Outline

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1 Lecture IV

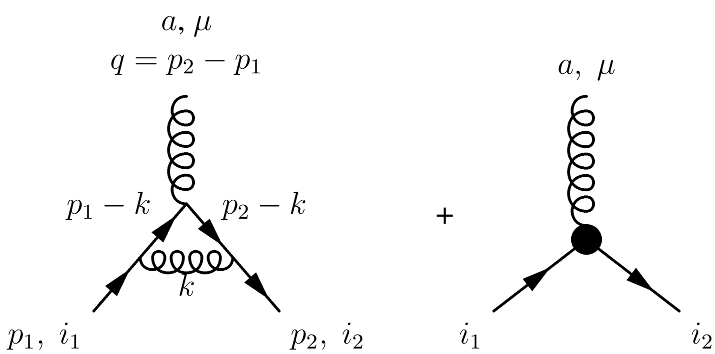
1.1 Soft/collinear divergences

Other divergences!

When computing α_s corrections to some processes, do we get rid of all **the divergent terms**? the answer is unfortunately **no**!!!

Vertex example

let us consider the following loop diagram



This diagram is computed in the Feynman gauge for simplicity and also by taking the fermions on their mass shell:
$p_2^2 = p_1^2 = m^2$ but $q^2 \neq 0$.

Vertex example I

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = -i\, e^2\, \mu^{(4-n)} \int \frac{d^n k}{(2\pi)^n}\, \gamma_\alpha\, \frac{\not{p}_2 - \not{k} + m}{(p_2 - k)^2 - m^2 + i\lambda}\, \gamma_\mu \times \frac{\not{p}_1 - \not{k} + m}{(p_1 - k)^2 - m^2 + i\lambda}\, \gamma^\alpha\, \frac{1}{k^2 + i\lambda}\, \left(T^b\, T^a\, T^b\right)_{i_2 i_1} \qquad (1)$$

To compute this diagram, the Feynman trick is used : the product of denominator is traded for one denominator but which depends on parameters over which we have to integrate, for example

$$\frac{1}{A\,B} = \int_0^1 dx\, \frac{1}{(x\,A + (1-x)\,B)^2}$$

This can be generalised to an arbitrary number of denominators.

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = -i\, e^2\, \mu^{(4-n)} \int_0^1 2\,y\,dy \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \times \frac{D}{[(k - y\,(p_2\,x + p_1\,(1-x)))^2 - y^2\,(p_2\,x + p_1\,(1-x))^2 + i\,\lambda]^3}.$$

Vertex example II

We make the following change of variable:

$$l = k - y\,(p_2\,x + p_1\,(1-x)). \qquad (2)$$

The denominator can be written in a simpler way and $\Lambda^{(1)}$ becomes:

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = -i\, e^2\, \mu^{(4-n)} \int_0^1 2\,y\,dy \int_0^1 dx \int \frac{d^n l}{(2\pi)^n} \times \frac{D}{[l^2 - R^2 + i\,\lambda]^3}. \qquad (3)$$

with

$$R^2 = y^2\,(m^2 - q^2\,x\,(1-x)) \qquad (4)$$

Vertex example III

The numerator D is a polynomial of degree one in l^2:

$$D = a\, l^2 + b.$$

Only the part in l^2 will give an **ultra-violet divergence**, the part constant will give an **infrared divergence**. So let's write $\Lambda^{(1)}$ as the sum of a part which only gives rise to ultraviolet divergences and part which only gives rise to infrared divergences:

$$\Lambda^{(1)}_\mu(p_2, p_1, q) = \Lambda^{(1)UV}_\mu + \Lambda^{(1)IR}_\mu \tag{5}$$

with

$$\Lambda^{(1)UV}_\mu = -i\, e^2\, \mu^{(4-n)} \int_0^1 2\,y\,dy \int_0^1 dx \int \frac{d^n l}{(2\pi)^n}\, a\, \frac{l^2}{(l^2 - R^2 + i\,\lambda)^3} \tag{6}$$

$$\Lambda^{(1)IR}_\mu = -i\, e^2\, \mu^{(4-n)} \int_0^1 2\,y\,dy \int_0^1 dx \int \frac{d^n l}{(2\pi)^n}\, b\, \frac{1}{(l^2 - R^2 + i\,\lambda)^3} \tag{7}$$

Vertex example IV

The integration on the 4-momentum l gives us:

$$\begin{aligned}\Lambda^{(1)UV}_\mu &= e^2\, \frac{\mu^{(4-n)}}{(4\,\pi)^{n/2}}\, \frac{(n-2)^2}{2}\, \gamma^\mu\, \Gamma(2 - \frac{n}{2}) \\ &\quad \times \int_0^1 dx\, (m^2 - q^2\, x\,(1-x) - i\,\lambda)^{n/2-2} \int_0^1 dy\; y^{n-3}\end{aligned} \tag{8}$$

and

$$\begin{aligned}\Lambda^{(1)IR}_\mu &= -\, e^2\, \frac{\mu^{(4-n)}}{(4\,\pi)^{n/2}}\, \Gamma(3 - \frac{n}{2}) \\ &\quad \times \int_0^1 dx\, (m^2 - q^2\, x\,(1-x) - i\,\lambda)^{n/2-3} \int_0^1 dy\; y^{n-5}\, F(y)\end{aligned}$$

with

$$F(y) = b_0 + b_1(x)\, y + b_2(x)\, y^2 \tag{9}$$

Integration on the variable y is easy. For the UV part, we get:

$$\Lambda_\mu^{(1)UV} = e^2 \frac{\mu^{(4-n)}}{(4\pi)^{n/2}} \frac{n-2}{2} \gamma^\mu \Gamma(2-\frac{n}{2}) \times \int_0^1 dx (m^2 - q^2 x (1-x) - i\lambda)^{n/2-2} \quad (10)$$

This integration will generate a **divergence** for the IR part because we have to integrate something of the type:

$$\int_0^1 dy \, (b_0 \, y^{n-5} + b_1 \, y^{n-4} + b_2 \, y^{n-3}) = \frac{b_0}{n-4} + \frac{b_1}{n-3} + \frac{b_2}{n-2}. \quad (11)$$

Soft divergence

The first term gives the divergence. As presented, the origin of this divergence is not clear. It appears at $y = 0$, at this value of y $k = l$, we have to integrate over l something like

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 + i\lambda)^3}$$

Using Wick rotation (making a change of variable to go to an Euclidean space), using the spherical coordinates to parametrise the $\bar{l}$ integration and studying only the radial part leads to

$$\int_0^{+\infty} dv \, v^{\frac{n}{2}-1-3}$$

where $v = \bar{l}^2$.

This integral diverges at $v = 0$ if $n = 4$. Thus this divergence appears at $y = 0$ and $l = 0$, this means $k = 0$. In addition, it is crucial to realise that if one of the external fermions is not on its mass shell, then the R^2 term will not vanish at $y = 0$ and thus the integrals on l (or k) will not diverge. To sum up, there will be a soft divergence if a massless (spin 1) boson is exchanged between two lines which are on their mass shell. It is not difficult to convince ourselves that the exchange of a massless fermion will not do the job because the fermion propagator behaves as $\not{k}/k^2$ while the boson propagator behaves as $1/k^2$ but a spin 0 boson will work.

Collinear divergence

This soft divergence also appears in QED, but in QCD it is worse. Indeed, coming back to the equation (7), the x integration also diverges if $m = 0$, this is not the case in QED where the fermions are massive but in QCD a virtual gluon can be exchanged between two on shell gluon lines and in this case $m = 0$! What is the origin of this divergence? Coming back to eq. (4) with $m = 0$ yields that R^2 vanishes at $x = 0$ and $x = 1$ in addition to the case $y = 0$. Let us fix $y \neq 0$ and look at the limit $x = 0$. Since in this limit, $R^2 = 0$, the l integral will diverge at $l = 0$ which means $k = y\, p_1$ (the limit $x = 1$ would lead to $k = y\, p_2$). The divergences at $x = 0$ or $x = 1$ originate from the fact that k becomes collinear to p_1 or p_2. These divergences are called collinear divergences. Note that there can be a pile of divergences "soft + collinear" at $y = 0$ et $x = 0$ ($x = 1$). Note also that, if the conditions are not gathered for having a soft divergence, there will be no collinear one.

Simple tool

A very simple way to test if a loop integral diverges in the soft region is to rescale the loop momentum and study the power of the rescaling parameter. As an illustration, let us take our example again, the integral, stripped from all the constants, was

$$\begin{aligned}
\Lambda_\mu^{(1)}(p_2, p_1, q) &\simeq \int d^n k \, \frac{H(k)}{((p_2 - k)^2 - m^2)\, ((p_1 - k)^2 - m^2)\, k^2} \\
&\simeq \int d^n k \, \frac{H(k)}{(k^2 - 2\, k \cdot p_2)\, (k^2 - 2\, k \cdot p_1)\, k^2}
\end{aligned}$$

Rescaling k by ρ leads to

$$\Lambda_\mu^{(1)}(p_2, p_1, q) \simeq \rho^{n-4} \int d^n k \, \frac{H(\rho\, k)}{(\rho\, k^2 - 2\, k \cdot p_2)\, (\rho\, k^2 - 2\, k \cdot p_1)\, k^2}$$

In general, the integral will behave as ρ^β. If $\beta \leq 0$ the integral will diverge when k becomes soft, if $\beta > 0$, the integral is convergent. It is easy to realise that they will be an infinite number of diagrams which will diverge, we cannot apply a renormalisation procedure.

Simple tool

How these divergences will disappear? To answer this question, it is important to think about what needs to be included when computing the α_s corrections to a reaction. To fix the idea, let us consider the Drell-Yan production at LHC. At lowest order, the associated partonic reactions are $q_i + \bar{q}_i \to \gamma^*$. But LHC is a hadronic collider, that means that one cannot expect to have events containing only a lepton pair and that's it! In this background, inclusive cross section is used. To measure it, all the events containing **at least** a lepton pai are collected. At theoretical level and at the order we work, one has also to consider reactions like $q_i + \bar{q}_i \to \gamma^* + g$ or $q_i + g \to \gamma^* + q_i$. That is to say, that the α_s corrections do not come only from the loop corrections but also the so-called "real" emission where an extra on shell parton is emitted.

1.2 Drell-Yan cross section

Notation

To begin with, the partons will be labelled by i_k where k is an integer and have a 4-momentum p_k. These parton labels belong to the set $S_p = \{u, \bar{u}, d, \bar{d}, \ldots, g\}$. All the cross section will be calculated in a space-time of dimension n. Let us start to compute the lowest order. Relying on the parton model to relate hadronic cross section to partonic one, the hadronic cross section is given by

$$\sigma_H = \sum_{i_1, i_2 \in S_p} \int dx_1 \, dx_2 \, F_{i_1}^{H_1}(x_1) \, F_{i_2}^{H_2}(x_2) \, \hat{\sigma}_{i_1 + i_2 \to \gamma^*} \tag{12}$$

It is implicitly assumed that the partonic cross section must fulfil conservation laws, thus if the sum selects a choice of partons i_1, i_2 which violates these laws the partonic cross section is set to zero, for example : the choice $i_1 = u$ and $i_2 = g$ is not possible due to spin conservation, or the choice $i_1 = u$ and $i_2 = \bar{d}$ is ruled out because of the electric charge conservation,etc. Note also that, for a couple of labels i_1, i_2 which verifies the conservation laws say d, $\bar{d}$, the following combinations has to take into account

$$F_d^{H_1}(x_1) \, F_{\bar{d}}^{H_2}(x_2) \, \hat{\sigma}_{d + \bar{d} \to \gamma^*}$$

and

$$F_{\bar{d}}^{H_1}(x_1) \, F_d^{H_2}(x_2) \, \hat{\sigma}_{\bar{d} + d \to \gamma^*}$$

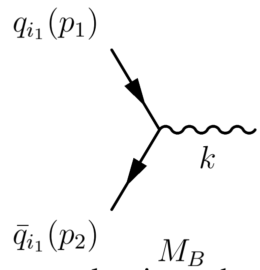
Hadronic cross section

Neglecting all the fermion masses, the partonic cross section is given by

$$\frac{d\hat{\sigma}_{i_1 + i_2 \to \gamma^*}}{dQ^2} = \frac{1}{4 \, p_1 . p_2} \int \frac{d^{n-1} p_3}{(2\,\pi)^{n-1} \, 2 \, E_3} \, (2\,\pi)^n \, \delta^n(p_1 + p_2 - p_3) \, |\overline{M}_B|^2$$

$|\overline{M}_B|^2$ is the squared amplitude M_B **averaged** over initial polarisations and colours and **summed** on the final polarisation and colours.

The amplitude M_B is described by the following Feynman diagram



M_B

and from the Feynman rules, it can be written as

$$M_B = -i\, e\, \mu^{\varepsilon}\, q_{i_1}\, \bar{v}_j(p_2)\, \gamma^{\mu}\, u_j(p_1)\, \epsilon_{\mu}(p_3)$$

where j represents the colour of the quarks (it is the same for the two lines since the $\gamma^{\star}$ is colourless!) and q_{i_1} the electric charge of the parton of type i_1 in unit of e. The squared matrix element is then given by

$$\begin{aligned} |M_B|^2 &= e^2\, q_{i_1}^2\, \mu^{2\,\varepsilon}\, \delta_{jj}\, \mathrm{Tr}\,[\not{p}_2\, \gamma_{\mu}\, \not{p}_1\, \gamma_{\nu}] \left(-g_{\mu\nu} + \frac{p_3^{\mu}\, p_3^{\nu}}{Q^2} \right) \\ &= 8\,(1-\varepsilon)\, e^2\, q_{i_1}^2\, \delta_{jj}\, p_1 \cdot p_2 \end{aligned}$$

with Q^2 is the virtuality of the photon (this is also the invariant mass of the lepton pair), Averaging over the initial colour and spin and summing over the final ones leads

$$|\overline{M}_B|^2 = \frac{2}{N}\,(1-\varepsilon)\, e^2\, q_{i_1}^2\, \mu^{2\,\varepsilon}\, p_1 \cdot p_2$$

The integration over the phase space can be done very easily by trading $d^{n-1}p_3/(2\,E_3)$ against $d^n p_3\, \delta^+(p_3^2 - Q^2)$ and integrating on $d^n p_3$ using the energy-momentum conservation $\delta^n(p_1 + p_2 - p_3)$ yielding

$$\frac{d\hat{\sigma}_{i_1+i_2\to\gamma^{\star}}}{dQ^2} = \frac{1}{4\, p_1.p_2}\,(2\,\pi)\, \delta^+((p_1+p_2)^2 - Q^2)\, |\overline{M}_B|^2$$

Let introduce some new variables. The available energy in the partonic centre of mass is $\sqrt{\hat{s}}$ with $\hat{s} = (p_1 + p_2)^2 = 2\, p_1 \cdot p_2$, we define $\tau = Q^2/s$ and $z = Q^2/\hat{s} = \tau/(x_1\, x_2)$. In terms of these new variables, the partonic cross section reads

$$\begin{aligned} \frac{d\hat{\sigma}_{i_1+i_2\to\gamma^{\star}}}{dQ^2} &= \frac{\pi}{Q^2}\, \frac{1}{N}\,(1-\varepsilon)\, e^2\, q_{i_1}^2\, \mu^{2\,\varepsilon}\, \delta(1-z) \\ &\equiv \hat{\sigma}_B(Q^2,\varepsilon)\, e^2\, q_{i_1}^2\, \delta(1-z) \end{aligned}$$

Note that this reaction is over constraint, the remaining Dirac distribution will disappear when integrating over x_1 and x_2. The hadronic cross section becomes

$$\frac{d\sigma_H}{dQ^2} = \sum_{i_1,i_2\in S_p} \int dx_1\, dx_2\, F_{i_1}^{H_1}(x_1)\, F_{i_2}^{H_2}(x_2)\, \hat{\sigma}_B(Q^2,\varepsilon)\, (e\, q_{i_1})^2\ \delta(1-z)$$
$$= e^2\, \frac{\hat{\sigma}_B(Q^2,\varepsilon)}{s} \sum_{i_1,i_2\in S_p} q_{i_1}^2 \int_{Q^2/S}^1 \frac{dx_1}{x_1}\, F_{i_1}^{H_1}(x_1)\, F_{i_2}^{H_2}\left(\frac{Q^2}{x_1\, S}\right)$$

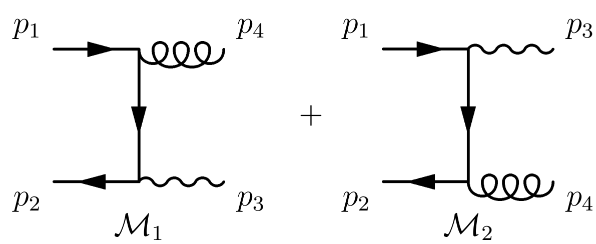
The lower bound of the x_1 integration is determined by requiring that

$$x_2 \leq 1 \quad \rightarrow \quad \frac{Q^2}{x_1\, S} \leq 1 \quad \rightarrow \quad x_1 \geq \frac{Q^2}{S}$$

1.3 α_s corrections : $q\,\bar{q}$ contribution

$q\,\bar{q}$ contribution

Let us focus on the reaction $q_i + \bar{q}_i \rightarrow \gamma^* + g$. It is described by two Feynman diagrams



The different amplitudes read

$$\mathcal{M}_1 = K\, \bar{v}(p_2)\, \gamma_\mu\, \frac{(\not{p}_1 - \not{p}_4)}{(p_1-p_4)^2 + i\,\lambda}\, \gamma_\nu\, u(p_1)\, \epsilon^\mu(p_3)\, \epsilon^\nu(p_4)$$
$$\mathcal{M}_2 = K\, \bar{v}(p_2)\, \gamma_\nu\, \frac{(\not{p}_4 - \not{p}_2)}{(p_4-p_2)^2 + i\,\lambda}\, \gamma_\mu\, u(p_1)\, \epsilon^\mu(p_3)\, \epsilon^\nu(p_4)$$

Soft approximation

Because of the mass shell conditions, the different denominators simplify as

$$(p_1 - p_4)^2 = -2\, p_1 \cdot p_4$$
$$(p_4 - p_2)^2 = -2\, p_2 \cdot p_4$$

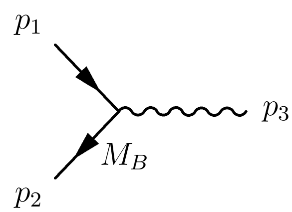
so they both go to zero when $p_4 \to 0$. One can write a soft approximation of these amplitudes by taking $p_4 = 0$ in the numerator and using the Dirac equations obey by the spinors $u(p_1)$ and $\bar{v}(p_2)$, namely $(\not p_1 - m)\, u(p_1) = 0$ and $\bar{v}(p_2)\,(\not p_2 + m) = 0$. One then gets

$$\mathcal{M}_{1\,\mathrm{soft}} = -K\, \frac{p_{1\,\nu}}{p_1 \cdot p_4}\, \bar{v}(p_2)\, \gamma_\mu\, u(p_1)\, \epsilon^\mu(p_3)\, \epsilon^\nu(p_4)$$
$$\mathcal{M}_{2\,\mathrm{soft}} = K\, \frac{p_{2\,\nu}}{p_2 \cdot p_4}\, \bar{v}(p_2)\, \gamma_\mu\, u(p_1)\, \epsilon^\mu(p_3)\, \epsilon^\nu(p_4)$$

that is to say

$$\mathcal{M}_{q\bar{q}\,\mathrm{soft}} \equiv M_{1\,\mathrm{soft}} + M_{2\,\mathrm{soft}} = \kappa \left[\frac{p_2 \cdot \epsilon(p_4)}{p_2 \cdot p_4} - \frac{p_1 \cdot \epsilon(p_4)}{p_1 \cdot p_4} \right] M_B$$

where M_B is the amplitude for the lower order diagram



Squared amplitude

The square matrix element is the given in this approximation

$$\overline{\Sigma}\, |\mathcal{M}|^2_{q\bar{q}\,\mathrm{soft}} = C\, \frac{p_1 \cdot p_2}{p_1 \cdot p_4\, p_2 \cdot p_4}\, |M_B|^2$$

Note that the **full amplitude squared** will have the following structure

$$\overline{\Sigma}\, |\mathcal{M}|^2_{q\bar{q}} = \left[H_{12}(p_4)\, \frac{p_1 \cdot p_2}{p_1 \cdot p_4\, p_2 \cdot p_4} + G(p_4) \right]$$

where the functions $H_{12}(p_4)$ and $G(p_4)$ are regular when $p_4 \to 0$ (or p_4 collinear to p_1 or p_2).It is clear, from this structure, that $|M_t|^2$ is singular in the soft limit ($p_4 \to 0$) and/or in the collinear limits ($p_4 = z_1\, p_1$ or $p_4 = z_2\, p_2$).

Since the extra gluon is not observed (in the sense that the sum over all the possibilities is taken). one has to integrate over the momentum p_4. The cross section is then given by

$$\frac{d\hat{\sigma}_{q_i+\bar{q}_i\to\gamma^*+g}}{dQ^2} = \frac{1}{4\,p_1.p_2}\int \frac{d^{n-1}p_3}{(2\,\pi)^{n-1}\,2\,E_3}\,\frac{d^{n-1}p_4}{(2\,\pi)^{n-1}\,2\,E_4} \times (2\,\pi)^n\,\delta^n(p_1+p_2-p_3-p_4)\,\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}}$$

At the hadronic level, the cross section is given by

$$\frac{d\sigma_H}{dQ^2} = \sum_{i_1,i_2\in S_p}\int dx_1\,dx_2\,F_{i_1}^{H_1}(x_1)\,F_{i_2}^{H_2}(x_2)\,\frac{d\hat{\sigma}_{q_i+\bar{q}_i\to\gamma^*+g}}{dQ^2} \tag{13}$$

The squared amplitude for $q\,\bar{q}$ case

The computation of the diagrams can be done easily (cf. Notes on the Compton effect in QED)

$$\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}} = (eq_{i_1}\mu^\varepsilon)^2(g\mu^\varepsilon)^2\,\frac{C_F}{N}2\,(1-\varepsilon)\left[(1-\varepsilon)(\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}})+2\,\frac{\hat{s}\,Q^2}{\hat{u}\,\hat{t}}-2\varepsilon\right],$$

Phase space integral I

In the centre of mass of initial partons, the parametrisation of the different 4-momenta are

$$p_1=(\sqrt{\hat{s}/2},0,\cdots,\sqrt{\hat{s}/2})\ ;\ p_2=(\sqrt{\hat{s}/2},0,\cdots,-\sqrt{\hat{s}/2})\ ;\ p_4=E_4\,(1,\cdots,\cos\theta_1). \tag{14}$$

To evaluate the integration over the phase space, we proceed as usual, keeping in mind that the virtual photon is massive

$$\begin{aligned} PS &= \int\frac{d^{n-1}p_4}{(2\pi)^{n-1}\,2\,E_4}\frac{d^{n-1}p_3}{(2\pi)^{n-1}\,2\,E_3}(2\pi)^n\delta^{(n)}(p_1+p_2-p_3-p_4) \\ &= (2\pi)^{2-n}\int\frac{d^{n-1}p_4}{2\,E_4}\,\delta^+((p_1+p_2-p_4)^2-Q^2) \\ &= \frac{(2\pi)^{2-n}}{4\sqrt{\hat{s}}}\left(\frac{\hat{s}-Q^2}{2\sqrt{\hat{s}}}\right)^{n-3}\int d\Omega_{n-2}. \end{aligned} \tag{15}$$

Phase space integral II

To perform the **angular integration**, the following change of variable is introduced $\cos\theta_1 = 2y - 1$, this leads to

$$PS = \frac{1}{8\pi} \left(\frac{4\pi}{Q^2}\right)^{\varepsilon} \frac{z^{\varepsilon}(1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 dy\, y^{-\varepsilon}(1-y)^{-\varepsilon}, \tag{16}$$

In terms of these dimensionless variables, the different invariants are

$$\hat{s} = \frac{Q^2}{z}\,;\ (p_1 - p_4)^2 = \hat{t} = -\frac{Q^2}{z}\,(1-y)(1-z)\,;\ (p_2 - p_4)^2 = \hat{u} = -\frac{Q^2}{z}\,(1-z)y \tag{17}$$

Extraction of divergent terms I

What is interesting is the coefficient H_{12}, it can be extracted by factorising $\hat{s}/(\hat{t}\,\hat{u})$, which is equal to E_{12}, in the right hand side of eq. (**??**)

$$H_{12}(y,z) = (eq_{i_1}\mu^{\varepsilon})^2 (g\mu^{\varepsilon})^2 \frac{C_F}{N} (1-\varepsilon) \left[(1-\varepsilon)\,\frac{\hat{t}^2 + \hat{u}^2}{\hat{s}} + 2\,Q^2\right]$$

In terms of the new variables y and z, we have

$$H_{12}(y,z) = (eq_{i_1}\mu^{\varepsilon})^2 (g\mu^{\varepsilon})^2 \frac{C_F}{N} (1-\varepsilon)\,\frac{Q^2}{z}\,\left\{(1-\varepsilon)\,(1-z)^2\,\left[(1-y)^2 + y^2\right] + 2\,z\right\}$$

which can also be written as

$$H_{12}(y,z) = (g\mu^{\varepsilon})^2\, C_F\, |\overline{M}_B|^2\, \frac{1}{z}\,\left\{(1-\varepsilon)\,(1-z)^2\,\left[(1-y)^2 + y^2\right] + 2\,z\right\}$$

The **eikonal factor** E_{12} can be also expressed in terms of the variables y and z

$$\begin{aligned} E_{12} &= \frac{2}{Q^2}\,\frac{z}{(1-z)^2}\,\frac{1}{y\,(1-y)} \\ &= \frac{2}{Q^2}\,\frac{z}{(1-z)^2}\,\left[\frac{1}{y} + \frac{1}{1-y}\right] \\ &\equiv E_{12}^{(1)} + E_{12}^{(2)} \end{aligned}$$

Extraction of divergent terms II

The key idea to extract the divergent parts is to write

$$H_{12}(y,z)\,E_{12} = [H_{12}(y,z) - H_{12}(y,1)]\,E_{12} + H_{12}(y,1)\,E_{12}$$

The first term in squared bracket does not diverge when $z \to 1$ but does diverge when $y \to 0$ or $y \to 1$ (Note that in this simple case $H_{12}(y,1)$ does not depend any more on y). To single out the y divergent parts, the product $H_{12}(y,z)\,E_{12}$ has to be written as

$$\begin{aligned} H_{12}(y,z)\,E_{12} = {} & [H_{12}(y,z) - H_{12}(y,1) - H_{12}(0,z) + H_{12}(0,1)]\,E_{12}^{(1)} \\ & + [H_{12}(y,z) - H_{12}(y,1) - H_{12}(1,z) + H_{12}(1,1)]\,E_{12}^{(2)} \\ & + H_{12}(y,1)\,E_{12} + [H_{12}(0,z) - H_{12}(0,1)]\,E_{12}^{(1)} \\ & + [H_{12}(1,z) - H_{12}(1,1)]\,E_{12}^{(2)} \end{aligned}$$

Using the fact that $H_{12}(y,1) = H_{12}(0,1) = H_{12}(1,1)$, the last equation can be simplified

$$\begin{aligned} H_{12}(y,z)\,E_{12} = {} & [H_{12}(y,z) - H_{12}(0,z)]\,E_{12}^{(1)} \\ & + [H_{12}(y,z) - H_{12}(1,z)]\,E_{12}^{(2)} \\ & + H_{12}(y,1)\,E_{12} + [H_{12}(0,z) - H_{12}(0,1)]\,E_{12}^{(1)} \\ & + [H_{12}(1,z) - H_{12}(1,1)]\,E_{12}^{(2)} \end{aligned}$$

The two first terms will give finite terms after the phase space integration, let us disregard them and focus on the divergent pieces. The phase space integration will lead to

$$\begin{aligned} PS\,\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}} = {} & PS\,\{H_{12}(y,1)\,E_{12} + [H_{12}(0,z) - H_{12}(0,1)]\,E_{12}^{(1)} \\ & + [H_{12}(1,z) - H_{12}(1,1)]\,E_{12}^{(2)}\} + \text{finite pieces} \\ = {} & PS\,\{H_{12}(0,z)\,E_{12}^{(1)} + H_{12}(1,z)\,E_{12}^{(2)}\} + \text{finite pieces} \end{aligned}$$

Let us first evaluate the function H_{12} with the different arguments

$$H_{12}(0,z) = (g\mu^{\varepsilon})^2\,C_F\,|\overline{M}_B|^2\,\frac{1}{z}\,\left[1 + z^2 - \varepsilon\,(1-z)^2\right] = H_{12}(1,z)$$

Let us compute the different pieces

$$
\begin{aligned}
PS\, H_{12}(0,z)\, E_{12}^{(1)} &= H_{12}(0,1)\, PS\, E_{12} \\
&= \frac{1}{4\,\pi\, Q^2} \left(\frac{4\,\pi}{Q^2}\right)^{\varepsilon} z^{1+\varepsilon}\,(1-z)^{-1-2\,\varepsilon}\, H_{12}(0,z) \\
&\quad \times \left(-\frac{1}{\varepsilon}\right) \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}
\end{aligned}
$$

"Plus" distributions I

Note that the integration over z does not show up because we are differential with respect to Q^2 but there is a problem at $z = 1$ due to the term $(1-z)^{-1-2\,\varepsilon}$. Nevertheless, we can make the pole in ε related to the singularity at $z = 1$ appear. For that, we have to realise that $(1-z)^{-1-2\,\varepsilon}$ at the limit $\varepsilon = 0$ is a distribution, remember that the Dirac distribution can be obtained as

$$
\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon\,\sqrt{\pi}}\, e^{-\left(\frac{x}{\varepsilon}\right)^2}
$$

Thus, to discuss its property we have to apply it to a test function. Let us introduce a test function $F(z)$ which is regular at $z - 1$

$$
\begin{aligned}
&\int_0^1 dz\, F(z)\,(1-z)^{-1-2\,\varepsilon} \\
&\quad = \int_0^1 dz\,(F(z)-F(1))\,(1-z)^{-1-2\,\varepsilon} + F(1)\int_0^1 dz\,(1-z)^{-1-2\,\varepsilon} \\
&\quad = \int_0^1 dz\, \frac{F(z)-F(1)}{1-z} \sum_{n=0}^{\infty} (-2\,\varepsilon)^n\, \ln^n(1-z) - F(1)\,\frac{1}{2\,\varepsilon}
\end{aligned}
$$

"Plus" distributions II

Thus, in the distribution sense, we can write that

$$
(1-z)^{-1-2\,\varepsilon} = -\frac{1}{2\,\varepsilon}\,\delta(1-z) + \frac{1}{(1-z)_+} - 2\,\varepsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ + O(\varepsilon^2)
$$

where the "plus" distributions are defined as

$$
\int_0^1 dz\,(g(z))_+\, F(z) \equiv \int_0^1 dz\, g(z)\,(F(z)-F(1)) \tag{18}
$$

where $g(z)$ is a function singular at $z = 1$ such that $(1 - z)\, g(z)$ is integrable and $F(z)$ is a regular one at the same point. Note that the lower bound 0 in the integral is purely conventional.

Let us make a remark. Due to this convention (zero as lower bound), it is easy to verify that

$$\int_0^1 dz\,(g(z))_+ = 0$$

To show that, choose a test function which is a constant! Sometimes, it appears, because of the kinematics or the cuts, that the lower bound is different from zero, thus we have to compute something like

$$\begin{aligned}
&\int_a^1 dz\, F(z)\,(g(z))_+ \\
&\quad = \int_a^1 dz\,(F(z) - F(1))\,g(z) + F(1)\,\int_a^1 dz(g(z))_+ \\
&\quad = \int_a^1 dz\,(F(z) - F(1))\,g(z) + F(1)\,\left[\int_0^1 dz\,(g(z))_+ - \int_0^a dz\,(g(z))_+\right] \\
&\quad = \int_a^1 dz\,(F(z) - F(1))\,g(z) - F(1)\,\int_0^a dz\, g(z)
\end{aligned}$$

Final result I

Let us set $a_{qq}^{(n)}(z) \equiv C_F\,(1 + z^2 - \varepsilon\,(1 - z)^2)$, the phase space integration becomes

$$\begin{aligned}
PS\, H_{12}(0,z)\, E_{12}^{(1)} &= \frac{\alpha_s}{Q^2}\left(\frac{4\,\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,|\overline{M}_B|^2 \\
&\times\left\{\frac{1}{2\,\varepsilon^2}\,\delta(1-z)\,a_{qq}^{(n)}(1) - \frac{1}{\varepsilon}\,\frac{a_{qq}^{(n)}(z)}{(1-z)_+} - \frac{a_{qq}^{(4)}(z)}{(1-z)_+}\,\ln(z)\right. \\
&\left.\quad + \; 2\,a_{qq}^{(4)}(z)\,\left(\frac{\ln(1-z)}{1-z}\right)_+\right\} + O(\varepsilon^2)
\end{aligned}$$

Since $H_{12}(0,z) = H_{12}(1,z)$ and since the phase space is **symmetric** $y \leftrightarrow 1 - y$, the contribution which diverges at $y = 1$ will be equal to the one which diverges at $y = 0$.

Final result II

Thus, the total contributions will be given by

$$\frac{1}{2\,\hat{s}}\,PS\,\left(H_{12}(0,z)\,E_{12}^{(1)}+H_{12}(1,z)\,E_{12}^{(2)}\right)$$
$$=z\,\frac{\alpha_s}{2\,\pi}\,\left(\frac{4\,\pi\mu^2}{Q^2}\right)^{\varepsilon}\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\hat{\sigma}_B(Q^2,\varepsilon)\,e^2\,q_i^2\,F_{q\bar{q}}(z,\varepsilon)$$

with

$$F_{q\bar{q}}(z,\varepsilon)=\frac{1}{\varepsilon^2}\,\delta(1-z)\,a_{qq}^{(n)}(1)-\frac{2}{\varepsilon}\,\frac{a_{qq}^{(n)}(z)}{(1-z)_+}-2\,\frac{a_{qq}^{(4)}(z)}{(1-z)_+}\,\ln(z)$$
$$+\,4\,a_{qq}^{(4)}(z)\,\left(\frac{\ln(1-z)}{1-z}\right)_+ + \text{finite terms}$$

Remarks

It is worthwhile to note that the coefficients in front of the divergences "factorise", in the sense that they can be written as a function of z (or taken at $z=1$) times the cross section at lowest order. This factorisation takes place also in a n dimensional phase space. The first term (proportional to $1/\varepsilon^2$) originates from the soft region while the second term (proportional to $1/\varepsilon$) has a collinear origin. Note also that the variable z plays the role of a "collinear" variable. Indeed, let us denote k_1, the 4-momentum of the quark after the emission of the gluon of 4-momentum p_4. The energy momentum conservation at each vertices impose that

$$p_3=k_1+p_2 \tag{19}$$
$$p_1=p_4+k_1 \tag{20}$$

We get from equation (20) that $k_1^2=-2\,p_1\cdot p_4=\hat{t}$ and from eq. (19) that $Q^2=k_1^2+2\,k_1\cdot p_2$, combining these two equalities leads to

$$Q^2=-\frac{Q^2}{z}\,(1-z)\,(1-y)+2\,k_1\cdot p_2$$

At $y=1$, $Q^2=2\,k_1\cdot p_2$, thus

$$\hat{s}=2\,p_1\cdot p_2=\frac{Q^2}{z}=\frac{2\,k_1\cdot p_2}{z}$$

implying that $k_1 = z\,p_1$. To sum up, when $y = 1$, p_4 is collinear to p_1 ($p_4 = (1-z)\,p_1$) and the variable z represent the fraction of 4-momentum carried away by the particle j_1 (having a 4-momentum k_1) from the particle i_1 (having a 4-momentum p_1). Using a similar argument, it is easy to show that the variable z plays a similar role when $y = 0$.

1.4 α_s corrections : $q\,g$ contribution

The squared amplitude for $q\,g$ case

The amplitude squared can be obtained from the preceding case by exchanging $\hat{s} \leftrightarrow \hat{t}$ and multiplying by -1 because an anti-fermion of the initial state becomes a fermion in the final state. Let us note that the colour factor changes because there is a gluon in the initial state instead of a quark, so the averaged changes. To get the right factor, we have to take the preceding one C_F/N and multiplies it by $N/(N^2-1)$

$$\frac{C_F}{N}\,\frac{N}{N^2-1} = \frac{N^2-1}{2\,N^2}\,\frac{N}{N^2-1} = \frac{1}{2\,N}$$

The squared amplitude for the reaction $q + g \to \gamma^\star + q$ is then

$$\overline{\Sigma}\,|\mathcal{M}|^2_{qg} = (1-\varepsilon)\,(ee_q\mu^\varepsilon)^2(g\mu^\varepsilon)^2\,\frac{1}{2\,N}\,2\,\Bigg[(1-\varepsilon)(-\frac{\hat{s}}{\hat{u}} - \frac{\hat{u}}{\hat{s}}) - 2\,\frac{\hat{t}\,q^2}{\hat{u}\,\hat{s}} + 2\varepsilon\Bigg]$$

Extraction of divergent terms I

In this case, the coefficient of the eikonal factor E_{12} can be easily extracted and is given by

$$H_{12}(y,z) = (1-\varepsilon)\,(e\,q_i\,\mu^\varepsilon)^2\,(g\,\mu^\varepsilon)^2\,\frac{1}{2\,N}\,\frac{Q^2}{z} \times \left[(1-\varepsilon)\,(1+(1-z)^2\,y^2)\,(1-y)\,(1-z) - 2\,z\,(1-y)\,(1-z)^2\right]$$

Note that in this case, $H_{12}(y,1) = 0 = H_{12}(0,1) = H_{12}(1,1)$ which is an expected result because, at lowest order, there is not such a initial state! Note also that $H_{12}(1,z) = 0$ telling us that there is no divergence when p_4 is collinear to p_1 in this case. The only divergence appears at $y = 0$

Extraction of divergent terms II

$$H_{12}(0,z) = (1-\varepsilon)\,(e\,q_i\,\mu^\varepsilon)^2\,(g\,\mu^\varepsilon)^2\,\frac{1}{2\,N}\,\frac{Q^2}{z}\,(1-z)\,\left[(1-z)^2+z^2-\varepsilon\right]$$

Let us introduce $a_{qg}^{(n)}(z) = 1/2\,(1-z)\,[(1-z)^2+z^2-\varepsilon]$, $H_{12}(0,z)$ becomes then

$$H_{12}(0,z) = (g\,\mu^\varepsilon)^2\,|\overline{M}_B|^2\,\frac{Q^2}{z}\,a_{qg}^{(n)}(z)$$

As in the preceding case, we want to pick up only the divergent part, for that, what we have to compute is the following

$$PS\,\overline{\Sigma}\,|\mathcal{M}|^2_{qg} = PS\,H_{12}(0,z)\,E_{12}^{(1)} + \text{finite pieces}$$

With the help of the results got in the preceding subsections, we obtain

$$\begin{aligned} PS\,H_{12}(0,z)\,E_{12}^{(1)} = \frac{\alpha_s}{Q^2}\,\left(\frac{4\,\pi\mu^2}{Q^2}\right)^\varepsilon\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,|\overline{M}_B|^2 \\ \times\left\{-\frac{1}{\varepsilon}\,\frac{a_{qg}^{(n)}(z)}{(1-z)_+} - \frac{a_{qg}^{(4)}(z)}{(1-z)_+}\,\ln(z)\right. \\ \left. + 2\,a_{qg}^{(4)}(z)\,\left(\frac{\ln(1-z)}{1-z}\right)_+\right\} + O(\varepsilon^2) \end{aligned}$$

Final result

Note that, as already seen, there is no divergence at $z=1$, thus $a_{qg}^{(n)}(1)$ must be zero, in agreement with its definition, this the reason why there is no term proportional to $1/\varepsilon^2$. But, we will keep the notation $a_{qg}^{(n)}(z)/(1-z)_+$ despite the fact that the factor $1-z$ cancels between the numerator and the denominator for matter of uniformity.

$$\begin{aligned} &\frac{1}{2\,\hat{s}}\,PS\,H_{12}(0,z)\,E_{12}^{(1)} \\ &\quad = z\,\frac{\alpha_s}{2\,\pi}\,\left(\frac{4\,\pi\mu^2}{Q^2}\right)^\varepsilon\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\hat{\sigma}_B(Q^2,\varepsilon)\,e^2\,q_i^2\,F_{qg}(z,\varepsilon) \end{aligned}$$

with

$$F_{qg}(z,\varepsilon) = -\frac{1}{\varepsilon}\,\frac{a_{qg}^{(n)}(z)}{(1-z)_+} - \frac{a_{qg}^{(4)}(z)}{(1-z)_+}\,\ln(z) + a_{qg}^{(4)}(z)\,\left(\frac{\ln(1-z)}{1-z}\right)_+ + \text{finite terms}$$

1.5 The virtual term

Virtual cross section

The details of the computation of the virtual contribution is postponed in appendix A. The result is

$$\begin{aligned} M_v &= M_B\,\frac{\alpha_S}{4\,\pi}\,C_F\,\left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\varepsilon}\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \\ &\quad \times \left\{-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\,\pi^2}{3} + i\,\pi\left\{-\frac{2}{\varepsilon} - 3\right\}\right\} \end{aligned} \tag{21}$$

Note that to take the same convention as for the real emission, we change $\varepsilon_{ir} = -\varepsilon$.

The virtual cross section is obtained by taking the **interference** between the lowest order amplitude and the virtual one

$$\begin{aligned} \sigma_v &= \frac{1}{4\,p_1.p_2}\int \frac{d^n\,p_3}{(2\,\pi)^{n-1}} \\ &\quad \times (2\,\pi)^n\,\delta^n(p_1+p_2-p_3-p_4)\,\delta^+(p_3^2) \\ &\quad \times 2\,Re(M_B\,M_v^*) \end{aligned} \tag{22}$$

equivalently

$$\begin{aligned} \sigma_v &= \hat{\sigma}_B(Q^2,\varepsilon)\,\frac{\alpha_s}{2\,\pi}\,C_F\,\left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\varepsilon}\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \\ &\quad \times \left\{-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\,\pi^2}{3}\right\} \end{aligned} \tag{23}$$

Total contribution I

Collecting all the different pieces, the differential hadronic cross section can be written as

$$\frac{d\sigma_H}{dQ^2} = \sum_{i\in S_q} (e\, q_i)^2 \int \frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\, F_{q_i}^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \left\{ \hat{\sigma}_B(Q^2,\varepsilon) \left[\delta(1-z) \right.\right.$$

$$\times \left(1 + \frac{\alpha_s}{2\,\pi} \left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \left\{ -\frac{2\,C_F}{\varepsilon^2} - \frac{3\,C_F}{\varepsilon} + \frac{a_{qq}^{(4)}(1)}{\varepsilon^2} \right\} \right)$$

$$\left. - \frac{\alpha_s}{2\,\pi} \left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\, \frac{2}{\varepsilon} \left(\frac{a_{qq}^{(a4)}(z)}{(1-z)_+} \right) \right]$$

$$\left. + \text{finite pieces} \right\}$$

$$- \left(F_{q_i}^{H_1}(x_1) + F_{\bar{q}_i}^{H_1}(x_1)\right) F_g^{H_2}(x_2)$$

$$\times \frac{\alpha_s}{2\,\pi} \left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\, \frac{1}{\varepsilon} \left(\frac{a_{qg}^{(4)}(z)}{(1-z)_+} \right)$$

$$\left. + \text{finite pieces } + \left[1 \leftrightarrow 2 \right] \right\} \qquad (24)$$

Disappearance of soft divergences

From the definition of $a^{(4)}(1) = 2\,C_F$, we see that the soft divergence (term proportional to $1/\varepsilon^2$) in eq. (24) cancels between the real emission and the virtual one. This is not specific to the example we treated, this is the "Lee – Kinoshita – Naurenberg" theorem which states that the soft divergences drop out when adding the real and virtual emission. Nevertheless, all the divergences do not disappear, the collinear ones still remain after combining the real and the virtual emission, so what to do?

Collinear contributions I

Let us re-write eq. (24)

$$\frac{d\sigma_H}{dQ^2} = \sum_{i\in S_q} (e\,q_i)^2\,\hat{\sigma}_B(Q^2,\varepsilon)\,\int \frac{dx_1}{x_1}\frac{dx_2}{x_2}\left\{\left[F_{q_i}^{H_1}(x_1)\,F_{\bar{q}_i}^{H_2}(x_2)\delta(1-z)\right.\right.$$
$$+\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\left(F_{q_i}^{H_1}(x_1)\,F_{\bar{q}_i}^{H_2}(x_2)\left(-\frac{2}{\varepsilon}\right)P_{qq}(z)\right.$$
$$\left.+\,(F_{q_i}^{H_1}(x_1)\,F_g^{H_2}(x_2)+F_g^{H_1}(x_1)\,F_{\bar{q}_i}^{H_2}(x_2))\left(-\frac{1}{\varepsilon}\right)P_{qg}(z)\right)$$
$$\left.\left.+\text{finite pieces}+\left[1\leftrightarrow 2\right]\right]\right\}$$

with

$$P_{qq}(z) = C_F\,\left[\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\,\delta(1-z)\right]$$
$$P_{qg}(z) = T_R\,\left[z^2+(1-z)^2\right]$$

and $T_R = 1/2$

Collinear contributions II

Let us focus on the divergent part only. By changing x_2 (or x_1) into $\tau/(x_1\,z)$ $(\tau/(x_2\,z))$, this divergent part can be written as

$$\frac{d\sigma_{H\,\text{div}}}{dQ^2} = \sum_{i\in S_q}(eq_i)^2\,\hat{\sigma}_B(Q^2,\varepsilon)\,\left\{\left[\left[\int_0^1\frac{dx_1}{x_1}\,F_{q_i}^{H_1}(x_1)\,F_{\bar{q}_i}^{H_2}\left(\frac{\tau}{x_1}\right)\right.\right.\right.$$
$$+\int_0^1\frac{dx_1}{x_1}\,F_{q_i}^{H_1}(x_1)\left(-\frac{1}{\varepsilon}\right)\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$
$$\times\int_{\frac{\tau}{x_1}}^1\frac{dz}{z}\left(F_{\bar{q}_i}^{H_2}\left(\frac{\tau/x_1}{z}\right)\,P_{qq}(z)+F_g^{H_2}\left(\frac{\tau/x_1}{z}\right)\,P_{qg}(z)\right)$$
$$+\int_0^1\frac{dx_2}{x_2}\,F_{\bar{q}_i}^{H_2}(x_2)\left(-\frac{1}{\varepsilon}\right)\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$
$$\left.\times\int_{\frac{\tau}{x_2}}^1\frac{dz}{z}\left(F_{q_i}^{H_1}\left(\frac{\tau/x_2}{z}\right)\,P_{qq}(z)+F_g^{H_1}\left(\frac{\tau/x_2}{z}\right)\,P_{qg}(z)\right)\right]$$
$$\left.+\left[1\leftrightarrow 2\right]\right\} \tag{25}$$

Scale dependent PDF

From this formula, by introducing the scale dependent partonic density function (PDF)

$$F_q^H(x, M^2) = F_q^H(x) - \frac{1}{\varepsilon}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \times \int_x^1 \frac{dz}{z}\left[F_q^H\left(\frac{x}{z}\right) P_{qq}(z) + F_g^H\left(\frac{x}{z}\right) P_{qg}(z)\right]$$

One can reabsorb the **collinear divergences** into a redefinition of **the "bare" parton densities** (the ones with no scale) and up to terms of order α_s^2, the divergent term can be written as

$$\frac{d\sigma_{H\,\mathrm{div}}}{dQ^2} = \sum_{i\in S_q}(eq_i)^2\,\hat{\sigma}_B(Q^2,\varepsilon)\left\{\int_0^1\frac{dx_1}{x_1}F_{q_i}^{H_1}(x_1,M^2)\,F_{\bar{q}_i}^{H_2}\left(\frac{\tau}{x_1},M^2\right) + \left[1\leftrightarrow 2\right]\right\} \tag{26}$$

The scale M^2 is arbitrary, it has been introduced by writing

$$\left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} = \left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\left(\frac{M^2}{Q^2}\right)^{\varepsilon} \simeq \left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\left[1+\varepsilon\,\ln\left(\frac{M^2}{Q^2}\right)+O(\varepsilon^2)\right]$$

Drell-Yan at α_s

Thus the Drell-Yan cross section, including the α_s **corrections**, can be written as

$$\frac{d\sigma_H}{dQ^2} = \sum_{i\in S_q}(e\,q_i)^2\,\hat{\sigma}_B(Q^2,\varepsilon)\left\{\int_0^1\frac{dx_1}{x_1}\int_0^1\frac{dx_2}{x_2}F_{q_i}^{H_1}(x_1,M^2)\,F_{\bar{q}_i}^{H_2}\left(x_2,M^2\right) + \frac{\alpha_s}{2\pi}\,[\text{finite pieces}] + \left[1\leftrightarrow 2\right]\right\}$$

Note that the procedure to get rid of the **collinear divergences** is very similar to the **renormalisation procedure**. As in the renormalisation, it exists **RGE for the PDF**.

RGE for PDF

Let us come back to the scale dependent PDF. the derivative of this function with respect to M^2 gives an expression which is finite when $\varepsilon \to 0$. It is easy to realise that by writing

$$\frac{1}{\varepsilon}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} = \frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma - \ln\left(\frac{M^2}{\mu^2}\right) + O(\varepsilon)$$

In addition, it can be shown that including the dominant contribution at each order in perturbation theory for the collinear divergence that

$$M^2\,\frac{dF_q^H(x,M^2)}{dM^2} = \frac{\alpha_s(M^2)}{2\,\pi}\sum_{j=q,g}\int_0^1 \frac{dy}{y}\,P_{qj}\left(\frac{x}{y}\right)\,F_q^H(y,M^2)$$

From the study of other processes, especially processes involving gluons in the initial state at higher order, leads to the general RGE (called "DGLAP")

$$\frac{d}{dt}\begin{pmatrix} F_q^H(x,t) \\ F_g^H(x,t)\end{pmatrix} = \frac{\alpha_s(t)}{2\,\pi}\int_x^1 \frac{dy}{y}\begin{pmatrix} P_{qq}^{(0)}(y)\;P_{qg}^{(0)}(y) \\ P_{gq}^{(0)}(y)\;P_{gg}^{(0)}(y)\end{pmatrix}\begin{pmatrix} F_q^H(x/y,t) \\ F_g^H(x/y,t)\end{pmatrix}$$

where $t = \ln(M^2/M_0^2)$.

DGLAP kernels

We put a superscript on the kernel to say that they are computed in the lowest order. Including more order in the calculation, leads to more complicated kernel

$$P_{ij}(z) = P_{ij}^{(0)}(z) + \frac{\alpha_s}{2\,\pi}\,P_{ij}^{(1)}(z) + \ldots$$

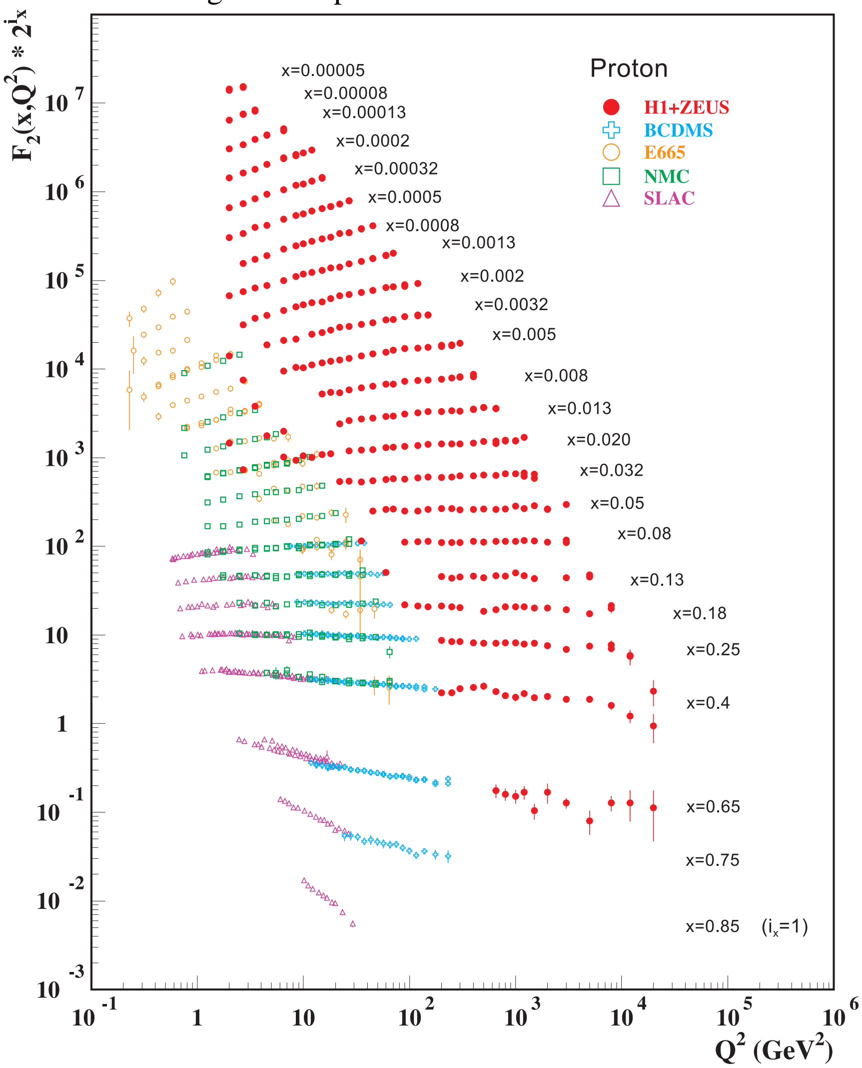
The value of these kernels are

$$\begin{aligned}
P_{qq}^{(0)}(y) &= C_F\left[\frac{1+y^2}{(1-y)_+} + \frac{3}{2}\,\delta(1-y)\right] \\
P_{qg}^{(0)}(y) &= \frac{N_F}{2}\,\frac{y^2+(1-y)^2}{y} \\
P_{gq}^{(0)}(y) &= C_F\left[\frac{1+(1-y)^2}{y}\right] \\
P_{gg}^{(0)}(y) &= 2\,N\left[\frac{1}{(1-y)_+} + \frac{1-y}{y} + y\,(1-y)\right] + \delta(1-y)\,\frac{b_0}{2\,\pi}
\end{aligned}$$

with $b_0 = (11\,N - 2\,N_F)/(12\,\pi)$

violation of the "scale invariance"

It is interesting to note that using scale dependent PDF induces a violation of



the scale invariance

The QCD improved parton model

Taking into account the QCD interactions of the partons between themselves leads to the "QCD improved" parton model. The formulation is the same as in the "naive" parton model, indeed

$$\sigma^{H_1 H_2} = \sum_{i,j} \int dx_1 dx_2 \, F_i^{H_1}(x_1, M^2) \, F_j^{H_2}(x_2, M^2) \, \alpha_s(\mu^2)^p \, \hat{\sigma}_{ij}(x_1, x_2, s).$$

where the evolved PDF obey to the DGLAP equations. They still have to be extracted from experiments at a certain scale and can be used to another scale.

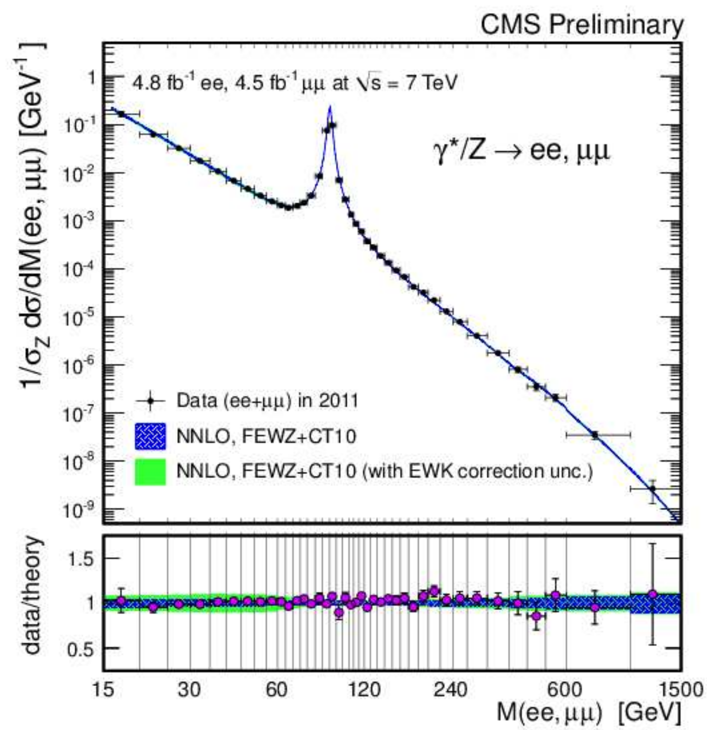
Higher order can be included

$$\hat{\sigma}_{ij}(x_1, x_2, s) = \hat{\sigma}_{ij}^{(0)}(x_1, x_2, s) + \frac{\alpha_s}{2\,\pi}\,\hat{\sigma}_{ij}^{(1)}(x_1, x_2, s) + \dots$$

The terminology is the following:

Leading Order (LO) approximation : compute $\hat{\sigma^{(0)}}_{ij}(x_1, x_2, s)$ and use $P_{ij}^{(0)}(y)$ of the DGLAP evolution **Next to Leading Order (NLO) approximation** : compute $\hat{\sigma^{(0)}}_{ij}(x_1, x_2, s)$ and $\hat{\sigma^{(1)}}_{ij}(x_1, x_2, s)$ and use $P_{ij}^{(0)}(y)$ and $P_{ij}^{(1)}(y)$ of the DGLAP evolution ....

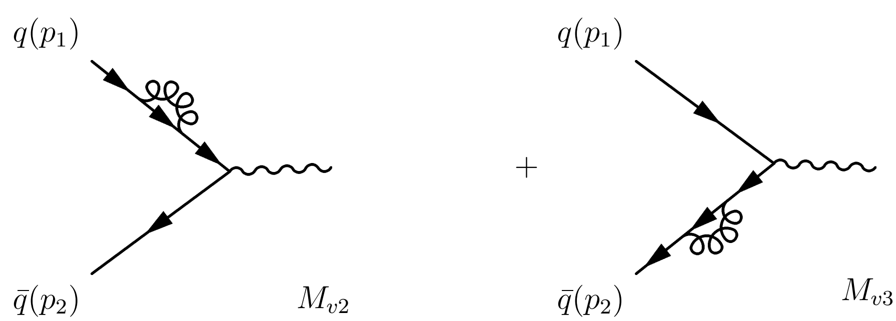
The more we include terms coming perturbative expansion, the more the theoretical results are precise : the uncertainty related to the renormalisation and factorisation scale is reduced.

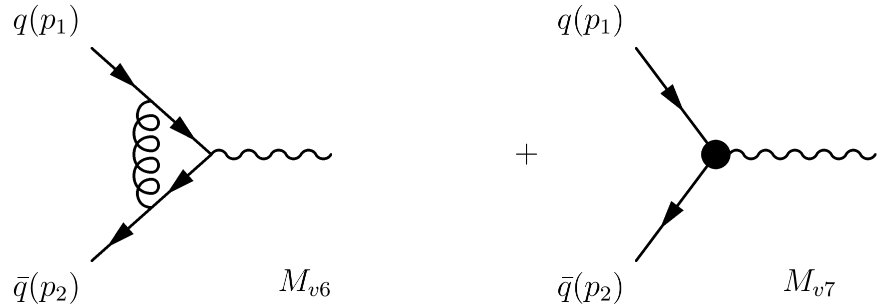


What we learnt in lecture IV

- It exists other **kinds of divergences** than the UV ones : the **soft divergences** when the energy of a massless boson goes to zero and the **collinear divergences** when two massless partons becomes parallel
- The **soft** divergences cancel when adding **virtual and real contributions**
- The **collinear** divergences (in the initial state) are absorbed into a **redefinition of the PDF**
- Leading to an evolution equation (similar to RGE) for the PDF
- The "QCD improved" parton model which take into account the interaction between parton (to a certain approximation!) leads to a **violation** of the scale invariance verified experimentally
- The "QCD improved" parton model to describe very well **the LHC data**

A Computation of the virtual contribution





The amplitudes M_{v1} et M_{v2} are not "real" Feynman amplitudes, they represent the correction to the quark and anti-quark wave functions. To compute them, we need the residue of the pole of the one loop quark propagator given by (in Feynman gauge)

$$\Sigma^1 = -\frac{\alpha_s}{(4\pi)}\, C_F\, \left[\frac{1}{\varepsilon_{ir}} + \gamma - \ln(4\pi)\right] = -\frac{\alpha_S}{(4\pi)}\, C_F\, (4\,\pi)^{-\varepsilon_{ir}}\, \frac{\Gamma(1-\varepsilon_{ir})}{\varepsilon_{ir}} \quad (27)$$

The wave function normalisation is $\sqrt{1/(1-\Sigma^1)} \simeq 1 + 1/2\,\Sigma^1$, thus the two amplitudes M_{v1} and M_{v2} are given by

$$M_{v1} = M_B \left[-\frac{\alpha_S}{(8\,\pi)}\, C_F\, \left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{-\varepsilon_{ir}} \left(\frac{Q^2}{\mu^2}\right)^{-\varepsilon_{ir}} \frac{\Gamma(1-\varepsilon_{ir})}{\varepsilon_{ir}}\right] \quad (28)$$

$$M_{v2} = M_B \left[-\frac{\alpha_S}{(8\,\pi)}\, C_F\, \left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{-\varepsilon_{ir}} \left(\frac{Q^2}{\mu^2}\right)^{-\varepsilon_{ir}} \frac{\Gamma(1-\varepsilon_{ir})}{\varepsilon_{ir}}\right] \quad (29)$$

artificially, the energy scale Q has been introduced.

A.0.1 Vertex corrections

The computation of the diagram M_{v3} is done with the help of the QED vertex correction $e^-\,e^+\,\gamma$. The limit $m \to 0$ has to be taken, but, we have to be careful, this limit must be taken before the development around $\varepsilon_{ir} = 0$. In the note blabla1, the following resuts have been obtained

$$\Lambda_\mu^{(1)UV} = \frac{\alpha}{4\,\pi}\, \gamma^\mu\, \left(\frac{1}{\varepsilon_{uv}} - \gamma + \ln(4\,\pi) - 1 - I_3\right) \quad (30)$$

with

$$I_3 = \int_0^1 dx\ \ln\left(\frac{m^2 - (q^2 + i\,\lambda)\,x\,(1-x)}{\mu^2}\right). \tag{31}$$

and

$$\begin{aligned}
\Lambda_\mu^{(1)IR} &= -\frac{\alpha}{4\,\pi}\ \left(4\,\pi\,\mu^2\right)^{-\varepsilon_{ir}}\ \Gamma(1-\varepsilon_{ir}) \\
&\times \int_0^1 dx\ (m^2 - (q^2 + i\,\lambda)\,x\,(1-x))^{-1+\varepsilon_{ir}} \left\{ (2\,m^2 - q^2)\ \frac{\gamma^\mu}{\varepsilon_{ir}} \right. \\
&+ \gamma^\mu \left[(m^2 - (q^2 + i\,\lambda)\,x\,(1-x)) - \frac{4}{1+2\,\varepsilon_{ir}}\ \left(2\,m^2 - \frac{q^2}{2}\right)\right] \\
&\left. + \frac{4}{1+2\,\varepsilon_{ir}}\ \frac{1}{2}(p_3 - p_4)^\mu\,m - m\,(p_3 - p_4)^\mu \right\}
\end{aligned} \tag{32}$$

With respect to the note blabla1, we have taken $p_1 = p_3$ et $p_2 = -p_4$. At the limit $m \to 0$, the integral I_3 becomes

$$I_3 = \ln\left(\frac{-(q^2 + i\,\lambda)}{\mu^2}\right) - 2 \tag{33}$$

and the equation (32) goes to

$$\begin{aligned}
\Lambda_\mu^{(1)IR} &= -\frac{\alpha}{4\,\pi}\ \left(4\,\pi\,\mu^2\right)^{-\varepsilon_{ir}}\ \Gamma(1-\varepsilon_{ir}) \\
&\times \int_0^1 dx\ (-(q^2 + i\,\lambda))^{-1+\varepsilon_{ir}}\,(x\,(1-x))^{-1+\varepsilon_{ir}} \left\{ -q^2\ \frac{\gamma^\mu}{\varepsilon_{ir}} \right. \\
&\left. + \gamma^\mu \left[(-(q^2 + i\,\lambda)\,x\,(1-x)) + \frac{4}{1+2\,\varepsilon_{ir}}\,\frac{q^2}{2}\right] \right\}
\end{aligned} \tag{34}$$

The x integral can be performed easily leading to

$$\begin{aligned}
\Lambda_\mu^{(1)IR} &= -\frac{\alpha}{4\,\pi}\ \left(4\,\pi\,\mu^2\right)^{-\varepsilon_{ir}}\ \Gamma(1-\varepsilon_{ir})\,(-(q^2 + i\,\lambda))^{\varepsilon_{ir}}\,\gamma^\mu \\
&\times \left\{ \frac{\Gamma^2(\varepsilon_{ir})}{\Gamma(2\,\varepsilon_{ir})}\ \left(\frac{1}{\varepsilon_{ir}} - \frac{2}{1+2\,\varepsilon_{ir}}\right) + \frac{\Gamma^2(1+\varepsilon_{ir})}{\Gamma(2+2\,\varepsilon_{ir})} \right\}
\end{aligned} \tag{35}$$

this result can be written neglecting terms which vanish when $\varepsilon_{ir} \to 0$

$$\begin{aligned}\Lambda_\mu^{(1)IR} &= -\frac{\alpha}{4\,\pi}\,\left(4\,\pi\,\mu^2\right)^{-\varepsilon_{ir}}\,\frac{\Gamma(1-\varepsilon_{ir})\,\Gamma^2(1+\varepsilon_{ir})}{\Gamma(1+2\,\varepsilon_{ir})}\\ &\quad\times(-(q^2+i\,\lambda))^{\varepsilon_{ir}}\,\gamma^\mu\left\{\frac{2}{\varepsilon_{ir}^2}-\frac{4}{\varepsilon_{ir}}+9\right\}\end{aligned} \tag{36}$$

A colour factor must be added, it is given by

$$\begin{aligned}\left(T^a\,T^a\right)_{k\,l} &= \frac{1}{2}\left(\delta_{k\,l}\,\delta_{i\,i}-\frac{1}{N}\,\delta_{k\,i}\,\delta_{i\,l}\right)\\ &= C_F\,\delta_{k\,l},\end{aligned} \tag{37}$$

We have also to interchange α and α_s and add a factor e_i. Furthermore, the correct counter term must be added (for the QCD correction to QED vertex) given by

$$Z^{(1)}-1 = C_F\,\frac{\alpha_S}{4\,\pi}\,\left(\frac{1}{\varepsilon_{uv}}-\gamma+\ln(4\,\pi)\right) \tag{38}$$

The vertex correction is then given by

$$\begin{aligned}\Lambda_\mu^{(1)} &= C_F\,\delta_{k\,l}\,e_i\,\frac{\alpha_S}{\alpha}\,(\Lambda_\mu^{(1)UV}+\Lambda_\mu^{(1)IR})\\ &= e_i\,\frac{\alpha_S}{4\,\pi}\,C_F\,\delta_{k\,l}\,\left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{-\varepsilon_{ir}}\,\left(\frac{-q^2-i\,\lambda}{Q^2}\right)^{\varepsilon_{ir}}\,\frac{\Gamma(1-\varepsilon_{ir})\,\Gamma^2(1+\varepsilon_{ir})}{\Gamma(1+2\,\varepsilon_{ir})}\\ &\quad\times\gamma^\mu\left\{-\frac{2}{\varepsilon_{ir}^2}+\frac{4}{\varepsilon_{ir}}-8-\ln\left(\frac{-q^2-i\,\lambda}{\mu^2}\right)\right\}\end{aligned} \tag{39}$$

where in the equation (39), again, here too, we have introduced the energy scale Q. Thus the amplitude M_{v3} is given by

$$\begin{aligned}M_{v3} &= M_B\,\frac{\alpha_S}{4\,\pi}\,C_F\,\left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{-\varepsilon_{ir}}\,\frac{\Gamma(1-\varepsilon_{ir})\,\Gamma^2(1+\varepsilon_{ir})}{\Gamma(1+2\,\varepsilon_{ir})}\\ &\quad\times\left\{-\frac{2}{\varepsilon_{ir}^2}+\frac{4}{\varepsilon_{ir}}-8-\frac{2}{\varepsilon_{ir}}\ln\left(\frac{-q^2-i\,\lambda}{Q^2}\right)+4\ln\left(\frac{-q^2-i\,\lambda}{Q^2}\right)\right.\\ &\quad\left.-\ln\left(\frac{-q^2-i\,\lambda}{\mu^2}\right)-\ln^2\left(\frac{-q^2-i\,\lambda}{Q^2}\right)\right\}\end{aligned} \tag{40}$$

where M_B is the amplitude at the lowest order.

The virtual amplitude is the sum of the three amplitudes M_{v1}, M_{v2} and M_{v3}, it is given by

$$
\begin{aligned}
M_v &= M_B \frac{\alpha_S}{4\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{-\varepsilon_{ir}} \frac{\Gamma(1-\varepsilon_{ir})\,\Gamma^2(1+\varepsilon_{ir})}{\Gamma(1+2\,\varepsilon_{ir})} \\
&\times \left\{ -\frac{2}{\varepsilon_{ir}^2} + \frac{3}{\varepsilon_{ir}} - 8 - \frac{2}{\varepsilon_{ir}} \ln\left(\frac{-q^2 - i\,\lambda}{Q^2}\right) + 4 \ln\left(\frac{-q^2 - i\,\lambda}{Q^2}\right) \right. \\
&\left. - \ln\left(\frac{-q^2 - i\,\lambda}{\mu^2}\right) - \ln^2\left(\frac{-q^2 - i\,\lambda}{Q^2}\right) + \ln\left(\frac{Q^2}{\mu^2}\right) \right\} \qquad (41)
\end{aligned}
$$

We have to be careful here too because the real part of the argument of the logarithms is negative! At the limit $\lambda \to 0$

$$
\begin{aligned}
\ln\left(\frac{-q^2 - i\,\lambda}{Q^2}\right) &= -i\,\pi \\
\ln^2\left(\frac{-q^2 - i\,\lambda}{Q^2}\right) &= -\pi^2
\end{aligned}
$$

so the result becomes simpler

$$
\begin{aligned}
M_v &= M_B \frac{\alpha_S}{4\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{-\varepsilon_{ir}} \frac{\Gamma(1-\varepsilon_{ir})\,\Gamma^2(1+\varepsilon_{ir})}{\Gamma(1+2\,\varepsilon_{ir})} \\
&\times \left\{ -\frac{2}{\varepsilon_{ir}^2} + \frac{3}{\varepsilon_{ir}} - 8 + \pi^2 + i\,\pi \left\{ \frac{2}{\varepsilon_{ir}} - 3 \right\} \right\} \qquad (42)
\end{aligned}
$$

In order to have the same combination of Γ functions as for the real emission, we have to expand around $\varepsilon = 0$ a factor $\Gamma(1+\varepsilon_{ir})\,\Gamma(1-\varepsilon_{ir})$

$$
\Gamma(1+\varepsilon_{ir})\,\Gamma(1-\varepsilon_{ir}) \simeq 1 + \frac{\pi^2}{6}\,\varepsilon_{ir}^2
$$

The virtual amplitude the becomes

$$
\begin{aligned}
M_v &= M_B \frac{\alpha_S}{4\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{-\varepsilon_{ir}} \frac{\Gamma(1+\varepsilon_{ir})}{\Gamma(1+2\,\varepsilon_{ir})} \\
&\times \left\{ -\frac{2}{\varepsilon_{ir}^2} + \frac{3}{\varepsilon_{ir}} - 8 + \frac{2\,\pi^2}{3} + i\,\pi \left\{ \frac{2}{\varepsilon_{ir}} - 3 \right\} \right\} \qquad (43)
\end{aligned}
$$