QCD Lectures

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Outline

1. Lecture II : QCD
   - Global $SU(3)$ transformations
   - Local $SU(3)$ transformations

Lie group reminder I

In most of the cases, the action of the Lie group can be realised by matrices

1. The special linear groups over $\mathbb{R}$ or $\mathbb{C}$: SL($n, \mathbb{R}$) or SL($n, \mathbb{C}$) consisting of $n \times n$ matrices with determinant one with entries in $\mathbb{R}$ of $\mathbb{C}$.

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2. The unitary groups and special unitary groups, U(n) and SU(n), consisting of $n \times n$ complex matrices satisfying $U^\dagger = U^{-1}$ (and also $\det(U) = 1$ in the case of SU(n))

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3. The orthogonal groups and special orthogonal groups, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, consisting of $n \times n$ real matrices satisfying $R^\mathrm{T} = R^{-1}$ (and also $\det(R) = 1$ in the case of $\mathrm{SO}(n)$)

Lie group reminder : examples

A well known example is the rotations, say in 3-dimension, Since a rotation is linear : its action can be describe by a matrix; we also know that rotations preserve angles and lengths : $R^T R = 1$, and rotations preserve orientation (and area) : $\det R = 1$ thus the 3-dimensional rotations can be described by the Lie group $SO(3)$.

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In Quantum Field Theories, discrete symmetries ($\neq$ space–time symmetries) are described by SU(n) groups : gauge symmetries, classification of hadrons,...

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Lie brackets : a specific operation that turns two tangent vectors into another tangent vector (way to construct the tangent vector of the composition of two elements of the Lie group $g \cdot h$ knowing the tangent vectors of g and h). This Lie brackets allow us to replicate the group multiplication entirely on the Lie algebra.

Kinetic and mass terms for spin 1/2 fields

Introducing the **triplet** $\Psi(x)$ which contains the spin 1/2 field in three colour states

$$\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \end{pmatrix} \tag{1}$$

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$$\mathcal{L} = \sum_{i,j=1}^{3} \bar{\Psi}_j(x) \, (i \, \partial\!\!\!/ - m) \, \delta_{ij} \, \Psi_i(x) \tag{2}$$

The $SU(3)$ transformations

$\Psi(x)$ belongs to the **fundamental representation** of the $SU(3)$ Lie group

$$\Psi(x) \to \Psi'(x) = U\,\Psi(x) \tag{3}$$

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The Lagrangian $\mathcal{L}$ is invariant under the transformations (3) and (4) because $\partial_\mu\,\Psi(x)$ **transforms** like $\Psi(x)$.

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Lie brackets

$$\left[T^a, T^b\right] = i\,f^{abc}\,T^c$$

where f^{abc} is the **structure constant** of the group.

Covariant derivative

$\alpha^a \to \alpha^a(x)$ the Lagrangian $\mathcal{L}$ is **no more invariant** under the local transformations

$$\Psi(x) \to \Psi'(x) = U(x)\,\Psi(x) \tag{6}$$

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$\partial_\mu \Psi(x)$ does not transform like $\Psi(x)$. Introduce vector fields $A^b_\mu(x)$ such that

$$\mathcal{L} = \bar{\Psi}(x)\;(i\,\not{D} - m)\;\Psi(x) \quad (8)$$

$D_\mu = \partial_\mu - i\,g\,A^b_\mu(x)\,T^b$, the transformation of the fields $A^b_\mu(x)$ are such that

$$D_\mu \Psi(x) \to D'_\mu \Psi'(x) = U(x)\,D_\mu \Psi(x) \quad (9)$$

The transformation of the gauge fields

$$\mathcal{A}_\mu(x) \equiv A^b_\mu(x)\, T^b$$

$$\begin{aligned} D'_\mu \Psi'(x) &= U(x)\,(\partial_\mu - i\,g\,\mathcal{A}_\mu(x))\,(U^{-1}(x)\,\Psi'(x)) \\ &= \left(\partial_\mu - i\,g\,\left[U(x)\,\mathcal{A}_\mu(x)\,U^{-1}(x) + \frac{1}{i\,g}\,(\partial_\mu\,U(x))\,U^{-1}(x)\right]\right) \\ &\quad \times\ \Psi'(x) \end{aligned}$$

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But $D'_\mu = \partial_\mu - i\,g\,\mathcal{A}'_\mu(x)$, thus, to have the equality the field $\mathcal{A}_\mu(x)$ must transform as

$$\mathcal{A}'_\mu(x) = U(x)\,\mathcal{A}_\mu(x)\,U^{-1}(x) + \frac{1}{i\,g}\,(\partial_\mu\,U(x))\,U^{-1}(x) \qquad (10)$$

Kinetic term for the gauge field I

A **kinetic term** for $A^b_\mu(x)$ **invariant** under local transformations of $SU(3)$, guess inspired by QED : $F_{\mu\nu}(x) = \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x)$

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How this tensor **transforms** under the local gauge transformations?
C.f. notes for details $\cdots$

$$F'_{\mu\nu}(x) = U(x) \left(F_{\mu\nu}(x) - i\,g\,[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)]\right) U^{-1}(x) + i\,g\,\left[\mathcal{A}'_\mu(x), \mathcal{A}'_\nu(x)\right] \quad (11)$$

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$\mathcal{G}_{\mu\nu}(x) \equiv F_{\mu\nu}(x) - i\, g\, [\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)]$ transforms under the local gauge transformations as

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This tensor is said to be **covariant** because it transforms as D_μ, indeed,

$$D'_\mu\, \Psi'(x) \to U(x)\, D_\mu\, \Psi(x) = U(x)\, D_\mu\, U^{-1}(x)\, \Psi'(x)$$

Kinetic term for the gauge field II

$$\mathcal{L}_c = -\frac{1}{2}\,\text{Tr}\left[\mathcal{G}_{\mu\nu}(x)\,\mathcal{G}^{\mu\nu}(x)\right] \tag{13}$$

A **colour singlet** : a scalar with respect to the colour indices

$$\begin{aligned}\text{Tr}\left[\mathcal{G}'_{\mu\nu}(x)\,\mathcal{G}'^{\,\mu\nu}(x)\right] &= \text{Tr}\left[U(x)\,\mathcal{G}_{\mu\nu}(x)\,U^{-1}(x)\,U(x)\,\mathcal{G}^{\mu\nu}(x)\,U^{-1}(x)\right] \\ &= \text{Tr}\left[\mathcal{G}_{\mu\nu}(x)\,\mathcal{G}^{\mu\nu}(x)\right] \end{aligned} \tag{14}$$

Crucial : cyclicity of the trace.

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$$\begin{aligned} \mathcal{G}_{\mu\nu}(x) &= F_{\mu\nu}(x) - i\,g\,\left[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)\right] \\ &= \left(\partial_\mu\, A^a_\nu(x) - \partial_\nu\, A^a_\mu(x)\right)\; T^a + g\, f^{bcd}\, A^b_\mu(x)\, A^c_\nu(x)\, T^d \\ &\equiv G^a_{\mu\nu}\, T^a \end{aligned} \tag{15}$$

where

$$G^a_{\mu\nu} = \partial_\mu\, A^a_\nu(x) - \partial_\nu\, A^a_\mu(x) + g\, f^{abc}\, A^b_\mu(x)\, A^c_\nu(x) \tag{16}$$

Kinetic term for the gauge field III

the **kinetic term** reads

$$\mathcal{L}_c = -\frac{1}{2}\, G^a_{\mu\nu}(x)\, G^{b\,\mu\nu}(x)\, \mathrm{Tr}\left[T^a\, T^b\right] \qquad (17)$$

but $\mathrm{Tr}[T^a\, T^b] = 1/2\, \delta^{ab}$

$$\mathcal{L}_c = -\frac{1}{4}\, G^a_{\mu\nu}(x)\, G^{a\,\mu\nu}(x) \qquad (18)$$

Note that each $G^a_{\mu\nu}(x)$ by itself is **not gauge invariant**, it is the **sum over the colour indices** which is gauge invariant.

The total Lagrangian

$$\mathcal{L} = -\frac{1}{4} G^a_{\mu\nu}(x)\, G^{a\,\mu\nu}(x) + \bar{\Psi}_i(x)\, \left(i\, \not{D}_{ij} - m\, \delta_{ij}\right)\, \Psi_j(x) \tag{19}$$

Infinitesimal transformations : $\alpha^a \ll 1$

$$U(x) \simeq 1 + i\, g\, \alpha^a\, T^a \tag{20}$$

Under these kind of transformations, the variation of the different fields are given by

$$\delta\Psi_i(x) = i\, g\, \alpha^a(x)\, (T^a)_{ij}\, \Psi_j(x) \tag{21}$$

$$\delta\bar{\Psi}_i(x) = -i\, g\, \bar{\Psi}_j(x)\, \alpha^a(x)\, (T^a)_{ji} \tag{22}$$

$$\delta\left(D_\mu \Psi(x)\right)_i = i\, g\, \alpha^a(x)\, (T^a)_{ik}\, (D_\mu)_{kj}\, \Psi_j(x) \tag{23}$$

$$\delta A^a_\mu(x) = (D_\mu)^{ab}\, \alpha^b(x) \tag{24}$$

$$\delta G^a_{\mu\nu}(x) = g\, f^{abc}\, G^b_{\mu\nu}(x)\, \alpha^c(x) \tag{25}$$

The covariant derivative in the fundamental representation

$(D_\mu)_{ij} = \partial_\mu \, \delta_{ij} - i \, g \, A^a_\mu(x) \, (T^a)_{ij}$ in the **fundamental** representation
$(D_\mu)^{ab} = \partial_\mu \, \delta^{ab} - g \, f^{abc} \, A^c_\mu(x)$ in the **adjoint** representation

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$$(\mathcal{T}^c)_{ab} = -i \, f^{cab} = -i \, f^{abc} \qquad (26)$$

A **general result** of the Lie group

$$\left[\mathcal{T}^a, \mathcal{T}^b\right] = i \, f^{abc} \, \mathcal{T}^c \qquad (27)$$

Insured by the **Jacobi identity**

$$\left[T^a, \left[T^b, T^c\right]\right] + \left[T^c, \left[T^a, T^b\right]\right] + \left[T^b, [T^c, T^a]\right] = 0 \qquad (28)$$

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The covariant derivative reads in the **adjoint** representation

$$\begin{aligned}(D_\mu)^{ab} &= \partial_\mu \, \delta^{ab} - i \, g \, (\mathcal{T}^c)_{ab} \, A_\mu^c(x) \\ &= \partial_\mu \, \delta^{ab} - g \, f^{abc} \, A_\mu^c(x)\end{aligned} \tag{29}$$

Quantum level

$$\mathcal{L} = -\frac{1}{4} G^{a}_{\mu\nu}(x)\, G^{a\,\mu\nu}(x) + \bar{\psi}_i(x) \left(i \not{D}_{ij} - m\, \delta_{ij} \right) \psi_j(x) - \frac{1}{2\,\xi} \left(\partial^{\mu} A^{a}_{\mu}(x) \right)^2 + \left(\partial^{\mu} \eta^{a\,\star}(x) \right) D^{ab}_{\mu}\, \eta^{b}(x) \tag{30}$$

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$-\frac{1}{2\,\xi} \left(\partial^\mu A^a_\mu(x)\right)^2$: **gauge fixing term**, whatever the way to quantify the field $A^a_\mu(x)$, some problems show up due to the gauge freedom. Need to fix the gauge : covariant gauge $\partial^\mu A^a_\mu(x) = 0$

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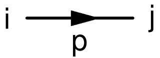
$\left(\partial^\mu \eta^{a\,\star}(x)\right)\, D^{ab}_\mu\, \eta^b(x)$: **ghost term** the price to pay for using covariant gauge, the propagator propagates non physical polarisation states! Introduce "strange" fields $\eta(x)$ (ghost : scalar field which obeys to Fermi-Dirac statistic!) whose role is to cancel these spurious polarisation states.

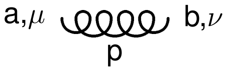
Feynman rules I

Quark propagator:



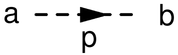
$$\frac{i\,\delta^{ij}\,(\not{p}+m)}{p^2-m^2+i\,\lambda}$$

Gluon propagator:



$$\frac{-i\,\delta^{ab}}{p^2+i\,\lambda}\left(g^{\mu\,\nu}-(1-\xi)\,\frac{p^{\mu}\,p^{\nu}}{p^2+i\,\lambda}\right)$$

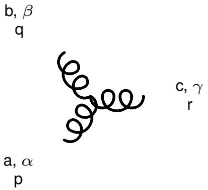
Ghost propagator:



$$\frac{i\,\delta^{ab}}{p^2+i\,\lambda}$$

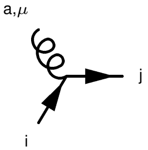
Feynman rules II

Vertex gluon-gluon-gluon (**all momentum are incoming**)



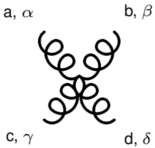
$$-g\, f^{abc} \left[g^{\alpha\,\beta}\,(p-q)^{\gamma} + g^{\beta\,\gamma}\,(q-r)^{\alpha} + g^{\gamma\,\alpha}\,(r-p)^{\beta} \right]$$

Vertex quark-quark-gluon



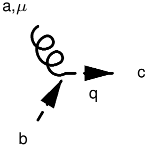
$$-i\,g\,(T^a)_{ji}\,\gamma^{\mu}$$

Vertex gluon-gluon-gluon-gluon



$$\begin{aligned} &-ig^2\, f^{eac}\, f^{ebd}\,(g^{\alpha\,\beta}\, g^{\gamma\,\delta} - g^{\alpha\,\delta}\, g^{\beta\,\gamma}) \\ &-ig^2\, f^{ead}\, f^{ebc}\,(g^{\alpha\,\beta}\, g^{\gamma\,\delta} - g^{\alpha\,\gamma}\, g^{\beta\,\delta}) \\ &-ig^2\, f^{eab}\, f^{ecd}\,(g^{\alpha\,\gamma}\, g^{\beta\,\delta} - g^{\alpha\,\delta}\, g^{\beta\,\gamma}) \end{aligned}$$

Vertex ghost-ghost-gluon



$$g\, f^{abc}\, q^{\mu}$$

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- The quantification breaks the classical gauge invariance, but it remains a quantum version of the gauge invariance : BRST symmetry