QCD Lectures

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CNRS/Université de Savoie

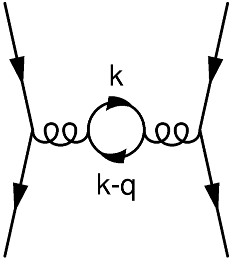
VSOP-30 presentation – July 2024

Outline

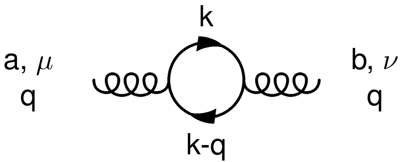
1. Lecture III : Renormalisation
   - Superficial degree of divergence
   - A specific example
   - The running coupling constant
   - Choice of the scale μ

The problem

Computation of the second order in perturbation of a QCD process,
$q_i\,\bar{q}_i \to q_k\,\bar{q}_k$



UV divergences



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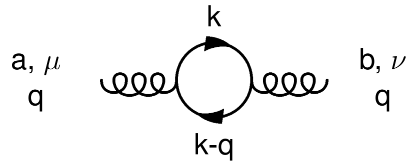

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$$\mathcal{P}^{(1)}_{\mu\nu}(q) \simeq \int \frac{d^4k}{(2\pi)^n}\, Tr\left[\gamma_\mu \frac{\not{k}+m}{(k^2-m^2+i\lambda)}\gamma_\nu \frac{(\not{k}-\not{q})+m}{((k-q)^2-m^2+i\lambda)}\right]$$

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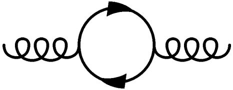

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$k \in$ **Minkowski space** $k^2 = k_0^2 - |\vec{k}|^2$: "Wick rotation" Minkowski space (k) $\rightarrow$ an Euclidean one ($\bar{k}$)

$$\int d^4k\, \frac{k_\mu\, k_\nu}{k^4} \sim \int_0^\infty d|\bar{k}|\, |\bar{k}| \rightarrow \infty$$ *UV divergence*

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We made the assumption, at the very beginning, that the $q-g$ interaction is **point-like** $(-i\,g\,T^a\,\gamma^\mu)$. But we cannot test at such high energies that the interaction $q-g$ is like that

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Back to our example:

$$\int_0^\infty |\bar{k}|^{n-1}\, d|\bar{k}|\, |\bar{k}|^{-2} = \left[\frac{|\bar{k}|^{n-2}}{n-2}\right]_0^\infty \qquad \text{convergent for } n < 2$$

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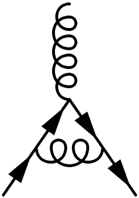

$$\int d^4k \, \frac{k_\mu \, k_\nu}{k^6} \to \int_0^\infty \frac{d|\bar{k}|}{|\bar{k}|}$$ logarithmic UV divergence

$$\int_0^\infty d|\bar{k}| \, |\bar{k}|^{n-5}$$ converge for $n < 4$

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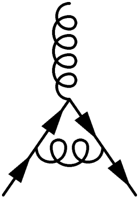

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But how many Green functions diverge?

A Simple tool

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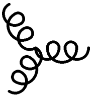

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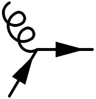

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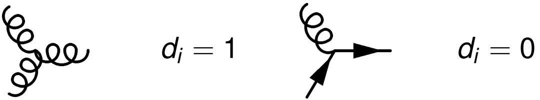

- L the number of independent four-momenta (number of loops), each term corresponds to k^4 (d^4k)

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- $E_B + 2\,I_B = \sum_i n_i\,b_i$ where b_i is the number of bosons attached to the vertex of type i

$$\omega(G) = 4 - E_B - \frac{3}{2}\,E_F + \sum_i n_i\,\left(b_i + d_i + \frac{3}{2}\,f_i - 4\right) \tag{2}$$

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But if the vertex of type i originates from a term in the Lagrangian of the type

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This term must have a dimension 4 as any element of the Lagrangian, introducing $[g_i]$ the dimension of the coupling constant g_i we have then that

$$[g_i] + b_i + d_i + \frac{3}{2} f_i = 4$$

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because the A field has dimension 1 as the derivative ∂_μ and the dimension of the ψ field is 3/2. Thus, the superficial degree of divergence can be written

$$\omega(G) = 4 - E_B - \frac{3}{2} E_F - \sum_i n_i \, [g_i] \quad (3)$$

Exercise

1)Rederive the formula for the superficial degree of divergence in a space-time of dimensions *n*

2) Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \Phi(x)\right) \left(\partial^\mu \Phi(x)\right) - \frac{m^2}{2} \Phi^2(x) - \frac{\lambda}{4!} \Phi^4(x)$$

where $\Phi(x)$ is a scalar field. Determine for which value of *n*, this theory is super renormalisable, renormalisable, non renormalisable.

QCD case

In the case of QCD, there is only one type of coupling constant whose dimension is zero! But the ghosts must be included, thus

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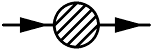
- $\vdots$

2-points, 3-points

$\omega(G) = 2$



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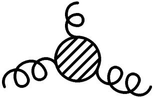

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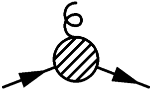

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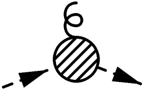
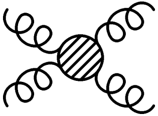


$\omega(G) = 1$

4-points



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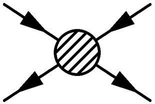
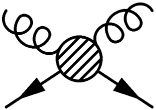
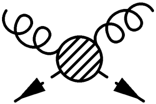


$\omega(G) = -2$



$\omega(G) = -1$



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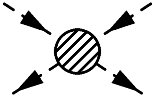

$\omega(G) = 0$

5-points and more : $\omega(G) < 0$

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Number of divergent Green functions is **finite** $\Rightarrow$ absorb them into a redefinition of the parameters of the Lagrangian.

At quantum level, $\mathcal{L}(\psi_B, A_B, \eta_B, m_B, g_B)$, **bare parameters** : not physical (∞) $\Rightarrow \mathcal{L}$ expressed in terms of the **renormalised parameters**

$$\psi_B(x) = Z_2^{1/2}\,\psi(x), \quad A^a_{B\,\mu}(x) = Z_3^{1/2}\,A^a_\mu(x), \quad \eta^a_B(x) = \tilde{Z}_3^{1/2}\,\eta^a(x),$$

$$m_B = \frac{Z_0}{Z_2}\,m, \quad g_B = \frac{Z_{1\,F}}{Z_2\,Z_3^{1/2}}\,g' = \frac{Z_1}{Z_3^{3/2}}\,g' = \frac{\tilde{Z}_1}{\tilde{Z}_3\,Z_3^{1/2}}\,g', \quad \xi_B = Z_3\,\xi$$

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All the couplings must be **equal** even the renormalised ones, a consequence of the gauge invariance

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This is what the Taylor-Slavnov identities tell us! For matter of convenience, we introduce

$$Z_i = 1 + \delta Z_i, \quad \tilde{Z}_i = 1 + \delta\,\tilde{Z}_i$$

The Lagrangian can be **expanded** in terms of the δZ_i and $\delta \tilde{Z}_i$

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Space time of dimensions n, the **dimension** (in term of energy) of the coupling constant g' is not **zero**!

$$\begin{array}{lll} [\mathcal{L}] = 4 & [g'] = 0 & [m] = 1 \\ [\mathcal{L}] = n & [g'] = 2 - \frac{n}{2} & [m] = 1 \end{array}$$

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$$g' \to g\,\mu^{2-n/2} \quad \text{with} \quad [g] = 0 \quad \text{and} \quad [\mu] = 1$$

To absorb the change of $[g']$, a **new energy scale** μ is introduced.

Whatever the way we regularise, the renormalisation procedure makes **the appearance of an energy scale.** The use of a cut-off Λ would lead to

$$A \ln\left(\frac{\Lambda}{Q}\right) + B$$

where Q is typical energy scale and A and B are two coefficients independent of Λ.

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An arbitrary energy scale can be introduced in such a way that

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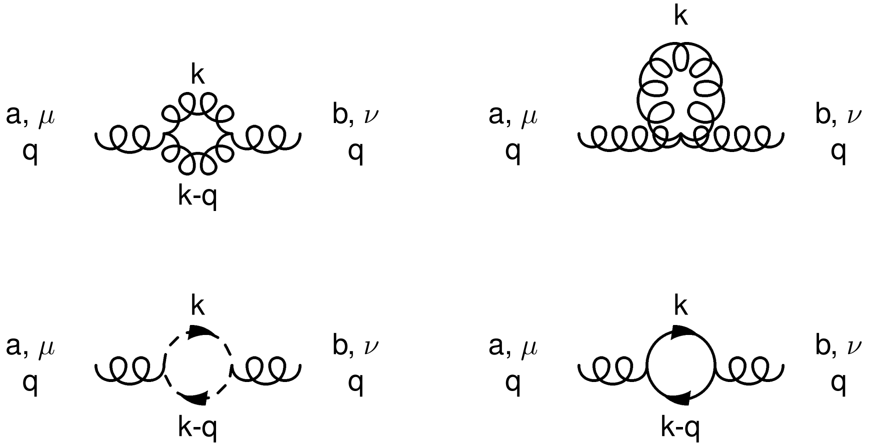
In the old textbooks on Quantum Field Theory, the scale μ is **the scale at which the UV divergences are subtracted**.
In addition, with the logarithmic dependence on the regulator, one can also absorb some finite terms (even not logarithmic), this defines the **renormalisation scheme**.

One loop corrections to the gluon propagator



Results (Feynman gauge $\xi = 1$)

$$
\begin{aligned}
\mathcal{P}^{(1)\,gg}_{\mu\nu}(q) &= \frac{1}{\varepsilon}\, N\, \delta^{ab}\, K(\varepsilon) \left[q^2\, g^{\mu\,\nu} \left(\frac{19}{12} + \frac{29\,\varepsilon}{9} \right) - q^{\mu}\, q^{\nu} \left(\frac{11}{6} + \frac{67\,\varepsilon}{18} \right) \right] \\
\mathcal{P}^{(1)\,ggg}_{\mu\nu}(q) &= 0 \\
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\mathcal{P}^{(1)\,qq}_{\mu\nu}(q) &= -\frac{1}{\varepsilon}\, T_F\, \delta^{ab}\, K(\varepsilon) \left[q^2\, g^{\mu\,\nu} \left(\frac{4}{3} + \frac{20\,\varepsilon}{9} \right) - q^{\mu}\, q^{\nu} \left(\frac{4}{3} + \frac{20\,\varepsilon}{9} \right) \right]
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\end{aligned}$$

$$K(\varepsilon) \simeq 1 + \varepsilon \left(\ln(4\,\pi) - \gamma + \ln\left(\frac{\mu^2}{-q^2 - i\,\lambda} \right) \right) + O(\varepsilon^2) \quad \text{and} \quad T_F = \frac{N_F}{2}$$

with γ is the **Euler constant** : $\gamma = 0.5772\cdots$ and $\varepsilon = (4-n)/2$.

More results

$\mathcal{P}$ the sum of four contributions

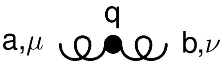
$$\mathcal{P}^{(1)}_{\mu\nu}(q) = \mathcal{P}^{(1)\,gg}_{\mu\nu}(q) + \mathcal{P}^{(1)\,ggg}_{\mu\nu}(q) + \mathcal{P}^{(1)\,GG}_{\mu\nu}(q) + \mathcal{P}^{(1)\,qq}_{\mu\nu}(q)$$

The **ghost contribution** is necessary in order that $\mathcal{P}^{(1)}_{\mu\nu}(q)$ is transverse : $q^\mu\,\mathcal{P}^{(1)}_{\mu\nu}(q) = q^\nu\,\mathcal{P}^{(1)}_{\mu\nu}(q) = 0$ as required by Slavnov-Taylor identities. Note that $\mathcal{P}^{(1)}_{\mu\nu}(q)$ is not the gluon propagator, it can be shown that

$$\mathcal{D}^{-1}_{\mu\nu} = D^{-1}_{\mu\nu} - i\,\mathcal{P}^{(1)}_{\mu\nu} \tag{5}$$

where $\mathcal{D}$ is the exact propagator (one loop in our case) and D the free propagator.

Counter term



$$- i \, \delta Z_3^{(1)} \, \delta^{ab} \left(q^2 \, g^{\mu\nu} - q^\mu q^\nu \right)$$

$$i \mathcal{P}^{(1)\ tot}_{\mu\nu,ab} = i \, \mathcal{P}^{(1)}_{\mu\nu,ab} - i \, \delta Z_3^{(1)} \, \delta^{ab} \left(q^2 \, g^{\mu\nu} - q^\mu q^\nu \right)$$

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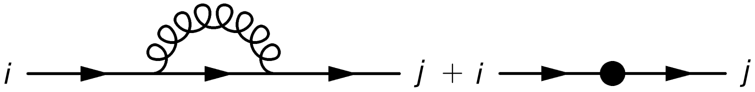
In the $\overline{MS}$ scheme and the Feynman gauge ($\xi = 1$)

$$\delta Z_3^{(1)} = \frac{\alpha_s}{4\,\pi} \left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma \right) \left(\frac{5}{3}\, N - \frac{4}{3}\, T_F \right)$$

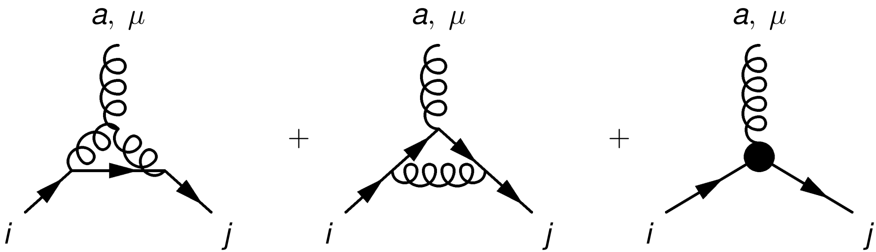
Other counter terms

Let us compute also the counter term associated to the quark wave function



$$\delta Z_2 = -\frac{\alpha_s}{4\,\pi}\, C_F \left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma \right)$$

as well as the counter term associated to the vertex quark – gluon



$$\delta Z_{1\,F} = -\frac{\alpha_s}{4\,\pi}\,(C_F + N)\,\left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma\right)$$

The renormalised α_s

Relation between the **bare** $\alpha_{s\,B}$ and the **renormalised** one α_s

$$\alpha_{s\,B} = \alpha_s \, \mu^{2\,\varepsilon} \, \frac{Z_{1\,F}^2}{Z_2^2 \, Z_3} \equiv \alpha_s \, \mu^{2\,\varepsilon} \, Z_\alpha$$

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$\alpha_{s\,B}$ does not depend on μ

$$\frac{\mu^2\,d\alpha_{s\,B}}{d\mu^2} = 0$$

ξ dependence

For **simplicity** reason, we choose the Feynman gauge to present the different results. Letting the *xi* parameter **free**, the results for the counter terms would have been

$$\delta Z_3^{(1)} = \frac{\alpha_s}{4\,\pi} \left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma \right) \left(N \left[\frac{13}{6} - \frac{\xi}{2} \right] - T_F\,\frac{4}{3} \right)$$

$$\delta Z_2^{(1)} = -\frac{\alpha_s}{(4\pi)}\,C_F\,\xi \left[\frac{1}{\varepsilon} - \gamma + \ln(4\pi) \right]$$

$$\delta Z_1^F = -\frac{\alpha_s}{4\,\pi} \left(\frac{1}{\varepsilon} + \ln(4\,\pi) - \gamma \right) \left(C_F\,\xi + N \left[\frac{3}{4} + \frac{\xi}{4} \right] \right)$$

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It is easy to verify that the ξ dependence drops out in Z_α, This is expected because Z_α is related to a physical quantity.

α_s must depend on μ

$$\mu^2 \frac{d}{d\mu^2} \left(\alpha_s \, \mu^{2\,\varepsilon} \, Z_\alpha \right) = 0$$

$$\beta(\alpha_s) \left[Z_\alpha + \alpha_s \frac{dZ_\alpha}{d\alpha_s} \right] + \varepsilon \, \alpha_s \, Z_\alpha = 0 \quad \text{with} \quad \beta(\alpha_s) = \mu^2 \frac{d\alpha_s}{d\mu^2}$$

that is to say

$$\beta(\alpha_s) = -\varepsilon \, \alpha_s - \alpha_s^2 \, \kappa(\varepsilon) \left(\frac{11\,N - 2\,N_F}{12\,\pi} \right)$$

with $\kappa(\varepsilon) = 1 + \varepsilon \, \ln(4\,\pi) - \varepsilon \, \gamma$.

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with $\kappa(\varepsilon) = 1 + \varepsilon \, \ln(4\,\pi) - \varepsilon \, \gamma$. $\beta(\alpha_s)$ is not singular when $\varepsilon \to 0$, limit $\varepsilon \to 0$,

$$\beta(\alpha_s) = -\alpha_s^2 \, b_0 \quad \text{with} \quad b_0 = \frac{11\,N - 2\,N_F}{12\,\pi}$$

The μ dependence of α_s

Solve the **differential equation** with initial condition

$$\frac{d\,\alpha_s(t)}{dt} = \beta(\alpha_s(t)) \quad \text{with} \quad t = \ln(\mu^2/\mu_0^2)$$

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The β function can be computed at **any order** in α_s

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An example of the **renormalisation group equations (RGE)**

$$t = \int_{\alpha_s(0)}^{\alpha_s(t)} \frac{dx}{\beta(x)}$$

Keeping only the first term

$$\alpha_s(t) = \frac{\alpha_s(0)}{1 + b_0\,t\,\alpha_s(0)}$$

Discussions

$b_0 > 0$ if $N_F \leq 16 \rightarrow d\,\alpha_s(t)/dt \leq 0$. So $\alpha_s(t) \searrow$ when $t \nearrow$

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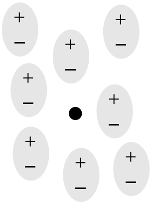

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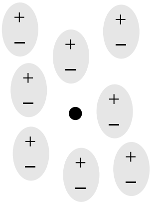
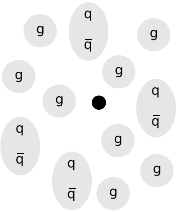


In QED, $b_0 < 0$ screening effect : an electric charge is screened by the virtual $\pm$ charge in the vacuum

In QCD, $b_0 > 0$ anti-screening effect : an colour charge is screened by the virtual $q\,\bar{q}$ but anti-screened by the g in the vacuum



The parameter Λ

A parameter Λ is defined such that:

$$\ln\left(\frac{\mu^2}{\Lambda^2}\right) = -\int_{\infty}^{\alpha_s(\mu^2)} \frac{dx}{\beta(x)}$$

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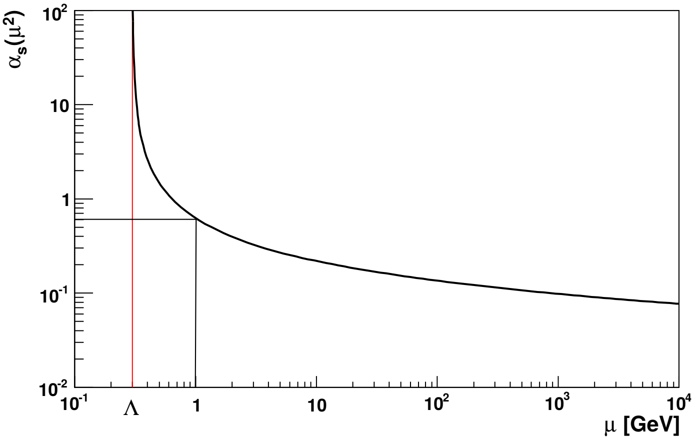
Taking the first term of the β function, one gets

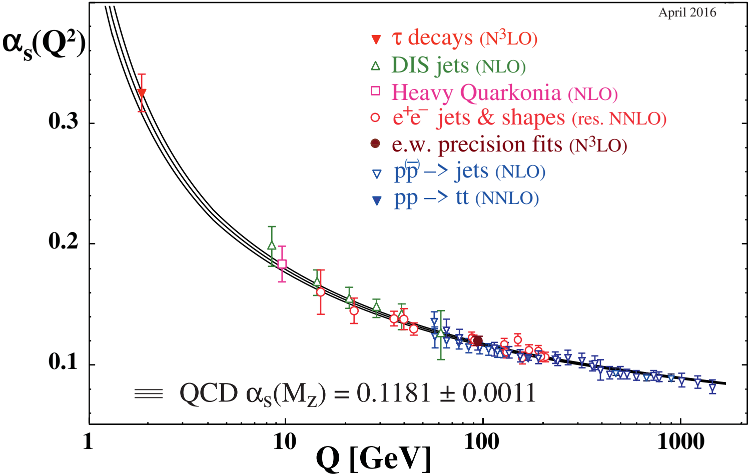
$$\alpha_s(\mu^2) = \frac{1}{b_0 \ln\left(\frac{\mu^2}{\Lambda^2}\right)} \quad \Rightarrow \quad \mu^2 = \Lambda^2 \qquad \alpha_s(\Lambda^2) = \infty$$

Λ : a scale which separate **perturbative** and **non perturbative** regime (Λ depends on the renormalisation scheme)

Plot of $\alpha_s(\mu)$



α_s Measurement



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At one loop

$$\bar{R}(\mu^2) = \frac{\alpha_s(\mu^2)}{\pi}$$

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$$\bar{R}(S) - \bar{R}(\mu^2) = O(\alpha_s^2(\mu^2))$$

A good choice for the scale $\mu \simeq \sqrt{S}$, more precisely "$\simeq$" means that $\alpha_s(\mu^2)\ \ln(S/\mu^2) \ll 1$.
The variation of the scale μ around $\sqrt{S}$ gives an error band for the theoretical prediction.

Remarks

The formula, we got, for $\alpha_s(\mu^2)$ is called at **Leading Logarithmic (LL) accuracy**.

$$\bar{R}(S) = \bar{R}(\mu^2) \sum_{n=0}^{\infty} a_n \left(\alpha_s(\mu^2)\, t\right)^n$$

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Including, the expression of the β function at two loop in the differential equation which drives the μ dependence of $\alpha_s(\mu^2)$,

$$\bar{R}(S) = \bar{R}(\mu^2) \sum_{n=0}^{\infty} [a_n \left(\alpha_s(\mu^2)\, t\right)^n + b_n\, \alpha_s(\mu^2)^n\, t^{n-1}]$$

Next to Leading Logarithmic (NLL) accuracy.

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- Renormalisation procedure : express the "bare" parameters of the Lagrangian in terms of the renormalised (physical) one plus some counter terms. These latter are adjusted to cancel the UV divergences. Work order by order in perturbation.
- As the outcome of renormalisation, an arbitrary energy scale appears. The renormalised parameters depend on it.

What we learnt in lecture III

- The independence of measurable quantities on this scale yields sets of differential equations which drive the dependence of these renormalised parameters on this scale