QCD Lectures

J.-Ph. Guillet

LAPTh
CNRS/Université de Savoie

VSOP-30 presentation – July 2024

Outline

1. Lecture IV
   - Soft/collinear divergences
   - Drell-Yan cross section
   - α_s corrections : $q\,\bar{q}$ contribution
   - α_s corrections : $q\,g$ contribution
   - The virtual term

Other divergences!

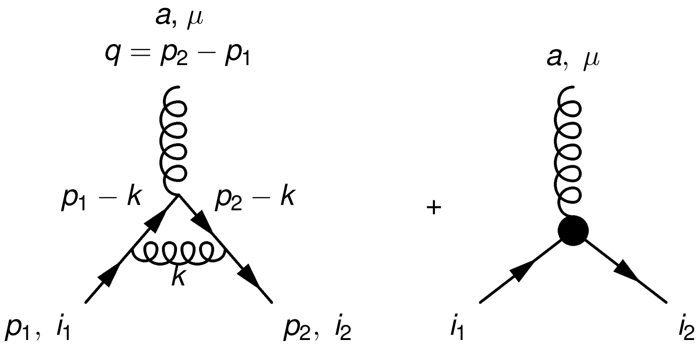
When computing α_s corrections to some processes, do we get rid of all **the divergent terms**?

Other divergences!

When computing α_s corrections to some processes, do we get rid of all **the divergent terms**? the answer is unfortunately **no**!!!

Vertex example

let us consider the following loop diagram



$p_2^2 = p_1^2 = m^2$ but $q^2 \neq 0$.

Vertex example I

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = -i\, e^2\, \mu^{(4-n)} \int \frac{d^n k}{(2\pi)^n}\, \gamma_\alpha\, \frac{\not{p}_2 - \not{k} + m}{(p_2 - k)^2 - m^2 + i\lambda}\, \gamma_\mu \times \frac{\not{p}_1 - \not{k} + m}{(p_1 - k)^2 - m^2 + i\lambda}\, \gamma^\alpha\, \frac{1}{k^2 + i\lambda} \left(T^b\, T^a\, T^b \right)_{i_2 i_1} \qquad (1)$$

Vertex example I

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = -i\, e^2\, \mu^{(4-n)} \int \frac{d^n k}{(2\pi)^n}\, \gamma_\alpha\, \frac{\not p_2 - \not k + m}{(p_2 - k)^2 - m^2 + i\lambda}\, \gamma_\mu \times \frac{\not p_1 - \not k + m}{(p_1 - k)^2 - m^2 + i\lambda}\, \gamma^\alpha\, \frac{1}{k^2 + i\lambda}\, \left(T^b\, T^a\, T^b\right)_{i_2 i_1} \quad (1)$$

Use the Feynman trick

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = -i\, e^2\, \mu^{(4-n)} \int_0^1 2\, y\, dy \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \times \frac{D}{[(k - y\,(p_2\, x + p_1\,(1-x)))^2 - y^2\,(p_2\, x + p_1\,(1-x))^2 + i\,\lambda]^3}.$$

Vertex example II

Change of variable

$$l = k - y\,(p_2\,x + p_1\,(1 - x)). \tag{2}$$

Vertex example II

Change of variable

$$l = k - y\,(p_2\,x + p_1\,(1 - x)). \tag{2}$$

This leads to

$$\begin{aligned}\Lambda_\mu^{(1)}(p_2, p_1, q) &= -i\,e^2\,\mu^{(4-n)} \int_0^1 2\,y\,dy \int_0^1 dx \int \frac{d^n l}{(2\pi)^n} \\ &\times \frac{D}{[l^2 - R^2 + i\,\lambda]^3}.\end{aligned} \tag{3}$$

with

$$R^2 = y^2\,(m^2 - q^2\,x\,(1 - x)) \tag{4}$$

Vertex example III

The numerator D is a polynomial of degree one in l^2:

$$D = a\,l^2 + b.$$

Vertex example III

The numerator D is a polynomial of degree one in l^2:

$$D = a\, l^2 + b.$$

Only the part in l^2 will give an **ultra-violet divergence**, the part constant will give an **infrared divergence.**

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = \Lambda_\mu^{(1)UV} + \Lambda_\mu^{(1)IR} \tag{5}$$

Vertex example III

The numerator D is a polynomial of degree one in l^2:

$$D = a\, l^2 + b.$$

Only the part in l^2 will give an **ultra-violet divergence**, the part constant will give an **infrared divergence.**

$$\Lambda_\mu^{(1)}(p_2, p_1, q) = \Lambda_\mu^{(1)\,UV} + \Lambda_\mu^{(1)\,IR} \tag{5}$$

$$\Lambda_\mu^{(1)\,IR} \;=\; -i\, e^2\, \mu^{(4-n)} \int_0^1 2\, y\, dy \int_0^1 dx \int \frac{d^n l}{(2\pi)^n}\, b\, \frac{1}{(l^2 - R^2 + i\,\lambda)^3}$$

Vertex example IV

$$\Lambda_\mu^{(1)\,IR} = -\,e^2\,\frac{\mu^{(4-n)}}{(4\,\pi)^{n/2}}\,\Gamma(3-\frac{n}{2})$$
$$\times \int_0^1 dx\,(m^2 - q^2\,x\,(1-x) - i\,\lambda)^{n/2-3}\,\int_0^1 dy\;y^{n-5}\,F(y)$$

with

$$F(y) = b_0 + b_1(x)\,y + b_2(x)\,y^2 \qquad (6)$$

Vertex example IV

$$\Lambda_\mu^{(1)\,IR} = - e^2 \, \frac{\mu^{(4-n)}}{(4\,\pi)^{n/2}} \, \Gamma(3 - \frac{n}{2}) \times \int_0^1 dx\,(m^2 - q^2\,x\,(1-x) - i\,\lambda)^{n/2-3} \int_0^1 dy\; y^{n-5}\,F(y)$$

with

$$F(y) = b_0 + b_1(x)\,y + b_2(x)\,y^2 \qquad (6)$$

This integration will generate a **divergence** for the IR part because we have to integrate something of the type:

$$\int_0^1 dy\,(b_0\,y^{n-5} + b_1\,y^{n-4} + b_2\,y^{n-3}) = \frac{b_0}{n-4} + \frac{b_1}{n-3} + \frac{b_2}{n-2}. \qquad (7)$$

Soft divergence

Divergence at $y = 0$, setting $y = 0$ before the l integration yields

$$\int \frac{d^n l}{(2\pi)^n} \, \frac{1}{(l^2 + i\,\lambda)^3}$$

Soft divergence

Divergence at $y = 0$, setting $y = 0$ before the l integration yields

$$\int \frac{d^n l}{(2\pi)^n} \, \frac{1}{(l^2 + i\,\lambda)^3}$$

Using Wick rotation + spherical coordinates for $\bar{l}$

$$\int_0^{+\infty} dv \, v^{\frac{n}{2}-1-3} \quad \text{where} \quad v = \bar{l}^2$$

This integral diverges at $v = 0$ if $n = 4$.

Soft divergence

Divergence at $y = 0$, setting $y = 0$ before the l integration yields

$$\int \frac{d^n l}{(2\pi)^n} \, \frac{1}{(l^2 + i\,\lambda)^3}$$

Using Wick rotation + spherical coordinates for $\bar{l}$

$$\int_0^{+\infty} dv \, v^{\frac{n}{2}-1-3} \quad \text{where} \quad v = \bar{l}^2$$

This integral diverges at $v = 0$ if $n = 4$.
Divergence at $y = 0$ and $l = 0 \Rightarrow k = 0$. If one of the external fermions is **not on its mass shell**, $R^2 \neq 0$ at $y = 0$ and thus the integrals on l (or k) will not diverge.

Soft divergence

Divergence at $y = 0$, setting $y = 0$ before the l integration yields

$$\int \frac{d^n l}{(2\pi)^n} \frac{1}{(l^2 + i\,\lambda)^3}$$

Using Wick rotation + spherical coordinates for $\bar{l}$

$$\int_0^{+\infty} dv\, v^{\frac{n}{2}-1-3} \quad \text{where} \quad v = \bar{l}^2$$

This integral diverges at $v = 0$ if $n = 4$.
Divergence at $y = 0$ and $l = 0 \Rightarrow k = 0$. If one of the external fermions is **not on its mass shell**, $R^2 \neq 0$ at $y = 0$ and thus the integrals on l (or k) will not diverge. **Soft divergence** if a **massless (spin 1) boson** is exchanged between two lines which are **on their mass shell.**

Collinear divergence

Soft divergence also appears in QED, but in QCD it is **worse**! Indeed, at $m = 0$, the x integration also **diverges**: in QCD a virtual gluon can be exchanged between two on shell gluon lines!

Collinear divergence

Soft divergence also appears in QED, but in QCD it is **worse**! Indeed, at $m = 0$, the x integration also **diverges**: in QCD a virtual gluon can be exchanged between two on shell gluon lines!
What is the **origin of this divergence**?

Collinear divergence

Soft divergence also appears in QED, but in QCD it is **worse**! Indeed, at $m = 0$, the x integration also **diverges**: in QCD a virtual gluon can be exchanged between two on shell gluon lines!
What is the **origin of this divergence**?
Let us **fix** $y \neq 0$ and look at the limit $x = 0$. Since in this limit, $R^2 = 0$, the l integral will diverge at $l = 0$ which means $k = y\, p_1$ (the limit $x = 1$ would lead to $k = y\, p_2$).

Collinear divergence

Soft divergence also appears in QED, but in QCD it is **worse**! Indeed, at $m = 0$, the x integration also **diverges**: in QCD a virtual gluon can be exchanged between two on shell gluon lines!

What is the **origin of this divergence**?

Let us **fix** $y \neq 0$ and look at the limit $x = 0$. Since in this limit, $R^2 = 0$, the l integral will diverge at $l = 0$ which means $k = y\, p_1$ (the limit $x = 1$ would lead to $k = y\, p_2$).

The divergences at $x = 0$ or $x = 1$ originate from the fact that k becomes **collinear** to p_1 or p_2. These divergences are called **collinear divergences**.

Collinear divergence

Soft divergence also appears in QED, but in QCD it is **worse**! Indeed, at $m = 0$, the x integration also **diverges**: in QCD a virtual gluon can be exchanged between two on shell gluon lines!
What is the **origin of this divergence**?
Let us **fix** $y \neq 0$ and look at the limit $x = 0$. Since in this limit, $R^2 = 0$, the l integral will diverge at $l = 0$ which means $k = y\, p_1$ (the limit $x = 1$ would lead to $k = y\, p_2$).
The divergences at $x = 0$ or $x = 1$ originate from the fact that k becomes **collinear** to p_1 or p_2. These divergences are called **collinear divergences**.
Note that there can be a **pile of divergences** "soft + collinear" at $y = 0$ et $x = 0$ ($x = 1$).

Simple tool

A very **simple way** to test a loop integral in the **soft region** : **rescale** the loop momentum and study the power of the rescaling parameter

$$\Lambda_\mu^{(1)}(p_2, p_1, q) \simeq \int d^n k \, \frac{H(k)}{((p_2 - k)^2 - m^2)\,((p_1 - k)^2 - m^2)\, k^2}$$

$$\simeq \int d^n k \, \frac{H(k)}{(k^2 - 2\, k \cdot p_2)\,(k^2 - 2\, k \cdot p_1)\, k^2}$$

Simple tool

A very **simple way** to test a loop integral in the **soft region** : **rescale** the loop momentum and study the power of the rescaling parameter

$$\Lambda_\mu^{(1)}(p_2, p_1, q) \simeq \int d^n k \, \frac{H(k)}{((p_2 - k)^2 - m^2)\,((p_1 - k)^2 - m^2)\, k^2}$$
$$\simeq \int d^n k \, \frac{H(k)}{(k^2 - 2\,k \cdot p_2)\,(k^2 - 2\,k \cdot p_1)\, k^2}$$

Rescaling k by ρ leads to

$$\Lambda_\mu^{(1)}(p_2, p_1, q) \simeq \rho^{n-4} \int d^n k \, \frac{H(\rho\, k)}{(\rho\, k^2 - 2\,k \cdot p_2)\,(\rho\, k^2 - 2\,k \cdot p_1)\, k^2}$$

Simple tool

A very **simple way** to test a loop integral in the **soft region** : **rescale** the loop momentum and study the power of the rescaling parameter

$$\Lambda_\mu^{(1)}(p_2, p_1, q) \simeq \int d^n k \, \frac{H(k)}{((p_2-k)^2 - m^2)\,((p_1-k)^2 - m^2)\, k^2}$$
$$\simeq \int d^n k \, \frac{H(k)}{(k^2 - 2\,k\cdot p_2)\,(k^2 - 2\,k\cdot p_1)\, k^2}$$

Rescaling k by ρ leads to

$$\Lambda_\mu^{(1)}(p_2, p_1, q) \simeq \rho^{n-4} \int d^n k \, \frac{H(\rho\, k)}{(\rho\, k^2 - 2\,k\cdot p_2)\,(\rho\, k^2 - 2\,k\cdot p_1)\, k^2}$$

Integral will **behave as** ρ^β. If $\beta \leq 0$ the integral **diverges**, if $\beta > 0$, the integral **converges.** There is an **infinite number of diagrams** which diverge, we cannot apply a renormalisation procedure.

Simple tool

How these divergences will **disappear**?

Simple tool

How these divergences will **disappear**?
To answer this question, **think** about what needs to be included when computing the α_s corrections to a reaction.

Simple tool

How these divergences will **disappear**?
To answer this question, **think** about what needs to be included when computing the α_s corrections to a reaction. To fix the idea, let us consider the **Drell-Yan production** at LHC.

Simple tool

How these divergences will **disappear**?
To answer this question, **think** about what needs to be included when computing the α_s corrections to a reaction. To fix the idea, let us consider the **Drell-Yan production** at LHC. At lowest order, the associated partonic reactions are $q_i + \bar{q}_i \to \gamma^*$.

Simple tool

How these divergences will **disappear**?
To answer this question, **think** about what needs to be included when computing the α_s corrections to a reaction. To fix the idea, let us consider the **Drell-Yan production** at LHC. At lowest order, the associated partonic reactions are $q_i + \bar{q}_i \to \gamma^*$. But LHC is **a hadronic collider**, that means that one cannot expect to have events containing only a lepton pair and that's it!

Simple tool

How these divergences will **disappear**?
To answer this question, **think** about what needs to be included when computing the α_s corrections to a reaction. To fix the idea, let us consider the **Drell-Yan production** at LHC. At lowest order, the associated partonic reactions are $q_i + \bar{q}_i \to \gamma^*$. But LHC is **a hadronic collider**, that means that one cannot expect to have events containing only a lepton pair and that's it! In this background, **inclusive cross section** is used.

Simple tool

How these divergences will **disappear**?
To answer this question, **think** about what needs to be included when computing the α_s corrections to a reaction. To fix the idea, let us consider the **Drell-Yan production** at LHC. At lowest order, the associated partonic reactions are $q_i + \bar{q}_i \to \gamma^*$. But LHC is **a hadronic collider**, that means that one cannot expect to have events containing only a lepton pair and that's it! In this background, **inclusive cross section** is used. To measure it, all the events containing **at least** a lepton pair are collected.

Simple tool

How these divergences will **disappear**?
To answer this question, **think** about what needs to be included when computing the α_s corrections to a reaction. To fix the idea, let us consider the **Drell-Yan production** at LHC. At lowest order, the associated partonic reactions are $q_i + \bar{q}_i \to \gamma^*$. But LHC is **a hadronic collider**, that means that one cannot expect to have events containing only a lepton pair and that's it! In this background, **inclusive cross section** is used. To measure it, all the events containing **at least** a lepton pair are collected. At theoretical level and at the order we work, one has also to consider reactions like $q_i + \bar{q}_i \to \gamma^* + g$ or $q_i + g \to \gamma^* + q_i$. That is to say, that the α_s corrections do not come only from the loop corrections but also the so-called **"real" emission** where an extra on shell parton is emitted.

Notation

The partons are **labelled** by i_k $k = 1, 2, \ldots$ and have a 4-momentum p_k, $i_k \in \{u, \bar{u}, d, \bar{d}, \ldots, g\} \equiv S_p$. Work in a **space-time of dimension** n.

Notation

The partons are **labelled** by i_k $k = 1, 2, \ldots$ and have a 4-momentum p_k, $i_k \in \{u, \bar{u}, d, \bar{d}, \ldots, g\} \equiv S_p$. Work in a **space-time of dimension** n. The lowest order in the parton model is given by

$$\sigma_H = \sum_{i_1, i_2 \in S_p} \int dx_1\, dx_2\, F_{i_1}^{H_1}(x_1)\, F_{i_2}^{H_2}(x_2)\, \hat{\sigma}_{i_1 + i_2 \to \gamma^*} \tag{8}$$

Notation

The partons are **labelled** by i_k $k = 1, 2, \ldots$ and have a 4-momentum p_k, $i_k \in \{u, \bar{u}, d, \bar{d}, \ldots, g\} \equiv S_p$. Work in a **space-time of dimension** n. The lowest order in the parton model is given by

$$\sigma_H = \sum_{i_1, i_2 \in S_p} \int dx_1 \, dx_2 \, F_{i_1}^{H_1}(x_1) \, F_{i_2}^{H_2}(x_2) \, \hat{\sigma}_{i_1 + i_2 \to \gamma^*} \tag{8}$$

The partonic cross section must fulfil **conservation laws**, thus if a choice of partons i_1, i_2 **violates** these laws the partonic cross section is set to **zero**. For a couple of labels i_1, i_2 which verifies the conservation laws say d, $\bar{d}$, the following combination

$$F_d^{H_1}(x_1) \, F_{\bar{d}}^{H_2}(x_2) \, \hat{\sigma}_{d + \bar{d} \to \gamma^*}$$

and

$$F_{\bar{d}}^{H_1}(x_1) \, F_d^{H_2}(x_2) \, \hat{\sigma}_{\bar{d} + d \to \gamma^*}$$

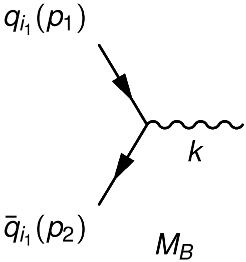
Hadronic cross section

Neglecting all the fermion masses, the partonic cross section is given by

$$\frac{d\hat{\sigma}_{i_1+i_2\to\gamma^*}}{dQ^2} = \frac{1}{4\,p_1.p_2} \int \frac{d^{n-1}p_3}{(2\,\pi)^{n-1}\,2\,E_3}\,(2\,\pi)^n\,\delta^n(p_1+p_2-p_3)\,|\overline{M}_B|^2$$

$|\overline{M}_B|^2$ is the squared amplitude M_B **averaged** over initial polarisations and colours and **summed** on the final polarisation and colours.

Lowest order amplitude



$$M_B = -i\, e\, \mu^{\varepsilon}\, q_{i_1}\, \bar{v}_j(p_2)\, \gamma^{\mu}\, u_j(p_1)\, \epsilon_{\mu}(p_3)$$

where j the **colour of the quarks** (it is the same for the two lines since the $\gamma^{\star}$ is colourless!) and q_{i_1} the **electric charge** of the parton of type i_1.

Squared amplitude

$$
\begin{aligned}
|M_B|^2 &= e^2\, q_{i_1}^2\, \mu^{2\,\varepsilon}\, \delta_{jj}\, \mathrm{Tr}\left[\not{p}_2\, \gamma_\mu\, \not{p}_1\, \gamma_\nu\right] \left(-g_{\mu\nu} + \frac{p_3^\mu\, p_3^\nu}{Q^2}\right) \\
&= 8\,(1-\varepsilon)\, e^2\, q_{i_1}^2\, \delta_{jj}\, p_1 \cdot p_2
\end{aligned}
$$

with Q^2 is the virtuality of the photon (this is also the invariant mass of the lepton pair),

Squared amplitude

$$
\begin{aligned}
|M_B|^2 &= e^2\, q_{i_1}^2\, \mu^{2\,\varepsilon}\, \delta_{jj}\, \mathrm{Tr}\left[\not{p}_2\, \gamma_\mu\, \not{p}_1\, \gamma_\nu\right] \left(-g_{\mu\nu} + \frac{p_3^\mu\, p_3^\nu}{Q^2}\right) \\
&= 8\,(1-\varepsilon)\, e^2\, q_{i_1}^2\, \delta_{jj}\, p_1 \cdot p_2
\end{aligned}
$$

with Q^2 is the virtuality of the photon (this is also the invariant mass of the lepton pair), **Averaging** over the initial colour and spin and **summing** over the final ones leads

$$
|\overline{M}_B|^2 = \frac{2}{N}\,(1-\varepsilon)\, e^2\, q_{i_1}^2\, \mu^{2\,\varepsilon}\, p_1 \cdot p_2
$$

Partonic cross section

The integration over the phase space can be done very easily by trading $d^{n-1}p_3/(2\,E_3)$ **against** $d^n p_3\,\delta^+(p_3^2 - Q^2)$ and integrating on $d^n p_3$ using the energy-momentum conservation $\delta^n(p_1 + p_2 - p_3)$ yielding

$$\frac{d\hat{\sigma}_{i_1+i_2\to\gamma^\star}}{dQ^2} = \frac{1}{4\,p_1.p_2}\,(2\,\pi)\,\delta^+((p_1 + p_2)^2 - Q^2)\,|\overline{M}_B|^2$$

Partonic cross section

The integration over the phase space can be done very easily by trading $d^{n-1}p_3/(2\,E_3)$ **against** $d^n p_3\,\delta^+(p_3^2 - Q^2)$ and integrating on $d^n p_3$ using the energy-momentum conservation $\delta^n(p_1 + p_2 - p_3)$ yielding

$$\frac{d\hat{\sigma}_{i_1+i_2\to\gamma^\star}}{dQ^2} = \frac{1}{4\,p_1.p_2}\,(2\,\pi)\,\delta^+((p_1+p_2)^2 - Q^2)\,|\overline{M}_B|^2$$

Let introduce some new variables. The available energy in the partonic centre of mass is $\sqrt{\hat{s}}$ with $\hat{s} = (p_1+p_2)^2 = 2\,p_1\cdot p_2$, we define $\boldsymbol{\tau = Q^2/s}$ and $\boldsymbol{z = Q^2/\hat{s} = \tau/(x_1\,x_2)}$. In terms of these new variables, the partonic cross section reads

$$\begin{aligned}\frac{d\hat{\sigma}_{i_1+i_2\to\gamma^\star}}{dQ^2} &= \frac{\pi}{Q^2}\,\frac{1}{N}\,(1-\varepsilon)\,e^2\,q_{i_1}^2\,\mu^{2\,\varepsilon}\,\delta(1-z)\\ &\equiv \hat{\sigma}_B(Q^2,\varepsilon)\,e^2\,q_{i_1}^2\,\delta(1-z)\end{aligned}$$

Hadronic cross section

The hadronic cross section becomes

$$\frac{d\sigma_H}{dQ^2} = \sum_{i_1, i_2 \in S_p} \int dx_1 \, dx_2 \, F_{i_1}^{H_1}(x_1) \, F_{i_2}^{H_2}(x_2) \, \hat{\sigma}_B(Q^2, \varepsilon) \, (e \, q_{i_1})^2 \; \delta(1 - z)$$

$$= e^2 \, \frac{\hat{\sigma}_B(Q^2, \varepsilon)}{s} \sum_{i_1, i_2 \in S_p} q_{i_1}^2 \int_{Q^2/S}^{1} \frac{dx_1}{x_1} \, F_{i_1}^{H_1}(x_1) \, F_{i_2}^{H_2}\left(\frac{Q^2}{x_1 \, S}\right)$$

Hadronic cross section

The hadronic cross section becomes

$$\frac{d\sigma_H}{dQ^2} = \sum_{i_1, i_2 \in S_p} \int dx_1\, dx_2\, F_{i_1}^{H_1}(x_1)\, F_{i_2}^{H_2}(x_2)\, \hat{\sigma}_B(Q^2, \varepsilon)\, (e\, q_{i_1})^2\; \delta(1 - z)$$

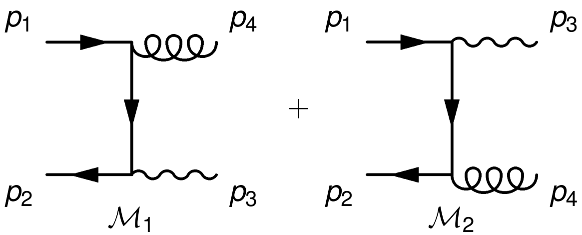
$$= e^2\, \frac{\hat{\sigma}_B(Q^2, \varepsilon)}{S} \sum_{i_1, i_2 \in S_p} q_{i_1}^2 \int_{Q^2/S}^{1} \frac{dx_1}{x_1}\, F_{i_1}^{H_1}(x_1)\, F_{i_2}^{H_2}\left(\frac{Q^2}{x_1\, S}\right)$$

The **lower bound** of the x_1 integration is determined by requiring that

$$x_2 \leq 1 \quad \rightarrow \quad \frac{Q^2}{x_1\, S} \leq 1 \quad \rightarrow \quad x_1 \geq \frac{Q^2}{S}$$

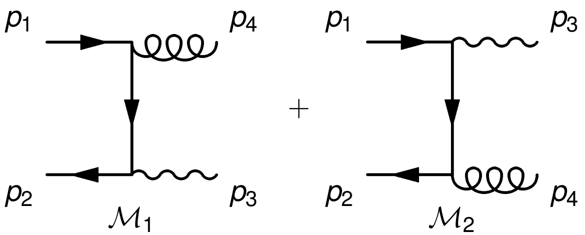
$q\,\bar{q}$ contribution

The reaction $q_i + \bar{q}_i \to \gamma^* + g$



$q\,\bar{q}$ contribution

The reaction $q_i + \bar{q}_i \to \gamma^* + g$



The different amplitudes read

$$\mathcal{M}_1 = K\,\bar{v}(p_2)\,\gamma_\mu\,\frac{(\not{p}_1 - \not{p}_4)}{(p_1 - p_4)^2 + i\,\lambda}\,\gamma_\nu\,u(p_1)\,\epsilon^\mu(p_3)\,\epsilon^\nu(p_4)$$

$$\mathcal{M}_2 = K\,\bar{v}(p_2)\,\gamma_\nu\,\frac{(\not{p}_4 - \not{p}_2)}{(p_4 - p_2)^2 + i\,\lambda}\,\gamma_\mu\,u(p_1)\,\epsilon^\mu(p_3)\,\epsilon^\nu(p_4)$$

Soft approximation

Using mass shell conditions

$$(p_1 - p_4)^2 = -2\,p_1 \cdot p_4$$
$$(p_4 - p_2)^2 = -2\,p_2 \cdot p_4$$

Denominators go to zero when $p_4 \to 0$.

Soft approximation

Using mass shell conditions

$$(p_1 - p_4)^2 = -2\, p_1 \cdot p_4$$
$$(p_4 - p_2)^2 = -2\, p_2 \cdot p_4$$

Denominators go to zero when $p_4 \to 0$.

$$\mathcal{M}_{1\,\text{soft}} = -K\, \frac{p_{1\,\nu}}{p_1 \cdot p_4}\, \bar{v}(p_2)\, \gamma_\mu\, u(p_1)\, \epsilon^\mu(p_3)\, \epsilon^\nu(p_4)$$
$$\mathcal{M}_{2\,\text{soft}} = K\, \frac{p_{2\,\nu}}{p_2 \cdot p_4}\, \bar{v}(p_2)\, \gamma_\mu\, u(p_1)\, \epsilon^\mu(p_3)\, \epsilon^\nu(p_4)$$

Soft approximation

Using mass shell conditions

$$(p_1 - p_4)^2 = -2\,p_1 \cdot p_4$$
$$(p_4 - p_2)^2 = -2\,p_2 \cdot p_4$$

Denominators go to zero when $p_4 \to 0$.

$$\mathcal{M}_{1\,\text{soft}} = -K\,\frac{p_{1\,\nu}}{p_1 \cdot p_4}\,\bar{v}(p_2)\,\gamma_\mu\,u(p_1)\,\epsilon^\mu(p_3)\,\epsilon^\nu(p_4)$$
$$\mathcal{M}_{2\,\text{soft}} = K\,\frac{p_{2\,\nu}}{p_2 \cdot p_4}\,\bar{v}(p_2)\,\gamma_\mu\,u(p_1)\,\epsilon^\mu(p_3)\,\epsilon^\nu(p_4)$$

that is to say

$$\mathcal{M}_{q\bar{q}\,\text{soft}} \equiv M_{1\,\text{soft}} + M_{2\,\text{soft}} = \kappa\left[\frac{p_2 \cdot \epsilon(p_4)}{p_2 \cdot p_4} - \frac{p_1 \cdot \epsilon(p_4)}{p_1 \cdot p_4}\right] M_B$$

Squared amplitude

The square matrix element is the given in this approximation

$$\overline{\sum} |\mathcal{M}|^2_{q\bar{q}\,\text{soft}} = C\,\frac{p_1 \cdot p_2}{p_1 \cdot p_4\, p_2 \cdot p_4}\,|M_B|^2$$

Squared amplitude

The square matrix element is the given in this approximation

$$\overline{\sum} |\mathcal{M}|^2_{q\bar{q}\,\text{soft}} = C\,\frac{p_1 \cdot p_2}{p_1 \cdot p_4\, p_2 \cdot p_4}\,|M_B|^2$$

Note that the **full amplitude squared** will have the following structure

$$\overline{\sum} |\mathcal{M}|^2_{q\bar{q}} = \left[H_{12}(p_4)\,\frac{p_1 \cdot p_2}{p_1 \cdot p_4\, p_2 \cdot p_4} + G(p_4) \right]$$

Squared amplitude

The square matrix element is the given in this approximation

$$\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}\,\mathrm{soft}} = C\,\frac{p_1 \cdot p_2}{p_1 \cdot p_4\, p_2 \cdot p_4}\,|M_B|^2$$

Note that the **full amplitude squared** will have the following structure

$$\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}} = \left[H_{12}(p_4)\,\frac{p_1 \cdot p_2}{p_1 \cdot p_4\, p_2 \cdot p_4} + G(p_4)\right]$$

$H_{12}(p_4)$ and $G(p_4)$ **regular** when $p_4 \to 0$ (or p_4 collinear to p_1 or p_2).$|M_t|^2$ is **singular** in the soft limit ($p_4 \to 0$) and/or in the collinear limits ($p_4 = z_1\,p_1$ or $p_4 = z_2\,p_2$).

Need to integrate over the momentum p_4

$$\frac{d\hat{\sigma}_{q_i+\bar{q}_i\to\gamma^*+g}}{dQ^2} = \frac{1}{4\,p_1.p_2} \int \frac{d^{n-1}p_3}{(2\,\pi)^{n-1}\,2\,E_3}\,\frac{d^{n-1}p_4}{(2\,\pi)^{n-1}\,2\,E_4} \times (2\,\pi)^n\,\delta^n(p_1+p_2-p_3-p_4)\,\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}}$$

At the hadronic level, the cross section is given by

$$\frac{d\sigma_H}{dQ^2} = \sum_{i_1,i_2\in S_p} \int dx_1\,dx_2\,F^{H_1}_{i_1}(x_1)\,F^{H_2}_{i_2}(x_2)\,\frac{d\hat{\sigma}_{q_i+\bar{q}_i\to\gamma^*+g}}{dQ^2} \quad (9)$$

The squared amplitude for $q\,\bar{q}$ case

The computation of the diagrams can be done easily

$$\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}} = (eq_{i_1}\mu^{\varepsilon})^2(g\mu^{\varepsilon})^2\,\frac{C_F}{N}2\,(1-\varepsilon)\left[(1-\varepsilon)(\frac{\hat{t}}{\hat{u}}+\frac{\hat{u}}{\hat{t}})+2\,\frac{\hat{s}\,Q^2}{\hat{u}\,\hat{t}}-2\varepsilon\right],$$

Phase space integral I

In the initial partons CMF

$$p_1 = \sqrt{\hat{s}/2}\,(1,0,\cdots,1)\;\;;\;\; p_2 = \sqrt{\hat{s}/2}\,(1,0,\cdots,-1)\;\;;$$
$$p_4 = E_4\,(1,\cdots,\cos\theta_1).$$

To evaluate the integration over the phase space

$$\begin{aligned} PS &= \int \frac{d^{n-1}p_4}{(2\pi)^{n-1}\,2\,E_4}\,\frac{d^{n-1}p_3}{(2\pi)^{n-1}\,2\,E_3}(2\pi)^n\delta^{(n)}(p_1+p_2-p_3-p_4) \\ &= \frac{(2\pi)^{2-n}}{4\sqrt{\hat{s}}}\left(\frac{\hat{s}-Q^2}{2\sqrt{\hat{s}}}\right)^{n-3}\int d\Omega_{n-2}. \end{aligned} \tag{10}$$

Phase space integral II

To perform the **angular integration**, the following change of variable is introduced $\cos\theta_1 = 2y - 1$, this leads to

$$PS = \frac{1}{8\pi} \left(\frac{4\pi}{Q^2}\right)^{\varepsilon} \frac{z^{\varepsilon}(1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 dy\; y^{-\varepsilon}(1-y)^{-\varepsilon}, \qquad (11)$$

Phase space integral II

To perform the **angular integration**, the following change of variable is introduced $\cos\theta_1 = 2y - 1$, this leads to

$$PS = \frac{1}{8\pi} \left(\frac{4\pi}{Q^2}\right)^{\varepsilon} \frac{z^{\varepsilon}(1-z)^{1-2\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 dy\; y^{-\varepsilon}(1-y)^{-\varepsilon}, \qquad (11)$$

In terms of these **dimensionless variables**, the different invariants are

$$\hat{s} = \frac{Q^2}{z}\;;\;\; (p_1 - p_4)^2 = \hat{t} = -\frac{Q^2}{z}(1-y)(1-z)\;;$$
$$(p_2 - p_4)^2 = \hat{u} = -\frac{Q^2}{z}(1-z)y$$

Extraction of divergent terms I

The coefficient H_{12}, in terms of the new variables y and z

$$H_{12}(y,z) = (g\mu^{\varepsilon})^2\, C_F\, |\overline{M}_B|^2\, \frac{1}{z}\, \left\{(1-\varepsilon)\,(1-z)^2\, \left[(1-y)^2 + y^2\right] + 2\,z\right\}$$

Extraction of divergent terms I

The coefficient H_{12}, in terms of the new variables y and z

$$H_{12}(y,z) = (g\mu^{\varepsilon})^2\, C_F\, |\overline{M}_B|^2\, \frac{1}{z}\, \left\{(1-\varepsilon)\,(1-z)^2\, \left[(1-y)^2 + y^2\right] + 2\,z\right\}$$

The **eikonal factor** E_{12} can be also expressed in terms of the variables y and z

$$\begin{aligned} E_{12} &= \frac{2}{Q^2}\, \frac{z}{(1-z)^2}\, \frac{1}{y\,(1-y)} \\ &= \frac{2}{Q^2}\, \frac{z}{(1-z)^2}\, \left[\frac{1}{y} + \frac{1}{1-y}\right] \\ &\equiv E_{12}^{(1)} + E_{12}^{(2)} \end{aligned}$$

Extraction of divergent terms II

Extract the divergent part from the squared amplitude

$$PS\,\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}} = PS\,\{H_{12}(0,z)\,E^{(1)}_{12} + H_{12}(1,z)\,E^{(2)}_{12}\} + \text{finite pieces}$$

Extraction of divergent terms II

Extract the divergent part from the squared amplitude

$$PS\,\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}} = PS\,\{H_{12}(0,z)\,E_{12}^{(1)} + H_{12}(1,z)\,E_{12}^{(2)}\} + \text{finite pieces}$$

Let us first evaluate the function H_{12} with the different arguments

$$H_{12}(0,z) = (g\mu^{\varepsilon})^2\,C_F\,|\overline{M}_B|^2\,\frac{1}{z}\,\left[1 + z^2 - \varepsilon\,(1-z)^2\right] = H_{12}(1,z)$$

Extraction of divergent terms II

Extract the divergent part from the squared amplitude

$$PS\,\overline{\Sigma}\,|\mathcal{M}|^2_{q\bar{q}} = PS\,\{H_{12}(0,z)\,E_{12}^{(1)} + H_{12}(1,z)\,E_{12}^{(2)}\} + \text{finite pieces}$$

Let us first evaluate the function H_{12} with the different arguments

$$H_{12}(0,z) = (g\mu^{\varepsilon})^2\,C_F\,|\overline{M}_B|^2\,\frac{1}{z}\,\left[1 + z^2 - \varepsilon\,(1-z)^2\right] = H_{12}(1,z)$$

Let us compute the different pieces

$$\begin{aligned} PS\,H_{12}(0,z)\,E_{12}^{(1)} &= H_{12}(0,1)\,PS\,E_{12} \\ &= \frac{1}{4\,\pi\,Q^2}\,\left(\frac{4\,\pi}{Q^2}\right)^{\varepsilon}\,z^{1+\varepsilon}\,(1-z)^{-1-2\,\varepsilon}\,H_{12}(0,z) \\ &\quad\times\left(-\frac{1}{\varepsilon}\right)\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \end{aligned}$$

"Plus" distributions I

Appearance of the pole in ε related to the singularity at $z = 1$, $(1-z)^{-1-2\,\varepsilon}$ at the limit $\varepsilon = 0$ is a distribution

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon\,\sqrt{\pi}}\, e^{-\left(\frac{x}{\varepsilon}\right)^2}$$

"Plus" distributions I

Appearance of the pole in ε related to the singularity at $z = 1$, $(1-z)^{-1-2\,\varepsilon}$ at the limit $\varepsilon = 0$ is a distribution

$$\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon\,\sqrt{\pi}}\, e^{-\left(\frac{x}{\varepsilon}\right)^2}$$

Thus, to discuss its property we have to apply it to a test function. Let us introduce a test function $F(z)$ which is regular at $z - 1$

$$\int_0^1 dz\, F(z)\,(1-z)^{-1-2\,\varepsilon}$$

$$= \int_0^1 dz\,(F(z) - F(1))\,(1-z)^{-1-2\,\varepsilon} + F(1)\int_0^1 dz\,(1-z)^{-1-2\,\varepsilon}$$

$$= \int_0^1 dz\,\frac{F(z)-F(1)}{1-z}\sum_{n=0}^{\infty}(-2\,\varepsilon)^n\,\ln^n(1-z) - F(1)\,\frac{1}{2\,\varepsilon}$$

"Plus" distributions II

Thus, in the distribution sense, we can write that

$$(1-z)^{-1-2\,\varepsilon} = -\frac{1}{2\,\varepsilon}\,\delta(1-z) + \frac{1}{(1-z)_+} - 2\,\varepsilon\left(\frac{\ln(1-z)}{1-z}\right)_+ + O(\varepsilon^2)$$

"Plus" distributions II

Thus, in the distribution sense, we can write that

$$(1-z)^{-1-2\,\varepsilon} = -\frac{1}{2\,\varepsilon}\,\delta(1-z) + \frac{1}{(1-z)_+} - 2\,\varepsilon\,\left(\frac{\ln(1-z)}{1-z}\right)_+ + O(\varepsilon^2)$$

where the "plus" distributions are defined as

$$\int_0^1 dz\,(g(z))_+\,F(z) \equiv \int_0^1 dz\,g(z)\,(F(z)-F(1)) \tag{12}$$

where $g(z)$ is a function singular at $z=1$ such that $(1-z)\,g(z)$ is integrable and $F(z)$ is a regular one at the same point. Note that the lower bound 0 in the integral is purely conventional.

Final result I

Introduce $a_{qq}^{(n)}(z) \equiv C_F\,(1 + z^2 - \varepsilon\,(1-z)^2)$

$$PS\,H_{12}(0,z)\,E_{12}^{(1)} = \frac{\alpha_s}{Q^2}\left(\frac{4\,\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,|\overline{M}_B|^2$$
$$\times\left\{\frac{1}{2\,\varepsilon^2}\,\delta(1-z)\,a_{qq}^{(n)}(1) - \frac{1}{\varepsilon}\,\frac{a_{qq}^{(n)}(z)}{(1-z)_+} - \frac{a_{qq}^{(4)}(z)}{(1-z)_+}\,\ln(z)\right.$$
$$\left. + 2\,a_{qq}^{(4)}(z)\,\left(\frac{\ln(1-z)}{1-z}\right)_+\right\} + O(\varepsilon^2)$$

Since $H_{12}(0,z) = H_{12}(1,z)$ and since the phase space is **symmetric** $y \leftrightarrow 1-y$, the contribution which diverges at $y=1$ will be equal to the one which diverges at $y=0$.

Final result II

Thus, the total contributions will be given by

$$\frac{1}{2\,\hat{s}}\,PS\,\left(H_{12}(0,z)\,E_{12}^{(1)} + H_{12}(1,z)\,E_{12}^{(2)}\right)$$
$$= z\,\frac{\alpha_s}{2\,\pi}\,\left(\frac{4\,\pi\mu^2}{Q^2}\right)^{\varepsilon}\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\hat{\sigma}_B(Q^2,\varepsilon)\,e^2\,q_i^2\,F_{q\bar{q}}(z,\varepsilon)$$

with

$$F_{q\bar{q}}(z,\varepsilon) = \frac{1}{\varepsilon^2}\,\delta(1-z)\,a_{qq}^{(n)}(1) - \frac{2}{\varepsilon}\,\frac{a_{qq}^{(n)}(z)}{(1-z)_+} - 2\,\frac{a_{qq}^{(4)}(z)}{(1-z)_+}\,\ln(z)$$
$$+ 4\,a_{qq}^{(4)}(z)\,\left(\frac{\ln(1-z)}{1-z}\right)_+ + \text{finite terms}$$

Remarks

- The coefficients in front of the divergences **"factorise"** : $f(z) \times \hat{\sigma}_B$, true in n dimensions

Remarks

- The coefficients in front of the divergences **"factorise"** : $f(z) \times \hat{\sigma}_B$, true in n dimensions
- At $y = 0$ or $y = 1$, the variable z plays the role of a **"collinear"** variable. Let us denote k_1, the 4-momentum of the quark after the emission of the gluon of 4-momentum p_4,

$$p_3 = k_1 + p_2 \, ; \; p_1 = p_4 + k_1$$

by momentum conservation $Q^2 = k_1^2 + 2\,k_1 \cdot p_2$, At $y = 1$, $Q^2 = 2\,k_1 \cdot p_2$, thus

$$\hat{s} = 2\,p_1 \cdot p_2 = \frac{Q^2}{z} = \frac{2\,k_1 \cdot p_2}{z}$$

implying that $k_1 = z\,p_1$.

The squared amplitude for $q\,g$ case

The amplitude squared obtained from the preceding case by exchanging $\hat{s} \leftrightarrow \hat{t}$ and multiplying by -1 because an anti-fermion of the initial state becomes a fermion in the final state.

$$\frac{C_F}{N}\,\frac{N}{N^2-1} = \frac{N^2-1}{2\,N^2}\,\frac{N}{N^2-1} = \frac{1}{2\,N}$$

The squared amplitude for $q\,g$ case

The amplitude squared obtained from the preceding case by exchanging $\hat{s} \leftrightarrow \hat{t}$ and multiplying by -1 because an anti-fermion of the initial state becomes a fermion in the final state.

$$\frac{C_F}{N}\,\frac{N}{N^2-1} = \frac{N^2-1}{2\,N^2}\,\frac{N}{N^2-1} = \frac{1}{2\,N}$$

The squared amplitude for the reaction $q + g \to \gamma^\star + q$ is then

$$\overline{\sum}|\mathcal{M}|^2_{qg} = (1-\varepsilon)\,(ee_q\mu^\varepsilon)^2(g\mu^\varepsilon)^2\,\frac{1}{2\,N}\,2\,\left[(1-\varepsilon)(-\frac{\hat{s}}{\hat{u}} - \frac{\hat{u}}{\hat{s}}) - 2\,\frac{\hat{t}\,q^2}{\hat{u}\,\hat{s}} + 2\varepsilon\right]$$

Extraction of divergent terms I

In this case, the coefficient of the eikonal factor E_{12} can be easily extracted and is given by

$$H_{12}(y,z) = (1 = \varepsilon)\,(e\,q_i\,\mu^{\varepsilon})^2\,(g\,\mu^{\varepsilon})^2\,\frac{1}{2\,N}\,\frac{Q^2}{z}$$
$$\times\left[(1-\varepsilon)\,(1+(1-z)^2\,y^2)\,(1-y)\,(1-z)\right.$$
$$\left.-\,2\,z\,(1-y)\,(1-z)^2\right]$$

Extraction of divergent terms I

In this case, the coefficient of the eikonal factor E_{12} can be easily extracted and is given by

$$
\begin{aligned}
H_{12}(y,z) = (1 = \varepsilon)\,(e\,q_i\,\mu^{\varepsilon})^2\,(g\,\mu^{\varepsilon})^2\,\frac{1}{2\,N}\,\frac{Q^2}{z} \\
\times \Big[(1-\varepsilon)\,(1+(1-z)^2\,y^2)\,(1-y)\,(1-z) \\
-\,2\,z\,(1-y)\,(1-z)^2\Big]
\end{aligned}
$$

Note that in this case, $H_{12}(y,1) = 0 = H_{12}(0,1) = H_{12}(1,1)$ which is an expected result because, at lowest order, there is not such a initial state! Note also that $H_{12}(1,z) = 0$ telling us that there is no divergence when p_4 is collinear to p_1 in this case. The only divergence appears at $y = 0$

Extraction of divergent terms II

Let us introduce $a^{(n)}_{qg}(z) = 1/2\,(1-z)\,[(1-z)^2 + z^2 - \varepsilon]$

$$H_{12}(0,z) = (g\,\mu^{\varepsilon})^2\,|\overline{M}_B|^2\,\frac{Q^2}{z}\,a^{(n)}_{qg}(z)$$

Extraction of divergent terms II

Let us introduce $a_{qg}^{(n)}(z) = 1/2\,(1-z)\,[(1-z)^2 + z^2 - \varepsilon]$

$$H_{12}(0,z) = (g\,\mu^{\varepsilon})^2\,|\overline{M}_B|^2\,\frac{Q^2}{z}\,a_{qg}^{(n)}(z)$$

Pick up only the **divergent part**

$$PS\,\overline{\Sigma}\,|\mathcal{M}|^2_{qg} = PS\,H_{12}(0,z)\,E_{12}^{(1)} + \text{finite pieces}$$

Extraction of divergent terms II

Let us introduce $a_{qg}^{(n)}(z) = 1/2\,(1-z)\,[(1-z)^2 + z^2 - \varepsilon]$

$$H_{12}(0,z) = (g\,\mu^{\varepsilon})^2\,|\overline{M}_B|^2\,\frac{Q^2}{z}\,a_{qg}^{(n)}(z)$$

Pick up only the **divergent part**

$$PS\,\overline{\Sigma}\,|\mathcal{M}|^2_{qg} = PS\,H_{12}(0,z)\,E_{12}^{(1)} + \text{finite pieces}$$

With the help of the preceding results

$$\begin{aligned} PS\,H_{12}(0,z)\,E_{12}^{(1)} &= \frac{\alpha_s}{Q^2}\left(\frac{4\,\pi\mu^2}{Q^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,|\overline{M}_B|^2 \\ &\times\left\{-\frac{1}{\varepsilon}\,\frac{a_{qg}^{(n)}(z)}{(1-z)_+} - \frac{a_{qg}^{(4)}(z)}{(1-z)_+}\,\ln(z)\right. \\ &\left.+\,2\,a_{qg}^{(4)}(z)\left(\frac{\ln(1-z)}{1-z}\right)_+\right\} + O(\varepsilon^2) \end{aligned}$$

Final result

No divergence at $z = 1 \Rightarrow a_{qg}^{(n)}(1) = 0$, this the reason why there is no term proportional to $1/\varepsilon^2$.

$$\frac{1}{2\,\hat{s}}\,PS\,H_{12}(0,z)\,E_{12}^{(1)}$$
$$= z\,\frac{\alpha_s}{2\,\pi}\,\left(\frac{4\,\pi\,\mu^2}{Q^2}\right)^{\varepsilon}\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\hat{\sigma}_B(Q^2,\varepsilon)\,e^2\,q_i^2\,F_{qg}(z,\varepsilon)$$

Final result

No divergence at $z = 1 \Rightarrow a_{qg}^{(n)}(1) = 0$, this the reason why there is no term proportional to $1/\varepsilon^2$.

$$\frac{1}{2\,\hat{s}}\,PS\,H_{12}(0,z)\,E_{12}^{(1)}$$
$$= z\,\frac{\alpha_s}{2\,\pi}\,\left(\frac{4\,\pi\mu^2}{Q^2}\right)^{\varepsilon}\,\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\,\hat{\sigma}_B(Q^2,\varepsilon)\,e^2\,q_i^2\,F_{qg}(z,\varepsilon)$$

with

$$F_{qg}(z,\varepsilon) = -\frac{1}{\varepsilon}\,\frac{a_{qg}^{(n)}(z)}{(1-z)_+} - \frac{a_{qg}^{(4)}(z)}{(1-z)_+}\,\ln(z)$$
$$+ a_{qg}^{(4)}(z)\,\left(\frac{\ln(1-z)}{1-z}\right)_+ + \text{finite terms}$$

Virtual cross section

The result of the computation of the virtual contribution is

$$M_v = M_B \frac{\alpha_S}{4\pi} C_F \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \times \left\{ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3} + i\pi \left\{ -\frac{2}{\varepsilon} - 3 \right\} \right\} \tag{13}$$

Virtual cross section

The result of the computation of the virtual contribution is

$$M_v = M_B \frac{\alpha_S}{4\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \times \left\{ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3} + i\pi \left\{ -\frac{2}{\varepsilon} - 3 \right\} \right\} \quad (13)$$

The virtual cross section is obtained by taking the **interference** between the lowest order amplitude and the virtual one

$$\sigma_V = \hat{\sigma}_B(Q^2,\varepsilon) \frac{\alpha_S}{2\pi} C_F \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \times \left\{ -\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \frac{2\pi^2}{3} \right\} \quad (14)$$

Total contribution I

Collecting all the different pieces and using $S_q = u, d, s, c, b$

$$
\begin{aligned}
\frac{d\sigma_H}{dQ^2} = \sum_{i \in S_q} (e\,q_i)^2 \int \frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\, F_{q_i}^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \Bigg\{ \hat{\sigma}_B(Q^2,\varepsilon) \Bigg[\delta(1-z) \\
\times \left(1 + \frac{\alpha_s}{2\,\pi} \left(\frac{4\,\pi\,\mu^2}{Q^2} \right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)} \left\{ -\frac{2\,C_F}{\varepsilon^2} - \frac{3\,C_F}{\varepsilon} + \frac{a_{qq}^{(4)}(1)}{\varepsilon^2} \right\} \right) \\
- \frac{\alpha_s}{2\,\pi} \left(\frac{4\,\pi\,\mu^2}{Q^2} \right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\, \frac{2}{\varepsilon} \left(\frac{a_{qq}^{(a4)}(z)}{(1-z)_+} \right) \Bigg] \\
+ \text{finite pieces} \Bigg\} \\
- \left(F_{q_i}^{H_1}(x_1) + F_{\bar{q}_i}^{H_1}(x_1) \right) F_g^{H_2}(x_2) \\
\times \frac{\alpha_s}{2\,\pi} \left(\frac{4\,\pi\,\mu^2}{Q^2} \right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\,\varepsilon)}\, \frac{1}{\varepsilon} \left(\frac{a_{qg}^{(4)}(z)}{(1-z)_+} \right) \\
+ \text{finite pieces} + \Big[1 \leftrightarrow 2 \Big] \Bigg\}
\end{aligned}
$$

Disappearance of soft divergences

From the definition of $a^{(4)}(1) = 2\, C_F$, the **soft divergence** (term prop. to $1/\varepsilon^2$) **cancels** between the **real emission** and **the virtual one.**

Disappearance of soft divergences

From the definition of $a^{(4)}(1) = 2\,C_F$, the **soft divergence** (term prop. to $1/\varepsilon^2$) **cancels** between the **real emission** and **the virtual one**.

"Lee – Kinoshita – Naurenberg" theorem : the **soft divergences drop out** when adding the **real and virtual emission**.

Disappearance of soft divergences

From the definition of $a^{(4)}(1) = 2\, C_F$, the **soft divergence** (term prop. to $1/\varepsilon^2$) **cancels** between the **real emission** and **the virtual one**.

"Lee – Kinoshita – Naurenberg" theorem : the **soft divergences drop out** when adding the **real and virtual emission**.

The collinear ones still remain after combining the real and the virtual emission, so what to do?

Collinear contributions I

$$\frac{d\sigma_H}{dQ^2} = \sum_{i \in S_q} (e\,q_i)^2\, \hat{\sigma}_B(Q^2, \varepsilon) \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \left\{ \left[F_{q_i}^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \delta(1-z) \right.\right.$$

$$+ \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(F_{q_i}^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \left(-\frac{2}{\varepsilon} \right) P_{qq}(z) \right.$$

$$+ \left(F_{q_i}^{H_1}(x_1)\, F_{g}^{H_2}(x_2) + F_{g}^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \right) \left(-\frac{1}{\varepsilon} \right) P_{qg}(z) \Big)$$

$$\left.\left. + \text{finite pieces} + \left[1 \leftrightarrow 2 \right] \right] \right\}$$

Collinear contributions I

$$\frac{d\sigma_H}{dQ^2} = \sum_{i \in S_q} (e\, q_i)^2\, \hat{\sigma}_B(Q^2, \varepsilon) \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \left\{ \left[F_{q_i}^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \delta(1-z) \right.\right.$$

$$+ \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2} \right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left(F_{q_i}^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \left(-\frac{2}{\varepsilon} \right) P_{qq}(z) \right.$$

$$+ \left(F_{q_i}^{H_1}(x_1)\, F_g^{H_2}(x_2) + F_g^{H_1}(x_1)\, F_{\bar{q}_i}^{H_2}(x_2) \right) \left(-\frac{1}{\varepsilon} \right) P_{qg}(z) \Big)$$

$$\left. + \text{finite pieces} + \Big[1 \leftrightarrow 2 \Big] \right\}$$

with

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\, \delta(1-z) \right]$$

$$P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]$$

and $T_R = 1/2$

Collinear contributions II

By changing x_2 (or x_1) into $\tau/(x_1\, z)$ ($\tau/(x_2\, z)$)

$$\begin{aligned}
\frac{d\sigma_{H\,\mathrm{div}}}{dQ^2} = \sum_{i \in S_q} (eq_i)^2 \, \hat{\sigma}_B(Q^2, \varepsilon) \Bigg\{ \Bigg[\int_0^1 \frac{dx_1}{x_1} \, F_{q_i}^{H_1}(x_1) \, F_{\bar{q}_i}^{H_2}\left(\frac{\tau}{x_1}\right) \\
+ \int_0^1 \frac{dx_1}{x_1} \, F_{q_i}^{H_1}(x_1) \left(-\frac{1}{\varepsilon}\right) \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \\
\times \int_{\frac{\tau}{x_1}}^1 \frac{dz}{z} \left(F_{\bar{q}_i}^{H_2}\left(\frac{\tau/x_1}{z}\right) P_{qq}(z) + F_g^{H_2}\left(\frac{\tau/x_1}{z}\right) P_{qg}(z) \right) \\
+ \int_0^1 \frac{dx_2}{x_2} \, F_{\bar{q}_i}^{H_2}(x_2) \left(-\frac{1}{\varepsilon}\right) \frac{\alpha_s}{2\pi} \left(\frac{4\pi\mu^2}{Q^2}\right)^{\varepsilon} \frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \\
\times \int_{\frac{\tau}{x_2}}^1 \frac{dz}{z} \left(F_{q_i}^{H_1}\left(\frac{\tau/x_2}{z}\right) P_{qq}(z) + F_g^{H_1}\left(\frac{\tau/x_2}{z}\right) P_{qg}(z) \right) \Bigg] \\
+ \Big[1 \leftrightarrow 2 \Big] \Bigg\} \qquad (15)
\end{aligned}$$

Scale dependent PDF

$$F_q^H(x, M^2) = F_q^H(x) - \frac{1}{\varepsilon}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$\times \int_x^1 \frac{dz}{z}\left[F_q^H\left(\frac{x}{z}\right) P_{qq}(z) + F_g^H\left(\frac{x}{z}\right) P_{qg}(z)\right]$$

Scale dependent PDF

$$F_q^H(x, M^2) = F_q^H(x) - \frac{1}{\varepsilon}\frac{\alpha_s}{2\pi}\left(\frac{4\pi\mu^2}{M^2}\right)^{\varepsilon}\frac{\Gamma(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$
$$\times \int_x^1 \frac{dz}{z}\left[F_q^H\left(\frac{x}{z}\right) P_{qq}(z) + F_g^H\left(\frac{x}{z}\right) P_{qg}(z)\right]$$

One can reabsorb the **collinear divergences** into a redefinition of **the "bare" parton densities** (the ones with no scale) and up to terms of order α_s^2, the divergent term can be written as

$$\frac{d\sigma_{H\,\text{div}}}{dQ^2} = \sum_{i \in S_q} (eq_i)^2\, \hat{\sigma}_B(Q^2, \varepsilon) \left\{ \int_0^1 \frac{dx_1}{x_1} F_{q_i}^{H_1}(x_1, M^2)\, F_{\bar{q}_i}^{H_2}\left(\frac{\tau}{x_1}, M^2\right) + \left[1 \leftrightarrow 2\right]\right\} \tag{16}$$

Drell-Yan at α_s

Thus the Drell-Yan cross section, including the α_s **corrections**, can be written as

$$\frac{d\sigma_H}{dQ^2} = \sum_{i \in S_q} (e\, q_i)^2 \, \hat{\sigma}_B(Q^2, \varepsilon) \left\{ \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \, F_{q_i}^{H_1}(x_1, M^2) \, F_{\bar{q}_i}^{H_2}\left(x_2, M^2\right) \right.$$
$$\left. + \frac{\alpha_s}{2\pi} \, [\text{finite pieces}] + \left[1 \leftrightarrow 2\right] \right\}$$

Drell-Yan at α_s

Thus the Drell-Yan cross section, including the α_s **corrections**, can be written as

$$\frac{d\sigma_H}{dQ^2} = \sum_{i \in S_q} (e\, q_i)^2 \, \hat{\sigma}_B(Q^2, \varepsilon) \left\{ \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \, F_{q_i}^{H_1}(x_1, M^2) \, F_{\bar{q}_i}^{H_2}\left(x_2, M^2\right) + \frac{\alpha_s}{2\pi} \, [\text{finite pieces}] + \left[1 \leftrightarrow 2\right] \right\}$$

Note that the procedure to get rid of the **collinear divergences** is very similar to the **renormalisation procedure.** As in the renormalisation, it exists **RGE for the PDF**.

RGE for PDF

Including the **dominant contribution** at each order in perturbation theory for the **collinear divergence** and studying other processes, especially processes involving **gluons in the initial state** at higher order, leads to the general RGE (called "DGLAP")

$$\frac{d}{dt}\begin{pmatrix} F_q^H(x,t) \\ F_g^H(x,t) \end{pmatrix} = \frac{\alpha_s(t)}{2\,\pi}\int_x^1 \frac{dy}{y}\begin{pmatrix} P_{qq}^{(0)}(y) & P_{qg}^{(0)}(y) \\ P_{gq}^{(0)}(y) & P_{gg}^{(0)}(y) \end{pmatrix}\begin{pmatrix} F_q^H(x/y,t) \\ F_g^H(x/y,t) \end{pmatrix}$$

where $t = \ln(M^2/M_0^2)$.

DGLAP kernels

Including more order in the calculation

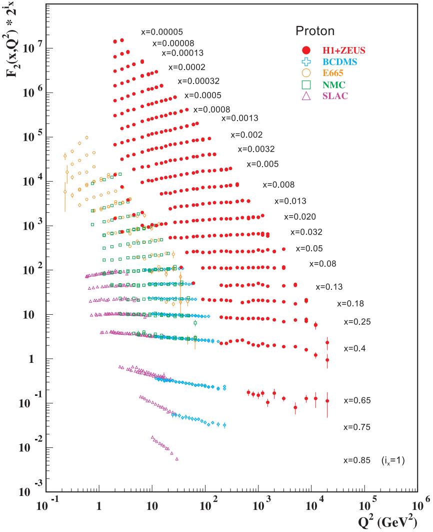
$$P_{ij}(z) = P_{ij}^{(0)}(z) + \frac{\alpha_s}{2\,\pi}\,P_{ij}^{(1)}(z) + \ldots$$

$$\begin{aligned}
P_{qq}^{(0)}(y) &= C_F \left[\frac{1+y^2}{(1-y)_+} + \frac{3}{2}\,\delta(1-y)\right] \\
P_{qg}^{(0)}(y) &= \frac{N_F}{2}\,\frac{y^2+(1-y)^2}{y} \\
P_{gq}^{(0)}(y) &= C_F \left[\frac{1+(1-y)^2}{y}\right] \\
P_{gg}^{(0)}(y) &= 2\,N \left[\frac{1}{(1-y)_+} + \frac{1-y}{y} + y\,(1-y)\right] + \delta(1-y)\,\frac{b_0}{2\,\pi}
\end{aligned}$$

with $b_0 = (11\,N - 2\,N_F)/(12\,\pi)$

violation of the "scale invariance"

Using scale dependent PDF induces a violation of the scale invariance



The QCD improved parton model

The QCD interactions between partons ⇒ the **"QCD improved" parton model**

$$\sigma^{H_1 H_2} = \sum_{i,j} \int dx_1 \, dx_2 \; F_i^{H_1}(x_1, M^2) \; F_j^{H_2}(x_2, M^2) \, \alpha_s(\mu^2)^p \; \hat{\sigma}_{ij}(x_1, x_2, s).$$

The QCD improved parton model

The QCD interactions between partons $\Rightarrow$ the **"QCD improved" parton model**

$$\sigma^{H_1 H_2} = \sum_{i,j} \int dx_1 dx_2 \; F_i^{H_1}(x_1, M^2) \; F_j^{H_2}(x_2, M^2) \, \alpha_s(\mu^2)^p \; \hat{\sigma}_{ij}(x_1, x_2, s).$$

Higher order can be included

$$\hat{\sigma}_{ij}(x_1, x_2, s) = \hat{\sigma}_{ij}^{(0)}(x_1, x_2, s) + \frac{\alpha_s}{2\,\pi} \, \hat{\sigma}_{ij}^{(1)}(x_1, x_2, s) + \ldots$$

The QCD improved parton model

The QCD interactions between partons $\Rightarrow$ the **"QCD improved" parton model**

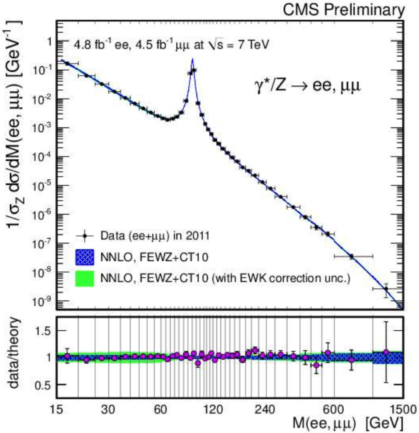
$$\sigma^{H_1 H_2} = \sum_{i,j} \int dx_1 dx_2 \; F_i^{H_1}(x_1, M^2) \; F_j^{H_2}(x_2, M^2) \, \alpha_s(\mu^2)^p \; \hat{\sigma}_{ij}(x_1, x_2, s).$$

Higher order can be included

$$\hat{\sigma}_{ij}(x_1, x_2, s) = \hat{\sigma}_{ij}^{(0)}(x_1, x_2, s) + \frac{\alpha_s}{2\,\pi} \, \hat{\sigma}_{ij}^{(1)}(x_1, x_2, s) + \ldots$$

Leading Order (LO) approximation : compute $\sigma^{\hat{(0)}}{}_{ij}(x_1, x_2, s)$ and use $P_{ij}^{(0)}(y)$ of the DGLAP evolution
Next to Leading Order (NLO) approximation : compute $\sigma^{\hat{(0)}}{}_{ij}(x_1, x_2, s)$ and $\sigma^{\hat{(1)}}{}_{ij}(x_1, x_2, s)$ and use $P_{ij}^{(0)}(y)$ and $P_{ij}^{(1)}(y)$ of the DGLAP evolution

....


CMS Preliminary
4.8 fb^{-1} ee, 4.5 fb^{-1} μμ at √s = 7 TeV
γ*/Z → ee, μμ
1/σ$_Z$ dσ/dM(ee, μμ) [GeV^{-1}]
Data (ee+μμ) in 2011
NNLO, FEWZ+CT10
NNLO, FEWZ+CT10 (with EWK correction unc.)
data/theory
M(ee, μμ) [GeV]

What we learnt in lecture IV

- It exists other **kinds of divergences** than the UV ones : the **soft divergences** when the energy of a massless boson goes to zero and the **collinear divergences** when two massless partons becomes parallel

What we learnt in lecture IV

- It exists other **kinds of divergences** than the UV ones : the **soft divergences** when the energy of a massless boson goes to zero and the **collinear divergences** when two massless partons becomes parallel
- The **soft** divergences cancel when adding **virtual and real contributions**

What we learnt in lecture IV

- It exists other **kinds of divergences** than the UV ones : the **soft divergences** when the energy of a massless boson goes to zero and the **collinear divergences** when two massless partons becomes parallel
- The **soft** divergences cancel when adding **virtual and real contributions**
- The **collinear** divergences (in the initial state) are absorbed into a **redefinition of the PDF**

What we learnt in lecture IV

- It exists other **kinds of divergences** than the UV ones : the **soft divergences** when the energy of a massless boson goes to zero and the **collinear divergences** when two massless partons becomes parallel
- The **soft** divergences cancel when adding **virtual and real contributions**
- The **collinear** divergences (in the initial state) are absorbed into a **redefinition of the PDF**
- Leading to an evolution equation (similar to RGE) for the PDF

What we learnt in lecture IV

- It exists other **kinds of divergences** than the UV ones : the **soft divergences** when the energy of a massless boson goes to zero and the **collinear divergences** when two massless partons becomes parallel
- The **soft** divergences cancel when adding **virtual and real contributions**
- The **collinear** divergences (in the initial state) are absorbed into a **redefinition of the PDF**
- Leading to an evolution equation (similar to RGE) for the PDF
- The "QCD improved" parton model which take into account the interaction between parton (to a certain approximation!) leads to a **violation** of the scale invariance verified experimentally

What we learnt in lecture IV

- It exists other **kinds of divergences** than the UV ones : the **soft divergences** when the energy of a massless boson goes to zero and the **collinear divergences** when two massless partons becomes parallel
- The **soft** divergences cancel when adding **virtual and real contributions**
- The **collinear** divergences (in the initial state) are absorbed into a **redefinition of the PDF**
- Leading to an evolution equation (similar to RGE) for the PDF
- The "QCD improved" parton model which take into account the interaction between parton (to a certain approximation!) leads to a **violation** of the scale invariance verified experimentally
- The "QCD improved" parton model to describe very well **the LHC data**