Flavors Symmetries in SM.

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Flavor Symmetry in SM

$3 \times 15 = 45$ quantum fields (in flavor space)

Particle Content (Matter)				$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	B	L
Q_L^i :	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\begin{pmatrix} c \\ s \end{pmatrix}_L$	$\begin{pmatrix} t \\ b \end{pmatrix}_L$	3	2	$\frac{1}{6}$	$\frac{1}{3}$	0
$(u^c)_L^i$:	$(u^c)_L$	$(c^c)_L$	$(t^c)_L$	$\bar{3}$	1	$-\frac{2}{3}$	$-\frac{1}{3}$	0
$(d^c)_L^i$:	$(d^c)_L$	$(s^c)_L$	$(b^c)_L$	$\bar{3}$	1	$\frac{1}{3}$	$-\frac{1}{3}$	0
E_L^i :	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	1	2	$-\frac{1}{2}$	0	1
$(e^c)_L^i$:	$(e^c)_L$	$(\mu^c)_L$	$(\tau^c)_L$	1	1	1	0	-1
Higgs	$\phi = \begin{pmatrix} \phi^+ \\ \phi \end{pmatrix}$			1	2	$\frac{1}{2}$	0	0
Gauge fields	$G_\mu^a \ (a=1,\cdots,8)$			8	1	1	0	0
	$W_\mu^i \ (i=1,2,3)$			1	3	0	0	0
	B_μ			1	1	0	0	0

Accidental global sym. (B, L)

$$\phi^c = i\sigma^2\phi^* = \epsilon\phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \phi^- \\ \phi^* \end{pmatrix} = \begin{pmatrix} \phi^* \\ -\phi^- \end{pmatrix}$$

$$\mathcal{L}_{SM} = \mathcal{L}_{Gauge} + \mathcal{L}_{Matter} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} + \mathcal{L}_{gauge\text{-}fixing} \quad (1)$$

* Actually B & L are anomalous violated by non-perturbative effects. Only B − L is conserved; B + L is violated by sphaleron physics.

$\mathcal{L}_{matter}$ contains K.E. & gauge interactions of the SM fermion fields:

$$\mathcal{L}^{Gauge}_{matter} = i\overline{Q^i_L}\,\not{D}\,Q^i_L + i\overline{(u^c)^i_L}\,\not{D}\,(u^c)^i_L + i\overline{(d^c)^i_L}\,\not{D}\,(d^c)^i_L$$

$$+ i\overline{E^i_L}\,\not{D}\,E^i_L + i\overline{(e^c)^i_L}\,\not{D}\,(e^c)^i_L \qquad (2)$$

i: generation index

$$\not{D} \equiv \gamma^\mu D_\mu \quad : \text{covariant derivative} \qquad (3)$$

The above $\mathcal{L}_{matter}$ is written using all left-handed fields. The left-handed charge-conjugate fields are related to the right-handed fields. Recall the following charge-conjugate field definition

(4) $\psi^c = C\,\overline{\psi}^T = C(\psi^\dagger\gamma^0)^T = C\gamma^{0T}\psi^* = C\gamma^0\psi^*$

(5) $\Rightarrow (\psi^c)_L = P_L\psi^c = P_L C\gamma^0\psi^* = C\gamma^0 P_R\psi^* = C\gamma^0\psi_R^* = C\overline{\psi}_R$

(where we have used $[\gamma_5, C]=0$ & $\{\gamma_5, \gamma^0\}=0$)

Taking the Dirac adjoint, we have

$$\overline{(\psi^c)_L} = \overline{C\gamma^0\psi_R^*} = (C\gamma^0\psi_R^*)^\dagger\gamma^0 = \psi_R^T\gamma^{0\dagger}C^\dagger\gamma^0$$

$$= \psi_R^T\gamma^0(C^{-1})\gamma^0 \qquad (C \text{ is unitary}, CC^\dagger = C\cdot C^{-1} = 1)$$

$$= -\psi_R^T C^{-1}(\gamma^0)^2$$

$$= -\psi_R^T C^{-1} \qquad (6)$$

$$= \psi_R^T C \qquad (6)'$$

$$\boxed{\begin{aligned}&\{C^{-1}, \gamma^0\}\\ &= -\{C, \gamma^0\}\\ &= 0\end{aligned}}$$

(In Weyl rep. $C = +i\gamma^0\gamma^2 = -i\gamma^2\gamma^0$
$= \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$
$C^2 = -1 \Rightarrow C^{-1} = C^\dagger = -C$)

(In Weyl rep. we also have
$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma_5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$)

Thus, we have

Grassmanian!

$$\overline{(\psi^c)_L}\gamma_\mu(\psi^c)_L = (-\psi_R^T C^{-1})\gamma_\mu(C\overline{\psi}_R^T)$$

$$= -\psi_R^T(-\gamma_\mu)^T\overline{\psi}_R^T \quad \leftarrow (C^{-1}\gamma_\mu C = -\gamma_\mu^T)$$

$$= \psi_R^T\gamma_\mu^T\overline{\psi}_R^T = -(\overline{\psi}_R\gamma_\mu\psi_R)^T = -\overline{\psi}_R\gamma_\mu\psi_R \qquad (7)$$

Exercise:

$$\Rightarrow \overline{(\psi^c)_L}\,\not{\partial}\,(\psi^c)_L = -(\overline{\psi}_R\overleftarrow{\not{\partial}}\psi_R) = -\partial_\mu(\overline{\psi}_R\gamma^\mu\psi_R) + \overline{\psi}_R\not{\partial}\psi_R \quad ! \checkmark$$

* Setting all interactions to zero $\mathcal{L}_{Matter} \longrightarrow \mathcal{L}^{K.E.}_{Matter}$ has $U(45)$ symmetry in flavor space.

We can then recast $\mathcal{L}_{matter}$ in standard form used in most text books:

$$\mathcal{L}_{matter} = i\overline{Q_L^i}\,\not{D}\,Q_L^i + i\overline{u_R^i}\,\not{D}\,u_R^i + i\,\overline{d_R^i}\,\not{D}\,d_R^i + i\,\overline{E_L^i}\,\not{D}\,E^i + i\,\overline{e_R^i}\,\not{D}\,e_R^i \tag{8}$$

* Note that (bare) fermion mass terms are forbidden in SM.

Dirac mass terms are forbidden because no fermion transforms under the complex-conjugate representation of another fermion.

Majorana mass terms are forbidden because

(a) all fermions in SM carry no-zero hypercharge; furthermore,

(b) some fermions transform under a complex irreps of $SU(3)_c$

(c) Some fermions transform under a pseudoreal irreps. of $SU(2)_L$.

If one extends SM by including sterile neutrinos N_R^i, then majorana mass term is possible for N_R^i

$$-\frac{1}{2} M_R^{ij} N_R^{iT} C N_R^j + h.c. \tag{9}$$

Actually, one can written a Majorana mass term for ν_L^i in the SM

$$-\frac{1}{2} M_L^{ij} \nu_L^{iT} C \nu_L^j + h.c. \tag{10}$$

Although this is Lorentz invariant, it <u>breaks</u> $SU(2)_L \times U(1)_Y$ and also the global accidental lepton number by two units. More precisely, it is forbidden by the low energy accidental symmetry $B-L$. Nevertheless, it can be induced by higher-dimension operator! (Later!)

The matter Lagrangian $\mathcal{L}_{matter}$ in (8) has a lot of accidental global symmetry,

$$\left.\begin{aligned} Q_L^i &\to \mathcal{U}_{Q_L}^{ij} Q_L^j \\ u_R^i &\to \mathcal{U}_{u_R}^{ij} u_R^j \\ d_R^i &\to \mathcal{U}_{d_R}^{ij} d_R^j \\ E_L^i &\to \mathcal{U}_{E_L}^{ij} E_L^j \\ e_R^i &\to \mathcal{U}_{e_R}^{ij} e_R^j \end{aligned}\right\} \qquad (11)$$

where $\mathcal{U}_{Q_L}, \mathcal{U}_{u_R}, \mathcal{U}_{d_R}, \mathcal{U}_{E_L}, \mathcal{U}_{e_R}$ are $\overset{5}{\hat{}}$ unitary 3×3 matrices (for 3 generations). Unitary Matrices $\leftrightarrow$ preserve K.E terms

This is the flavor symmetry group $[\mathcal{U}(3)]^5$ for 3 generations of quarks & leptons.

However, this huge global accidential flavor symmetry is violated by the Yukawa couplings of the fermions with the Higgs field ϕ: — ϕ or $\tilde{\phi}$

$$\mathcal{L}_{Yukawa} = -Y_u^{ij}\overline{Q_L^i}\,\widetilde{\epsilon\phi^*}\,u_R^j - Y_d^{ij}\overline{Q_L^i}\,\phi\, d_R^j - Y_e^{ij}\overline{E_L^i}\,\phi\, e_R^j + h.c. \qquad (12)$$

$\phi^c \equiv$ $\boxed{\begin{aligned}\tilde{\phi} &\equiv \epsilon\phi^* \\ &= i\sigma_2\phi^* \\ &\text{is often used!}\end{aligned}}$

Only a small subset of $[\mathcal{U}(3)]^5$ is respected by the Yukawa interaction (12). They are the baryon & lepton number symmetries in SM.

$*(Y_{u,d,e})^{ij}$ carry generation indices, whereas $i\not{D}$ doesn't!

In (12), Y_u, Y_d, Y_e are 3 complex 3×3 matrices in the generation space. Thus we have introduced

$$3\times(3\times3)\times2 = 54 \text{ real parameters} \tag{13}$$

in the model. This would render SM unpredictive with such large numbers of free parameters. We will come back to this issue in a little bit.

First, go back to B & L. As mentioned earlier, only a small subset of $[U(3)]^5$ is respected by $\mathcal{L}_{\text{Yukawa}}$. They are phase transformations!

$$B = 1/3 \qquad \left.\begin{aligned} Q_L^i &\longrightarrow e^{i\frac{1}{3}\theta} Q_L^i \\ u_R^i &\longrightarrow e^{i\frac{1}{3}\theta} u_R^i \\ d_R^i &\longrightarrow e^{i\frac{1}{3}\theta} d_R^i \end{aligned}\right\} \quad \begin{gathered}\text{Baryon \#} \\ B \subset [U(3)]^5\end{gathered} \tag{14}$$

and

$$L = 1 \qquad \left.\begin{aligned} E_L^i &\longrightarrow e^{i\phi} E_L^i \\ e_R^i &\longrightarrow e^{i\phi} e_R^i \end{aligned}\right\} \quad \begin{gathered}\text{Lepton \#} \\ L \subset [U(3)]^5\end{gathered} \tag{15}$$

* $|\text{proton}\rangle = |uud\rangle$ has $\frac{1}{3}+\frac{1}{3}+\frac{1}{3} = 1$ lightest baryon # ← stable
* $|\text{neutron}\rangle = |udd\rangle$ also has $3\times(\frac{1}{3}) = 1$ baryon #
* On the other hand, pions

  $\pi^\pm, \pi^0 \sim u\bar{d}/\bar{u}d, (u\bar{u}+d\bar{d})/\sqrt{2}$ have 0

  baryon numbers. They are mesons.
* Since B & L are accidental sym (or precisely just $B-L$), nature should satisfy the selection rules that they impose. Indeed, proton, $e^\pm$, $\nu_{e,\mu,\tau}$ (massless) are stable in SM. $p \not\to e^+\gamma$, $\mu^\pm \to e^\pm\gamma$ etc, are strictly forbidden in SM. As a matter of fact, $\mu^\pm \to e^\pm\gamma$ violates L_e & L_μ, but conserves $L = L_e + L_\mu + L_\tau$!

Next, as demonstrated previously, SSB of the Higgs field

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \Rightarrow \langle \epsilon \phi^* \rangle = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \quad (16)$$

will provide fermion masses from $\mathcal{L}_{Yukawa}$ in a gauge-inv. fashion. That is, for the quark sector, we have

$$\mathcal{L}^{quark}_{Mass} = - M_u^{ij} \overline{u_L^i} u_R^j - M_d^{ij} \overline{d_L^i} d_R^j + h.c. \quad (17)$$

& for lepton sector, we have

$$\mathcal{L}^{lepton}_{Mass} = - M_e^{ij} \overline{e_L^i} e_R^j + h.c. \quad (18)$$

with $\boxed{M^{ij}_{u,d,e} = \frac{Y^{ij}_{u,d,e} \cdot v}{\sqrt{2}}}$ ← Complex 3×3 matrices. (19).

(Can be diagonized by bi-unitary transformation!)

* Neutrinos remain massless since there are no ν_R^i to pair up with ν_L^j to form masses. And Majorana masses like (10) are forbidden by B−L.

* $M_{u,d,e}$ are Dirac masses matrices.

* from (13) & (19), we know

M_u, M_d contain $2 \times (3 \times 3) \times 2 = 36$ free real parameters

& M_e contains $(3 \times 3) \times 2 = 18$ free real parameters.

* Not all of these 54 free parameters are physically relevant.

* We still have the freedom of re-definition of the fermion fields to 'shrink' the parameter space.

* Again, in order to preserve the K.E. terms, the re-definition must be implemented as unitary matrices. There is 7 of them in SM:

$$\left.\begin{array}{ll} u_L \longrightarrow V_L^u u_L, & u_R \longrightarrow V_R^u u_R \\ d_L \longrightarrow V_L^d d_L, & d_R \longrightarrow V_R^d d_R \\ e_L \longrightarrow V_L^e e_L, & e_R \longrightarrow V_R^e e_R \\ \nu_L \longrightarrow V_L^\nu \nu_L & \end{array}\right\} \quad (20)$$

* Physics computation can be carried out in principle using (12). But diagonalization of the Yukawa matrices $\{Y_u, Y_d, Y_e\}$ and hence $\{M_u, M_d, M_e\}$, which is allowed by field-redefinition, make the computation much easier & the physical meaning is more clear (in a physical basis).

* But what happen to the rest of the SM Lagrangian under these field redefinitions?

Apparently, $\mathcal{L}_{gauge}$ & $\mathcal{L}_{Higgs}$ are unaffected.

How about $\mathcal{L}_{matter}$ which has $(U(3))^5$ flavor symmetry

Certainly, for the field-redefinition matrices

$$\left.\begin{aligned} V_L^u = V_L^d &= U_{Q_L} \\ V_L^e = V_L^\nu &= U_{E_L} \\ V_R^u &= U_{u_R} \\ V_R^d &= U_{d_R} \\ V_R^e &= U_{e_R} \end{aligned}\right\} \in [U(3)]^5 \qquad (21)$$

belong to $[U(3)]^5$. These should leave $\mathcal{L}_{matter}$ unchanged!

* Now, $V_L^u \neq V_L^d$ is tenable because gauge invariant is broken by the Higgs Mechanism.

Thus performing the transformations in (20) is O.K.

However,

$\Rightarrow$ One should subtract away the parameters that are related to the flavor $[U(3)]^5$, <u>except</u> the global $U(1)_B$ baryon number or $U(1)_L$ lepton number. Recall that $U(1)_B$ & $U(1)_L$ are accidental global symmetries in SM. Hence they are physical, reflected by the selection rules in all processes in SM. i.e. the θ & ϕ in (14) & (15) are parameters that can't be used to do field-redefinitions!

First, we have the following countings of 'free' parameters in the Yukawa couplings Y_u, Y_d, Y_e using $(U(3))^5$.
For the quark case, we have

$$\underbrace{2\times(3\times3)\times2}_{\substack{\#\text{ of free parameters}\\ \text{in } Y_u \,\&\, Y_d\\ (\text{or } M_u \,\&\, M_d)}} - (\underbrace{3\,(3\times3)}_{\substack{\#\text{ of free}\\ \text{parameters in}\\ U_{Q_L},\, U_{u_R},\, U_{d_R}}} - 1) \qquad (23)$$

$\hookrightarrow$ Baryon # parameter doesn't count.

$$= \underbrace{36}_{18\text{ real}+18\text{ phases}} - \underbrace{26}_{9\text{ real}+18\text{ phases}-1\text{ global phases}} = 10 \text{ indep. free parameters}$$

Exercise: Show that a $N\times N$ unitary matrix has N^2 indep. free real parameters, $\frac{1}{2}N(N-1)$ real & $\frac{1}{2}N(N+1)$ phases.

$\boxed{*\text{Note } SO(N) \subset SU(N)}$

The 10 parameters can be chosen as the 6 quark masses $\{m_u, m_d, m_s, m_c, m_b, m_t\}$ plus the 3 angles & 1 CP phase in the CKM mixing matrix $V_{CKM} = (V_u^L)^\dagger (V_d^L)$.
Repeating the counting exercise for the lepton case, we have

$$\underbrace{(3\times3)\times2}_{\substack{\#\text{ of free parameters}\\ \text{in } Y_e\,(\text{or } M_e)}} - (\underbrace{2\times(3\times3)}_{\substack{\#\text{ of free}\\ \text{parameters in } U_{e_L} \,\&\, U_{e_R}}} - 1) = 1 \qquad (24)$$

$\hookrightarrow$ lepton # parameter doesn't count

This counting is incorrect, since we have at least three masses m_e, m_μ, & m_τ which are all independent!

* CKM: Cabbibo, $\underbrace{\text{Kobyashi - Maskawa}}_{\text{3 Generations of quarks} \rightarrow \text{CP}}$

θ_c

SM accidental global sym. : $U(1)_B \times U_{L_e}(1) \times U_{L_\mu}(1) \times U_{L_\tau}(1)$

In fact, for the lepton case,

$$L = L_e + L_\mu + L_\tau .$$

And we have individual lepton # symmetries L_e, L_μ & L_τ. i.e. We should have subtracted 3 instead of 1 in (24). Namely, (24) becomes

$$(3\times 3)\times 2 - (2\times(3\times 3) - \overbrace{3}^{L_e, L_\mu, L_\tau}) = 3 \qquad (24)'$$

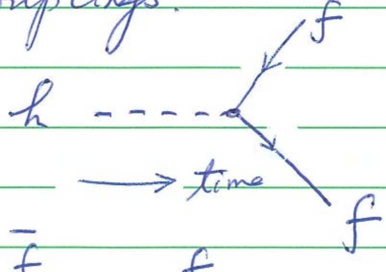
These 3 free parameters are the three charged lepton masses m_e, m_μ & m_τ. There are no mixing angles & CP phase in SM for massless neutrinos.

Altogether, we have 18+1 free parameters in SM:

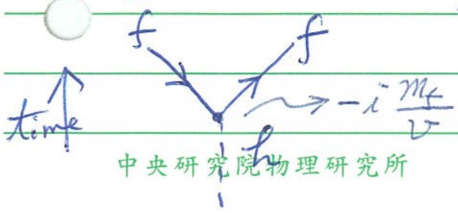
$$18 = \underset{\substack{\text{(quark sector)}\\ (m_d, \cdots, y_b)\\ + \text{CKM}}}{10} + \underset{\substack{\text{(lepton sector)}\\ (m_e, m_\mu, m_\tau)}}{3} + \underbrace{(g, g', g_s)}_{3 \text{ gauge couples}} + \underset{\substack{\text{Fermi const.}\\ (v)\\ \text{Vacuum expectation value } \langle\phi\rangle}}{G_F} + \underset{\substack{\text{Higgs self coupling}}}{\lambda} + \text{`` } \theta_{QCD} \text{ ''} \qquad (25)$$

$\oplus$ 1

* Now, back to the Yukawas. Using field redefinitions, one can perform diagonalization of the Yukawa matrices (or equiv. Mass matrices). From previous lectures, we know, after diagonalization, the physical Higgs h couples to the SM fermions diagonally, i.e. there's no FCNC in the Higgs' couplings. (26)



$$\mathcal{L}_Y = -\sum_f m_f \bar f f \left(1 + \frac{h}{v}\right) \quad \{\text{Flv diagonal}\}$$



$$\Gamma(h \to f\bar f) = N_c(f)\left(\frac{\alpha\, m_h}{8\sin^2\theta_W}\right)\frac{m_f^2}{m_W^2}\left(1 - \frac{4m_f^2}{m_h^2}\right)^{3/2}$$

$$N_c(f) = \begin{cases} 1 & \text{lepton} \\ 3 & \text{quark (Color factor)} \end{cases} \qquad (27)$$

We now know that neutrinos are ~~not~~ massless. Experimentalists observed neutrinos can oscillate from one species to another which requires massive neutrinos to do so. The simplest thing to do in SM is to introduce right-handed sterile neutrinos N_R^i ($i=1,2,3$) which carry no SM quantum number but has a lepton # +1. Then, just like the $(Y_u)^{ij}$ Yukawa coupling for the u-type quarks, we have an additional gauge inv. Yukawa couplings for the neutrinos

$$-(Y_\nu)^{ij}\,\overline{E_L^i}\,\epsilon\,\phi^*\,N_R^j \quad + h.c. \qquad (28)$$

After symmetry breaking, neutrinos can pick up Dirac masses as

$$-(Y_\nu)^{ij}\,\overline{E_L^i}\begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} N_R^j + h.c$$

$$\longrightarrow \; -M_\nu^{ij}\,\overline{\nu_L^i}\,N_R^j + h.c. \;, \qquad M_\nu^{ij} \equiv + \frac{Y_\nu}{\sqrt{2}}\, v \qquad (29)$$

Now, we can repeat the counting of indep. parameters for the lepton sector with massive Dirac neutrinos, (which is exactly analogous the quark sector)

$$\underbrace{2\times(3\times 3)\times 2}_{\#\text{ of free parameters in } M_e \,\&\, M_\nu} \;-\; \underbrace{(3(3\times 3)}_{\#\text{ of free parameters in } \mathcal{U}_{E_L},\, \mathcal{U}_{e_R},\, \mathcal{U}_{N_R}\text{ (unitary matrices)}} - \overset{\to\text{ lepton }\#}{1}) \;=\; 10 \qquad (30)$$

gauge inv.

$V_e^L = V_\nu^L$ (before sym. breaking)

* These 10 parameters are (31)

$$\{m_e, m_\mu, m_\tau, m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\} + V_{PMNS} = (V_\nu^L)^\dagger V_e^L)$$

(3 angles plus 1 Dirac CP phase)

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Since N_R^i is sterile, gauge symmetry doesn't forbid a Majorana mass term

$$-\frac{1}{2} M_R^{ij} N_R^{iT} C N_R^j + h.c. \quad , (M_R^{ij} = M_R^{ji}) \quad (32)$$

which is renormalizable with dimension 3. However it violates lepton number by 2 units since N_R has lepton +1, according to (28). But lepton number (and baryon number) is an accidental global sym. in SM, there's no reason to regard it as an exact symmetry if one extends SM (like introducing N_R^i and hence (32)!)

So let's include (32) in our extension of SM by introducing N_R^i. The Lagrangian for N_R is now

$$\mathcal{L}_{N_R} = i \overline{N}_R \not{\partial} N_R \left(- \overline{E}_L Y_\nu \epsilon \phi^* N_R - \frac{1}{2} N_R^T M_R C N_R + h.c. \right) \quad (33)$$

where we have suppressed the generation indices.

Suppose now that M_R is much larger than its kinetic term so that we can ignore the 1^{st} term in $\mathcal{L}_{N_R}$. Then the EOM for N_R is simply (34)

$$\frac{\partial \mathcal{L}_{N_R}}{\partial N_R} = 0 \quad \Rightarrow + \overline{E}_L Y_\nu \epsilon \phi^* - \frac{1}{2} M_R C N_R + \frac{1}{2} N_R^T M_R C = 0$$

Here we have flipped sign due to Grassmanian nature when $\partial/\partial N_R$ moves to the right to act on the rightmost N_R. Now the last two terms can be combined:

$$-\frac{1}{2}(M_R C N_R)_\alpha^i = -\frac{1}{2} M_R^{ij} C_{\alpha\beta} N_{R\beta}^j = -\frac{1}{2} N_{R\beta}^j (-C_{\beta\alpha}) M_R^{ji}$$

$$= +\frac{1}{2}(N_R^T M_R C)_\alpha^i \text{ same as the last term.} \quad (35)$$

$$\Rightarrow \frac{\delta \mathcal{L}}{\delta N_R} = 0 \Rightarrow \overline{E}_L Y_\nu \epsilon \phi^* + \qquad N_R^T M_R C = 0 \quad (36)$$

Taking the transpose, $-\phi^\dagger \epsilon Y_\nu^T \gamma^0 \overline{E}_L^* - C M_R^T N_R = 0 \quad (36)'$

where we have used $\boxed{\epsilon^T = -\epsilon,\ C^T = -C}$ & $\boxed{\gamma^{0T} = \gamma^0}$

Solving N_R from (36)$'$:

$$C M_R^T N_R = -\phi^+ \epsilon Y_\nu^T \gamma^0 E_L^*$$

$$(M_R^T)^{-1} C^{-1} (C M_R^T N_R) = (M_R^T)^{-1} C^{-1} (-\phi^+ \epsilon Y_\nu^T \gamma^0 E_L^*)$$

$$\Rightarrow N_R = -\phi^+ \epsilon C^{-1} \gamma^0 (M_R^T)^{-1} Y_\nu^T E_L^*$$

$$\boxed{C^{-1} = -C} \quad = \phi^+ \epsilon C \gamma^0 (Y_\nu M_R^{-1})^T E_L^* \qquad (37)$$

Plugging (37) back into the Lagangian (33), we obtain a dim. 5 operator $\mathcal{L}_5$.

$$\mathcal{L}_5 = -\bar{E}_L Y_\nu \epsilon \phi^* N_R - \frac{1}{2} N_R^T M_R C N_R \quad + h.c.$$

$$= -\bar{E}_L Y_\nu \epsilon \phi^* \cdot \phi^+ \epsilon C \gamma^0 (Y_\nu M_R^{-1})^T E_L^* \qquad + h.c.$$

$$- \frac{1}{2} E_L^+ (Y_\nu M_R^{-1}) \gamma^0 (-C)(-\epsilon) \phi^* \cdot M_R C \cdot \phi^+ \epsilon C \gamma^0 (Y_\nu M_R^{-1})^T E_L^* \qquad + h.c.$$

Now using $\{\gamma^0, C\} = 0$, $C^{-1} = -C$, $\epsilon^2 = -1$, and be careful about the indices groupings, we have

$$\mathcal{L}_5 = + \bar{E}_L \gamma^0 \epsilon \phi^* C Y_\nu (Y_\nu M_R^{-1})^T \phi^+ \epsilon E_L^* \qquad + h.c.$$

$$- \frac{1}{2} E_L^+ (\gamma^0)^2 \epsilon \phi^* C Y_\nu (Y_\nu M_R^{-1})^T \phi^+ \epsilon E_L^* \qquad + h.c.$$

$\because \bar{E}_L \gamma^0 = E_L^+$ & $(\gamma^0)^2 = 1$, we have

$$\mathcal{L}_5 = + \frac{1}{2} E_L^+ \epsilon \phi^* \cdot C Y_\nu (Y_\nu M_R^{-1})^T \cdot \phi^+ \epsilon E_L^* \quad + h.c. \qquad (38)$$

$$\equiv + \quad (E_L^{+i} \epsilon \phi^* C) \frac{C^{ij}}{M} \cdot (\phi^+ \epsilon E_L^{*j}) \qquad + h.c.$$

where

$$\frac{C^{ij}}{M} = + \frac{1}{2} \left(Y_\nu (Y_\nu M_R^{-1})^T \right)^{ij} \overset{\because M_R^{-1} = (M_R^{-1})^T}{=} \frac{C^{ji}}{M} \qquad (39)$$

Symmetric in generation space.

When ϕ takes its VEV $v/\sqrt{2}$, (38) gives rise to a Majorana mass term to the left-handed neutrinos ν_L^i:

$$\mathcal{L}_5 \longrightarrow +\nu_L^{\dagger i} C\left(\frac{v^2}{2}\frac{c^{ij}}{M}\right)\nu_L^{*j} + h.c.$$

$$\equiv +\frac{1}{2}\nu_L^{\dagger i} C (M_L^\dagger)^{ij}\nu_L^{*j} + \underset{\Downarrow}{h.c.}$$

$$= +\frac{1}{2}\nu_L^\dagger C M_L^\dagger \nu_L^* - \frac{1}{2}\nu_L^T C M_L \nu_L \qquad (\text{note } \overset{=C^{-1}}{C^\dagger} = -C). \quad (40)$$

with $M_L = \dfrac{v^2}{M} c^\dagger$, with $\frac{c}{M}$ given by (39). (41)

Eq. (38) is known as Weinberg's operator in the literature. It's the only dim 5 gauge-invariant operator in the so-called SM effective field theory (SMEFT).

$$\text{SMEFT} = \underset{(d\le 4)}{\mathcal{L}_{SM}} + \underbrace{\frac{1}{\Lambda}\mathcal{L}_5 + \frac{1}{\Lambda^2}\mathcal{L}_6 + \cdots}_{\text{SM gauge invariant}} \qquad (42)$$

"fitted in a T-shirt"

While there's only 1 operator at dim = 5, there are many dim = 6 operators, (80 ?) ...

Now the majorana mass term (40) for the left-handed neutrinos was induced by a higher-dim. gauge-invariant Weinberg operator. Recall that (40) by itself was not allowed at tree level due to gauge invariance. And the Weinberg operator can be viewed as a low-energy operator as a result of integrating out the heavy right-handed neutrinos N_R's. In fact, (41) is the famous see-saw mechanism.

Now we will take an effective theory point of view for the SM. If one truncate the SMEFT at Dim 5, we have

$$\mathcal{L}_{SMEFT} = \mathcal{L}_{SM} + \mathcal{L}_5 \qquad (43).$$

where $\mathcal{L}_5$ is the Weinberg operator given by (38) & (39).
The # of indep. parameters for the lepton sector is now different.

$$\underbrace{(3\times 3)\times 2}_{Y_e^{ij}\,(\text{No } Y_\nu^{ij}!)} + \underbrace{3\times 2\times 2}_{C^{ij}=C^{ji}} - \underbrace{(3\times 3)\times 2}_{2\text{ Unitary matrices } \mathcal{U}_{E_L}\ \&\ \mathcal{U}_{e_R}\ (\text{No } \mathcal{U}_{N_R}!)} = 12 \qquad (44).$$

These 12 parameters for the lepton sector can be taken as

$$\{m_e, m_\mu, m_\tau, m_{\nu_1}, m_{\nu_2}, m_{\nu_3}\} + \underbrace{V_{PMNS}}_{3 \text{ angles } + 3\ \underbrace{\text{CP phases}}_{1 \text{ Dirac, } 2 \text{ Majorana}}} \qquad (45).$$

* Note that we don't subtract the lepton number in (44) because lepton number is no-longer a symmetry in $\mathcal{L}_5$.
* From (41), assuming $(C)^{ij}$ is of order 1, to achieve neutrino mass of order 10^{-2}eV, the high scale M is of order 10^{15} GeV which is close to the Grand Unification scale M_{GUT}. In other words, M_R the right-handed neutrino mass scale M_R is also close to the M_{GUT} scale according to (39).
* PMNS : Pontecorvo, Maki-Nakagawa-Sakata ((1962)
((1957)

Bruno Pontecorvo: Italian & Soviet nuclear physicist, assistant of Fermi.
A convinced communist, deflected to Soviet Union in 1950, ... (See Wikipedia)
He predicted neutrino oscillations in 1957 to solve the solar-neutrino problem
$\nu_e \to \nu_\mu$, finally observed in 1998 by the Super-Kamiokande exp.

April 2, 2024

Diagonalization of the Yukawa couplings matrices

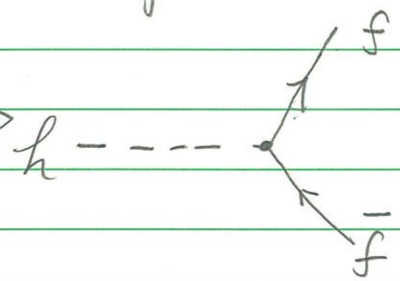
Recall that the fermion mass matrices in SM are related to the Yukawa couplings matrices as

$$(M_{u,d,e})^{ij} = \frac{v}{\sqrt{2}} (Y_{u,d,e})^{ij} \qquad (1)$$

Thus $M_{u,d,e}$ are arbitrary 3×3 complex matrices, which can be diagonalized by bi-unitary transformations. These transformations are the redefinition unitary matrices that leaves the K.E. terms invariance too!

Since the physical Higgs field $h(x)$ is always come along with v as the combination $(v+h)/\sqrt{2}$, once $(Y_{u,d,e})$ are diagonalized, the Higgs fermion couplings are diagonalized too!

$\Longrightarrow$ There's no FCNC Higgs couplings with the SM fermions in SM.

$\Longrightarrow$ 

$= -i\frac{m_f}{v}$ (Diagonal in flavor!) (2)

This can be easily seen as follows

$$\frac{1}{\sqrt{2}}(v + h(x)) = \frac{v}{\sqrt{2}}\left(1 + \frac{h(x)}{v}\right) \qquad (3)$$

Thus

$$-m_f \bar{f} f = -\frac{v}{\sqrt{2}} y_f \bar{f} f \longrightarrow -\frac{v}{\sqrt{2}}\left(1 + \frac{h(x)}{v}\right) y_f \bar{f} f$$
$$= -m_f \bar{f} f - \frac{m_f}{v} h \bar{f} f$$

Exercise: What are the couplings between the Higgs boson & the $W^\pm$ & Z gauge bosons? (4)

Gauge Interaction:

While the redefinition unitary transf.s help us to diagonalize the Yukawas, what will happens to the gauge interactions for the matter fields?

Recall

$$D_\mu = \partial_\mu - i g_s T^a G^a_\mu - i g T^i W^i_\mu - i g' Y B_\mu \qquad (5)$$

Field redefinition unitary transformations

$$\left.\begin{array}{ll} u_L \to V^u_L u_L, & u_R \to V^u_R u_R \\ d_L \to V^d_L d_L, & d_R \to V^d_R d_R \\ e_L \to V^e_L e_L, & e_R \to V^e_R e_R \\ \nu_L \to V^\nu_L \nu_L, & (\nu_R \to V^\nu_R \nu_R) \end{array}\right\} \qquad (6)$$

absent in SM

* Obviously, the kinetic terms are invariant under (6).

* So is the QCD part.

$$\begin{aligned} \bar q \gamma^\mu T^a G^a_\mu q &= \bar q_L \gamma^\mu T^a G^a_\mu q_L + \bar q_R \gamma^\mu T^a G^a_\mu q_R \\ &\longrightarrow \bar q_L (V^q_L)^\dagger \gamma^\mu T^a G^a_\mu (V^q_L) q_L \\ &\quad + \bar q_R (V^q_R)^\dagger \gamma^\mu T^a G^a_\mu (V^q_R) q_R \qquad (7) \\ &\longrightarrow \bar q \gamma^\mu T^a G^a_\mu q. \end{aligned}$$

(QCD is flavor blind!)

$\Rightarrow$ Only the electroweak part needs to be concerned.

$$\begin{aligned} & g T^i W^i_\mu + g' Y B_\mu \qquad (8) \\ &= g(T^1 W^1_\mu + T^2 W^2_\mu) + g T^3 W^3_\mu + g' Y B_\mu \\ &= g \tfrac{1}{\sqrt 2}(\tau^+ W^+_\mu + \tau^- W^-_\mu) + g T^3 (Z_\mu c_\theta + A_\mu s_\theta) + g' Y(-Z_\mu s_\theta + A_\mu c_\theta) \\ &= g \tfrac{1}{\sqrt 2}(T^+ W^+_\mu + T^- W^-_\mu) + Z_\mu (g c_\theta T^3 - g' s_\theta (Q - T^3)) + A_\mu (g s_\theta T^3 + g' c_\theta (Q - T^3)) \end{aligned}$$

Recall that $g s_\theta = g' c_\theta = e$, the last term is simply $e Q A_\mu$; and the middle term is $\frac{g}{c_\theta}(T^3 - s^2_\theta Q) Z_\mu$

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$$\Rightarrow g T^i W^i_\mu + g' Y B_\mu = \frac{g}{\sqrt{2}}(\tau^+ W^+_\mu + \tau^- W^-_\mu) + \frac{g}{c_\theta}(T^3 - s_\theta^2 Q) Z_\mu + e Q A_\mu \qquad (9)$$

with
$$\tau^+ \equiv \frac{1}{2}(\tau^1 + i\tau^2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\tau^- \equiv \frac{1}{2}(\tau^1 - i\tau^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Clearly, like the gluon case in QCD, Z_μ & A_μ couple to the matter fields are invariant under (6).

$\Rightarrow$ There is no flavor changing neutral currents in SM at tree-level, which is a phenomenological desired feature.

For the $W^\pm$, due to the $\tau^\pm$ acting on the flavor up & down entries in the doublets, charged current is not invariant under (6) $\Rightarrow$ possible crossed generation for the $W^\pm$ interaction!

(i) lepton sector

$$\bar{E}\left(-i\frac{g}{\sqrt{2}}\gamma^\mu(\tau^+ W^+_\mu + \tau^- W^-_\mu) - i\frac{g}{c_\theta}(T^3 - s_\theta^2 Q)\gamma^\mu Z_\mu - ie Q \gamma^\mu A_\mu\right) E \qquad (10)$$
$$+ \bar{e}_R\left(-i\frac{g}{\sqrt{2}}\gamma^\mu \cdot 0 - i\frac{g}{c_\theta}(0 - s_\theta^2 Q)\gamma^\mu Z_\mu - ie Q\gamma^\mu A_\mu\right) e_R$$

Note that

$$\left.\begin{aligned} \bar{E}\cdots\tau^+\cdots E &= (\bar{\nu}^i\ \bar{e}^i)_L \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nu^i \\ e^i \end{pmatrix}_L = \bar{\nu}^i_L \cdots e^i_L \\ \& \quad \bar{E}\cdots\tau^-\cdots E &= (\bar{\nu}^i\ \bar{e}^i)_L \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \nu^i \\ e^i \end{pmatrix}_L = \bar{e}^i_L \cdots \nu^i_L \end{aligned}\right\} \qquad (11)$$

$$\Rightarrow \bar{E}\gamma^\mu(\tau^+ W^+_\mu + \tau^- W^-_\mu) E = \bar{\nu}^i_L \gamma^\mu W^+_\mu e^i_L + \bar{e}^i_L \gamma^\mu W^-_\mu \nu^i_L$$

Under $e_L \to V^e_L e_L$ & $\nu_L \to V^\nu_L \nu_L$, the above Charged current go to

$$\bar{\nu}_L \underbrace{(V^\nu_L)^\dagger V^e_L}_{V_{PMNS} \text{ matrix}} \gamma^\mu W^+_\mu e_L + \text{h.c.} \qquad (12)$$

Since ν_L's are massless in SM, we can just rename the neutrinos ν_L

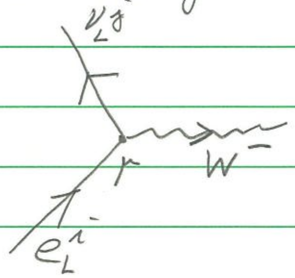
$$\nu_L \longrightarrow \nu_L' = (V_L^e)^\dagger V_L^\nu \, \nu_L \equiv V_{PMNS}^\dagger \, \nu_L \qquad (13)$$

without affecting any other terms in $\mathcal{L}_{SM}$. Now dropped the prime for $\nu_L' \to \nu_L$. So the charged current for the lepton sector is simply

$$-i\frac{g}{\sqrt{2}}\left(\bar{e}_L^i \gamma^\mu W_\mu^- \nu_L^i + \bar{\nu}_L^i \gamma^\mu W_\mu^+ e_L^i\right) \qquad (14)$$

which is diagonal in the generation space!

$\Longrightarrow$



$$= i\frac{g}{2\sqrt{2}}\gamma^\mu(1-\gamma_5)\,\delta^{ji}$$

$$\mathcal{L}_{CC} \supset \frac{g}{2\sqrt{2}}\bar{\nu}^i\gamma^\mu(1-\gamma_5)e^i W_\mu^+ + h.c. \qquad (15)$$

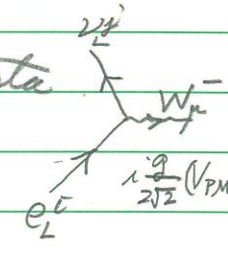
(massless ν_L's)

Since the neutrinos are known to be massive now, (15) should be modified to

$$\mathcal{L}_{CC} \supset \frac{g}{2\sqrt{2}}\Big[\bar{e}^i (V_{PMNS}^\dagger)_{ij}\gamma^\mu(1-\gamma_5)\nu^j W_\mu^- + \bar{\nu}^i (V_{PMNS})_{ij}\gamma^\mu(1-\gamma_5)e^j W_\mu^+\Big] \qquad (16)$$

with V_{PMNS} = unitary matrix $\equiv (V_L^\nu)^\dagger (V_L^e)$ (17)

*PMNS = Pontecorvo - Maki - Nakagawa - Sakata



$i\frac{g}{2\sqrt{2}}(V_{PMNS})^{ji}(1-\gamma_5)$

The neutral current can be worked out similarly.

$$\bar E\left((T^3 - s_\theta^2 Q)\not Z + eQ\not A\right)E$$

$$= (\bar\nu\ \ e)_L\left[\left(\begin{pmatrix}1/2 & \\ & -1/2\end{pmatrix} - s_\theta^2\begin{pmatrix}0 & \\ & -1\end{pmatrix}\right)\not Z + e\begin{pmatrix}0 & \\ & -1\end{pmatrix}\not A\right]\begin{pmatrix}\nu\\ e\end{pmatrix}$$

$$= \bar\nu_L^i\left(\tfrac12\right)\not Z\,\nu_L^i + \bar e_L^i\left(-\tfrac12 + s_\theta^2\right)\not Z\, e_L^i + \bar e_L^i(-e)\not A\, e_L^i \qquad (18)$$

& $\bar e_R Q\not A e_R = -\bar e_R^i\not A e_R^i \qquad \bar e_R(-s_\theta^2 Q)\not Z e_R = +s_\theta^2\,\bar e_R\not Z e_R \qquad (19)$

$$\Rightarrow \mathcal{L}_{NC} \supset \frac{g}{c_\theta}\left[\bar\nu_L^i\,\tfrac12\not Z\,\nu_L^i + \bar e_L^i\left(-\tfrac12 + s_\theta^2\right)\not Z\, e_L^i + s_\theta^2\,\bar e_R^i\not Z\, e_R^i\right] - e\underbrace{\left[\bar e_L^i\not A\, e_L^i + \bar e_R^i\not A\, e_R^i\right]}_{\bar e^i\not A e^i\ \text{(Vector like)}} \qquad (20)$$

Clearly $\mathcal{L}_{NC}$ is uncharged under (6)

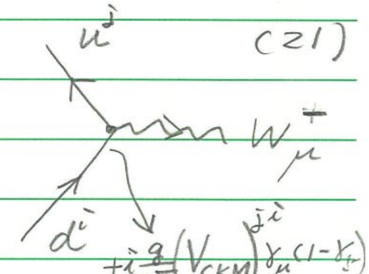
(ii) Quark sector.

The calculation is similar to the lepton case with

$\nu_L \to u_L$ & $e_L \longrightarrow d_L$ & $u_R \longrightarrow \nu_R$ & $d_R \to e_R$.

The charged current interaction is

$$\mathcal{L}_{CC} \supset \frac{g}{2\sqrt2}\left[\bar d^j\left(V_{CKM}^\dagger\right)^{ji}\gamma^\mu(1-\gamma_5)\,u^i\,W_\mu^- + \bar u^j\left(V_{CKM}\right)^{ji}\gamma^\mu(1-\gamma_5)\,d^i\,W_\mu^+\right] \qquad (21)$$



(22) with $V_{CKM} \equiv (V_L^u)^\dagger(V_L^d)$ = unitary matrix

[Note that $(V_{CKM})_{PDG} = (V_L^u)(V_L^d)^\dagger$]

CKM = Cabibbo-Kobayashi-Maskawa

And the neutral current is

$$\mathcal{L}_{NC} \supset \frac{g}{c_\theta}\Big[\bar u_L^i\left(\tfrac12 - s_\theta^2\tfrac23\right)\not Z\,u_L^i + \bar d_L^i\left(-\tfrac12 - s_\theta^2\left(-\tfrac13\right)\right)\not Z\,d_L^i + \bar u_R^i\left(-s_\theta^2\tfrac23\right)\not Z\,u_R^i + \bar d_R^i\left(-s_\theta^2\left(-\tfrac13\right)\right)\not Z\,d_R^i\Big] \qquad (23)$$

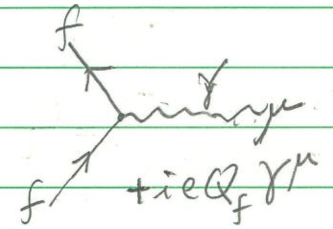
$$+e\left[\bar u_L^i\tfrac23\not A\,\bar u_L^i + \bar u_R^i\tfrac23\not A\,u_R^i + \bar d_L^i\left(-\tfrac13\right)\not A\,d_L^i + \bar d_R^i\left(-\tfrac13\right)\not A\,d_R^i\right]$$

which are also invariant under (6)'s redefinition transformations.

Summarizing the neutral current here

$$\mathcal{L}_{NC} = \mathcal{L}_\gamma + \mathcal{L}_Z$$

$$\mathcal{L}_\gamma = e \sum_f Q_f \bar{f} \gamma^\mu A_\mu f$$



(24)

$$\mathcal{L}_Z = \frac{g}{\cos\theta} \sum_f \Big[\bar{f}_L \gamma^\mu \big(T_3(f_L) - \sin^2\theta\, Q(f) \big) Z_\mu f_L$$
$$+ \bar{f}_R \gamma^\mu \big(-\sin^2\theta\, Q(f) \big) f_R Z_\mu \Big]$$

$f \in \{ \nu^i, e^i, u^i, d^i \}$

* The neutral current interactions are <u>flavor diagonal</u>!

* $\mathcal{L}_Z$ can be written in terms of vector & axial-vector currents as follows.

$$\mathcal{L}_Z = \sum_f \left(\bar{f}_L C_L^f \gamma^\mu Z_\mu f_L + \bar{f}_R C_R^f \gamma^\mu Z_\mu f_R \right)$$

$$= \sum_f \bar{f} \gamma^\mu (C_V^f + C_A^f \gamma_5) Z_\mu f$$

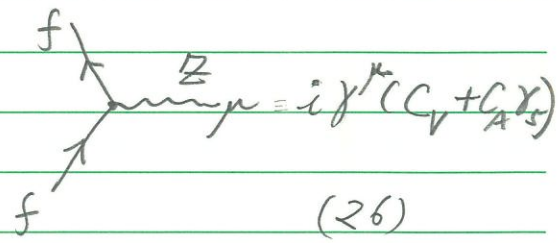
{Note some used $C_V - C_A\gamma_5$!}

(25)

where $$\left.\begin{aligned} C_L^f &\equiv \frac{g}{\cos\theta} \left[T_3(f) - \sin^2\theta\, Q(f) \right] \\ C_R^f &= \frac{g}{\cos\theta} \left(-\sin^2\theta\, Q(f) \right) \end{aligned}\right\}$$

&

$$\left.\begin{aligned} C_V^f &= \frac{1}{2}\left(C_L^f + C_R^f \right) \\ &= \frac{1}{2}\left(T_3(f) - 2\sin^2\theta\, Q(f) \right) \frac{g}{\cos\theta} \\ C_A^f &= \frac{1}{2}\left(-C_L^f + C_R^f \right) \\ &= \frac{1}{2}\frac{g}{\cos\theta}\left(-T_3(f) \right) \end{aligned}\right\}$$



(26)

* $C_A^f \neq C_V^f$ in general. This also leads to <u>parity</u> violations in Z-exchange too!

4/4/2024
清明節

Accidental Global Symmetry Group of SM

We want to show that

$$G = U(1)_B \otimes U(1)_{L_e} \otimes U(1)_{L_\mu} \otimes U(1)_{L_\tau} \quad (1)$$

is the global symmetry group of the SM Lagrangian

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{matter} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} + \mathcal{L}_{G.F.} + \mathcal{L}_{ghosts}$$

Recall that, except for $\mathcal{L}_{Yukawa}$, $\mathcal{L}_{SM}$ has a flavor $(U(3))^5$ (or $(U(N))^5$ for N-generations) global symmetry

$$\left.\begin{array}{l} Q_L \to U_{Q_L} Q_L \\ u_R \to U_{u_R} u_R \\ d_R \to U_{d_R} d_R \end{array}\right\} \quad \left.\begin{array}{l} E_L \to U_{E_L} E_L \\ e_R \to U_{e_R} e_R \end{array}\right\} \quad (2)$$

$$\boxed{N_R \to U_{N_R} N_R \in BSM}$$

(The gauge interactions break the $U(45)$ (or $U(15N)$) symmetry in the free kinetic terms into $(U(3))^5$ (or $(U(N))^5$))

The question is what global symmetry left by introducing $\mathcal{L}_{Yuk}$?

$$(U(3))^5 \xrightarrow{\mathcal{L}_{Yukawa}} ? \quad (3)$$

Recall
$$\mathcal{L}_{Yukawa} = -\overline{Q}_L Y_u \widetilde{\Phi} u_R - \overline{Q}_L Y_d \Phi d_R - \overline{E}_L Y_e \Phi e_R + h.c. \quad (4)$$

$$\left(-\overline{E}_L Y_\nu \widetilde{\Phi} N_R + h.c.\right)$$

BSM

Under (6),

$$\left.\begin{aligned}
&\bar{Q}_L Y_u \phi^c u_R \longrightarrow \bar{Q}_L (U_{Q_L}^\dagger Y_u U_{u_R}) \phi^c u_R \\
&\bar{Q}_L Y_d \phi\, d_R \longrightarrow \bar{Q}_L (U_{Q_L}^\dagger Y_d U_{d_R}) \phi\, d_R \\
&\bar{E}_L Y_e \phi\, e_R \longrightarrow \bar{E}_L (U_{E_L}^\dagger Y_e U_{e_R}) \phi\, e_R \\
&(\bar{E}_L Y_\nu \tilde{\phi} N_R \longrightarrow \bar{E}_L (U_{E_L}^\dagger Y_\nu U_{N_R}) \tilde{\phi} N_R)
\end{aligned}\right\} \quad (5)$$

Thus, $\mathcal{L}_Y$ is not invariant unless

$$\left.\begin{aligned}
U_{Q_L}^\dagger Y_u U_{u_R} &= Y_u \\
U_{Q_L}^\dagger Y_d U_{d_R} &= Y_d \\
U_{E_L}^\dagger Y_e U_{e_R} &= Y_e \\
(\; U_{E_L}^\dagger Y_\nu U_{N_R} &= Y_\nu \;)
\end{aligned}\right\} \quad (6).$$

Taking $(6)^\dagger$ and consider $(6)\times(6)^\dagger$ leads us to

$$\left.\begin{aligned}
U_{Q_L}^\dagger Y_u Y_u^\dagger U_{Q_L} &= Y_u Y_u^\dagger \\
U_{Q_L}^\dagger Y_d Y_d^\dagger U_{Q_L} &= Y_d Y_d^\dagger \\
U_{E_L}^\dagger Y_e Y_e^\dagger U_{E_L} &= Y_e Y_e^\dagger \\
(U_{E_L}^\dagger Y_\nu Y_\nu^\dagger U_{E_L} &= Y_\nu Y_\nu^\dagger)
\end{aligned}\right\} \quad (7).$$

Consider $(6)^\dagger \times (6)$ leads us to

$$\left.\begin{aligned}
\mathcal{U}_{u_R}^\dagger Y_u^\dagger Y_u \mathcal{U}_{u_R} &= Y_u^\dagger Y_u \\
\mathcal{U}_{d_R}^\dagger Y_d^\dagger Y_d \mathcal{U}_{d_R} &= Y_d^\dagger Y_d \\
\mathcal{U}_{e_R}^\dagger Y_e^\dagger Y_e \mathcal{U}_{e_R} &= Y_e^\dagger Y_e \\
(\mathcal{U}_{N_R}^\dagger Y_\nu^\dagger Y_\nu \mathcal{U}_{N_R} &= Y_\nu^\dagger Y_\nu)
\end{aligned}\right\} \quad (8)$$

Now, in the basis where the Yukawa coupling Matrices are diagonal, i.e $(YY^\dagger)_{u,d,e,\nu}$ & $(Y^\dagger Y)_{u,d,e,\nu}$ are real & diagonal, eqs. (7) & (8) imply

$$\left.\begin{aligned}
&\mathcal{U}_{E_L}, \mathcal{U}_{e_R}, (\mathcal{U}_{N_R}) \\
&\mathcal{U}_{Q_L}, \mathcal{U}_{u_R}, \mathcal{U}_{d_R}
\end{aligned}\right\} \text{ must be diagonal phase matrices} \quad (9)$$

Furthermore, in the basis where $Y_{u,d,e,(\nu)}$ are diagonal, eq (6) implies

$$\left.\begin{aligned}
\mathcal{U}_{Q_L} &= \mathcal{U}_{u_R} = \mathcal{U}_{d_R} \\
\mathcal{U}_{E_L} &= \mathcal{U}_{e_R} = (\mathcal{U}_{N_R})
\end{aligned}\right\} \quad (10)$$

Thus the phase matrices

(11) $\mathcal{U}_{Q_L} = \mathcal{U}_{u_R} = \mathrm{Diag}(e^{i\theta_u}, e^{i\theta_c}, e^{i\theta_t}) = \mathcal{U}_{d_R} = \mathrm{diag}(e^{i\theta_d}, e^{i\theta_s}, e^{i\theta_b})$

& (12) $\mathcal{U}_{E_L} = \mathcal{U}_{e_R} = \mathrm{Diag}(e^{i\theta_e}, e^{i\theta_\mu}, e^{i\theta_\tau}) (= \mathcal{U}_{N_R} = \mathrm{diag}(e^{i\theta_{\nu_e}}, e^{i\theta_{\nu_\mu}}, e^{i\theta_{\nu_\tau}}))$

Eq.(11) is the quark flavor symmetry in the Yukawas, just like QCD, QED, & $\mathcal{L}_{NC}$ but not $\mathcal{L}_{CC}$! For the charged current interaction, we have the V_{CKM} linking up type & down-type quarks in different generations. The only global symmetry that commutes with the generic V_{CKM} is $\mathcal{U}_{Q_L} = \mathcal{U}_{u_R} = \mathcal{U}_{d_R} = \mathrm{diag}(e^{i\frac{1}{3}\theta_B}, e^{i\frac{1}{3}\theta_B}, e^{i\frac{1}{3}\theta_B})$! (13)

This is the global baryon number symmetry $U_B(1)$.
We choose 1/3 in the phase such that all quarks have baryon # 1/3, which implies proton & neutron have baryon # +1.

In the SM where there are <u>no</u> N_R, we have

$$U_{E_L} = U_{e_R} = \mathrm{diag}(e^{i\theta_e}, e^{i\theta_\mu}, e^{i\theta_\tau}) \qquad (14)$$

which corresponds to L_e & L_μ & L_τ 3 lepton #s symmetry.

In BSM, where N_Rs exist, we then have V_{PMNS} that connecting ν_i & e_j in different generations. In this case, we don't have the individual L_e, L_μ & L_τ symmetry. The only possibility is

$$U_{E_L} = U_{e_R} = U_{N_R} = \mathrm{diag}(e^{i\theta_L}, e^{i\theta_L}, e^{i\theta_L}) \qquad (15)$$

which corresponds to total lepton number $L = L_e + L_\mu + L_\tau$ conservation. Thus we have proved

$\Rightarrow$ In SM, the global flavor symmetry of $\mathcal{L}_{SM}$

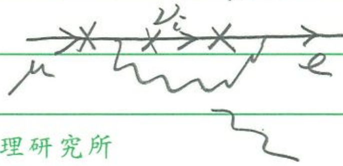
is $\quad U_B(1) \otimes U_{L_e}(1) \otimes U_{L_\mu}(1) \otimes U_{L_\tau}(1)$. $\qquad (16)$

This symmetry probids $\mu \not\to e\gamma$.

$\Rightarrow$ In BSM (with right-handed neutrinos), the global flavor symmetry is

$$U_B(1) \otimes U_L(1) \qquad (17)$$

This symmetry allows $\mu \to e\gamma$ but one needs neutrino masses to make things work. Indeed, it's well known that the loop diagram



$$\propto m_{\nu_i}^2 \Rightarrow Br \propto 10^{-55}! \qquad (18)$$

* However non-perturbative (nontrivial topological) solutions for the SM electroweak theory exists due to

$\pi_3(G) = \mathbb{Z}$ for any compact connected simple group G.

[See 't Hooft, PRL 37, 8 (1972)]

This non-perturbative solutions can violate B & L such that only $(B-L)$ remains unbroken.

* $(B-L)$ emerges from some GUTs as a gauge $U(1)_{B-L}$ generator. Since no massless gauge particle has ever been observed other than the photon, this $U(1)_{B-L}$ must be broken at some high scale.

* Summary for SM with 3 generations of quarks & leptons

pure kinetic terms $i\not{D}$

$$U(45) \xrightarrow{i\not{D}} (U(3))^5 \xrightarrow{\text{Yukawa}} U(1)_B \otimes U(1)_{L_e} \otimes U(1)_{L_\mu} \otimes U(1)_{L_\tau}$$

$$\xrightarrow[\text{NP physics}]{} U(1)_{B-L}$$

CKM Matrix: $V_{CKM} \equiv (V_L^u)^\dagger (V_L^d)$ a unitary matrix in the generation space enters in the charged current interaction

$$\mathcal{L}_{cc} = g(W_\mu^+ J_W^{\mu +} + W_\mu^- J_W^{\mu -}) \quad (1)$$

where
$$J_W^{\mu +} = \frac{1}{\sqrt{2}} \bar{u}_L \gamma^\mu V_{CKM} d_L = \frac{1}{\sqrt{2}} \bar{u}_L^i \gamma^\mu (V_{CKM})_{ij} d_L^j \quad (2)$$

$\Rightarrow$ In SM, only $W^\pm$ can change flavors of the fermions. All other neutral current interactions (γ, Z^0, h) are diagonal in flavors.

Now a $N \times N$ unitary matrix $\mathcal{U}$ has N^2-parameters in general. For real unitary matrix, $\mathcal{U}^\dagger \mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1$ reduces to $\mathcal{U}^T \mathcal{U} = \mathcal{U}\mathcal{U}^T = 1$ which implies $\mathcal{U}$ is orthogonal. Thus, there are $\frac{1}{2}N(N-1)$ real parameters, the rest of them $N^2 - \frac{1}{2}N(N-1) = \frac{1}{2}N(N+1)$ must be complex phases. Not all these phases are physical. For N generations, we have $\{u_{i_{L,R}}, d_{i_{L,R}}\}$ $i = 1, \cdots, N$, so we have $(2N-1)$ global quark phases to absorb the phases in $\mathcal{U}$.

Note One overall phase is kept for the fermion number (baryon number)! Thus the number of physical phases in the CKM matrix for N-generation of SM fermions is

$$\frac{1}{2}N(N+1) - (2N-1) = \frac{1}{2}(N-1)(N-2) \quad (3)$$

$\Rightarrow$ For $N = 3$, V_{CKM} has 1 phase, which implies $\cancel{CP}$ in SM. This is the only source of $\cancel{CP}$ in SM.

$\Rightarrow$ No $\cancel{CP}$ violation for 1 & 2 generations.

One can show explicitly how things work for $N=2$.
In this case, the V_{CKM} can have generically 1 angle & 3 phases:

$$V_{2\times2} = \begin{pmatrix} e^{i\delta_1}\cos\theta & e^{i\delta_2}\sin\theta \\ -e^{i\delta_3}\sin\theta & e^{i\delta_4}\cos\theta \end{pmatrix} \qquad (4)$$

unitarity $VV^\dagger = V^\dagger V = 1$ implies $\delta_1 - \delta_3 = \delta_2 - \delta_4$. (5)
The current is proportional to

$$J_W^{+\mu} \sim \cos\theta\,(\bar{u}_L\gamma^\mu e^{i\delta_1} d_L + \bar{c}_L\gamma^\mu e^{i\delta_4} s_L)$$
$$+\sin\theta\,(\bar{u}_L\gamma^\mu e^{i\delta_2} s - \bar{c}_L\gamma^\mu e^{i\delta_3} d_L)$$

Let the quark fields changed by global phases as

$(U(1))^3$

$$u_{L,R} \to e^{i(\delta_T+\delta_u)} u_{L,R} \qquad c_{L,R} \to e^{i(\delta_T+\delta_c)} c_{L,R}$$
$$d_{L,R} \to e^{i(\delta_T+\delta_d)} d_{L,R} \qquad s_{L,R} \to e^{i(\delta_T+\delta_s)} s_{L,R}$$

where δ_T is a total global phase.
Clearly, if one choose

$$\left.\begin{aligned} -\delta_u + \delta_d + \delta_1 &= 0 \\ -\delta_c + \delta_s + \delta_4 &= 0 \\ -\delta_u + \delta_s + \delta_2 &= 0 \\ -\delta_c + \delta_d + \delta_3 &= 0 \end{aligned}\right\}$$

Note: $\delta_1 - \delta_3 = \delta_2 - \delta_4$ is satisfied.

all the 4 phases in $V_{2\times2}$ can be absorbed away!

(6)

$$\Longrightarrow J_W^{+\mu} \to \cos\theta\,(\bar{u}_L\gamma^\mu d_L + \bar{c}_L\gamma^\mu s_L) \quad \leftarrow \text{same generation}$$
$$+\sin\theta\,(\bar{u}_L\gamma^\mu s - \bar{c}_L\gamma^\mu d_L) \quad \leftarrow \text{cross generation}$$

(7)

i.e. $$V_{2\times2} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ no CP for 2 generations.

Experimentally, θ is the Cabbibo angle θ_c: $\sin\theta_c \simeq 0.22 \Rightarrow \theta_c \simeq 13^\circ$!

(Euler)
For $N=3$, we have 3 rotation angles and 1 CP-phase. The CKM matrix is parameterized as

$$V_{CKM} \equiv \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (8)$$

$c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$
θ_{ij} all lie in 1st quadrant.
δ : CP-phase

Numerous experimental data showed that
$s_{13} \ll s_{23} \ll s_{12} \ll 1$

This motivated Wolfenstein's parameterization the following.

Define $s_{12} = \lambda$, $s_{23} = A\lambda^2$, $s_{13}e^{i\delta} = A\lambda^3(\rho + i\eta)$ (9)

i.e. $(s_{12}, s_{23}, s_{13}, \delta) \longrightarrow (\lambda, A, \rho, \eta)$

From (8) we see that

(i) $s_{13}e^{-i\delta} = V_{ub} \Rightarrow s_{13} = |V_{ub}|$ (10)

(ii) $s_{12}c_{13} = |V_{us}| \Rightarrow \lambda = \dfrac{|V_{us}|}{c_{13}} = \dfrac{|V_{us}|}{\sqrt{1-s_{13}^2}} = \dfrac{|V_{us}|}{\sqrt{1-|V_{ub}|^2}}$ (11)

$$= \frac{|V_{us}|}{\sqrt{|V_{ud}|^2 + |V_{us}|^2}}$$

where unitary condition $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$ has been imposed.

(iii) $\dfrac{|V_{cb}|}{|V_{us}|} = \dfrac{s_{23}c_{13}}{s_{12}c_{13}} = \dfrac{s_{23}}{s_{12}}$

(12)

$$\Rightarrow s_{23} = s_{12}\left|\frac{V_{cb}}{V_{us}}\right| = \lambda\left|\frac{V_{cb}}{V_{us}}\right| = A\lambda^2$$

Numerically, $\lambda \simeq \theta_c \simeq 13^\circ \simeq \frac{\pi}{14} \simeq 0.22$. Thus $|V_{CKM}|$ is of order

$$|V_{CKM}| = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ -\lambda & 1 & \lambda^2 \\ \lambda^3 & -\lambda^2 & 1 \end{pmatrix} \qquad (13)$$

Wolfenstein took this as 0-order & by taking A & $\rho - i\eta$ as order one parameters, he wrote down

$$V_{CKM} \simeq \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4) \qquad (14)$$

This is the Wolfenstein's parameterization of the CKM quark mixing matrix in SM.

* This Wolfenstein parameterization for V_{CKM} is <u>not</u> unitary and <u>not</u> phase convention independent. It is possible to define a new set of $(\bar\rho, \bar\eta)$ such that V_{CKM}, if written in terms of $(\lambda, A, \bar\rho, \bar\eta)$, can be unitary to all order in λ. The definitions of $\bar\rho$ & $\bar\eta$ are

$$\rho + i\eta = \frac{\bar\rho + i\bar\eta}{\sqrt{1-\lambda^2}} \frac{\sqrt{1-A^2\lambda^4}}{[1-A^2\lambda^4(\bar\rho + i\bar\eta)]} \qquad (15)$$

This relation together with (9) − (12) ensure that

(16). $\bar\rho + i\bar\eta = -\dfrac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$ is phase convention independant.

We will not got into this matter here.
For details, see Wolfenstein, PRL 51, 1945 (1983); Buras et al, PRD50, 3433 (1994); CKMfitter group, EPJC 41, 1, 1 (2005).

V_{CKM} is unitary, so it satisfies $VV^\dagger = V^\dagger V = 1$.

$$(VV^\dagger) = 1 \implies \sum_{k=d,s,b} V_{ik} V_{jk}^* = \delta_{ij}, \quad (i,j) = (u,c,t) \quad (16)$$

$$(V^\dagger V) = 1 \implies \sum_{k=u,c,t} V_{ki}^* V_{kj} = \delta_{ij}, \quad (i,j) = (d,s,b) \quad (17)$$

(16) & (17) simply mean that the three rows & three columns in the CKM matrix are two sets of orthonormal vectors.

Thus, (16) implies for $i=j=u,c,t$,

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$

Sizes: $\sim 1 \qquad \sim \lambda^2 \qquad \sim \lambda^6$

$$|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1 \qquad (18)$$

$\sim \lambda^2 \qquad \sim 1 \qquad \sim \lambda^4$

&

$$|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2 = 1$$

$\sim \lambda^6 \qquad \sim \lambda^4 \qquad \sim 1$

For $i \neq j$, $(i,j) = (u,c), (u,t), (c,t)$, we have

$$V_{ud}V_{cd}^* + V_{us}V_{cs}^* + V_{ub}V_{cb}^* = 0 \qquad c \to u \text{ transition}$$

$\sim 1 \cdot \lambda \qquad \sim \lambda \cdot 1 \qquad \sim \lambda^3 \cdot \lambda^2$

$$(19) \quad V_{ud}V_{td}^* + V_{us}V_{ts}^* + V_{ub}V_{tb}^* = 0 \qquad t \to u \text{ transition}$$

$\sim 1 \cdot \lambda^3 \qquad \sim \lambda \cdot \lambda^2 \qquad \sim \lambda^3 \cdot 1$

$$V_{cd}V_{td}^* + V_{cs}V_{ts}^* + V_{cb}V_{tb}^* = 0 \qquad t \to c \text{ transition.}$$

$\sim \lambda \cdot \lambda^3 \qquad \sim 1 \cdot \lambda^2 \qquad \sim \lambda^2 \cdot 1$

* the other 3 cases of (c,u), (t,u) & (t,c) are just complex conjugate of the above conditions.

* No top quark bound state! $t \to Wb \sim 100\%$ rapidly!

Similarly, (17) implies for $(i=j=d, s, b)$, we have

$$
\begin{aligned}
&|V_{ud}|^2 + |V_{cd}|^2 + |V_{td}|^2 = 1 \\
&\sim 1 \qquad \sim \lambda^2 \qquad \sim \lambda^6 \\
&|V_{us}|^2 + |V_{cs}|^2 + |V_{ts}|^2 = 1 \\
&\sim \lambda^2 \qquad \sim 1 \qquad \sim \lambda^4 \\
&|V_{ub}|^2 + |V_{cb}|^2 + |V_{tb}|^2 = 1 \\
&\sim \lambda^6 \qquad \sim \lambda^4 \qquad \sim 1
\end{aligned}
\tag{20}
$$

For $i \neq j$, $(i, j) = (d, s), (d, b), (s, b)$, we have

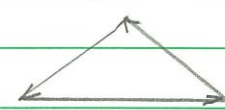
$$V_{ud} V_{us}^* + V_{cd} V_{cs}^* + V_{td} V_{ts}^* = 0$$
$$\sim 1 \cdot \lambda \qquad \sim \lambda \cdot 1 \qquad \sim \lambda^3 \cdot \lambda^2$$

$s \to d$

(21)

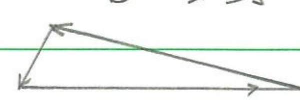
$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$$
$$\sim 1 \cdot \lambda^3 \qquad \sim \lambda \cdot \lambda^2 \qquad \sim \lambda^3 \cdot 1$$

$b \to d$



$$V_{us} V_{ub}^* + V_{cs} V_{cb}^* + V_{ts} V_{tb}^* = 0$$
$$\sim \lambda \cdot \lambda^3 \qquad \sim 1 \cdot \lambda^2 \qquad \sim \lambda^2 \cdot 1$$

$b \to s$

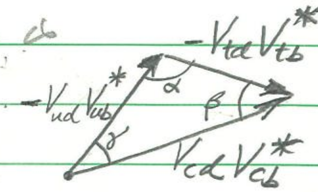


* (s, d), (b, d), (b, s) are complex conjugate of the above case.

One can see that the $b \to d$ transition is the most interesting case since the unitarity condition implies a triangle (the three complex terms define a triangle in the complex plane) with roughly equal length on the three sides. This triangle, together with others[5], are called the unitarity triangles in the B-physics community. All 6 unitarity triangles have the same area.

$b \to d$ transition: Its associated unitarity condition is

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0, \qquad (22)$$

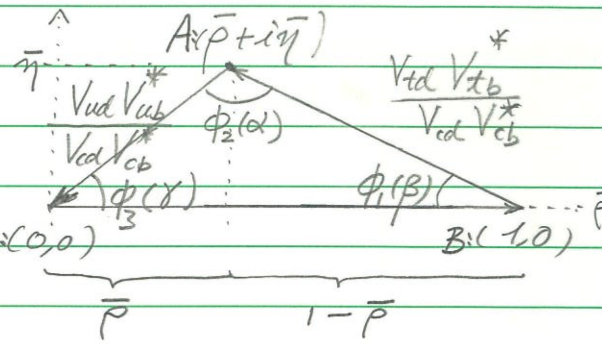


Divided by $V_{cd}V_{cb}^*$:

$$\underbrace{\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}}_{\substack{\text{from (16)} \\ = -(\bar\rho + i\bar\eta)}} + 1 + \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} = 0 \qquad (23)$$

Rescale

$$\Rightarrow \quad 1 = +(\bar\rho + i\bar\eta) - \frac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*} \quad \Rightarrow$$



Buras & companies also defined

$$R_u \equiv \frac{|V_{ud}V_{ub}^*|}{|V_{cd}V_{cb}^*|} = \left(\bar\rho^2 + \bar\eta^2\right)^{1/2}$$

$$R_t \equiv \frac{|V_{td}V_{tb}^*|}{|V_{cd}V_{cb}^*|} = \left((1-\bar\rho)^2 + \bar\eta^2\right)^{1/2} \qquad (24)$$

$$V_{ub} = |V_{ub}|\, e^{-i\gamma} = s_{13}\, e^{-i\delta} \qquad (\text{i.e. } \gamma = \delta)$$

$$V_{td} = |V_{td}|\, e^{-i\beta}$$

$$\alpha + \beta + \gamma = 2\pi \quad \text{for unitarity triangle.}$$

Alternative notations are ϕ_1, ϕ_2 & ϕ_3, and one can show that

<u>Exercise</u>:

$$\left.\begin{aligned} \phi_1 &= \beta = \arg\left(-\frac{V_{cd}V_{cb}^*}{V_{td}V_{tb}^*}\right) \\ \phi_2 &= \alpha = \arg\left(-\frac{V_{td}V_{tb}^*}{V_{ud}V_{ub}^*}\right) \\ \phi_3 &= \gamma = \arg\left(-\frac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}\right) \end{aligned}\right\} \qquad (25)$$

[C. Jarlskog, PRL55, 1039(1985);
Z.Phys.C 29 491(1985)]

Jarlskog Invariant J

To measure CP in phase-convention independent way, the Jarlskog invariant J was introduced as

$$\mathrm{Im}[V_{ij}V_{kl}V_{il}^{*}V_{kj}^{*}] = J\sum_{m,n}\epsilon_{ikm}\,\epsilon_{jln} \qquad (26)$$

$(i, j, k, l, m, n = 1,2,3)$

The area of all unitarity triangles are all the same, and equal to $|J|/2$.

Sometimes, J is also written as

$$J = (-)^{r+s}\,\mathrm{Im}[V_{ij}V_{kl}V_{il}^{*}V_{kj}^{*}] \qquad (27)$$

For example, $r=2$, $s=2$.

$$V_{CKM} = \begin{array}{c c} & \begin{array}{ccc} j & & l \end{array} \\ \begin{array}{c} i \\ \\ k \end{array} & \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \end{array} \longrightarrow \begin{pmatrix} V_{ud} & V_{ub}^{*} \\ V_{td}^{*} & V_{tb} \end{pmatrix} \rightarrow J = \mathrm{Im}[V_{ud}V_{tb}V_{ub}^{*}V_{td}^{*}]$$

✓ (28)

while for $r=1$, $s=2$

$$V_{CKM} = \begin{array}{c c} \begin{array}{c} \\ i \\ k \end{array} & \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \\ & \begin{array}{ccc} j & & l \end{array} \end{array} \longrightarrow \begin{pmatrix} V_{cd} & V_{cb}^{*} \\ V_{td}^{*} & V_{tb} \end{pmatrix}$$

$$\Rightarrow \quad J = -\mathrm{Im}[V_{cd}V_{tb}V_{cb}^{*}V_{td}^{*}] \quad \text{etc.} \qquad (29)$$

* $J = s_{12}c_{12}s_{23}c_{23}s_{13}c_{13}^{2}\sin\delta$ for all nine possible (r, s).
* If δ (or η in Wolfenstein's parameterization) vanishes, $J \to 0$.
* In terms of the Wolfenstein's parameterization in (14), one can check that $J \sim A^2\lambda^6\eta$. Despite η, A are of order 1, J is small because it is suppressed by λ^6.

Experimentally one has the best fit $J = (3.12^{+0.13}_{-0.12})\times 10^{-5}$ (30)
(2024, PDG)

J is rephasing invariant. To show this, we recall

$$V_{CKM} = (V_L^u)^\dagger (V_L^d).$$

where V_L^u & V_L^d are defined by

$$\begin{pmatrix} u \\ c \\ t \end{pmatrix}_L \to V_L^u \begin{pmatrix} u \\ c \\ t \end{pmatrix} \quad \& \quad \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L \longrightarrow V_L^d \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L. \qquad (31)$$

Suppose we perform phase transformations to the quark fields

$$\begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} \longrightarrow P^u \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} = \begin{pmatrix} e^{i\alpha_u} & & \\ & e^{i\alpha_c} & \\ & & e^{i\alpha_t} \end{pmatrix} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} \qquad (32)$$

and

$$\begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R} \longrightarrow P^d \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R} = \begin{pmatrix} e^{i\beta_d} & & \\ & e^{i\beta_s} & \\ & & e^{i\beta_b} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R}.$$

The V_{CKM} will then transform as

$$V_{CKM} \longrightarrow (P^u)^\dagger \, V_{CKM} \, P^d$$

i.e. $$(V_{CKM})_{ij} \longrightarrow (P^u)^\dagger_{im} (V_{CKM})_{mn} (P^d)_{nj}$$

$$= e^{-i\alpha_{u_i}} \delta_{im} (V_{CKM})_{mn} e^{i\beta_{d_n}} \delta_{nj}$$

$$= e^{-i(\alpha_{u_i} - \beta_{d_j})} (V_{CKM})_{ij} \qquad (33)$$

We can check J is indeed rephasing invariant by looking at the special cases of (r, s) in (28) & (29) for J.

$$V_{ud} V_{tb} V_{ub}^* V_{td}^* \to e^{-i(\alpha_u - \beta_d)} V_{ud} \, e^{-i(\alpha_t - \beta_b)} V_{tb} \cdot e^{+i(\alpha_u - \beta_b)} V_{ub}^* \, e^{+i(\alpha_t - \beta_d)} V_{td}^*$$

$$\longrightarrow V_{ud} V_{tb} V_{ub}^* V_{td}^*$$

$\Rightarrow J = \mathrm{Im}[V_{ud} V_{tb} V_{ub}^* V_{td}^*]$ is rephasing invariant.

Similarly, $J = -\mathrm{Im}[V_{cd} V_{tb} V_{cb}^* V_{td}^*]$, etc are all rephasing invariant!

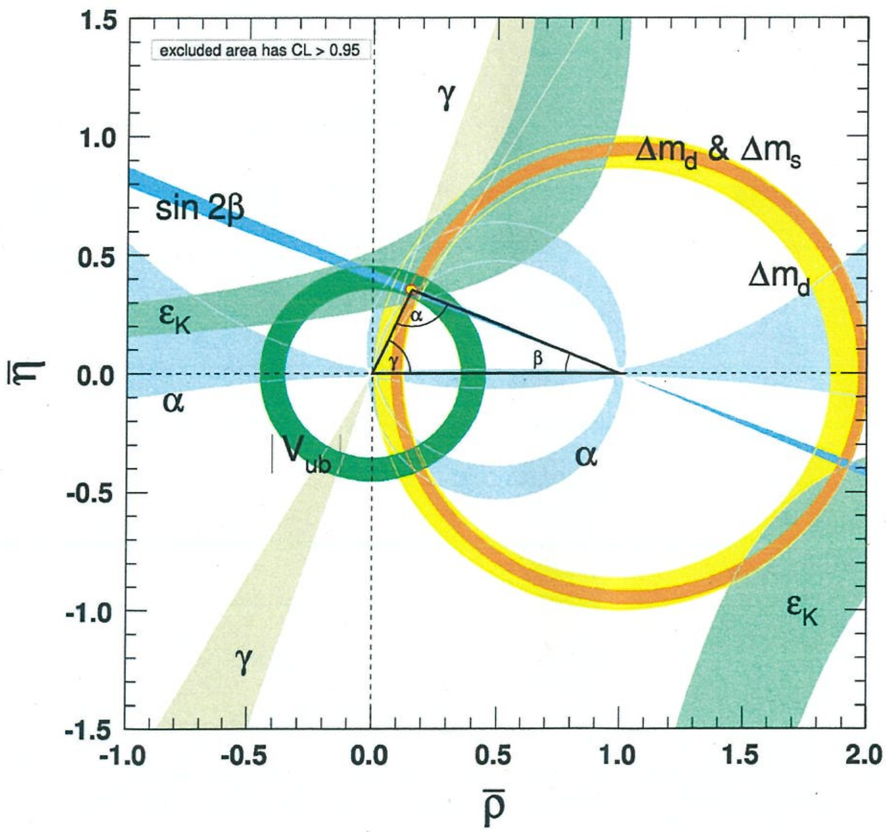
According to the PDG(2024), the best fit values for the CKM matrix elements are

$$\left.\begin{aligned}\sin\theta_{12} &= 0.22501 \pm 0.00068 \\ \sin\theta_{13} &= 0.003732^{+0.000090}_{-0.000085} \\ \sin\theta_{23} &= 0.04183^{+0.00079}_{-0.00069} \\ \delta &= 1.147 \pm 0.026\end{aligned}\right\}$$

Wolfenstein's parameter best fit values

$$\left.\begin{aligned}\lambda &= 0.22501 \pm 0.00068 \\ A &= 0.826^{+0.016}_{-0.015} \\ \bar{\rho} &= 0.1591 \pm 0.0094 \\ \bar{\eta} &= 0.3523^{+0.0073}_{-0.0071}\end{aligned}\right\}$$

$$|V_{\mathrm{CKM}}| = \begin{pmatrix} 0.97435 \pm 0.00016 & 0.22501 \pm 0.00068 & 0.003732^{+0.000090}_{-0.000085} \\ 0.22487 \pm 0.00068 & 0.97349 \pm 0.00016 & 0.04183^{+0.00079}_{-0.00069} \\ 0.00858^{+0.00019}_{-0.00017} & 0.04111^{+0.00077}_{-0.00068} & 0.999118^{+0.000029}_{-0.000034} \end{pmatrix},$$



excluded area has CL > 0.95
γ
Δm_d & Δm_s
sin 2β
Δm_d
ε_K
α
β
γ
α
|V_ub|
α
ε_K
γ
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
η̄
ρ̄