Standard Model

3/25/2024 Huế
@ Đê-Pô Cafe

The SM was first constructed for the leptons by Weinberg in 1967 based on the gauge group $SU(2) \times U(1)$ that unifies weak & EM interactions. Similar ideas were also pursed by Salam & Glashow independently
$\Rightarrow$ SM $\longrightarrow$ GSW SM

First, Let's review a little bit of the history of weak interaction.

Fermi's theory of β-decay:

Nuclear β-decay was discovered by Becquerel in 1896

$$(A, Z) \longrightarrow (A, Z+1) + e^-$$

In 2-body decay $A \to B + e^-$, all kinematical variables are fixed!

$$p_A = p_B + p_{e^-} \quad \text{(Energy-Momentum Conservation)}$$

$$\Rightarrow p_B = p_A - p_{e^-} \Rightarrow 2 p_A \cdot p_{e^-} = m_A^2 + m_e^2 - m_B^2$$

At the rest frame of A, we have

$$E_e = \frac{m_A^2 + m_e^2 - m_B^2}{2 m_A} \quad \text{which is a fixed number!}$$

However, experimentalists observed a continuous energy spectrum for the process!!

* Bohr suggested energy is not conserved
* Pauli suggested another particle he called 'neutron' was emitted along with e^-. [1930, private communication with Hahn & Meitner]
* Fermi renamed Pauli's 'neutron' the neutrino.
* Neutron was discovered by Chadwick at 1932.
* β-decay was interpreted as the decay of a neutron (n) inside the nucleus into a proton (p), plus a neutrino-electron pair: $n \to p + e^- + \bar{\nu}_e$

In the β-decay $\overset{1/2^+}{n} \rightarrow \overset{1/2^+}{p} + e^- + \bar{\nu}_e$ (with $\tau = 885.7 \pm 0.8\ s$)
one has to assign the new particle $\bar{\nu}_e$ a spin 1/2.
Furthermore, L_e, the electron lepton number is also conserved with the following quantum number assignment

L_e	particles/antiparticles
+1	e^-, ν_e
−1	e^+, $\bar{\nu}_e$
0	others particles

Since $\mu^\pm \rightarrow e^\pm \gamma$ is never observed even though it is kinematically allowed, one can introduce muon lepton number (L_μ) as well in an anologous way.
The third one L_τ, for the third generation can also be introduced!

$\Rightarrow$ **Total lepton number $L = L_e + L_\mu + L_\tau$**

Other weak decays allowed by lepton number conservation are

$$\pi^- \rightarrow \mu^- \bar{\nu}_\mu \qquad \tau = 2.6 \times 10^{-8}\ sec$$

$$\mu^- \rightarrow e^- \bar{\nu}_e \nu_\mu \qquad \tau = 2.2 \times 10^{-6}\ sec$$

* Since for typical strong & electromagnetic reaction rates are of order 10^{-23} sec & 10^{-16} sec respectively, the above processes indicate 'new' physics
$\longrightarrow$ Weak interaction!
* Experiments indicated that neutrinos can have tiny masses.

Classifications of Weak decays:

Leptonic decays

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

$$\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau, \quad e^- + \bar{\nu}_e + \nu_\tau$$

$\vdots$

Semi-leptonic decays

$$K^+ \to \mu^+ \nu_\mu, \quad e^+ \nu_e \qquad |\Delta S| = 1$$

$$K^+ \to \pi^0 \mu^+ \nu_\mu, \quad \pi^0 e^+ \nu_e$$

$\vdots$

Non-leptonic decays

$$K^\pm \to \pi^\pm \pi^0, \pi^\pm \pi^+ \pi^-, \pi^+ \pi^0 \pi^0 \qquad |\Delta S| = 1$$

$\vdots$

Inverse β-decay:

$$p\Big|_{\text{Bound}} \to n\, e^+ \nu_e$$

* A free proton can't decay in SM due to the global baryon number conservation. (Experimentally

$$\tau_{\text{proton}} \gtrsim 10^{32} \text{ yrs} \quad (\text{from } p \to e^+ \pi^0 \text{ search } (\Delta B \neq 0))$$

which is much larger than the age of our universe

$$\tau_{\text{universe}} \simeq 13.7 \text{ billion years } (\Lambda CDM)$$

from Planck's Satellite.

* Grand Unified Theories like SU(5), SO(10), ... predicted proton can decay, e.g. $p \to e^+ \pi^0, e^+ \gamma, \ldots$ But nobody has seen it yet!

Fermi's theory of β-decay is to replace the photon field A_μ in QED by the (νe) pair:

$$\mathcal{L}_{QED} = +e\, j_\mu^{EM} A^\mu = +e\,\bar{\psi} Q \gamma_\mu \psi A^\mu$$ (Peskin Schroeder's convention.)

$$\downarrow$$

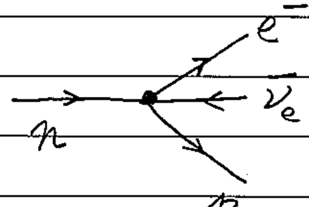
$$\mathcal{H}_{QED} = -e \int d^3\vec{x}\, \bar{\psi} Q \gamma_\mu \psi A^\mu$$

$$\downarrow \text{ Fermi}$$

$$\mathcal{H}_W \sim -\frac{G_F}{\sqrt{2}} \int d^3\vec{x} \left(\bar{\psi}_p \gamma_\mu \psi_n\right)\left(\bar{\psi}_e \gamma^\mu \psi_{\nu_e}\right) + h.c.$$

($\bar{\psi}_e$: create e^-; ψ_{ν_e}: creates $\bar{\nu}_e$; $\bar{\psi}_p$: creates p; ψ_n: annihilates n)

~~* First QFT~~



$\psi_{p,n,e,\nu}$ = Dirac fields

⟹ Parity conserving Int.

In general, for β-decay $A \to B e^- \bar{\nu}_e$, we can have the following parity conserving interaction Hamiltonian (+ non-conserving)?

$$\mathcal{H}_W = -\frac{G_F^\beta}{\sqrt{2}} \int d^3\vec{x} \sum_{i=S,V,T,A,P} C_i \left(\bar{\psi}_B \Gamma_i \psi_A\right)\left(\bar{\psi}_e \Gamma_i \psi_{\nu_e}\right)$$

with

Γ_i	$\bar{\psi}_1 \Gamma_i \psi_2$ (Bilinear)	Tensor Structure	NR limits
$\mathbb{1}$	$\bar{\psi}_1 \psi_2$	S (Scalar)	$\phi_1^\dagger \phi_2$
γ^μ	$\bar{\psi}_1 \gamma^\mu \psi_2$	V (Vector)	$\phi_1^\dagger \phi_2$
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	$\bar{\psi}_1 \sigma^{\mu\nu} \psi_2$	T (anti-sym. Tensor)	$\phi_1^\dagger \vec{\sigma} \phi_2$
$\gamma^\mu \gamma_5$	$\bar{\psi}_1 \gamma^\mu \gamma_5 \psi_2$	A (axial vector)	$\phi_1^\dagger \vec{\sigma} \phi_2$
$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$	$\bar{\psi}_1 \gamma_5 \psi_2$	P (pseudo scalar)	0

(ϕ : Large Comp. of ψ)

Exercise: Explain why we do not have the following billinear

$$\bar{\psi}_1 \sigma_{\mu\nu} \gamma_5 \psi_2$$

in the interaction Hamiltonian $\mathcal{H}_W$?

Out of the 5 possible billinears, in 1957, numerous experiments had finally determined the only allowed couplings are V & A, which is parity-violating! (θ-τ puzzle (1956)). The final form to describe nucleon β-decay is

$$\mathcal{H}_W = -\frac{G_F^\beta}{\sqrt{2}} \int d^3\vec{x} \, (\bar{\psi}_p \gamma^\mu (C_V - C_A \gamma_5) \psi_n)(\bar{\psi}_e \gamma_\mu (1-\gamma_5) \psi_{\nu_e}) + h.c.$$

C_V can be absorbed in G^β so we can set $C_V = 1$.
Through the angular distribution of $\bar{p}_e$ w.r.t. neutron's spin in $n \to p e^- \bar{\nu}_e$, C_A is determined to be

$$C_A = 1.25 \pm 0.009$$

And G^β is determined to be

$$G_F^\beta = (1.136 \pm 0.003) \times 10^{-5} \, GeV^{-2}$$

* Note that $[G^\beta] = [Mass]^{-2} = -2$

Muon decay: Muon was discovered by Anderson in cosmic rays in the late 1930s. Muon can decay weakly too $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ which can be described similarly by 4-fermi interaction

$$\mathcal{H}_W = -\frac{G_F^\mu}{\sqrt{2}} (\bar{\psi}_{\nu_\mu} \gamma^\mu (1-\gamma_5) \psi_\mu)(\bar{\psi}_e \gamma_\mu (1-\gamma_5) \psi_{\nu_e}) + h.c.$$

Setting $e, \bar{\nu}_e$ & ν_μ masses to zeros, we can obtain

$$\Gamma = \frac{1}{\tau} = \frac{(G_F^\mu)^2 m_\mu^5}{192 \pi^3}. \quad \text{With } m_\mu = 106 \text{ MeV}, \ \tau_\mu = 2.2 \times 10^{-6} s,$$

One can deduce G_F^μ, which is very close to G_F^β!

$\Longrightarrow$ Hypothesis of Fermi-interaction universality.

Today, G_F^μ is a basic input parameter in SM.

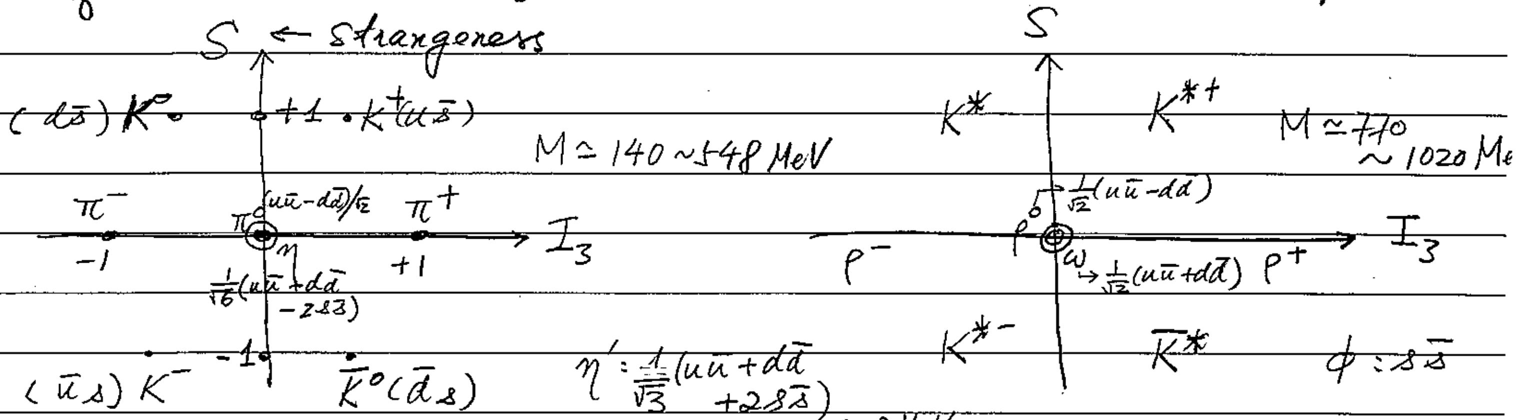
$$\boxed{G_F^\mu = 1.1663788(6) \times 10^{-5} \, GeV^{-2}} \quad \text{(PDG 2023)}$$

Quark Model

Isospin was introduced by Heisenberg to unify proton & neutron:

$$\begin{pmatrix} p \\ n \end{pmatrix} \in SU(2) \quad \underset{\sim}{2} \quad (\text{Global})$$

In the early 1960's, experiments observed 2 octets of mesons with $J^P = 0^-$ & 1^- (orbital excitation)

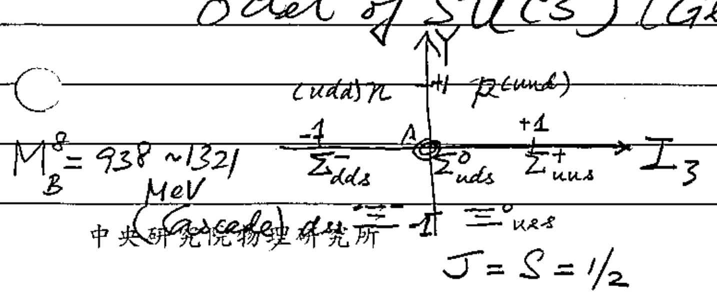


$J^P = 0^-$ (Masses nearly degenerated) $\qquad$ $J^P = 1^-$ (Masses nearly degenerated)

* Strangeness was introduced Kaons & hyperons to account for their unusual lower decay rates compared with the pions & nucleons. Recall that Cabibbo angle θ_c which has been used to suppress the $\Delta S = 1$ current!

* The 2 octets of mesons with nearly degenerated masses suggested to unify isospin & hypercharge into a SU(3), since SU(3) has an adj. irresp. of dim 8.

* In 1964, Gell-Mann & Neeman proposed that the eight baryons of $J^P = 1/2^+$ also belong to an octet of SU(3) (Global)



Y = hypercharge $\equiv B + S$
B = baryon #
S = strangeness

Gell-Mann–Nishijima formula
$Q = I_3 + Y/2$ (in unit of e)

* Also, in 1964, Gell-Mann & Zweig proposed hadronic octets as bound states of triplet-antitriplet or 3 triplets. This triplet fields are called quarks

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix} \in \underset{\sim}{3} \text{ of } SU(3) \quad \text{Fundamental}$$

$$\& \quad \bar{q} = \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix} \in \underset{\sim}{3}^* \text{ of } SU(3) \quad \text{anti-fundamental}$$

Since $3 \otimes \bar{3} = 1 \oplus 8$

& $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$.

One can have

Mesons $M = q\bar{q}$ bound states

Baryons $B = qqq$ or $\bar{q}\bar{q}\bar{q}$ bound states

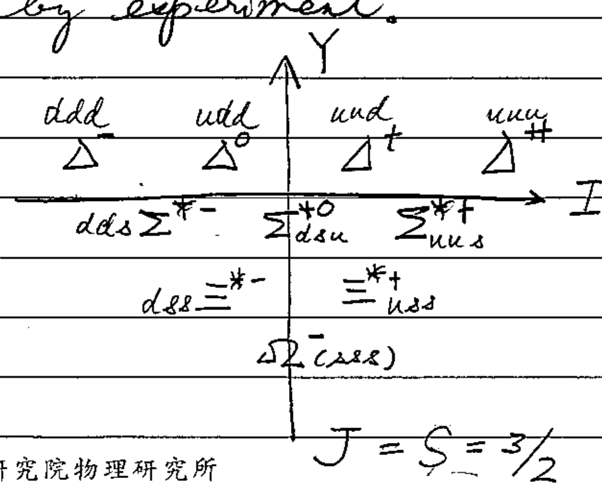
* If $B(q) = 1/3$, $B(\bar{q}) = -1/3$, $\Rightarrow B(\text{baryon}) = +1$,

* q has fractional charge
$Q(u) = \frac{2}{3}$, $Q(d) = -1/3$, $Q(s) = -1/3$

$B(\text{antibaryon}) = -1$

$B(\text{Meson}) = 0 = B(\text{anti-meson})$

* The decuplet $\underset{\sim}{10}$ was also predicted by the quark model. Indeed, the $\Omega^-(sss)$, composed by 3 strange quarks, was predicted by Gell-Mann. This earned him a Nobel prize after its confirmed by experiment.



p : uud , n : udd

$\frac{2}{3}\ \frac{2}{3}\ -\frac{1}{3}$ $\quad$ $\frac{2}{3}\ -\frac{1}{3}\ -\frac{1}{3}$

$M_{\underset{\sim}{10}} = 1232 \sim 1672$ MeV

Current - Current Interaction:

4-Fermi interaction was generalized in 1958 by Feynman & Gell-Mann and independently by Sudarshan & Marshak to the so-called current-current interaction hypothesis:

$$\mathcal{H}_W = -\frac{G_F}{\sqrt{2}} \int d^3\vec{x}\; J^\mu(x)\, J_\mu^\dagger(x)$$ → (adjoint)

with $J^\mu \equiv J^{+\mu} = J_\ell^\mu + J_h^\mu$, (charge raising current)
(Peskin-Schroeder)

& $J_\mu^\dagger \equiv J_\mu^- = (J_\mu)^\dagger$

Later

$$J_\ell^\mu = \bar{\nu}_e \gamma^\mu (1-\gamma_5) e + \bar{\nu}_\mu \gamma^\mu (1-\gamma_5) \mu + \text{“}\bar{\nu}_\tau \gamma^\mu (1-\gamma_5) \tau\text{”}$$

& $$J_h^\mu = \bar{p} \gamma^\mu (C_V - C_A \gamma_5) n + \text{“}\bar{\Lambda} \gamma^\mu (C_V - C_A \gamma_5) p\text{”} + \text{Meson-piece}$$
↳ hyperon (π, K, ⋯)

The form of lepton current J_ℓ^μ implies 3 distinct global lepton number conservation laws: (Noether's theorem)

Electron #:

$$L_e = N_e + N_{\nu_e} - (\bar{N}_e + \bar{N}_{\nu_e})$$

Muon #

$$L_\mu = N_\mu + N_{\nu_\mu} - (\bar{N}_\mu + \bar{N}_{\nu_\mu})$$

“tau #” $$L_\tau = N_\tau + N_{\nu_\tau} - (\bar{N}_\tau + \bar{N}_{\nu_\tau})$$

This hypothesis has supports experimentally since

$\mu^- \longrightarrow e^- \gamma$ $< 4.2 \times 10^{-13}$

$\mu^- \longrightarrow e^- e^+ e^-$ $< 1.0 \times 10^{-12}$

$\mu^- \longrightarrow e^- 2\gamma$ $< 7.2 \times 10^{-11}$

⋮

have never been observed!

Branching ratio best limits nowadays (2020) PDG

Although individual lepton #s are conserved in weak interaction, isospin & strangeness (in general flavor in modern terminology) are not! As seen in the hadronic current J_h^μ, it contains $p \to n$ as well as $p \to \Lambda$ transitions.

The current-current interaction hypothesis implies the following three categories of weak processes:

- Pure leptonic processes ($J_e^{\mu\dagger} J_{e\mu}$)

  e.g. $\mu^- \to e^- \nu_e \bar{\nu}_\mu$

  $\mu^+ \to e^+ \bar{\nu}_e \nu_\mu$

  $\nu(\bar{\nu})$ scatterings (elastic)

  $\nu_e e^- \to \nu_e e^-$

  $\bar{\nu}_e e^+ \to \bar{\nu}_e e^+$

- Semi-leptonic processes ($J_e^{\mu\dagger} J_{h\mu} + h.c.$)

  (i) $\Delta S = 0$ transitions

$n \to p e^- \bar{\nu}_e$	β decay
$e^- p \to n \nu_e$	e-capture
$\mu^- p \to n \nu_\mu$	μ-capture
$\nu_\mu P \to \nu_\mu P$	neutrino reactions
$\pi^\pm \to \mu^\pm \nu_\mu$	pion decay
$\Sigma^+ \to \Lambda^0 e^+ \nu_e$	strange decay

  (ii) $\Delta S = 1$ transitions

$\Lambda^0 \to p e^- \bar{\nu}_e$	hyperon decay
$K^+ \to \mu^+ \nu_\mu$	Kaon decay
$K^0 \to \mu^+ \mu^-$	
$\bar{\nu}_\mu p \to \mu^+ \Lambda^0 (\Sigma^0)$	neutrino reactions

- Non-leptonic processes ($J_h^{\mu\dagger} J_{h\mu}$)

  (i) $\Delta S = 0$

  parity violation in nuclei $np \to np$

  (ii) $\Delta S = 1$ Kaon decay $K \to \pi\pi, 3\pi$ (θ-τ puzzle)

The hadronic $\Delta S = 0$ current (as $\bar{p}\gamma_\mu(C_V - C_A\gamma_5)n$) & the $\Delta S = 1$ current (as $\bar{\Lambda}\gamma_\mu(C_V - C_A\gamma_5)p$) have been assumed having the same couplings C_V & C_A.
Experimentally, one observes

$$(\Delta S = 0 \text{ rate}) \sim 20\,(\Delta S = 1 \text{ rate})$$

Cabibbo proposed to maintain the universality by assuming J_h^μ as a normalized combination as

$$J_h^\mu = J^\mu_{\Delta S=0}\cos\theta_C + J^\mu_{\Delta S=1}\sin\theta_C$$

with the Cabibbo angle $\sin\theta_C = 0.21 \pm 0.03$

Thus, we have

$$\boxed{G_F^\beta = G_F^\mu \cos\theta_C}$$

Which is observed in data!

<u>Summary</u> Current-Current interaction hypothesis for weak processes take the form

$$\mathcal{H}_W = -\frac{G_F}{\sqrt{2}}\int d^3\vec{x}\; J_\mu^{\dagger} J^\mu$$

with

$$J^\mu \equiv J^{\mu+}$$

$$= J_\ell^\mu + \cos\theta_C\, J^\mu_{\Delta S=0} + \sin\theta_C\, J^\mu_{\Delta S=1}$$

This Hamiltonian describes <u>low-energy data</u> for all weak processes quite successful!
However the 4-fermion interaction fails at higher energies.

Problems in 4-fermi interaction theories:

In QFT, we know for a interaction Lagrangian of the form

$$\mathcal{L}_{int} = \sum_i g_i O_i \qquad \{O_i\} \text{ local operators}$$

$[g_i]$ = dimension of coupling g_i corresponding to O_i.

$$[g_i] \begin{cases} < 0 & \text{Super-renormalizable} \\ = 0 & \text{Renormalizable} \\ < 0 & \text{Non-renormalizable} \end{cases}$$

Since $[G_F] = -2$, it belongs to the class of non-renormalizable theory. $\Rightarrow$ $\mathcal{H}_W$ at best an effective field theory (EFT)

Aside: Dimensionalities of field theory objects

Action S shows up in the exponential $e^{iS/\hbar}$ in the path integral formulation of QM. In the natural units of $\hbar = c = 1$, S is dimensionless.

$$\Rightarrow S = \int \mathcal{L}\, d^4x \Rightarrow [\mathcal{L}] = 4 \text{ since } [d^4x] = -4.$$

For example,

(i) $\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ + eA\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$\Rightarrow [\psi] = 3/2, \ [A_\mu] = 1, \ [m] = 1, \ [e] = 0.$

(ii) $\mathcal{L}_{SQED} = (D_\mu\phi)^\dagger(D^\mu\phi)$, $\quad D_\mu \equiv \partial_\mu - ieQA_\mu$

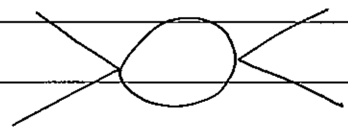
$\Rightarrow [\phi] = 1$

Thus, the 4-fermion interaction $(\bar{\psi}\psi)^2$ has dimension $4 \times \frac{3}{2} = 6$, which implies $[G_F] = -2$!

Theories with negative dimensionalities in the couplings are non-renormalizable, according to QFT.

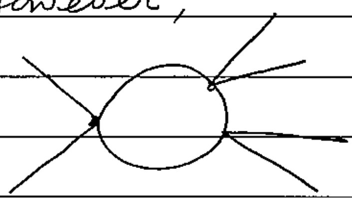
In non-renormalizable theories, one can't remove all the divergences appeared in higher order corrections by a finite # of counter terms.

Take the 4-fermi-interaction as an example:

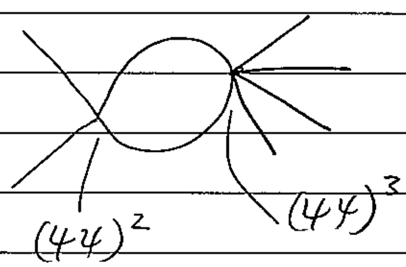


diverges; needs counter-term $(\psi\psi)^2$ which is O.K. since it has the same structure as the tree-level term.

However,



diverges; needs $(\psi\psi)^3$ counterterm which is not in the original Hamiltonian.



diverges; needs $(\psi\psi)^4$ counterterm

- - -

$\Rightarrow$ Unitarity violation
$\Rightarrow$ Non-renormalizable

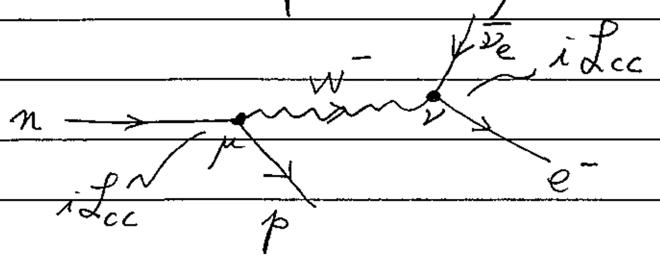
} → These two features are closely related in EFT

Conclusion: $\mathcal{H}_W$ 4-fermi theory (or its generation of current-current interaction hypothesis) can only be viewed as an EFT (in modern terminology)

$\Rightarrow$ Needs a better theory for β-decay (at high energy and more accurate description of the weak processes).

Intermediate massive vector boson (Schwinger; Bludman, Glashow)

First attempt was assumed weak interactions are mediated by a spin 1 charged massive field W_μ:



$$\mathcal{L}_{cc} = +\frac{g}{2\sqrt{2}}\left(J_\mu W^{+\mu} + J_\mu^\dagger W^{-\mu}\right)$$

$$= +\frac{g}{2\sqrt{2}}\left(J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu}\right)$$

Peskin-Schroeder's notation ( )

Massive spin 1 Vector particle propagator (Unitary gauge) $= \frac{-i}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right)$

The amplitude for β-decay becomes $\left(i\mathcal{M}_{fi} = \frac{1}{2!}\int \langle f | (i\mathcal{L}_{cc})^2 | i\rangle\right)$

$$\mathcal{M}_{fi} \sim \langle p | J_\mu^h | n\rangle \frac{g^2}{8} \frac{1}{q^2 - m_W^2}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{m_W^2}\right)\langle \bar{\nu}_e e | J^\dagger_{\nu e} | 0\rangle$$

$$\xrightarrow{q^2 \ll m_W^2} -\frac{g^2}{8 m_W^2}\langle p | J_\mu^h | n\rangle \langle \bar{\nu}_e e | J^\dagger_{\nu e} | 0\rangle$$

Comparing with the EFT (fermi theory), we obtain

$$\boxed{\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 m_W^2}}$$

* Charged massive W boson was first introduced by Schwinger.
* Neutral massive boson was later introduced by Bludman & Glashow.

Argument for a neutral massive vector boson:

From the current $J_{\ell\mu}$ & $J_{\ell\mu}^{\dagger}$ ($J_{\ell\mu}^{+}$ & $J_{\ell\mu}^{-}$), one can look at their time-components and construct their charge operators (Noether's theorem in work !)

$$Q^{+}(t) \equiv \frac{1}{2}\int d^3x\, J_{\ell 0}(\vec{x},t) \;,\; Q^{-}(t) \equiv \frac{1}{2}\int d^3\vec{x}\, J_{\ell 0}^{\dagger}(\vec{x},t)$$

where

$$J_{\ell 0} \equiv \psi_{\nu_e}^{\dagger}(1-\gamma_5)\psi_e = \nu_e^{\dagger}(1-\gamma_5)e$$

$$J_{\ell 0}^{\dagger} \equiv e^{\dagger}(1-\gamma_5)\nu_e \quad \text{(Shift to modern notations)}$$

One can show, by using canonical quantization relations, that

$$[\,Q^{+}(t),\, Q^{-}(t)\,] = Q^{3}(t) \quad \text{with}$$

$$Q^{3}(t) \equiv \frac{1}{2}\int d^3\vec{x}\, J_0^{3} \;, \text{ where}$$

$$J_0^{3} \equiv \nu_e^{\dagger}(1-\gamma_5)\nu_e - e^{\dagger}(1-\gamma_5)e$$

This implies there exists another neutral current J_μ^3

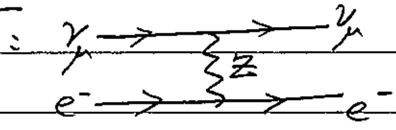
$$J_\mu^3 = \overline{\nu_e}\,\gamma_\mu(1-\gamma_5)\nu_e - \overline{e}\,\gamma_\mu(1-\gamma_5)e$$

which is different from the electromagnetic current

$J_\mu^{EM} \sim \bar{e}\gamma_\mu e$ that we have been familiar with in QED.

* This new neutral current implies a massive neutral boson Z, which couples to electron and neutrino as well !

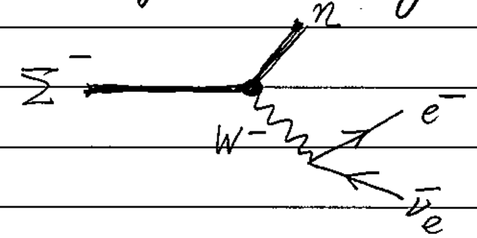
* Discovery of neutral current at CERN in 1973 by elastic scattering $\nu_\mu e^- \to \nu_\mu e^-$:



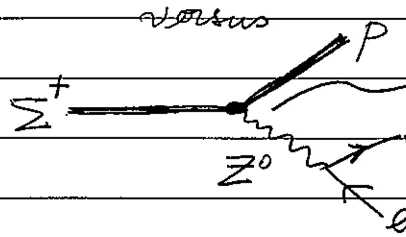
Since the first observation of neutral current at CERN, many other experiments had further confirmed its existence.
However, $\Delta S = 1$ neutral current processes are found to be greatly suppressed as compared to $\Delta S = 1$ charged current processes.

For example:

(i) Hyperon decays $\dfrac{\Gamma(\Sigma^+ \to p e^+ e^-)}{\Gamma(\Sigma^- \to n e^- \bar{\nu}_e)} < 1.3 \times 10^{-2}$

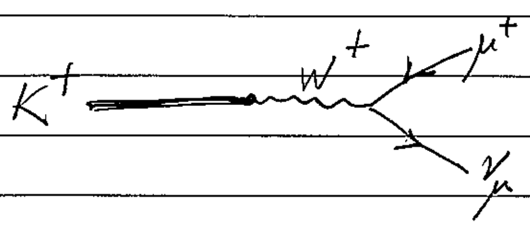


versus

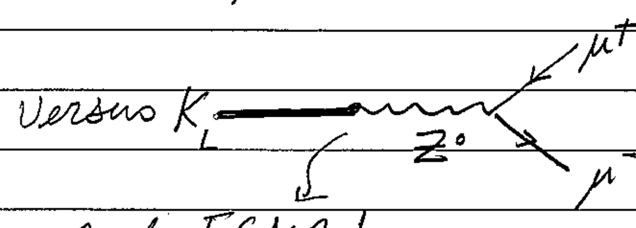


→ Needs flavor changing neutral current (FCNC)

(ii) Kaon decays: $\dfrac{\Gamma(K_L \to \mu^+ \mu^-)}{\Gamma(K^+ \to \mu^+ \nu_\mu)} < 1.3 \times 10^{-2}$



Versus

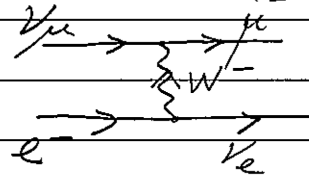


⟹ FCNC must be suppressed (i.e. absent at tree level)

⟹ SM must be built with this property at tree level.

The intermediate massive W boson did improves high energy behaviors of some but not all weak processes. For example

(i) $\nu_\mu e^- \to \mu^- \nu_e$



$s \equiv (p_{\nu_\mu} + p_e)^2$

has a cross section

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{G_F^2 m_W^4}{4\pi^2 s}\left(\frac{s}{s - m_W^2}\right)^2 .$$

Thus

$$\left(\frac{d\sigma}{d\Omega}\right) = \begin{cases} \dfrac{G_F^2 s}{4\pi^2}, & s \ll m_W^2 \\ \dfrac{G_F^2 m_W^4}{4\pi^2 s}, & s \gg m_W^2 \end{cases}$$

* The bad behavior from the $q^\mu q^\nu$ in the W-propagator is dropped out when it hits the external lepton lines giving rise to lepton masses that can be omitted at high energies. (EOM!)

(ii) $e^+e^- \to W^+W^-$, $\nu\bar{\nu} \to W^+W^-$ --

One can show that for these processes with external Ws, the cross sections grow like $s \sim (\text{Energy})^2$. This is because we have the bad behavior term $q^\mu q^\nu$ from the external Ws polarization sum!

$\Rightarrow$ Intermediate massive W-boson theory violates unitarity and thus is non-renormalizable according to QFT.

Three new ideas are needed

(a) Gauge theories

(b) Spontaneous Symmetry breaking] $\to$ Compatible with renormalizability

(c) Higgs Mechanism

to construct renormalization weak theory!

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Weinberg's Model of Leptons PRL 19(21), p1264-1266 (1967)

Return to the 4-fermion interaction

$$\mathcal{L}_W = +\frac{G_F}{\sqrt{2}} J_\mu J^{\mu\dagger}, \quad \begin{aligned} J_\mu = J_\mu^+ &= \bar{\nu}_e \gamma_\mu (1-\gamma_5) e \\ J_\mu^\dagger = J_\mu^- &= \bar{e} \gamma_\mu (1-\gamma_5) \nu_e \end{aligned} \Big\} \text{(Historic)}$$

$$= +\frac{4G_F}{\sqrt{2}} j_\mu^- j^{+\mu}, \quad \begin{aligned} j_\mu^+ &\equiv \frac{1}{2} J_\mu = \bar{\nu}_e \gamma_\mu P_L e \\ &= \bar{\nu}_{e_L} \gamma_\mu e_L \\ \& \quad j_\mu^- &\equiv \frac{1}{2} J_\mu^\dagger = \bar{e}_L \gamma_\mu \nu_{e_L} \end{aligned} \Big\} \text{(Now)}$$

Introducing the $W^\pm$ intermediate boson, we have

$$\mathcal{L}_{cc} = +\frac{g}{2\sqrt{2}} \left(J_\mu^+ W^{+\mu} + J_\mu^- W^{-\mu} \right) = +\frac{g}{\sqrt{2}} \left(j_\mu^+ W^{+\mu} + j_\mu^- W^{-\mu} \right)$$

Define a doublet of SU(2), i.e. isodoublet

$$\left(I_3 = \begin{matrix} +1/2 \\ -1/2 \end{matrix} \right) \quad E = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \equiv P_L \begin{pmatrix} \nu_e \\ e \end{pmatrix} = \frac{1}{2}(1-\gamma_5) \begin{pmatrix} \nu_e \\ e \end{pmatrix}$$

Now

$$j_\mu^- = \bar{e}_L \gamma_\mu \nu_{e_L} = \bar{E} \gamma_\mu \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} E = \bar{E} \gamma_\mu \tau^- E$$

where $\tau^- \equiv \frac{1}{2}(\tau^1 - i\tau^2)$ $\tau^{1,2,3}$ = Pauli Matrix, τ^- = Ladder operator (lowering)

Similarly $j_\mu^+ = \bar{E} \gamma_\mu \tau^+ E$, $\tau^+ = \frac{1}{2}(\tau^1 + i\tau^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$\Rightarrow \mathcal{L}_{cc} = +\frac{g}{\sqrt{2}} \left[\bar{E} \gamma_\mu \tau^+ E W^{+\mu} + \bar{E} \gamma_\mu \tau^- E W^{-\mu} \right]$$

$$= +\frac{g}{\sqrt{2}} \bar{E} \gamma_\mu \left(\tau^+ W^{+\mu} + \tau^- W^{-\mu} \right) E$$

$$= +g \bar{E} \gamma_\mu \left(\frac{1}{2}\tau^1 W^{1\mu} + \frac{1}{2}\tau^2 W^{2\mu} \right) E$$

$$\equiv +g \left(j_\mu^1 W^{1\mu} + j_\mu^2 W^{2\mu} \right)$$

where $\boxed{W^\pm \equiv \frac{1}{\sqrt{2}} \left(W^1 \mp i W^2 \right)}$

For the QED, we have $\mathcal{L}_{QED} = +e\, j_\mu^{EM} A^\mu$ with

$$j_\mu^{EM} = -\bar{e}\gamma_\mu e = -\bar{e}_L\gamma_\mu e_L - \bar{e}_R\gamma_\mu e_R$$

Now

$$-\bar{e}_L\gamma_\mu e_L = \frac{1}{2}\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L\gamma_\mu\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L - \frac{1}{2}\overline{\begin{pmatrix}\nu_e\\ e\end{pmatrix}}_L\gamma_\mu\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L, \quad E = \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L$$

$$= \bar{E}\gamma_\mu\frac{\tau_3}{2}E - \frac{1}{2}\bar{E}\gamma_\mu \mathbb{1} E$$

$$\Rightarrow j_\mu^{EM} = \underbrace{\frac{1}{2}\bar{E}\gamma_\mu\tau_3 E}_{} \underbrace{\left(-\frac{1}{2}\right)\bar{E}\gamma_\mu\mathbb{1}E \;(-)\,\bar{e}_R\gamma_\mu e_R}_{} \equiv j_\mu^3 + j_\mu^Y$$

couples to W_μ^3, third comp. of isovector $(W^1, W^2, W^3)_\mu$

$\Rightarrow j_\mu^Y$: Isosinglet (SU(2) singlet) current

* couples to hypercharge U(1) gauge field B_μ

Hypercharge assignment of E & e_R

$$Y(E) = -1, \quad Y(e_R) = -2$$

Thus the Gell-Nishijima relation holds

$$\left(F_3 \equiv \tau_3/2\right) \quad \hat{Q} = F_3 + \frac{\hat{Y}}{2} \Rightarrow Q(\nu_{eL}) = 0,\ Q(e_L) = -1 \ \&\ Q(e_R) = -1$$

Following Peskin-Schroeder's (and many others) conventions, we will adopt the following assignment

$$\boxed{Y(E) = -1/2, \quad Y(e_R) = -1}$$

such that $\boxed{\hat{Q} = T_3 + \hat{Y}}$ $\quad (T_3 \equiv \tau_3/2)$

i.e. electromagnetic current is a combination of the third comp. of a isovector current plus a new hypercharge current! $\Rightarrow \partial_\mu \to \partial_\mu - ig' Y B_\mu$ for hypercharge gauge field

Thus we have 4 currents

$$\vec{j}_\mu \equiv \bar{E}\gamma_\mu \frac{1}{2}\vec{\tau} E \qquad \text{isovector current}$$

$$j_\mu^Y = -\frac{1}{2}\bar{E}\gamma_\mu E - \bar{e}_R\gamma_\mu e_R \qquad \text{hypercharge current}$$

$$= \bar{E}\gamma_\mu \hat{Y} E + \bar{e}_R \gamma_\mu \hat{Y} e_R$$

Then their charges are

$$\left.\begin{aligned} Q^i(t) &= \int d^3\vec{x}\; j_0^i(\vec{x}, t) \\ Q^Y(t) &= \int d^3\vec{x}\; j_0^Y(\vec{x}, t) \end{aligned}\right\} \text{operators}$$

Using equal time (anti) commutation relations, one can show that

$$[Q^i, Q^j] = i\epsilon^{ijk}Q^k, \quad [Q^i, Q^Y] = 0.$$

$$\Longrightarrow \quad SU_L(2) \otimes U(1)_Y \quad \text{Algebra}$$

$$Q^i \to T^i, \quad Q^Y \to Y$$

Furthermore, one can prove the following results:

$$i\delta_i E_\alpha = [Q^i, E_\alpha] = -\left(\frac{\tau^i}{2}\right)_{\alpha\beta} E_\beta \Rightarrow E \text{ is iso-doublet}$$

$$i\delta_i e_R = [Q^i, e_R] = 0 \qquad \Rightarrow e_R \text{ is isosinglet}$$

$$i\delta_Y E_\alpha = [Q^Y, E_\alpha] = +\frac{1}{2}E_\alpha \qquad \Rightarrow \qquad Y(E) = -\frac{1}{2}$$

$$i\delta_Y e_R = [Q^Y, e_R] = +1 \qquad \Longrightarrow \qquad Y(e_R) = -1$$

The above calculations lead us to covariant derivatives

local gauge symmetry

$$\left.\begin{aligned} D_\mu E &= \left(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu - ig' Y(E) B_\mu\right) E \\ D_\mu e_R &= \left(\partial_\mu - ig' Y(e_R) B_\mu\right) e_R \end{aligned}\right\}$$

The following Lagrangian for E & e_R is invariant under a local gauge sym of $SU(2)_L \otimes U(1)_Y$

$$\mathcal{L} = \bar{E}_L i\gamma^\mu D_\mu E_L + \bar{e}_R i\gamma^\mu D_\mu e_R$$

* $SU(2)\otimes U(1)$ <u>prohibits</u> bare mass term for the leptons. [Because E_L & e_R transf. differently under $SU(2)\times U(1)$]

$$\Rightarrow \mathcal{L}_{int} = g\,\vec{j}_\mu \cdot \vec{W}^\mu + g' j^Y_\mu B^\mu$$

$$= \mathcal{L}_{CC} + \mathcal{L}_{NC}$$

$\mathcal{L}_{CC}$ is same as before $\mathcal{L}_{CC} = \frac{g}{\sqrt{2}}(j^+_\mu W^{+\mu} + j^-_\mu W^{-\mu})$
but $\mathcal{L}_{NC}$ consists of two pieces

$$\mathcal{L}_{NC} = g\, j^3_\mu W^{3\mu} + g' j^Y_\mu B^\mu$$

Weinberg introduced the weak mixing angle θ (Weinberg angle)

$$W^3_\mu = Z_\mu \cos\theta + A_\mu \sin\theta$$
$$B = -Z\sin\theta + A\cos\theta$$

i.e. $\begin{pmatrix} W^3 \\ B \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$

or reversing we have $\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}$

$$= \begin{pmatrix} W^3_\mu \cos\theta - B_\mu \sin\theta \\ W^3_\mu \sin\theta + B_\mu \cos\theta \end{pmatrix}$$

$$\Rightarrow \mathcal{L}_{NC} = (g\sin\theta\, j^3_\mu + g'\cos\theta\, j^Y_\mu) A^\mu$$
$$+ (g\cos\theta\, j^3_\mu - g'\sin\theta\, j^Y_\mu) Z^\mu$$

The first term we can identify with the j^{EM}_μ

$$g\sin\theta\, j^3_\mu + g'\cos\theta\, j^Y_\mu = e\, j^{EM}_\mu$$

We have derived $j^{EM}_\mu = j^3_\mu + j^Y_\mu$. Thus we must have

$\boxed{g\sin\theta = g'\cos\theta = e}$ $\Rightarrow$ $\boxed{\tan\theta = \frac{g'}{g}}$ or $\boxed{\frac{1}{e^2} = \frac{1}{g^2} + \frac{1}{g'^2}}$.

The second term in $\mathcal{L}_{NC}$ is new!

$$g\cos\theta\, j^3_\mu - g'\sin\theta\, j^Y_\mu = g\cos\theta\, j^3_\mu - g'\sin\theta\,(j^{EM}_\mu - j^3_\mu)$$

$$= (g\cos\theta + g'\sin\theta)\, j^3_\mu - g'\sin\theta\, j^{EM}_\mu$$

$$= (g\cos\theta + g\tan\theta\sin\theta)\, j^3_\mu - g\tan\theta\sin\theta\, j^{EM}_\mu \qquad \boxed{\text{Recall } \frac{g'}{g} = \tan\theta}$$

$$= \frac{g}{\cos\theta}(\cos^2\theta + \sin^2\theta)\, j^3_\mu - \frac{g}{\cos\theta}\sin^2\theta\, j^{EM}_\mu$$

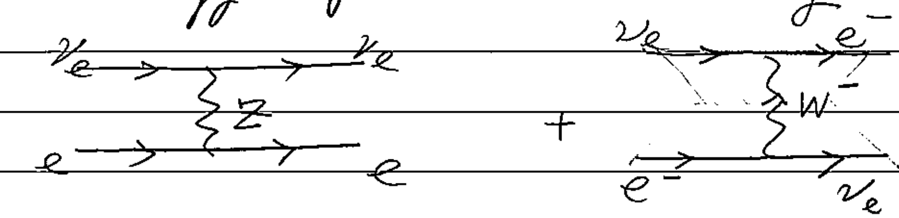
$$= \frac{g}{\cos\theta}\left[j^3_\mu - \sin^2\theta\, j^{EM}_\mu \right]$$

$$\Rightarrow \mathcal{L}_{NC} = e\, j^{EM}_\mu A^\mu + \frac{g}{\cos\theta}\, j^Z_\mu Z^\mu$$

with $j^Z_\mu \equiv j^3_\mu - \sin^2\theta\, j^{EM}_\mu$

j^Z_μ is a new neutral current differ from the j^{EM}_μ.

Low energy experimental data of ($\nu_e e \to \nu_e e$ elastic scattering)



fixed $\boxed{\sin^2\theta \cong 0.23}$ at the early 70s.

The next step is to give masses to the $W^\pm$, Z & the leptons in Weinberg's model.

Besides Weinberg's classic papers, we also have

(1) Glashow, Nucl. Phy. 22 (4), p579-588 (1961).

(2) Salam, in Elementary Particle Physics: Relativistic Groups & Analyticity, 8th Nobel Symposium, (1968) p.367 Edited by Svartholm, N. Almquist and Wiksell.

Spontaneously Symmetry breaking (SSB) in SM

Physical system has an underlying symmetry.
However, the solutions (in particular, the ground state lowest energy solutions) do not have the symmetry.
For field theories, it means the Lagrangian is invariant under the symmetry group, whereas the vacuum doesn't.

Goldstone theorem: For a continuous global symmetry in a Lagrangian field theory, every broken generators has a massless particle associated with it.

Higgs Mechanism: In the cases of gauge symmetries, these massless modes are absorbed by the longitudinal component of the gauge bosons associated by the broken generators.

Weinberg adopted Higgs Mechanism in his model to generate masses for the gauge bosons & charged lepton.

Classic papers:

(1) Goldstone, Salam & Weinberg, Phys. Rev. 127 (3), 965-970, (1962)

(2) Higgs, Phys. Rev. Lett. 13 (16), p508-509 (1964)
Englert & Brout PRL 13 (9) p321-23 (1964)
Guralnik, Hagen & Kibble, PRL 13 (20) p585-587 (1964)

In SSB, we need an order parameter.

	Order parameter	Goldstone Mode
Magnetism	local Magnetization M	Magnon/Spin wa[ve]
Superconductivity	Cooper pair (e^-e^-) bound states due to phonon interaction (Collective State)	$U(1)_{EM}$ is broken, plasmon gets a mass expressed as magnetic flux exclusion from a superconductor.

In particle physics, since we don't want our vacuum to break Lorentz invariance, the only possibility for the order parameter is a scalar field ϕ. Weinberg picked a $SU(2)_L$ ϕ_i to do the job, where ϕ_i $(i=1,2)$ are complex.
What is the hypercharge of ϕ?
Let

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Recall that $Q = T_3 + Y$ and we don't want electromagnetism to be spontaneously broken. So if we want $\langle \phi \rangle \neq 0$, say $\langle \phi_2 \rangle \neq 0$, $\langle \phi_1 \rangle = 0$, then we have to require $Q(\phi_2) = 0 \Rightarrow Y = -T_3 = +\frac{1}{2}$.
And thus $Q(\phi_1) = T_3 + Y = \frac{1}{2} + \frac{1}{2} = 1$.

$$\Rightarrow \phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \text{ belongs } (SU(2)_L, U(1)_Y) \sim (2, \tfrac{1}{2})$$

$$\Rightarrow D_\mu \phi = \left(\partial_\mu - ig\frac{\vec{\tau}}{2}\cdot\vec{W}_\mu - ig'\frac{1}{2}B_\mu\right)\phi$$

* ϕ doesn't carry color since we don't want to break color symmetry (later)

⇒ The general $SU(2)\times U(1)$ gauge invariant Lagrangian for the doublet field Φ is

$$\mathcal{L}_{Higgs} = (D_\mu\phi)^\dagger (D^\mu\phi) - V(\phi)$$

where $V(\phi)$ is an invariant potential. Restricted by renormalizability,

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

Now a doublet Φ has 4 real components, which can be expressed in two different ways:

Linear : $\phi_{Linear} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ v + h + i\eta\end{pmatrix}$, ϕ_1, ϕ_2, h, η are real fields, v = const.

or

Nonlinear : $\phi_{non\text{-}linear} = \underbrace{\exp\left(i\frac{\vec{\xi}\cdot\vec{\tau}}{v}\right)}_{\in SU(2) \text{ group element}}\begin{pmatrix}0 \\ \frac{v+h}{\sqrt{2}}\end{pmatrix}$, $\vec{\xi} = \{\xi_1, \xi_2, \xi_2\}$ & h are real fields

From small fluctuations around the vacuum v, we have

$$\exp\left(i\frac{\vec{\xi}\cdot\vec{\tau}}{v}\right)\begin{pmatrix}0 \\ \frac{v+h}{\sqrt{2}}\end{pmatrix} \simeq \left(1 + i\frac{\vec{\xi}\cdot\vec{\tau}}{v} + \cdots\right)\begin{pmatrix}0 \\ \frac{v+h}{\sqrt{2}}\end{pmatrix}$$

$$\simeq \frac{1}{\sqrt{2}}\begin{pmatrix}\xi_2 + i\xi_1 \\ v + h - i\xi_3\end{pmatrix}$$

$$\Rightarrow \left\{\begin{matrix}\phi_1 \leftrightarrow \xi_2 \\ \phi_2 \leftrightarrow \xi_1\end{matrix},\quad h \leftrightarrow h \quad \& \quad \eta \leftrightarrow -\xi_3\right\}$$

In fact, one can make a gauge transformation to turn $\phi_{non\text{-}linear}$ into ϕ_{linear} completely with $\phi_1 = \phi_2 = \eta = 0$!

$$\phi_{non\text{-}linear} \to \phi' = \mathcal{U}\, \phi_{non\text{-}linear} \text{ with } \mathcal{U} = \exp\left(-i \frac{\vec{\xi}\cdot\vec{\tau}}{v}\right)$$

$$= \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+h) \end{pmatrix} = \phi_{linear} \text{ with } \phi_1 = \phi_2 = \eta = 0$$

&

$$\vec{W}_\mu \cdot \vec{\tau}/2 \to \vec{W}'_\mu \cdot \vec{\tau}/2 = \mathcal{U}\, \vec{W}_\mu \cdot \frac{\vec{\tau}}{2}\, \mathcal{U}^{-1} - \frac{i}{g}(\partial_\mu \mathcal{U})\, \mathcal{U}^{-1}$$

as we have shown before in previous lecture on gauge theory.

& $B_\mu \to B'_\mu = B_\mu$ (hypercharge gauge field unchanged!)

* Now drop the primes in ϕ', $\vec{W}'_\mu$ & B'_μ.
* Thus the $\vec{\xi}$ fields are 'gone' completely. They are the 3 Goldstone bosons corresponding to the three broken generators of $SU(2)_L$.
* Hypercharge is also broken in $\langle \phi \rangle$ since ϕ also carries $Y = 1/2$
* Only the combination $Q = T_3 + Y$ is unbroken generator, and we recognize this is the electric charge generator. ⇒ QED is still exact!
* The linear realization

$$\phi \equiv \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + i G^0) \end{pmatrix}$$

* Rename field variables here!

is related to the non-linear realization for small fields $\vec{\xi}$ fluctuations!

* $$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v+h \end{pmatrix}$$ is a gauge fixed result! with h being a physical Higgs field!

Higgs Mechanism:

Let's go back to the renormalizable scalar potential

$$V = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger\phi)^2$$

Substitute ϕ by $\langle\phi\rangle = (0, \frac{1}{\sqrt{2}}v)^T$, the unitary gauge result, into V, we have $V = -\frac{1}{2}\mu^2 v^2 + \frac{1}{4}\lambda v^4$. $\frac{\partial V}{\partial v} = 0 \Rightarrow + v(-\mu^2 + \lambda v^2) = 0$

$\Rightarrow$ we have 2 solutions for v: $\boxed{v = 0}$ or $\boxed{v = \mu^2/\lambda}$ (Minimization condition)

for $\mu^2 > 0$. $\Rightarrow$ SSB

* Note that for the usual case with $V = \mu^2|\phi|^2 + \lambda|\phi|^4$ & $\mu^2 > 0$. $v = 0$ is the only solution and hence no SSB.

Next substitute the linear realization of

$$\phi = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v + h + iG^0) \end{pmatrix}$$

into V, after some algebra; we obtain, using $\phi^\dagger\phi = \frac{1}{2}v^2 + v\cdot h + G^+\cdot G^- + \frac{1}{2}(h^2 + G^{0\,2})$

$$V = -\frac{1}{4}\mu^2 v^2 + v\cdot h\,(-\mu^2 + \lambda v^2) + \left(\frac{1}{2}(h^2 + G^{0\,2}) + G^+\cdot G^-\right)(-\mu^2 + \lambda v^2)$$

A constant term which is irrelevant for particle physics. ($-\frac{1}{4}\mu^2 v^2$)

* due to minimization condition ($-\mu^2 + \lambda v^2 = 0$)

$$+ \lambda v^2 h^2 + 2\lambda v h\left(G^+\cdot G^- + \frac{1}{2}(h^2 + G^{0\,2})\right)$$
$$+ \lambda\left(G^+\cdot G^- + \frac{1}{2}(h^2 + G^{0\,2})\right)^2$$

(cubic coupling; quartic couplings)

$$\Rightarrow \boxed{\begin{array}{l} m_h = 2\lambda v^2 \\ \text{and } m_{G^+} = m_{G^-} = m_{G^0} = 0 \end{array}}$$

Plus some cubic couplings & quartic couplings from the last two terms. [Note that these cubic couplings are determined by the quartic couplings.

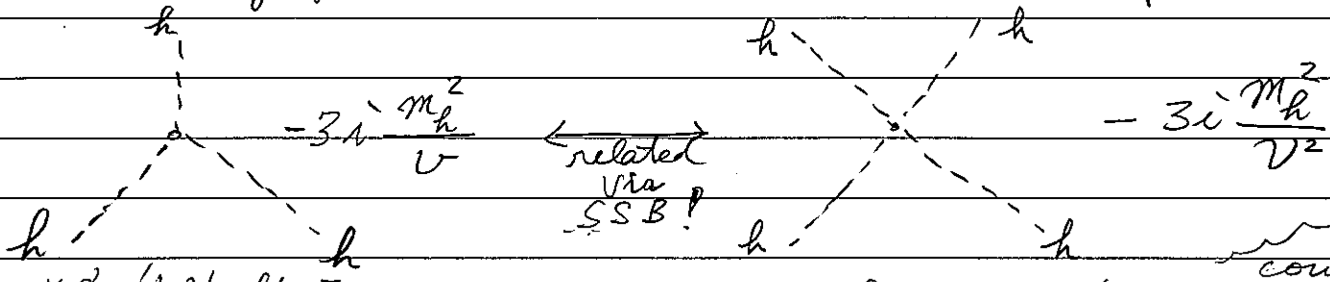
Thus $G^\pm$, G^0 like $\vec{\xi}$ have indeed massless; they are the Goldstone bosons in the linear realization. This result is sometimes referred as 't Hooft-Feynman gauge result. For unitary gauge with $\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+h \end{pmatrix}$, simply set $G^\pm = G^0 = 0$ in the above result.

Thus in the unitary gauge, we simply obtain

$$V = \text{const.} + \frac{1}{2} m_h^2 h^2 + \frac{1}{2}\frac{m_h^2}{v} h^3 + \frac{1}{8}\frac{m_h^2}{v^2} h^4$$

with $m_h^2 = 2\lambda v^2$.

The theory predicts cubic & quartic Higgs self-couplings.



* In 't Hooft-Feynman gauge, we also have cubic/quartic couplings among h & $G^\pm$, G^0 as well. ($G^\pm$ are unphysical but they are needed to include in internal lines)

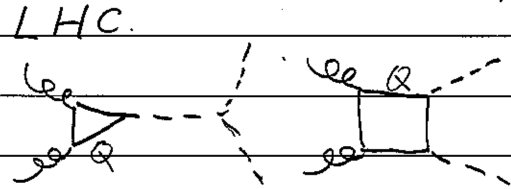
* Higgs boson h was discovered in 2012 (after almost 50 years since its first discussed in 1964) at the LHC by two exps. ATLAS & CMS.

The current most accurate Higgs mass measurement is

$$\boxed{m_h = 125.25 \pm 0.17 \text{ GeV}} \quad (2023 \text{ PDG}).$$

With $v = 246.22$ GeV,
the quartic coupling $\lambda = m_h^2/2v^2$ is completely fixed by the theory to be $\lambda \cong 0.36$!

Currently, experimentalists are working hard to determine the shape of the Higgs potential, i.e. determining the cubic & quartic couplings of the Higgs boson to check if they are as predicted by the SM.
To do so, one has to look for double Higgs production at the LHC.



$W^\pm, Z, \gamma$ masses:

$$\langle 0 |(D_\mu \phi)^\dagger (D^\mu \phi)| 0 \rangle = \left| \left(-ig \frac{\vec{\tau}}{2}\cdot \vec{W}_\mu - ig' \frac{1}{2} B_\mu \right) \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix} \right|^2$$

$$= \frac{1}{8} v^2 \left| \begin{pmatrix} gW_\mu^3 + g'B_\mu & g(W_\mu^1 - iW_\mu^2) \\ g(W_\mu^1 + iW_\mu^2) & -gW_\mu^3 + g'B_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} \right|^2$$

$$= \frac{1}{8} g^2 v^2 \left((W_\mu^1)^2 + (W_\mu^2)^2 \right) + \frac{1}{8} v^2 (g' B_\mu - g W_\mu^3)^2$$

$$= \frac{1}{4} g^2 v^2 W_\mu^+ W^{-\mu} + \frac{1}{8} v^2 (g' B_\mu - g W_\mu^3)^2$$

The first term indicates a mass term for the charged $W^\pm$:

$$M_W^2 = \frac{1}{4} g^2 v^2$$

Previous in our discussions of Fermi's β-decay theory, we obtain

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2}$$

$$\Rightarrow \boxed{\frac{G_F}{\sqrt{2}} = \frac{1}{2v^2}}$$ For $G_F \simeq 10^{-5}\,\mathrm{GeV}^{-2}$, we got $\boxed{v \simeq 246\ \mathrm{GeV}}$

The second term $\frac{1}{8} v^2 (g' B_\mu - g W_\mu^3)^2$ can be rewritten as the following more suggestive form

$$\frac{1}{8} (W_\mu^3 \ B_\mu) \begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix} \begin{pmatrix} W^{\mu 3} \\ B^\mu \end{pmatrix} = \frac{1}{8} (W_\mu^3 B_\mu) M_N^2 \begin{pmatrix} W^{\mu 3} \\ B^\mu \end{pmatrix}$$

The Mass matrix M_N^2 has eigenvalues 0 & $(g^2 + g'^2)$.
Thus the second term can be expressed as

$$= \frac{1}{2} M_Z^2 Z_\mu Z^\mu + \frac{1}{2} M_A^2 A_\mu A^\mu$$

with $Z_\mu = \frac{g W_\mu^3 - g' B_\mu}{\sqrt{g^2 + g'^2}}$, $M_Z = \frac{1}{2} v \sqrt{g^2 + g'^2}$

& $A_\mu = \frac{g' W_\mu^3 + g B_\mu}{\sqrt{g^2 + g'^2}}$, $M_A = 0$

(Weak mixing angle)

Introduce the Weinberg angle θ : $\tan\theta = \frac{g'}{g}$

$$\Rightarrow \left. \begin{aligned} A_\mu &= \cos\theta\, B_\mu + \sin\theta\, W_\mu^3 \\ Z_\mu &= -\sin\theta\, B_\mu + \cos\theta\, W_\mu^3 \end{aligned} \right\}$$

or
$$\left. \begin{aligned} B_\mu &= \cos\theta\, A_\mu - \sin\theta\, Z_\mu \\ W_\mu^3 &= \sin\theta\, A_\mu + \cos\theta\, Z_\mu \end{aligned} \right\}$$

Which coincide with previous results from preliminary considerations. Thus A_μ is identified as the massless photon field for QED.

* $$\frac{M_W}{M_Z} = \frac{\frac{1}{2} g v}{\frac{1}{2}\sqrt{g^2+g'^2}\, v} = \frac{g}{\sqrt{g^2+g'^2}} = \cos\theta$$

* From neutral current experimental result, we know $\sin\theta \cong 0.23$. Therefore,
$M_Z \neq M_W$ but not far away.
& $\boxed{M_W = M_Z \cos\theta} \Rightarrow \boxed{M_Z > M_W}$

* $M_Z \neq M_W$ because T_3 is now mixed with hypercharge Y to form $Q = T_3 + Y$ a conserved charge.

* Experimental data: (Global fit, PDG 2023)

*W & Z were discovered by Carlo Rubbia & Simon van der Meer in 1984 at the UA1 exp at CERN.

$M_Z = 91.1876 \pm 0.0021$ GeV

$$M_W = \begin{cases} 80.377 \pm 0.012 & \text{(PDG 2022 pre CDF)} \\ 80.4335 \pm 0.0094 & \text{(CDF 2022)} \end{cases}$$

$\sin^2\theta\big|_{\text{on-shell}} = 0.22339 \pm 0.00010$

* ρ-parameter $$\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta} = \begin{cases} 1 & \text{(SM)} \\ 1 & \text{*GM model} \\ \text{others} & \text{(NP)} \end{cases}$$

lepton masses: To obtain lepton mass in gauge-invariant fashion, we rely upon Yukawa interaction.

Since, $2\times 2 = 1 \oplus 3$, the following Yukawa coupling is a possible singlet

$$\mathcal{L}_Y = -Y_e \bar{E} \Phi e_R + h.c.$$

$$= -Y_e (\bar{\nu}_e \ \bar{e})_L \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v+h+iG^0) \end{pmatrix} e_R + h.c$$

$$= -Y_e \left\{ \bar{\nu}_e G^+ e_R + \frac{1}{\sqrt{2}}(\bar{e}_L e_R)(v+h+iG^0) \right\} + h.c.$$

$$= -\frac{Y_e}{\sqrt{2}} (\bar{e}_L e_R)(v+h+iG^0) + \cdots \ h.c.$$

$$\Rightarrow \boxed{m_e = \frac{Y_e}{\sqrt{2}} v, \quad m_{\nu_e} = 0} \quad (\text{Strict SM})$$

* Y_e can be complex # in general. But its phase can be absorbed away by a phase in e_R. $\Rightarrow Y_e$ can be chosen to be real!

Y_e is a Yukawa coupling. It can be reexpressed as $Y_e = \sqrt{2}\,\frac{m_e}{v}$. Also, there's a he^+e^- coupling $\propto \frac{m_e}{v}$ but no $h\bar{\nu}_e \nu_e$ coupling in SM.

* If one extends SM by including a right-handed sterile ν_{eR}, then the following Yukawa coupling can give rise a Dirac mass to the neutrino as well.

$$\mathcal{L}_Y' = -Y_\nu \bar{E} \tilde{\Phi} \nu_{eR} + h.c., \quad \tilde{\Phi} = i\tau_2 \Phi^* = \begin{pmatrix} \frac{1}{\sqrt{2}}(v+h-iG^0) \\ -G^- \end{pmatrix} = \Phi^c$$

$$= -Y_\nu (\bar{\nu}_e \ \bar{e})_L \begin{pmatrix} \frac{1}{\sqrt{2}}(v+h-iG^0) \\ -G^- \end{pmatrix} \nu_{eR} + h.c.$$

$$= -Y_\nu \left\{ \bar{\nu}_{eL} \nu_{eR} \frac{1}{\sqrt{2}}(v+h-iG^0) - \bar{e}_L \nu_{eR} G^- + h.c. \right\}$$

$$= -\frac{Y_\nu}{\sqrt{2}} \bar{\nu}_{eL} \nu_{eR} (v+h-iG^0) + \cdots + h.c.$$

$$\Rightarrow \boxed{m_{\nu_e} = \frac{Y_\nu}{\sqrt{2}} v} \quad \& \quad \boxed{Y_\nu = \frac{\sqrt{2}\, m_{\nu_e}}{v}}$$

* Similarly, Y_ν can be complex; but its phase can be absorbed by ν_{eR}. $\Rightarrow$ Pick Y_ν real

$\Rightarrow h\bar{\nu}_e \nu_e$ coupling is proportional to m_{ν_e}/v !

Weinberg's model of leptons can be readily generalized to multiple generations & include the quarks as well. The quantum numbers of the SM based on the gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ is given by

		$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
$Q_L^i = \begin{pmatrix} u \\ d \end{pmatrix}_{Li}$	: $\begin{pmatrix} u \\ d \end{pmatrix}_L, \begin{pmatrix} c \\ s \end{pmatrix}_L, \begin{pmatrix} t \\ b \end{pmatrix}_L, \cdots$	3	2	$\frac{1}{6}$
u_R^i	: $u_R, c_R, t_R, \cdots$	3	1	$\frac{2}{3}$
d_R^i	: $d_R, s_R, b_R,$	3	1	$-\frac{1}{3}$
$E_L^i = \begin{pmatrix} \nu \\ e \end{pmatrix}_{Li}$	: $\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \cdots$	1	2	$-\frac{1}{2}$
e_R^i	$e_R, \mu_R, \tau_R, \cdots$	1	1	-1

(i = generation index $1, 2, \cdots, N$)

$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} G^+ \\ \frac{1}{\sqrt{2}}(v+h+iG^0) \end{pmatrix}$	1	2	$\frac{1}{2}$

($\uparrow$ linear realization)

The SM Lagrangian can fit into a T-shirt!

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{Higgs} + \mathcal{L}_{Matter} + \mathcal{L}_{Yukawa} + \mathcal{L}_{GF}$$

where

$$\mathcal{L}_{gauge} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{4} W_{\mu\nu}^i W^{i\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$\mathcal{L}_{Higgs} = |D_\mu \phi|^2 - V(\phi), \quad V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

$$\mathcal{L}_{matter} = \overline{E_L^i}\, i \not{D} E_L^i + \overline{e_R^i}\, i \not{D} e_R^i + \overline{Q_L^i}\, i \not{D} Q_L^i + \overline{u_R^i}\, i \not{D} u_R^i + \overline{d_R^i}\, i \not{D} d_R^i$$

with

$$D_\mu \equiv \partial_\mu - i g_s \underbrace{T^a}_{\{\frac{\lambda^a}{2}\},\ 0} G_\mu^a - i g \underbrace{T^i}_{\{\frac{\tau^i}{2}\},\ 0} W_\mu^i - i g' Y B_\mu$$

and $\mathcal{L}_{Yukawa} = - Y_e^{ij} \bar{E}_L^i \phi e_R^j + h.c.$

$$- Y_u^{ij} \bar{Q}_L^i \phi^c u_R^j - Y_d^{ij} \bar{Q}_L^i \phi d_R^j + h.c.$$

where $(Y_u)^{ij}$ $(Y_d)^{ij}$ $(Y_e)^{ij}$ are complex $N \times N$ matrices in the generation space.

* If sterile N_Rs exist, we have $-(Y_\nu)^{ij} \bar{E}_L^i \phi^c N_R^j$ as well

* $\mathcal{L}_{SM}$ was written down in the so-called weak basis. Due to the fact that Y_e, Y_u, & Y_d are arbitrary complex matrices of Yukawa couplings, these weak basis is not the mass eigen-basis. One can proceed to calculate physical processes with this basis, However the interpretation of the results will not be simple. It's better to diagonalize the Yukawa matrices and work with mass-eigen basis.

* For sterile N_Rs, one can always write down a $SU(2) \times U(1)$ invariant Majorana mass term

$$\mathcal{L}_{N_R}^{mass} = - \frac{1}{2} M_R^{ij} N_R^{iT} C N_R^j + h.c.$$

where C is the charge conjugate matrix.

* Renormalizability of Weinberg's model was proved later by 't Hooft & Veltman in the early 70's. Crucial point is the use the regularization scheme that can respect the gauge symmetry. This is the dimensional regularization method, which is now widely used in the literature.

* Higher order of electroweak corrections can now be perfomed in PT.

$\Longrightarrow$ EW precision Tests became a reality!