial_\mu\, \eta^{\star\, a}(x))\, (\partial_\mu\, \eta^a(x)) + i\, \delta Z_2\, \bar{\psi}_i(x)\, \not{\partial}\, \psi_i(x) \\ &- \frac{1}{2}\, \delta Z_1\, g'\, \left(\partial_\mu\, A^a_\nu(x) - \partial_\nu\, A^a_\mu(x)\right)\, f^{abc}\, A^b_\mu(x)\, A^c_\nu(x) \\ &- \frac{1}{4}\, (2\,\delta Z_1 - \delta Z_3)\, g'\, f^{abc}\, A^b_\mu(x)\, A^c_\nu(x)\, f^{ade}\, A^{d\,\mu}(x)\, A^{e\,\nu}(x) \\ &- \delta\tilde{Z}_1\, g'\, (\partial_\mu\, \eta^{\star\, a}(x))\, f^{abc}\, A^c_\mu(x)\, \eta^b(x) \\ &+ \delta Z_1\, g'\, \bar{\psi}_j(x)\, \not{A}^a(x)\, T^a_{ji}\, \psi_i(x)\end{aligned}$$

Leading to

$$\mathcal{L}(\psi_B, A_B, \eta_B, m_B, g_B) = \mathcal{L}(\psi, A, \eta, m, g') + \delta\mathcal{L}(\psi, A, \eta, m, g')$$

Note that in the expansion, there are leftover terms, indeed, this renormalisation procedure works order by order in the coupling constant expansion, that is to say that $\delta Z_i = \sum_k g'^{2k}\, \delta Z_i^{(k)}$, only terms with the lowest degree in g' has been retained, especially

$$\frac{Z_1^2}{Z_3} = \frac{(1+\delta Z_1)}{1+\delta Z_3} \simeq (1 + 2\,\delta Z_1)\,(1-\delta Z_3) \simeq 1 + (2\,\delta Z_1 - \delta Z_3)$$

Note also that since we are working in a space time of dimensions n, the dimension (in term of energy) of the coupling constant g' is not zero! Indeed

$$\begin{array}{lll} [\mathcal{L}] = 4 & [g'] = 0 & [m] = 1 \\ [\mathcal{L}] = n & [g'] = 2 - \frac{n}{2} & [m] = 1 \end{array}$$

$$g' \to g\,\mu^{2-n/2} \qquad \text{with} \quad [g] = 0 \qquad \text{and} \quad [\mu] = 1$$

To absorb the change of the dimension of the coupling constant, a new energy scale μ is introduced.

The way this scale appear seems a bit artificial and seems to be related to the dimensional regularisation, but whatever the way we regularise, the renormalisation procedure makes the appearance of this energy scale. Indeed, if we have used a cut-off Λ to regularise the divergent integrals, we would have got a result for a divergent Green function of the type

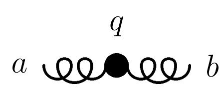
$$A\,\ln\left(\frac{\Lambda}{Q}\right) + B$$

where Q is typical energy scale and A and B are two coefficients independent of Λ. We want to absorb the dependence on the regulator into a renormalisation of parameters of the Lagrangian. But because of the logarithmic dependence, this procedure is ambiguous! Indeed, an arbitrary energy scale can be introduced in such a way that

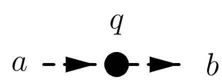
$$A\,\ln\left(\frac{\Lambda}{\mu}\right) + A\,\ln\left(\frac{\mu}{Q}\right) + B$$

In the old textbooks on Quantum Field Theory, the scale μ is **the scale at which the UV divergences are subtracted**. Note the key role plays by the fact that the UV divergences are logarithmic divergences!

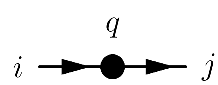
In addition, with the logarithmic dependence on the regulator, one can also absorb some finite terms (even not logarithmic), this defines the **renormalisation scheme**.



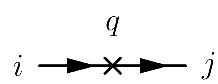
$$\delta Z_3\, i\, \delta^{ab}\, (q_\mu\, q_\nu - q^2\, g_{\mu\nu})$$



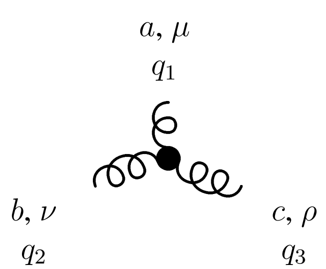
$$\delta \tilde{Z}_3\, i\, \delta^{ab}\, q^2$$



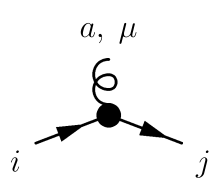
$$\delta Z_2\, i\, \delta_{ij}\, \not{q}$$



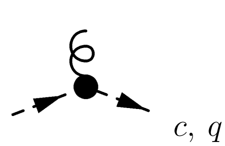
$$-\,\delta Z_0\, i\, \delta_{ij}\, m$$



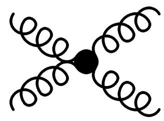
$$\delta Z_1\, g'\, f^{abc}\, V_{\mu\nu\rho}(q_1, q_2, q_3)$$



$$\delta Z_{1\,F}\, g'\, T^a_{ji}\, \gamma^\mu$$



$$\delta \tilde{Z}_1\, g'\, f^{abc}\, q_\mu$$

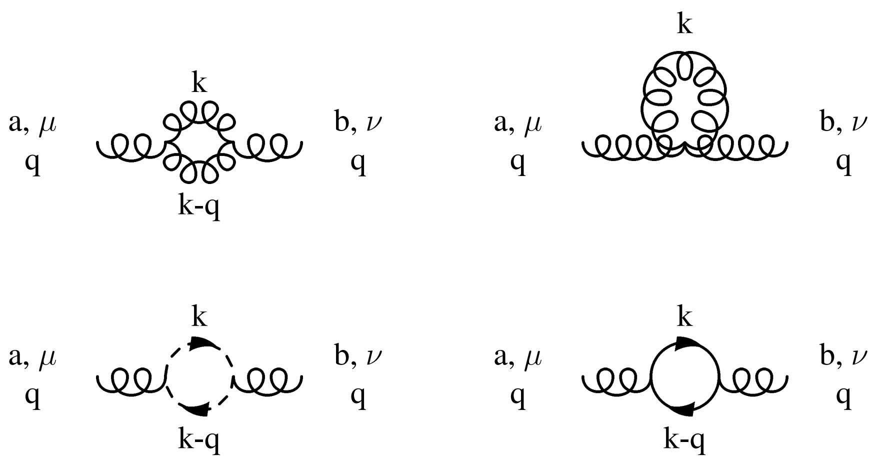


$$(2\,\delta Z_1 - \delta Z_3)\, g^2\, \mu^{2\,\varepsilon}\, \bar{V}^{abcd}_{\mu\nu\rho\sigma}$$

1.2 A specific example

One loop corrections to the gluon propagator

Let us treat an example : the corrections to the gluon propagator at one loop. To do that, it requires to compute the following contributions. Note that it can be shown that only the 1 particle irreducible diagrams have to be considered : a digram which cannot be separated in two by cutting only one line.



The way to compute these one loop diagrams is rather standard and can be find in textbooks, some notes are available in the directory. Applying the technics, we get

Results (Feynman gauge $\xi = 1$)

$$\begin{aligned}
\mathcal{P}^{(1)\,gg}_{\mu\nu}(q) &= \frac{1}{\varepsilon}\, N\, \delta^{ab}\, K(\varepsilon)\, \left[q^2\, g^{\mu\,\nu}\, \left(\frac{19}{12} + \frac{29\,\varepsilon}{9}\right) - q^{\mu}\, q^{\nu}\, \left(\frac{11}{6} + \frac{67\,\varepsilon}{18}\right)\right] \\
\mathcal{P}^{(1)\,ggg}_{\mu\nu}(q) &= 0 \\
\mathcal{P}^{(1)\,GG}_{\mu\nu}(q) &= \frac{1}{\varepsilon}\, N\, \delta^{ab}\, K(\varepsilon)\, \left[q^2\, g^{\mu\,\nu}\, \left(\frac{1}{12} + \frac{2\,\varepsilon}{9}\right) - q^{\mu}\, q^{\nu}\, \left(-\frac{1}{6} - \frac{5\,\varepsilon}{18}\right)\right] \\
\mathcal{P}^{(1)\,qq}_{\mu\nu}(q) &= -\frac{1}{\varepsilon}\, T_F\, \delta^{ab}\, K(\varepsilon)\, \left[q^2\, g^{\mu\,\nu}\, \left(\frac{4}{3} + \frac{20\,\varepsilon}{9}\right) - q^{\mu}\, q^{\nu}\, \left(\frac{4}{3} + \frac{20\,\varepsilon}{9}\right)\right]
\end{aligned}$$

with $T_F = N_F/2$ and

$$K(\varepsilon) = \frac{\alpha_s}{4\,\pi} \left(\frac{4\,\pi\,\mu^2}{-q^2 - i\,\lambda} \right)^{\varepsilon} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}$$
$$\simeq 1 + \varepsilon \left(\ln(4\,\pi) - \gamma + \ln\left(\frac{\mu^2}{-q^2 - i\,\lambda} \right) \right) + O(\varepsilon^2)$$

Γ is the so called Euler Gamma function which generalises the factorial to complex variable : $\Gamma(1+z) = z\,\Gamma(z)$, more properties in [1] and $\varepsilon = (4-n)/2$

More results

Let us introduce $\mathcal{P}$ the sum of four contributions

$$\mathcal{P}^{(1)}_{\mu\nu}(q) = \mathcal{P}^{(1)\,gg}_{\mu\nu}(q) + \mathcal{P}^{(1)\,ggg}_{\mu\nu}(q) + \mathcal{P}^{(1)\,GG}_{\mu\nu}(q) + \mathcal{P}^{(1)\,qq}_{\mu\nu}(q)$$

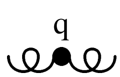
The **ghost contribution** is necessary in order that $\mathcal{P}^{(1)}_{\mu\nu}(q)$ is transverse : $q^\mu\,\mathcal{P}^{(1)}_{\mu\nu}(q) = q^\nu\,\mathcal{P}^{(1)}_{\mu\nu}(q) = 0$ as required by Slavnov-Taylor identities. Note that $\mathcal{P}^{(1)}_{\mu\nu}(q)$ is not the gluon propagator, it can be shown that

$$\mathcal{D}^{-1}_{\mu\nu} = D^{-1}_{\mu\nu} - i\,\mathcal{P}^{(1)}_{\mu\nu} \tag{5}$$

where $\mathcal{D}$ is the exact propagator (one loop in our case) and D the free propagator.

Counter term

We have to add the counter term

a,μ  b,ν $\qquad - i\,\delta Z_3^{(1)}\,\delta^{ab}\left(q^2\,g^{\mu\nu} - q^\mu q^\nu\right)$

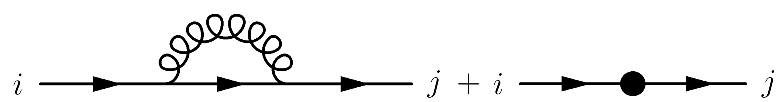
$$i\mathcal{P}^{(1)\,tot}_{\mu\nu,ab} = i\,\mathcal{P}^{(1)}_{\mu\nu,ab} - i\,\delta Z_3^{(1)}\,\delta^{ab}\left(q^2\,g^{\mu\nu} - q^\mu q^\nu\right)$$

In the $\overline{MS}$ scheme and the Feynman gauge ($\xi = 1$)

$$\delta Z_3^{(1)} = \frac{\alpha_s}{4\,\pi}\left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma\right)\left(\frac{5}{3}\,N - \frac{4}{3}\,T_F\right)$$

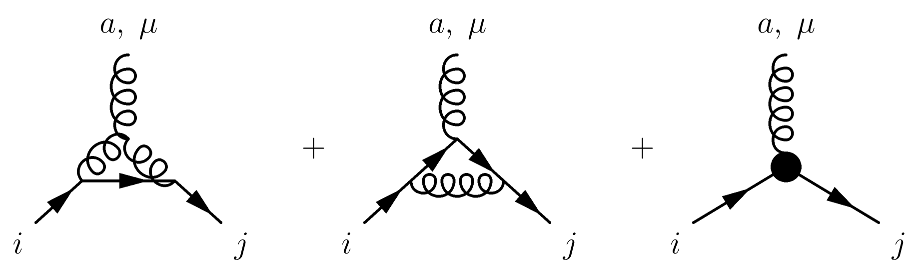
Other counter terms

Let us compute also the counter term associated to the quark wave function



$$\delta Z_2 = -\frac{\alpha_s}{4\,\pi}\,C_F\,\left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma\right)$$

as well as the counter term associated to the vertex quark – gluon



$$\delta Z_{1\,F} = -\frac{\alpha_s}{4\,\pi}\,(C_F + N)\,\left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma\right)$$

1.3 The running coupling constant

The renormalised α_s

Recalling the relation between the bare coupling constant g_B and the renormalised one $g' \equiv g\,\mu^{\varepsilon}$ and the relation $\alpha_s = \frac{g^2}{4\,\pi}$, we have the following relation between the bare $\alpha_{s\,B}$ and the renormalised one α_s

$$\begin{aligned} \alpha_{s\,B} &= \alpha_s\,\mu^{2\,\varepsilon}\,\frac{Z_{1\,F}^2}{Z_2^2\,Z_3} \\ &\equiv \alpha_s\,\mu^{2\,\varepsilon}\,Z_\alpha \end{aligned} \qquad (6)$$

where

$$
\begin{aligned}
Z_\alpha &\simeq 1 + 2\,\delta Z_{1\,F} - 2\,\delta Z_2 - \delta Z_3 + O(\alpha_s^2) \\
&= 1 - \frac{\alpha_s}{4\,\pi}\left[\frac{11}{3}\,N - \frac{2\,N_F}{3}\right]\left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma\right)
\end{aligned}
$$

But remember that $\alpha_{s\,B}$ does not depend on μ, the dependence on the scale μ appears once we expressed the Lagrangian in terms of the renormalised parameters, thus

$$
\frac{\mu^2\,d\alpha_{s\,B}}{d\mu^2} = 0
$$

ξ **dependence**

For **simplicity** reason, we choose the Feynman gauge to present the different results. Letting the xi parameter **free**, the results for the counter terms would have been

$$
\begin{aligned}
\delta Z_3^{(1)} &= \frac{\alpha_s}{4\,\pi}\left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma\right)\left(N\left[\frac{13}{6} - \frac{\xi}{2}\right] - T_F\,\frac{4}{3}\right) \\
\delta Z_2^{(1)} &= -\frac{\alpha_s}{(4\pi)}\,C_F\,\xi\,\left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi)\right] \\
\delta Z_1^{F} &= -\frac{\alpha_s}{4\,\pi}\left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma\right)\left(C_F\,\xi + N\,\left[\frac{3}{4} + \frac{\xi}{4}\right]\right)
\end{aligned}
$$

It is easy to verify that the ξ dependence drops out in Z_α, This is expected because Z_α is related to a physical quantity.

But, in order that the relation (6) is fulfilled the renormalised quantity α_s must depend on μ

$$
\begin{aligned}
&\mu^2\,\frac{d}{d\mu^2}\left(\alpha_s\,\mu^{2\,\varepsilon}\,Z_\alpha\right) = 0 \\
&\qquad = \beta(\alpha_s)\,\mu^{2\,\varepsilon}\,Z_\alpha + \varepsilon\,\alpha_s\,\mu^{2\,\varepsilon}\,Z_\alpha + \alpha_s\,\mu^{2\,\varepsilon}\,\mu^2\,\frac{dZ_\alpha}{d\mu^2} \qquad (7)
\end{aligned}
$$

with

$$
\beta(\alpha_s) = \mu^2\,\frac{d\alpha_s}{d\mu^2}
$$

But Z_α does not depend explicitly on μ^2 (note that this proper to $\overline{\text{MS}}$ scheme), thus

$$\mu^2 \frac{dZ_\alpha}{d\mu^2} = \frac{dZ_\alpha}{d\alpha_s} \mu^2 \frac{d\alpha_s}{d\mu^2} = \beta(\alpha_s) \frac{dZ_\alpha}{d\alpha_s}$$

Plug-in this last result into the right hand member of equation (7) leads to

$$\beta(\alpha_s) \left[Z_\alpha + \alpha_s \frac{dZ_\alpha}{d\alpha_s} \right] + \varepsilon \, \alpha_s \, Z_\alpha = 0$$

that is to say

$$\begin{aligned} \beta(\alpha_s) &= \frac{-\varepsilon \, \alpha_s}{1 + \frac{\alpha_s}{Z_\alpha} \frac{dZ_\alpha}{d\alpha_s}} \\ &\simeq -\varepsilon \, \alpha_s \left(1 - \alpha_s \frac{dZ_\alpha}{d\alpha_s} \right) \\ &= -\varepsilon \, \alpha_s - \alpha_s^2 \, \kappa(\varepsilon) \left(\frac{11 \, N - 2 \, N_F}{12 \, \pi} \right) \end{aligned}$$

with $\kappa(\varepsilon) = 1 + \varepsilon \, \ln(4\,\pi) - \varepsilon \, \gamma$. Let us remark that $\beta(\alpha_s)$ is not singular when $\varepsilon \to 0$, we can take the limit $\varepsilon \to 0$, this yields

$$\beta(\alpha_s) = -\alpha_s^2 \, b_0 \tag{8}$$

with

$$b_0 = \frac{11 \, N - 2 \, N_F}{12 \, \pi}$$

The μ dependence of α_s

To determine the μ dependence of α_s, we have to solve the differential equation given by eq. (8). More generally, introducing an initial condition, the running coupling constant obeys to the following differential equation

$$\frac{d \, \alpha_s(t)}{dt} = \beta(\alpha_s(t)) \quad \text{with} \quad t = \ln(\mu^2/\mu_0^2)$$

Note that the β function can be computed at any order in α_s and has the following perturbative expansion

$$\beta(\alpha_s(t)) = -b_0 \, \alpha_s(t)^2 \, (1 + b_1 \, \alpha_s(t) + \cdots)$$

This kind of differential equation is an example of the renormalisation group equations (RGE). These set of differential equations describe how a physical observable has to behave under a variation of the unphysical scale μ. Solving the differential equation amounts to integrate the inverse of the β function

$$t = \int_{\alpha_s(0)}^{\alpha_s(t)} \frac{dx}{\beta(x)}$$

Keeping only the first term of the perturbative expansion of the β function gives

$$t = \frac{1}{b_0} \left(\frac{1}{\alpha_s(t)} - \frac{1}{\alpha_s(0)} \right)$$

that is to say

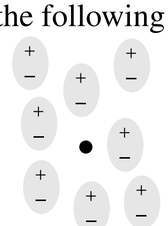
$$\alpha_s(t) = \frac{\alpha_s(0)}{1 + b_0\, t\, \alpha_s(0)}$$

This is the expression of running coupling constant at Leading Logarithmic (LL) accuracy. Knowing the coupling constant at a certain energy scale μ_0^2, this expression enables to compute the value of α_s at a scale μ^2

Discussions

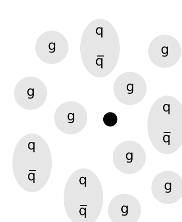
Let us discuss the dependence on μ^2 (t) of the running coupling constant. Let us remark that this dependence is related to the sign of b_0. Indeed, there is two contributions in the expression of b_0 : one positive proportional to N, i.e. coming from the gluon self interaction and another one, negative, proportional to N_F coming from the quark – gluon interaction. If $N_F \leq 16$, then $b_0 > 0$ and the variation of α_s with t is negative, thus $\alpha_s(t)$ decreases when t increases, this is called the asymptotic freedom.

If $N_F \geq 16$, $b_0 < 0$ and the variation of α_s with t is positive, thus $\alpha_s(t)$ increases when t increases, this the case of QED for example. The physical picture is the following



In QED, $b_0 < 0$ screening effect : an electric charge is screened by the virtual $\pm$ charge in the vacuum

In QCD, $b_0 > 0$ anti-screening effect : an colour charge is screened by the virtual $q\,\bar{q}$ but anti-screened by the g in the vacuum



The parameter Λ

Note that the initial condition can be defined with the help of parameter Λ such that

$$\ln\left(\frac{\mu^2}{\Lambda^2}\right) = -\int_{\infty}^{\alpha_s(\mu^2)} \frac{dx}{\beta(x)}$$

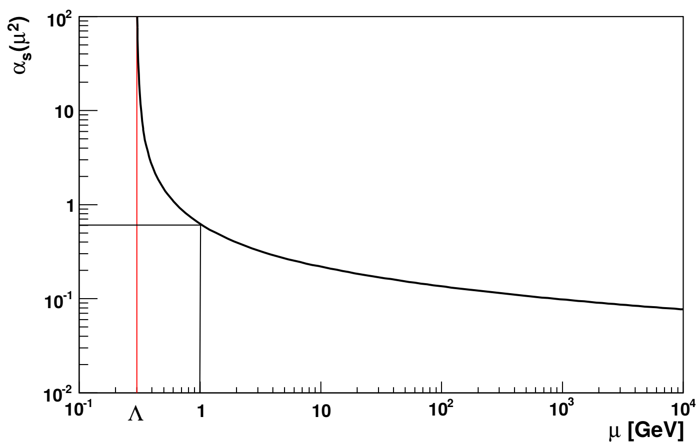
Taking the first term of the β function, one gets

$$\alpha_s(\mu^2) = \frac{1}{b_0 \, \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \tag{9}$$

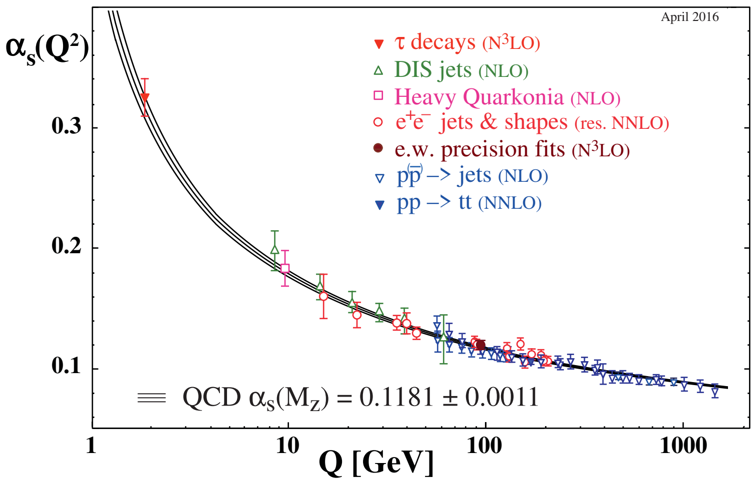
Λ : a scale which separate **perturbative** and **non perturbative** regime (Λ depends on the renormalisation scheme)

Plot of $\alpha_s(\mu)$

Let us plot the running coupling constant as given by eq. (9) as a function of μ.



α_s Measurement



1.4 Choice of the scale μ

The ration R

Let us consider the ratio R

$$R(\mu^2) \equiv \frac{\sigma(e^+ e^- \to \text{hadrons})}{\sigma(e^+ e^- \to \mu^+ \mu^-)}$$

At lowest order, $R(\mu^2) = R_B = N \sum_i q_i^2$. Let us define the quantity

$$\bar{R}(\mu^2) = \frac{R(\mu^2) - R_B}{R_B}$$

At one loop

$$\bar{R}(\mu^2) = \frac{\alpha_s(\mu^2)}{\pi}$$

How to choose this scale μ?

Since all the final state is integrated over, the only scale is $\sqrt{S}$: the available energy in the centre of mass frame $e^+ e^-$ with $S = (p_1 + p_2)^2$, thus

$$\bar{R}(S) = \frac{\alpha_s(S)}{\pi}$$

If another choice for the scale is made, say μ_0, since the relation between $\alpha_s(S)$ and $\alpha_s(\mu_0^2)$ is known, the quantity $\bar{R}(S)$ can be expressed in terms of the scale μ_0, it yields

$$\bar{R}(S) = \frac{\alpha_s(\mu_0^2)}{\pi} \left[1 - \alpha_s(\mu_0^2)\, b_0\, t + \alpha_s^2(\mu_0^2)\, (b_0\, t)^2 + \cdots\right]$$

with $t = \ln(S/\mu_0^2)$. Thus,

$$\bar{R}(S) - \bar{R}(\mu^2) = O(\alpha_s^2(\mu^2))$$

the difference between the two results is of order $\alpha_s^2(\mu^2)$, that is to say to not calculated terms (remember that we only compute the α_s correction). Nevertheless, if $\mu \simeq \sqrt{S}$, $\alpha_s(\mu^2)\, t \ll 1$ none of the terms in the square bracket will be large, the perturbative series is not expected to be spoiled. On the contrary, if $\mu \ll \sqrt{S}$ (or $\mu \gg \sqrt{S}$), $\alpha_s(\mu^2)\, t \simeq 1$, the perturbative series will be badly convergent.

To sum up, a good choice for the scale μ is around the "natural" scale $\sqrt{S}$, more precisely "around" means that $\alpha_s(\mu^2)\, \ln(S/\mu^2)$ is small compared to 1. A choice where μ is very different of $\sqrt{S}$ leads to unstable predictions, in the sense that the higher order corrections to $\bar{R}(\mu^2)$ will be large. Note that the variation of the scale μ around $\sqrt{S}$ gives an error band for the theoretical prediction, the more the number of terms of the perturbative series are included, the less this error band will be.

Remarks

Note also that it can be understood why the formula, we got, for $\alpha_s(\mu^2)$ is called at Leading Logarithmic (LL) accuracy. Indeed, it can be seen in the expression of $\bar{R}(S)$ in terms of μ_0^2, that the series is in terms of $(\alpha_s(\mu^2)\, t)^n$, that is to say that each power of α_s is multiplied the same power of t. Including, the expression of the β function at two loop in the differential equation which drives the μ dependence of $\alpha_s(\mu^2)$, solving it and expanding $\alpha_s(S)$ in terms of $\alpha_s(\mu_0^2)$ would have lead to a series which contains, besides the terms of the type $(\alpha_s(\mu^2)\, t)^n$, terms like $\alpha_s(\mu^2)^n\, t^{n-1}$. This expression of $\alpha_s(\mu^2)$ is called at Next to Leading Logarithmic accuracy (NLL).

What we learnt in lecture III

- The loop calculation may generate UV divergences (when the 4-momentum running in the loop goes to infinity)

- In the QCD case, the number of divergent Green functions is finite (because $[g] = 0$ in four-dimension space time)
- Due to the symmetries of the Lagrangian, all the divergences are logarithmic
- Renormalisation procedure : express the "bare" parameters of the Lagrangian in terms of the renormalised (physical) one plus some counter terms. These latter are adjusted to cancel the UV divergences. Work order by order in perturbation.
- As the outcome of renormalisation, an arbitrary energy scale appears. The renormalised parameters depend on it.

What we learnt in lecture III

- The independence of measurable quantities on this scale yields sets of differential equations which drive the dependence of these renormalised parameters on this scale