Supplementary Materials

Classical Groups: SO(N), SU(N), Sp(2N)

SO(N): Rotation group in N-dim. real vector space.

$\vec{x} = \{x_1, x_2, \cdots, x_N\}$, $\vec{y} = \{y_1, y_2, \cdots, y_N\}$

Scalar product ~~Norm~~: $\vec{x}^T \cdot \vec{y} = \sum_{i=1}^{N} x_i y_i$ is invariant under SO(N).

$\Rightarrow \quad \vec{x} \to O\vec{x}, \quad \vec{y} \to O\vec{y}$

$$\vec{x}^T \cdot \vec{y} = x^T \cdot y \to x O^T \cdot O y = x^T \cdot y$$

requires $O^T \cdot O = 1_{N\times N}$ or $O \cdot O^T = 1_{N\times N}$.
(for consistency!)

$N = 1$ trivial

$N = 2$. Rotation in $\hat{x}\hat{y}$. 2 dim. plane $\Rightarrow$ 1 angle $\Rightarrow$ 1 parameter

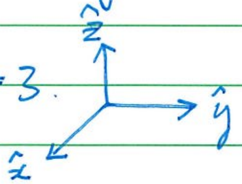
$$SO(2): \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \leftarrow \text{Exercise}$$

$\exp(i T \theta) \in SO(2)$

$$T \text{ generator}: -i\begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} = \sigma_2$$

$\sim$ Pauli: 2^{nd} matrix

$N = 3$.



One has $(\hat{x}\hat{y})$, $(\hat{y}, \hat{z})$, $(\hat{z}, \hat{x})$ 3 different planes

So we need 3 diff. angles (parameters) to parameterize SO(3).

e.g. Euler-angles, many other possibilities.

In general, for SO(N), we have $\frac{1}{2}N(N-1)$ different planes for rotations. $\Rightarrow \frac{1}{2}N(N-1)$ parameters / generators for SO(N).

$$O^T O = 1_{N\times N} \Rightarrow O_{j'i} O_{j'j} = \delta_{ij}$$

$$\downarrow$$

$$O_{i'i} O_{j'j} \delta_{i'j'} = \delta_{ij}$$

$$O O^T = 1_{N\times N} \Rightarrow O_{ii'} O_{ji'} = O_{ii'} O_{jj'} \delta_{i'j'} = \delta_{ij}$$

δ_{ij} is an invariant tensor of SO(N).

The condition $O O^T = 1 = O^T O$ imposes $(N + \frac{1}{2}N(N-1))$ real conditions on the matrix O. Since O, a $N \times N$ real matrix, has N^2 real elements, thus the number of indep. real parameters for an orthogonal $N \times N$ matrix is

$$N^2 - \left(N + \frac{1}{2}N(N-1)\right) = \frac{1}{2}N^2 - \frac{N}{2} = \frac{N}{2}(N-1)$$

which agrees with previous counting from numbers of indep. 2-planes to do rotations!

$$* \quad \mathrm{Det}(O^T \cdot O) = \mathrm{Det}(O \cdot O^T) = (\mathrm{Det}\, O)^2 = 1$$

$$\Rightarrow \mathrm{Det}\, O = \pm 1$$

$\mathrm{Det}\, O = 1$ special $SO(N)$

$= -1$ not connected with small transformation. e.g. $N=2$, we have spatial inversion $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ take $\{y \to -y\}$ & $\{x \to -x,\ y \to y\}$ respectively.

$$\Rightarrow O(N) \sim SO(N) \otimes \text{Inversion}$$

(odd # of spatial inversions)

$*$ rank of $SO(N)$: $\begin{cases} n & \text{for } N = 2n+1 \quad B_n \\ n & \text{for } N = 2n \quad D_n \end{cases}$ in Cartan's notation.

↑ # of maximal commuting generators.

$Sp(2N)$: $u = (x_1, \cdots, x_N, y_1, \cdots y_N)$ }
$v = (x_1', \cdots, x_N', y_1', \cdots y_N')$ } real components

Billinear $u^T \cdot \eta \cdot v$, where $\eta = \begin{pmatrix} 0 & \mathbb{I}_{N\times N} \\ -\mathbb{I}_{N\times N} & 0 \end{pmatrix}$ antisym.
is invariant under $Sp(2N)$.

Let $S \in Sp(2N)$.

$$u \to Su, \quad v \to Sv$$

$$u^T \cdot \eta \cdot v \to u^T S^T \eta S v = u^T \cdot \eta \cdot v$$

requires

$$S^T \cdot \eta S = \eta \text{ or equivalently } S \cdot \eta \cdot S^T = \eta$$

How many indep. parameters (generators) to describe $Sp(2N)$,
Let $S = \mathbb{I} + \theta + O(\theta^2)$, θ small
Write

$$\theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ where } A, B, C, D \text{ are } N\times N \text{ real matrices.}$$

$$\theta^T = \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}, \quad S^T = 1 + \theta^T + O(\theta^2)$$

$$S^T \cdot \eta \cdot S = (1 + \theta^T + \cdots)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}(1 + \theta + \cdots)$$

$$= \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\eta} + \underbrace{(\theta^T \cdot \eta + \eta \cdot \theta)}_{0} + O(\theta^2)$$

$$\theta^T \cdot \eta + \eta \cdot \theta = 0 \Rightarrow \begin{pmatrix} -C^T + C & A^T + D \\ -D^T - A & B^T - B \end{pmatrix} = 0 \Rightarrow \left.\begin{array}{l} A = -D^T \\ B = B^T \\ C = C^T \end{array}\right\}$$

Thus A has N^2 real comp., while B & C each, being symmetric, has $\frac{1}{2}N(N+1)$ indep. real comp. and D is determined by A.
$\Rightarrow$ # of indep. generator of $Sp(2N) = N^2 + 2 \cdot \frac{1}{2} N(N+1) = N(2N+1)$
#

* $\mathrm{Det}(S^T \cdot \eta \cdot S) = (\mathrm{Det}\, S)^2\, \mathrm{Det}\, \eta = \mathrm{Det}\, \eta$

$\Rightarrow (\mathrm{Det}\, S) = \pm 1$ — special $\mathrm{Det}\, S = +1$

* Hamiltonian approach has symplectic structure in phase space.
* $Sp(2N)$ has rank N, denoted by C_N by Cartan

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SU(N) : Unitary Group. All $N \times N$ complex matrices acting on N complex dim. space which preserves the following billinear form

$$\langle q', q \rangle = q'^{\dagger} \cdot q = (q_1'^*, q_2'^*, \cdots, q_N'^*) \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}$$

$$= \sum_{i=1}^{N} q_i'^* q_i$$

Introduce $q^i = q_i^*$, $\Rightarrow \langle q', q \rangle = \sum_i q'^i q_i$.

$\Rightarrow (q)_i \sim N$, $(q)^i \sim \bar{N}$ fundamental / anti-fundamental irreps.

Let $U \in U(N)$.

$q \to U q$, $q^{\dagger} \to q^{\dagger} U^{\dagger}$ $\quad$ $\dagger$: Hermitian Adjoint (complex conjugation + Transpose)

$\langle q', q \rangle \longrightarrow \langle q' U^{\dagger}, U q \rangle = \langle q', q \rangle$

$\Rightarrow U^{\dagger} U = 1$ or $U U^{\dagger} = 1$ (for consistency.)

(In components, $q_i \to U_i^{\ j} q_j$, $q^i = q^j U_j^{\ i *}$ we have $U_k^{\ i*} U_j^{\ k} = \delta^i_j$ or $U^i_{\ i'} \delta^{i'}_j U^{j'}_{\ j} = \delta^i_j$

i.e. δ^i_j is an invariant tensor for $U(N)$.)

Since $\mathrm{Det}\, U^{\dagger} U = \mathrm{Det}\, U^{\dagger} \cdot \mathrm{Det}\, U = (\mathrm{Det}\, U^T)^* (\mathrm{Det}\, U)$

$= (\mathrm{Det}\, U)^* (\mathrm{Det}\, U) = |\mathrm{Det}\, U|^2 = 1$ $\quad$ $U(N)$

$\Rightarrow \mathrm{Det}\, U = e^{i\delta}$ (a phase) $\Rightarrow$ Eigenvalues of U lie on S^1 circle!

* For SU(N), $\mathrm{Det}\, U = +1$ (Special Unitary)

* SU(N) was denoted as A_{n-1} by Cartan, rank $= n-1$.

Now for an arbitrary $N\times N$ complex matrice, it has $2N^2$ real elements. How many conditions are imposed by $UU^\dagger = U^\dagger U = 1$ for unitary matrix U?

$$\sum_k U^{i*}_k U^k_j = \delta^i_j = \begin{cases} 1 & i=j \Rightarrow N \text{ real conditions} \\ 0 & i\neq j \Rightarrow 2\left(\frac{1}{2}N(N-1)\right) \text{ real conditions} \end{cases}$$

$\Rightarrow$ An unitary matrix U has

$$2N^2 - \left(N + 2\tfrac{1}{2}\cdot N(N-1)\right) = N^2 \text{ real parameter.}$$

$\Rightarrow$ For $SU(N)$, $\mathrm{Det}\, U = 1$ imposes one real condition.

therefore $SU(N)$ has N^2-1 parameters
Whereas $U(N)$ has N^2 parameters

Since restricting to real case, unitary transf. reduces to orthogonal transf., out of the N^2 parameters, $\frac{1}{2}N(N-1)$ of them can be identified as rotation angles, while

$$N^2 - \frac{1}{2}N(N-1) = \frac{1}{2}N(N+1)$$

are the remaining phases for unitary transf.

$N=1$: 0 angle, 1 phase $\rightarrow U(1) : e^{i\theta}$
$N=2$: 1 angle, 3 phase

Below is a common 2×2 unitary matrix expression:

$$U = \begin{pmatrix} a & b \\ -e^{i\phi}b^* & e^{i\phi}a^* \end{pmatrix} \quad \text{with } |a|^2+|b|^2 = 1$$

One can easily check that $UU^\dagger = U^\dagger U = 1$ & $\mathrm{Det}\, U = e^{i\phi}$.
Let $a = e^{i\phi/2} e^{i\alpha}\cos\theta$ $\quad b = e^{i\phi/2}e^{i\beta}\sin\theta$ one obtains

$$U = e^{i\phi/2}\begin{pmatrix} e^{i\alpha}\cos\theta & e^{i\beta}\sin\theta \\ -e^{-i\beta}\sin\theta & e^{-i\alpha}\cos\theta \end{pmatrix}.$$

Furthermore, introduce $\alpha = \psi + \delta$ & $\beta = \psi - \delta$ one can rewrite U as

$$U = e^{i\phi/2}\begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix} \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \sin\theta \end{pmatrix}}_{\in SO(2)} \begin{bmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{bmatrix}!$$

* Not all the $\frac{1}{2}N(N+1)$ phases in the unitary matrix is physically relevant! In SM, the unitary matrix U is usually sandwiched between Dirac fermions of one type on the right and Dirac anti-fermions of another type on the left. Thus, we have $(2N-1)$ phases from the fermion fields to absorb away those from the unitary matrix.
We need to minus 1 because that's the global phase of all Dirac fermions which is associated with a global current like fermion #, baryon #, lepton #.
Thus, in physics, the # of indep. phases in an Unitary matrix is

$$\frac{1}{2}N(N+1) - (2N-1) = \frac{1}{2}(N^2 - 3N + 2)$$

$$= \frac{1}{2} \times (N-1)(N-2) \quad \text{(Dirac)}$$

* Thus, there are no physical phases for $N=1$, & $N=2$.
* In SM, N is the # of generations, & the phase is associated with CP phase. The minimal # of generations to have CP is three. $\rightarrow$ V_{CKM} has 3 angles & 1 CP phase.

* For 4 generations, we will have 6 angles & 3 CP phases.
* For Majorana fermions, since particle = antiparticle, we will have only N phases in the field to absorb those in the unitary matrix. Thus, for Majorana case, we have

$$\frac{1}{2}N(N+1) - N = \frac{1}{2}N(N-1) \quad \text{(Majorana)}$$

indep. CP phases.

$\Rightarrow$ Two generation of Majorana fermions can have 1 CP phase!

Examples:

(i) SU(2)

$$T^A = \frac{1}{2}\sigma^A, \quad A=1,2,3, \quad \sigma^A \text{ Pauli-Matrices}$$

→ 2-dim. defining rep. (fundamental) rep. of SU(2).

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

only σ^3 is diagonal! ⇒ rank(SU(2)) = 1.

Obviously $(\sigma^A)^\dagger = \sigma^A$, hermitian.

$\boxed{\text{Note: } 2^* \sim 2}$

Actually, all irreps of SU(2) are pseudo-real.

→ 3-dim. irreps. is

$$(T^A)_{BC} = -i\,\epsilon_{ABC}$$

This is the <u>adjoint</u> rep. of SU(2). # of irrep. = # of Lie Algebra

Same matrices as rotation group SO(3).

$$T^1 = -i\begin{pmatrix} & & \\ & & 1 \\ & -1 & \end{pmatrix}, \quad T^2 = -i\begin{pmatrix} & & -1 \\ & & \\ 1 & & \end{pmatrix}, \quad T^3 = -i\begin{pmatrix} & 1 & \\ -1 & & \\ & & \end{pmatrix}$$

Rotates around $\hat{x}$ (small angle θ)

$$\left.\begin{array}{l} x \to x \\ y \to y\cos\theta + z\sin\theta \\ z \to -y\sin\theta + z\cos\theta \end{array}\right\}$$

Rotates around $\hat{y}$ (small angle ξ)

$$\left.\begin{array}{l} x \to x\cos\xi - z\sin\xi \\ y \to y \\ z \to x\sin\xi + z\cos\xi \end{array}\right\}$$

Rotates around $\hat{z}$ (small angle ϕ)

$$\left.\begin{array}{l} x \to x\cos\phi + y\sin\phi \\ y \to -x\sin\phi + y\cos\phi \\ z \to z \end{array}\right\}$$

Note that $T^{1,2,3}$ are <u>not</u> diagonal in the adj. irresp.

* Another three dim. irrep. of SU(2) can be obtained using method of highest weights. In fact, <u>all</u> irreps. of SU(2) can be obtained using highest weight method. See Georgi's group theory book.

(ii) SU(3) Fundamental (defining) irrep. is

$$T^A = \frac{1}{2}\lambda^A \quad , \quad A = \{1, 2, .., 8\} \quad \text{Gell-Mann Matrices.}$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

* λ^3, λ^8 are real & diagonal $\Rightarrow$ rank$(SU(3)) = 2$.
* $\lambda^1, \lambda^4, \lambda^6$ are symmetric, pure real.
* $\lambda^2, \lambda^5, \lambda^7$ are anti-sym. * $\{2, 5, 7\}$ index set is special! * pure imag.
* $\mathrm{Tr}\,\lambda^A = 0 \quad \forall A = 1, \cdots, 8$

$\{\lambda^A\}$ satisfies
$$\left.\begin{aligned} &[\lambda^A, \lambda^B] = 2i f^{ABC} \lambda^C \\ \text{and} \quad &\{\lambda^A, \lambda^B\} = \frac{4}{3}\delta^{AB} \mathbb{1}_{3\times3} + 2d^{ABC}\lambda^C \end{aligned}\right\}$$

f^{ABC} = anti-sym. structural constants

$f^{123} = 1$

$f^{147} = -f^{156} = f^{246} = f^{257} = f^{345} = -f^{367} = \frac{1}{2}$

$f^{458} = f^{678} = \frac{\sqrt{3}}{2}$

* other f's <u>not</u> related by permutation symmetry are zeros.

d^{ABC} = totally symmetric coefficients

$d^{118} = d^{228} = d^{338} = -d^{888} = \frac{1}{\sqrt{3}}$

$d^{448} = d^{558} = d^{668} = d^{778} = -\frac{1}{2\sqrt{3}}$

$d^{344} = d^{355} = -d^{366} = -d^{377} = -d^{247} = d^{146} = d^{157} = d^{256} = \frac{1}{2}$

* Others vanish if # of indices from the set $\{2, 5, 7\}$ is odd.

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@ Lê Pô Cafe.
Huế

Biunitary Transformation

Given a arbitrary $N\times N$ complex matrix M, one can diagonalize it by 2 unitary matrices V_L & V_R such that

$$V_L \mathcal{M} V_R^\dagger = \mathrm{Diag}(m_1, m_2, \cdots, m_N) = \mathcal{M}_D \qquad (1)$$

where $\mathcal{M}_{Diag}$ is a diagonal $N\times N$ matrix.

The proof of this is as follows. Like a complex number $z = \rho e^{i\theta}$, a complex $N\times N$ matrix M can be decomposed as

$$M = H\,\mathcal{U} \qquad \text{(Polar decomposition)} \qquad (2)$$

where H is a Hermitian & positive definite matrix given by

$$H = H^\dagger = \sqrt{MM^\dagger} = \sqrt{M^\dagger M} \qquad (3)$$

& $\mathcal{U}$ is an arbitrary unitary matrix.

Thus $M^\dagger = \mathcal{U}^\dagger H^\dagger$, & $MM^\dagger = H \underbrace{\mathcal{U}\mathcal{U}^\dagger}_{\mathbb{1}} H^\dagger = H^2$.

Since $(MM^\dagger)^\dagger = MM^\dagger$ is Hermitian,

$\Rightarrow H = \sqrt{MM^\dagger}$ is also Hermitian.

Let V_L diagonalize H as $V_L H V_L^\dagger = H_{diag}$

$$\Rightarrow H_{diag} = V_L H V_L^\dagger = V_L M \underbrace{\mathcal{U}^\dagger V_L^\dagger}_{\equiv V_R^\dagger} \equiv V_L M V_R^\dagger = \mathcal{M}_D$$

(QED)

* Biunitary transformations have been used to diagonalize the complex Yukawa coupling matrices in the SM to give rise to Dirac masses for SM fermions.

$$\text{i.e.}\quad \overline{\psi}_L Y \psi_R = \underbrace{\overline{\psi}_L V_L^\dagger}_{\overline{\psi}'_L} \underbrace{V_L Y V_R^\dagger}_{Y_{diag}} \underbrace{V_R \psi_R}_{\psi'_R} = \underbrace{\overline{\psi}'_L Y_{diag} \psi'_R}_{\text{(Physical basis)}} \qquad (4)$$

In the special case that M is also symmetric, we can diagonalize M by a unitary matrix $\mathcal{U}$ such that

$$\mathcal{U}^T M \mathcal{U} = \text{diag}(m_1, m_2, \cdots, m_N) = M_D \tag{5}$$

The proof of this seems to be more involved.
See the follow references: (Takagi's diagonalization)

(i) Dreiner, Haber & Martin, phy. Rep. 494(5), 1-196 (2010). arXiv: 0812.1594v6 [hep-ph]

For a simpler proof, see Appendix C in

(ii) Borisov & Isaev, arXiv: 2312.17714v1 [hep-ph]

* This diagonalization method is used for Majorana mass matrices in neutrino physics.

Diagonalization of a complex symmetric square matrix

Here we will follow the simpler proof by Borisov & Isaev, arXiv:2312:17714v1 [hep-ph].

Theorem: For any complex symmetric $N\times N$ matrix M, there exists an unitary matrix $\mathcal{U}$ such that

$$\mathcal{U}^T M \mathcal{U} = \mathrm{diag}(m_1, m_2, \cdots, m_N) \equiv M_D \quad (1)$$

with m_i are all real & positive (≥ 0).

Proof: We will construct $\mathcal{U}$ explicitly in the proof. Note that M can be decomposed as

$$M = X + iY \quad (2)$$

where X, Y are $N\times N$ real symmetric matrices, so they can be diagonalized by orthogonal matrix. Instead of considering X & Y separately consider the following $2N\times 2N$ real symmetric matrix $\tilde{M}$,

$$\tilde{M} = \begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix} = \sigma_3 \otimes X - \sigma_1 \otimes Y = \tilde{M}^T \quad (3)$$

where σ_1 & σ_3 are the Pauli-matrices: $\sigma_1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

From linear algebra, we know $\tilde{M}$ can be diagonalized by an $2N\times 2N$ orthogonal matrix $\mathcal{O}$ with real eigenvalues:

$$\left.\begin{aligned} \mathcal{O}^T \tilde{M} \mathcal{O} &= \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{2N}) \\ \mathcal{O}^T \mathcal{O} &= \mathbb{1}_{2N\times 2N} \Rightarrow (\mathcal{O}^T)^{-1} = \mathcal{O} \end{aligned}\right\} \quad (4)$$

In terms of components, we have

$$\left.\begin{aligned} (\tilde{M}\mathcal{O})_{ij} &= (\mathcal{O}\lambda)_{ij} \\ \mathcal{O}_{ki}\mathcal{O}_{kj} &= \delta_{ij} \end{aligned}\right\} \quad i, j, k = 1, \cdots, 2N \quad (4)'$$

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(4)'s first eq can be simplified to

$$\tilde{M}_{ik} O_{kj} = O_{ik} \lambda_{kj} = O_{ik} \lambda_k \delta_{kj} = O_{ij} \lambda_j \qquad (4a)'$$

with fixed i & j.

Define the $2N$-dim. vectors ($2N$ of them)

$$\underset{\sim}{v}^{(j)} = \begin{pmatrix} O_{1j} \\ O_{2j} \\ \vdots \\ O_{2Nj} \end{pmatrix} \qquad j = 1, \cdots, 2N \qquad (5)$$

Eq. (4a)' can be viewed as the following matrix form

$$\tilde{M} \cdot v^{(j)} = (\sigma_3 \otimes X - \sigma_1 \otimes Y)\, v^{(j)} \qquad (4a)''$$

$$= \lambda_j v^{(j)} \qquad j = 1, 2, \cdots, 2N$$

And the orthogonality of O in (4)' can be viewed as

$$(v^{(i)}, v^{(j)}) = v^{(i)T} \cdot v^{(j)} = O_{ki} O_{kj} = \delta_{ij} \qquad (6)$$

i.e. $v^{(i)}$ is an orthonormal basis for $\mathbb{R}^{2N}$.

Next, construct another $2N \times 2N$ matrix ϵ

$$\epsilon \equiv i \sigma_2 \otimes I_N = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \qquad (7)$$

Note that $\epsilon \tilde{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix} = \begin{pmatrix} -Y & -X \\ -X & Y \end{pmatrix}$ and

$$\tilde{M} \epsilon = \begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} Y & X \\ X & -Y \end{pmatrix} \Rightarrow \epsilon \tilde{M} = -\tilde{M} \epsilon \qquad (8)$$

$$\Rightarrow \tilde{M} \epsilon v^{(j)} = -\epsilon \tilde{M} v^{(j)} = -\lambda_j \epsilon v^{(j)}$$

$\Rightarrow$ If λ_j is an eigenvalue of $\tilde{M}$ for $v^{(j)}$, $\epsilon v^{(j)}$ is another eigenvector with eigenvalue $-\lambda_j$!

$\Longrightarrow$ The $2N$ eigenvalues of $\underset{\sim}{M}$ break into pairs $(\lambda_j, -\lambda_j)$

Since λ_j are real, we can pick out the N non-negative of them and denote them as $m_k \geq 0$. Their corresponding $2N$-dim eigenvectors are denoted as $V^{(k)}$ $(k=1,2,\ldots,N)$

i.e. $\{V^{(1)}, V^{(2)}, V^{(3)}, \ldots V^{(N)}\} \longrightarrow \{m_1, m_2, m_3, \ldots, m_N\}$

* N $2N$-dim. eigenvectors with N eigenvalues ≥ 0.

We can break these $2N$-dim eigenvectors into two N-dim. vectors $u^{(k)}$ & $w^{(k)}$ such that

$$\underset{\sim}{M}\, V^{(k)} = m_k V^{(k)}$$

$$\Rightarrow \begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix}\begin{pmatrix} u^{(k)} \\ w^{(k)} \end{pmatrix} = m_k \begin{pmatrix} u^{(k)} \\ w^{(k)} \end{pmatrix} \qquad (9)$$

The orthonormality now become

$$\delta_{k\ell} = (V^{(k)}, V^{(\ell)}) = V^{(k)T} \cdot V^{(\ell)} = (u^{(k)}, w^{(k)})\begin{pmatrix} u^{(\ell)} \\ w^{(\ell)} \end{pmatrix}$$

$$= (u^{(k)}, u^{(\ell)}) + (w^{(k)}, w^{(\ell)}) \qquad (10)$$

and no crossed product between $u^{(k)}$ & $w^{(k)}$! Thus the choice of $u^{(k)}$ & $v^{(k)}$ can be arbitrary!

Now, introduce the <u>N</u> <u>N-dim</u> <u>complex</u> vectors

$$\left.\begin{aligned} & Z^{(k)} = u^{(k)} + i w^{(k)}, \quad Z^{(k)*} = u^{(k)} - i w^{(k)} \\ & \text{which implies} \\ & u^{(k)} = (Z^{(k)} + Z^{(k)*})/2, \quad w^{(k)} = (Z^{(k)} - Z^{(k)*})/2i \end{aligned}\right\} \qquad (11)$$

and

$$(Z^{(k)*}, Z^{(\ell)}) = \delta_{k\ell} \quad \left(\text{Follow immediately from (10)!}\right) \qquad (12)$$

Now the RHS of (9) can be rewritten as

$$\begin{pmatrix} X & -Y \\ -Y & -X \end{pmatrix}\begin{pmatrix} u^{(k)} \\ w^{(k)} \end{pmatrix} = \begin{pmatrix} Xu^{(k)} - Yw^{(k)} \\ -Yu^{(k)} - Xw^{(k)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(X+iY)Z^{(k)} + \frac{1}{2}(X-iY)Z^{(k)*} \\ \frac{i}{2}(X+iY)Z^{(k)} - \frac{i}{2}(X-iY)Z^{(k)*} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} MZ^{(k)} + M^* Z^{(k)*} \\ iMZ^{(k)} - iM^* Z^{(k)*} \end{pmatrix} \qquad (13)$$

The LHS of (9) is simply $m_k \begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} = \frac{1}{2} m_k \begin{pmatrix} z^{(k)} + z^{(k)*} \\ (z^{(k)} - z^{(k)*})/i \end{pmatrix}$ (14)

Equating (13) & (14) $\Rightarrow$

$$\left.\begin{aligned} M z^{(k)} + M^* z^{(k)*} &= m_k (z^{(k)} + z^{(k)*}) \\ M z^{(k)} - M^* z^{(k)*} &= -m_k (z^{(k)} - z^{(k)*}) \end{aligned}\right\} \quad (15)$$

$$\Rightarrow \quad \left.\begin{aligned} M \cdot z^{(k)} &= m_k z^{(k)*} \\ M^* z^{(k)*} &= m_k z^{(k)} \end{aligned}\right\} \quad (16)$$

Define a $N \times N$ matrix $\mathcal{U}$ by the N complex vector $z^{(k)}$ $(k=1,\dots,N)$

$$\mathcal{U}_{ik} = z_i^{(k)} \quad (i, k = 1, \cdots, N) \quad (17)$$

Then the first eq of (16) can be viewed as

$$M \cdot \mathcal{U} = \mathcal{U}^* \cdot \mathrm{diag}(m_1, m_2, \cdots, m_N) \quad (18)$$

From the orthonormality condition (12), we have

$$\delta_{kl} = (z^{(k)*}, z^{(l)}) = z^{(k)*} \cdot z^{(l)} = \mathcal{U}^*_{kj} \mathcal{U}_{kl} = (\mathcal{U}^\dagger)_{jk} \mathcal{U}_{kl}$$

i.e. $\mathcal{U}^\dagger \mathcal{U} = 1 \Rightarrow \mathcal{U}$ is unitary!

$$\Rightarrow \mathcal{U}^\dagger = \mathcal{U}^{-1} \Rightarrow \mathcal{U}^{*T} = \mathcal{U}^{-1} \Rightarrow \mathcal{U}^* = (\mathcal{U}^{-1})^T$$

$$\Rightarrow \mathcal{U}^T \mathcal{U}^* = \mathcal{U}^T (\mathcal{U}^{-1})^T = (\mathcal{U}^{-1} \cdot \mathcal{U})^T = 1^T = \mathbb{I}$$

Thus, multiply $\mathcal{U}^T$ to both sides of (18), we prove

$$\mathcal{U}^T \cdot M \cdot \mathcal{U} = \mathrm{diag}(m_1, m_2, \cdots, m_N)$$

with a unitary matrix $\mathcal{U}$ defined by the complex eigenvectors (17), and $m_k \geq 0$.

QED.

Triangle Diagrams

Consider a free massless Dirac fermion with the following action

$$S = \int d^4x \, i\bar{\psi}\not{\partial}\psi .$$

We know that this action has two abelian $U(1)$ symmetries one $U_V(1)$ with current $J_V^\mu = \bar{\psi}\gamma^\mu\psi$, the other one $U_A(1)$ with current $J_A^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi$. Classically, we have $\partial_\mu J_V^\mu = 0$ & $\partial_\mu J_A^\mu = 0$, as consequences of Nöether's theorem.

Now consider the 3-point function

$$i\Gamma^{\mu\nu\rho}(x_1, x_2, x_3) = (i)^3 \langle 0 | T[J_V^\mu(x_1) J_V^\nu(x_2) J_A^\rho(x_3)] | 0 \rangle \qquad (1)$$

Its fourier transform is

$$i\tilde{\Gamma}^{\mu\nu\rho}(p_1, p_2, q) \underbrace{\delta^4(p_1 + p_2 + q)}_{\text{due to translational invariance}} (2\pi)^4 \qquad (2)$$

$$= (i)^3 \int d^4x_1 \, d^4x_2 \, d^4x_3 \, \Gamma^{\mu\nu\rho}(x_1, x_2, x_3) \exp[ip_1\cdot x_1 + ip_2\cdot x_2 + iq\cdot x_3]$$

Here p_1, p_2 are associated with x_1, x_2 of the vector current, while q is for axial current.

Consider

$$p_{1\mu}\tilde{\Gamma}^{\mu\nu\rho}(p_1, p_2, q) \overset{(+)}{=} -i\int \Gamma^{\mu\nu\rho}(x_1, x_2, x_3)\, \partial_\mu^{x_1} e^{ip_1\cdot x_1 + ip_2\cdot x_2 + iq\cdot x_3}$$

Drop boundary term ⇒ $$\overset{(+)}{=} +i\int \partial_\mu^{x_1}\Gamma^{\mu\nu\rho}(x_1, x_2, x_3)\cdot e^{ip_1\cdot x_1 + ip_2\cdot x_2 + iq\cdot x_3} \qquad (2)'$$

Ward identity implies $\langle \partial_\mu J_V^\mu \rangle = 0 \Rightarrow \partial_\mu^{x_1}\Gamma^{\mu\nu\rho}(x) = 0$. Thus, conserved vector current means

$$p_{1\mu}\tilde{\Gamma}^{\mu\nu\rho}(p) = 0 . \qquad (3)$$

Similarly, $$p_{2\nu}\tilde{\Gamma}^{\mu\nu\rho}(p) = 0 \qquad (4)$$

&

$$q_\rho\tilde{\Gamma}^{\mu\nu\rho}(p) = \underbrace{-(p_1 + p_2)_\rho}_{\delta\text{-function from (2)}}\tilde{\Gamma}^{\mu\nu\rho} \overset{?}{=} 0 \qquad (5)$$

Note not implied by (3) & (4)!

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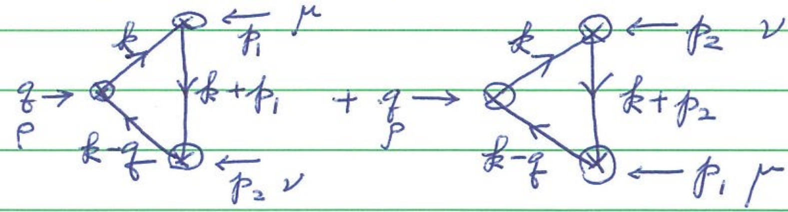
Perturbative analysis of the triangle diagrams.

$$+i\,\tilde{\Gamma}^{\mu\nu\rho}(p_1,p_2,q) = \qquad (6)$$



From Wick's theorem, one can write down the Feynman amplitude immediately (with 2 possible contractions lead to (6)!)

$$+i\,\tilde{\Gamma}^{\mu\nu\rho}(p_1,p_2,q) = \overset{(-i)^3?}{-}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\frac{i}{\not k}\gamma^\rho\gamma^5\frac{i}{\not k-\not q}\gamma^\nu\frac{i}{\not k+\not p_1}\gamma^\mu\right] + \begin{pmatrix}p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu\end{pmatrix} \qquad (7)$$

Fermion closed loop

over Dirac trace

We would like to check the 3 Ward identities (3), (4), (5) as expected from Ward identities. Let's consider (5) first, since it is the most suspicious one. Contracting with q_ρ,

$$+i q_\rho\tilde{\Gamma}^{\mu\nu\rho} = \overset{(-i)^3?}{+i}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\frac{1}{\not k}\not q\gamma^5\frac{1}{\not k-\not q}\gamma^\nu\frac{1}{\not k+\not p_1}\gamma^\mu\right] + \begin{pmatrix}p_1\leftrightarrow p_2\\ \mu\leftrightarrow\nu\end{pmatrix} \qquad (8)$$

Now write $\not q\gamma^5$ as

$$\begin{aligned}\not q\gamma^5 &= -\gamma^5\not q = \gamma^5(\not k-\not q-\not k)\\ &= \gamma^5(\not k-\not q) - \gamma^5\not k \quad \Leftarrow \{\gamma^5,\gamma^\mu\}=0\\ &= \gamma^5(\not k-\not q) + \not k\gamma^5.\end{aligned}$$

$$\Rightarrow \frac{1}{\not k}\not q\gamma^5\frac{1}{\not k-\not q} = \frac{1}{\not k}\left[\gamma^5(\not k-\not q)+\not k\gamma^5\right]\frac{1}{\not k-\not q}$$

$$= \frac{1}{\not k}\gamma^5 + \gamma^5\frac{1}{\not k-\not q}$$

$$\Longrightarrow$$

$$+i q_\rho\tilde{\Gamma}^{\mu\nu\rho} = \overset{(-i)^3?}{i}\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{1}{\not k}\gamma^5+\gamma^5\frac{1}{\not k-\not q}\right)\gamma^\nu\frac{1}{\not k+\not p_1}\gamma^\mu\right]$$

$$+\begin{pmatrix}\mu\leftrightarrow\nu\\ p_1\leftrightarrow p_2\end{pmatrix}$$

Writing out the other term, we have

$$+iq_\rho \tilde{\Gamma}^{\mu\nu\rho} = \overset{(-i)^3 ?}{i} \int \frac{d^4k}{(2\pi)^4}\, tr\Big[\Big(\frac{1}{\not k}\gamma^5\gamma^\nu \frac{1}{\not k+\not p_1}\gamma^\mu + \gamma^5 \frac{1}{\not k-\not q}\gamma^\nu \frac{1}{\not k+\not p_1}\gamma^\mu \Big)$$

$$+ \Big(\frac{1}{\not k}\gamma^5\gamma^\mu \frac{1}{\not k+\not p_2}\gamma^\nu + \gamma^5 \frac{1}{\not k-\not q}\gamma^\mu \frac{1}{\not k+\not p_2}\gamma^\nu \Big)\Big]$$

$$\equiv (-i)^3 ?\, \big(\Delta_1^{\mu\nu} + \Delta_2^{\mu\nu} \big) \qquad (9)$$

with

$$\Delta_1^{\mu\nu} = i\, Sp\Big[\frac{1}{\not k}\gamma^5\gamma^\nu \frac{1}{\not k+\not p_1}\gamma^\mu + \gamma^5 \frac{1}{\not k-\not q}\gamma^\mu \frac{1}{\not k+\not p_2}\gamma^\nu \Big] \qquad \{\gamma^5, \gamma^\mu\} = 0$$

$$= i\, Sp\Big[\frac{1}{\not k}\gamma^5\gamma^\nu \frac{1}{\not k+\not p_1}\gamma^\mu - \frac{1}{\not k-\not q}\gamma^\mu \frac{1}{\not k+\not p_2}\gamma^5\gamma^\nu \Big]$$

$$(10) \qquad = i\, Sp\Big[\frac{1}{\not k}\gamma^5\gamma^\nu \frac{1}{\not k+\not p_1}\gamma^\mu - \frac{1}{\not k+\not p_2}\gamma^5\gamma^\nu \frac{1}{\not k-\not q}\gamma^\mu \Big] \qquad \text{cylicity of trace}$$

and

$$\Delta_2^{\mu\nu} = i\, Sp\Big[\frac{1}{\not k}\gamma^5\gamma^\mu \frac{1}{\not k+\not p_2}\gamma^\nu - \frac{1}{\not k+\not p_1}\gamma^5\gamma^\mu \frac{1}{\not k-\not q}\gamma^\nu \Big] \qquad (11)$$

where $Sp \equiv \int \frac{d^4k}{(2\pi)^4} tr$.

Naively, if one shifts $k \to k+p_2$ in $\Delta_1^{\mu\nu}$, the first integral is the same as the second one. One might conclude $\Delta_1^{\mu\nu}$ should vanish. Similarly, $\Delta_2^{\mu\nu}$ should vanish too if one shifts $k \to k+p_1$ in the first integral. However, these shifts of integration variables are legal only if the integrals converge. The trouble is both $\Delta_1^{\mu\nu}$ & $\Delta_2^{\mu\nu}$ have superficial degree of divergence $D = 4-2 = 2$, i.e. they are quadratic divergent. Later, we will see that the Dirac trace will help to kill one k-momentum and make $\Delta_{1,2}^{\mu\nu}$ suffer only linear divergencies. At any rate, we have to be more careful about the momentum shift due to those divergencies.

(Minkowski space)

Consider $\tilde{\Delta}(a) = i\int\frac{d^4k}{(2\pi)^4}\left[f(k) - f(k+a)\right]$ (12)

where $f(k)$ is only linearly divergent. For small a, we have

$$\tilde{\Delta}(a) = -i\int\frac{d^4k}{(2\pi)^4}\left[a^\mu\partial_{k^\mu}f + \frac{1}{2}a^\mu a^\nu\partial_{k^\mu}\partial_{k^\nu}f + \cdots\right]$$

Each term is boundary term!

Each of these terms in $\tilde{\Delta}(a)$ is less & less divergent compared with $f(k)$. For linearly divergent $f(k)$, we truncate at the first term,

$$\tilde{\Delta}(a) = -i\int\frac{d^4k}{(2\pi)^4}\,a^\mu\partial_{k^\mu}f$$

Recall that in vector analysis, we have the divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{F}\,dV = \int_{S=\partial V}\vec{F}\cdot d\vec{S}$$

Let $\vec{F} = f\vec{a}$ where $\vec{a}$ is a constant vector. Since $\vec{\nabla}\cdot\vec{F} = \vec{\nabla}\cdot(f\vec{a}) = f\underbrace{\vec{\nabla}\cdot\vec{a}}_{0} + \vec{a}\cdot\vec{\nabla}f = \vec{a}\cdot\vec{\nabla}f$, we have $\int_V\vec{a}\cdot\vec{\nabla}f\,dV = \vec{a}\cdot\int_{\partial V}f\,d\vec{S}$.

$\because$ $\vec{a}$ is arbitrary const. vector, we therefore derive

$$\int_V\vec{\nabla}f\,dV = \int_{\partial V}f\,d\vec{S} = \int_{\partial V}f\,\hat{n}\,dS \quad \text{(Surface term)}$$

This expression can be generalized to higher dimensional space. In our case, we have, after Wick rotate to Euclidean space (S^4) & integrate over a large surface fixed at $|k|\to\infty$,

$$\int d^4k\,\partial_{k^\mu}f = i\int_{S^4}d^4k_E\,\partial_\mu f_E = i\int_{\partial S^4}d^3S_\mu\,f_E$$

✓ agree with Jackiw's!

Surface area factor

Thus

$$\tilde{\Delta}(a) = +\int\frac{d\Omega_4}{(2\pi)^4}\,a^\mu k_\mu k^2 f(k) \qquad (13)$$

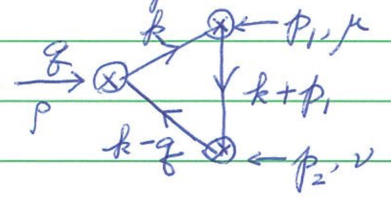
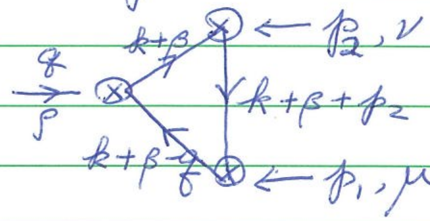
$d^3S_\mu = k^2k_\mu\,d\Omega_4$ — surface area of S^4

Suppressed E subscript for clarity

a 3-sphere at infinity $|k|\to\infty$.

We can apply (13) to $\Delta_1^{\mu\nu}$ & $\Delta_2^{\mu\nu}$ which have similar form like (12).

To see the similarity, let's allow a general offset β^μ between the two triangle diagrams in (6). Therefore

$$+i\Gamma^{\mu\nu\rho}(p_1,p_2,q) = \xrightarrow[\rho]{q}$$  $$+ \xrightarrow[\rho]{q}$$  (14)

$\boxed{-q = p_1+p_2}$

Using this offset momentum, we now rewrite the amplitude as

$$+iq_\rho\tilde{\Gamma}^{\mu\nu\rho}(p_1,p_2,q) = {}^{(-i)^3?}\,\tilde{\Delta}_1^{\mu\nu} + \tilde{\Delta}_2^{\mu\nu} \qquad (15)$$

where

$$\tilde{\Delta}_1^{\mu\nu} = iSp\left[\frac{1}{\not{k}}\gamma^5\gamma^\nu\frac{1}{\not{k}+\not{p}_1}\gamma^\mu - \frac{1}{\not{k}+\not{\beta}+\not{p}_2}\gamma^5\gamma^\nu\frac{1}{\not{k}+\not{\beta}-\not{q}}\gamma^\mu\right]$$

$$= iSp\left[\frac{1}{\not{k}}\gamma^\nu\gamma^\nu\frac{1}{\not{k}+\not{p}_1}\gamma^\mu - (k \to k+\beta+p_2)\right]$$

and

$$\tilde{\Delta}_2^{\mu\nu} = iSp\left[\frac{1}{\not{k}+\not{\beta}}\gamma^5\gamma^\mu\frac{1}{\not{k}+\not{\beta}+\not{p}_2}\gamma^\nu - \frac{1}{\not{k}+\not{p}_1}\gamma^5\gamma^\mu\frac{1}{\not{k}-\not{q}}\gamma^\nu\right]$$

$$= iSp\left[\frac{1}{\not{k}+\not{\beta}}\gamma^5\gamma^\mu\frac{1}{\not{k}+\not{\beta}+\not{p}_2}\gamma^\nu - (k \to k-\beta+p_1)\right]$$

i.e.

(16)
$$\tilde{\Delta}_1^{\mu\nu}: f \to f_1^{\mu\nu}(k) = \mathrm{tr}\left[\frac{1}{\not{k}}\gamma^5\gamma^\nu\frac{1}{\not{k}+\not{p}_1}\gamma^\mu\right], \quad a = \beta+p_2$$

$$\tilde{\Delta}_2^{\mu\nu}: f \to f_2^{\mu\nu}(k) = \mathrm{tr}\left[\frac{1}{\not{k}+\not{\beta}}\gamma^5\gamma^\mu\frac{1}{\not{k}+\not{\beta}+\not{p}_2}\gamma^\nu\right], \quad a = -\beta+p_1$$

Thus we can apply (13) to $\tilde{\Delta}_{1,2}^{\mu\nu}$.

(17)
$$\tilde{\Delta}_1^{\mu\nu} = + \int_{\partial S^4}\frac{d\Omega_4}{(2\pi)^4}(\beta+p_2)^\rho k_\rho k^2 f_1^{\mu\nu}$$

&

$$\tilde{\Delta}_2^{\mu\nu} = + \int_{\partial S^4}\frac{d\Omega_4}{(2\pi)^4}(-\beta+p_1)^\rho k_\rho k^2 f_2^{\mu\nu}$$

Now $f_1^{\mu\nu} = tr\left(\frac{1}{\not k}\gamma^5\gamma^\nu\frac{1}{\not k+\not p_1}\gamma^\mu\right)$

$$=(-)\frac{1}{k^2}\frac{1}{(k+p_1)^2}tr\left(\gamma^5 \not k\gamma^\nu(\not k+\not p_1)\gamma^\mu\right)$$

$tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta) = -4i\epsilon^{\mu\nu\alpha\beta}$

$$= -\frac{1}{k^2}\frac{1}{(k+p_1)^2}\left(-4i\,\epsilon^{\rho\nu\sigma\mu}k_\rho(k+p_1)_\sigma\right)$$

$$= 4i\frac{1}{k^2(k+p_1)^2}\epsilon^{\rho\nu\sigma\mu}k_\rho p_{1\sigma}$$

Note that in the last step, the ϵ-tensor kills the $k_\rho k_\sigma$ term. Hence the integrals are actually linearly divergent rather than quadratically divergent!

Similarly, we have

$$f_2^{\mu\nu} = tr\left(\frac{1}{\not k+\not\beta}\gamma^5\gamma^\mu\frac{1}{\not k+\not\beta+\not p_2}\gamma^\nu\right)$$

$$= \frac{1}{(k+\beta)^2}\frac{1}{(k+\beta+p_2)^2}4i\,\epsilon^{\rho\mu\sigma\nu}(k+\beta)_\rho(k+\beta+p_2)_\sigma$$

$$= \frac{1}{(k+\beta)^2}\frac{1}{(k+\beta+p_2)^2}4i\,(k+\beta)_\rho\, p_{2\sigma}\,\epsilon^{\rho\mu\sigma\nu}$$

$$\Rightarrow \tilde\Delta_1^{\mu\nu} = +\int_{\partial S^4}\frac{d\Omega_4}{(2\pi)^4}(\beta+p_2)^\lambda k_\lambda k^2\, 4i\frac{1}{k^2(k+p_1)^2}\epsilon^{\rho\nu\sigma\mu}k_\rho p_{1\sigma}$$

&

$$\tilde\Delta_2^{\mu\nu} = +\int_{\partial S^4}\frac{d\Omega_4}{(2\pi)^4}(-\beta+p_1)^\lambda k_\lambda k^2\, 4i\cdot\frac{1}{(k+\beta)^2}\frac{1}{(k+\beta+p_2)^2}\cdot\epsilon^{\rho\mu\sigma\nu}(k+\beta)_\rho p_{2\sigma}$$

* One can also write $d\Omega_4 k_\lambda = d\hat{k}_\lambda\,|k|\,d\Omega_4$. (Dong's notation.)

* $\Omega_n = \int d\Omega_n$ = surface area of n-sphere

$$= \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}. \Rightarrow \Omega_4 = \frac{2\cdot\pi^{4/2}}{\Gamma(2)} = 2\pi^2.$$

Since S^3_∞ is at infinity, $|k|\to\infty$. We can rewrite $\tilde{\Delta}^{\mu\nu}_{1,2}$ as

$$\tilde{\Delta}^{\mu\nu}_1 = +4i\int\frac{d\hat{k}_\lambda}{(2\pi)^4}(\beta+p_2)^\lambda |k|^3 \frac{1}{k^4}\epsilon^{\mu\nu\rho\sigma}k_\rho p_{1\sigma}$$

& $$\tilde{\Delta}^{\mu\nu}_2 = +4i\int\frac{d\hat{k}_\lambda}{(2\pi)^4}(-\beta+p_1)^\lambda |k|^3\frac{(-1)}{k^4}\epsilon^{\mu\nu\rho\sigma}k_\rho p_{2\sigma} \tag{18}$$

We now need to evaluate $\int_{S^3_\infty} d\hat{k}_\lambda k_\rho$. Let $\int_{S^3_\infty} d\hat{k}_\lambda k_\rho = C g_{\lambda\rho}$.

Contract with $g^{\lambda\rho}$, one gets $4C = \int_{\partial S^4} d\hat{k}_\lambda k^\lambda = |k|\int_{\partial S^4_1} = |k|\int d\Omega_4$.

Recall that $\int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \Rightarrow \int d\Omega_4 = 2\pi^2$. $\Rightarrow C = \frac{1}{4}|k|2\pi^2$. (19)

Therefore $$\tilde{\Delta}^{\mu\nu}_1 = \frac{+i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}(\beta+p_2)_\rho p_{1\sigma} = -\frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}p_{1\rho}(\beta+p_2)_\sigma$$

& $$\tilde{\Delta}^{\mu\nu}_2 = -\frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}(-\beta+p_1)_\rho p_{2\sigma} = +\frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}p_{2\rho}(p_1-\beta)_\sigma \tag{20}$$

$$\Rightarrow +iq_\rho\tilde{\Gamma}^{\mu\nu\rho} = (-1)^3\,\tilde{\Delta}^{\mu\nu}_1 + \tilde{\Delta}^{\mu\nu}_2 = -\frac{(-1)^{3?}\,i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\left(2p_{1\rho}p_{2\sigma} + (p_1+p_2)_\rho\beta_\sigma\right) \tag{21}$$

So there's an ambiguity in the momentum assignment since the result of $+iq_\rho\tilde{\Gamma}^{\mu\nu\rho}$ depends on β!

Question: How do we fix β? To resolve this ambiguity, one needs to study the other two Ward identities for the vector current. Repeating the above exercises, one would find

$$+ip_{1\mu}\tilde{\Gamma}^{\mu\nu\rho} = \frac{-i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}p_{1\mu}(p_2-\beta)_\sigma \quad i^3\,?$$

$$+ip_{2\nu}\tilde{\Gamma}^{\mu\nu\rho} = \frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}p_{2\nu}(p_1+\beta)_\sigma \quad i^3\,? \tag{22}$$

which also depend on β too! (Exercises)

For a free Dirac fermion, we really don't care much about the violations of these Ward identities. However, if one wants to couple to background gauge field, one would insist to have Ward identities satisfied by the vector current.

Clearly, with the choice of $\beta = p_2 - p_1$, we have the vector Ward identities (cancelling the factor i on both sides of (21) & (22))

$$+\, p_{1\mu}\, \widetilde{\Gamma}^{\mu\nu\rho} \overset{\alpha}{=} +\frac{1}{8\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\, p_{1\mu} p_{1\sigma} = 0$$

$$+\, p_{2\nu}\, \widetilde{\Gamma}^{\mu\nu\rho} \overset{\alpha}{=} +\frac{1}{8\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\, p_{2\nu} p_{2\sigma} = 0 \qquad (23)$$

while for the axial Ward identity

$$+\, q_\rho\, \widetilde{\Gamma}^{\mu\nu\rho} = -\frac{(-i)^3?\,1}{8\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\left(2 p_{1\rho} p_{2\sigma} + (p_1+p_2)_\rho (p_2-p_1)_\sigma\right)$$

$$= -\frac{(-1)^3?\,1}{2\pi^2}\, \epsilon^{\mu\nu\rho\sigma}\, p_{1\rho} p_{2\sigma} \qquad (24).$$

$(-i)\frac{1}{2\pi^2}\epsilon^{\mu\nu}(p, q)$

which is anomalous.

Eq.(24) is the anomaly for a free Dirac fermion in the momentum space.

In coordinate space, (24) becomes, when couples to $U(1)$ gauge field,

$$\partial_\mu J^\mu_A = -\frac{e^2}{16\pi^2}\, \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \qquad (25).$$

To check (25), let's couple the vector current to a $U(1)$ gauge field A_μ. The action now reads

$$S = \int d^4x\, i(\bar\psi \not{D} \psi) \quad , \quad \not{D} = \gamma^\mu D_\mu,\ D_\mu = \partial_\mu - ieQA_\mu$$

(We will set $Q = -1$ in below)

$$(26) \qquad = \int d^4x\,(i\bar\psi \not{D}\psi - eJ^\mu_V A_\mu)$$

with $J^\mu_V = \bar\psi \gamma^\mu \psi$.

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From (1) & (2)', we have

$$(-i)\, q_\rho \tilde{\Gamma}^{\mu\nu\rho} \frac{\delta^4(p_1+p_2+q)}{(2\pi)^4} = \int d^4x_1\, d^4x_2\, d^4x_3 \langle 0 | T[J_V^\mu(x_1) J_V^\nu(x_2) \partial_\rho J_A^\rho(x_3)] | 0 \rangle$$
$$\times \exp[i p_1\cdot x_1 + i p_2 \cdot x_2 + i q \cdot x_3]$$

Plugging in (25) for $\partial_\rho J_A^\rho$ we obtain

$$(-i)\, q_\rho \tilde{\Gamma}^{\mu\nu\rho} \frac{\delta^4(p_1+p_2+q)}{(2\pi)^4} = \int \langle 0 | T[J_V^\mu J_V^\nu \frac{(-)e^2}{16\pi^2} \epsilon^{\rho\sigma\lambda\tau}\, 4\, \partial_\rho A_\sigma \partial_\lambda A_\tau] | 0 \rangle \cdot e^{i\cdots}$$
$$= \frac{(-)e^2}{4\pi^2} \epsilon^{\rho\sigma\lambda\tau} \int \langle 0 | T[J_V^\mu J_V^\nu \partial_\rho A_\sigma \partial_\lambda A_\tau] | 0 \rangle e^{i\cdots}$$

Now

$$T[J_V^\mu J_V^\nu \partial_\rho A_\sigma \partial_\lambda A_\tau]$$
$$= \underbrace{J_V^\mu \partial_\rho A_\sigma}\, \underbrace{J_V^\nu \partial_\lambda A_\tau} + \underbrace{J_V^\mu \partial_\lambda A_\tau}\, \underbrace{J_V^\nu \partial_\rho A_\sigma} + \text{normal ordering terms}$$

$$\Rightarrow \langle 0 | T[J_V^\mu J_V^\nu \partial_\rho A_\sigma \partial_\lambda A_\tau] | 0 \rangle$$
$$= \langle 0 | \underbrace{J_V^\mu \partial_\rho A_\sigma} | 0 \rangle \langle 0 | \underbrace{J_V^\nu \partial_\lambda A_\tau} | 0 \rangle + \langle 0 | \underbrace{J_V^\mu \partial_\lambda A_\tau} | 0 \rangle \langle 0 | \underbrace{J_V^\nu \partial_\rho A_\sigma} | 0 \rangle$$

From (26), one can obtain the 2-pt. function of the vector current & gauge field.

$$\langle 0 | (+\, \underbrace{J_V^\mu(x_1)\, A_\sigma(x_3)}) | 0 \rangle = -i \delta^\mu_{\ \sigma}\, \delta^4(x_1 - x_3) \frac{1}{e}$$

$$\Rightarrow \langle 0 | (+\, \underbrace{J_V^\mu(x_1)\, \partial_\rho A_\sigma(x_3)}) | 0 \rangle = -i \delta^\mu_{\ \sigma}\, \partial_\rho^{x_3} \delta^4(x_1 - x_3) \frac{1}{e} \rightarrow +\frac{i}{e} \delta^\mu_{\ \sigma} \partial_\rho^{x_1} \delta^4(x_1 - x_3)$$

etc..

$$\Rightarrow \int \langle 0 | T[e J_V^\mu e J_V^\nu \partial_\rho A_\sigma \partial_\lambda A_\tau] | 0 \rangle e^{i\cdots} \rightarrow \int d^4x_1 d^4x_2 d^4x_3 \cdot \{ i^4 \delta^\mu_{\ \sigma} \delta^\nu_{\ \tau} p_{1\rho} p_{2\lambda}\, \delta^4(x_1 - x_3) \delta^4(x_2 - x_3)$$
$$+ i^4 \delta^\mu_{\ \tau} \delta^\nu_{\ \sigma} p_{1\lambda} p_{2\rho} \delta^4(x_1 - x_3) \delta^4(x_2 - x_3) \} \cdot e^{i\cdots}$$

$$\longrightarrow \delta^4(p_1+p_2+q) [\delta^\mu_{\ \sigma} \delta^\nu_{\ \tau} p_{1\rho} p_{2\lambda} + \delta^\mu_{\ \tau} \delta^\nu_{\ \sigma} p_{1\lambda} p_{2\rho}]$$

$$\Rightarrow \epsilon^{\rho\sigma\lambda\tau} \int \langle 0 | T[\cdots] | 0 \rangle e^{i\cdots} \longrightarrow \delta^4(p_1+p_2+q) \left(\epsilon^{\rho\mu\lambda\nu} p_{1\rho} p_{2\lambda} + \epsilon^{\rho\nu\lambda\mu} p_{1\lambda} p_{2\rho} \right)$$

$$\rightarrow \delta^4(p_1+p_2+q)\,\epsilon^{\mu\nu\rho\lambda}(-p_{1\rho}p_{2\lambda}+p_{1\lambda}p_{2\rho})$$

$$\rightarrow -2\,\delta^4(p_1+p_2+q)\,\epsilon^{\mu\nu\rho\lambda}\,p_{1\rho}p_{2\lambda} \,.$$

$$\Longrightarrow \quad (-i)\; q_\rho \tilde{\Gamma}^{\mu\nu\rho}\,\delta^4(p_1+p_2+q) = \frac{(-)1}{4\pi^2}(-2)\,\delta^4(p_1+p_2+q)\,\epsilon^{\mu\nu\rho\lambda}\,p_{1\rho}p_{2\lambda}$$

i.e.

$$(-i)\quad q_\rho \tilde{\Gamma}^{\mu\nu\rho} = -\frac{1}{2\pi^2}\,\epsilon^{\mu\nu\sigma\lambda}\,p_{1\sigma}p_{2\lambda} \times(-1)$$

which is the same as (24). This establishes the two anomalous Ward identities for the axial current in (24) & (25) are 'equivalent'.

If one wants to promote the background gauge field to be dynamical, one may wonder if there are higher corrections to the above one-loop anomalous Ward identity. It turns out that there are none! The one-loop result is exact, according to the deep theorem by Adler & Bardeen.

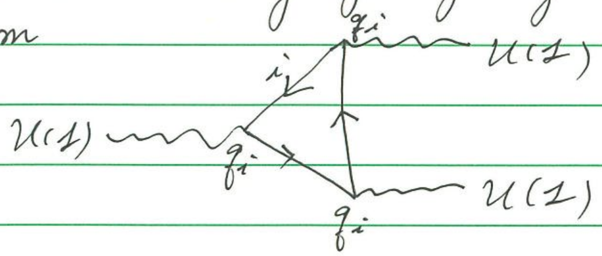
The above derivation of the anomaly has nothing to do with the mass. Thus adding a mass to the free Dirac fermion doesn't affect the anomaly. Since a Dirac mass term breaks the axial $U(1)$ symmetry explicitly, the anomalous Ward identity for the axial current is now

$$\partial_\mu J^\mu_A = +2im\bar{\psi}\gamma^5\psi - \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \qquad (27).$$

with $J^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$.

Abelian Gauge Anomaly

Consider a bunch of Weyl fermions $\{\psi_{Li}\}$ each carries charge q_i under a $U(1)$ gauge sym. group. The following triangle diagram



+ crossing.

is proportional to

$$\sum_i q_i^3 \qquad (1)$$

Alternately, if one keep the theory written in terms of left-handed & right-handed Weyl fermions, then (1) can be rewritten as

$$\sum_{left} q_i^3 - \sum_{right} q_i^3 \qquad (2)$$

This is because for a right-handed fermion of charge q, one can always take complex conjugate to make it into a left-handed fermion with charge $-q$. Thus (1) and (2) are equivalent.

Now for chiral massless fermions, there's obstruction for them to couple to gauge fields, here the abelian $U(1)$ gauge fields. To have a consistent QFT, one must require the following anomaly cancellation condition for $U(1)$:

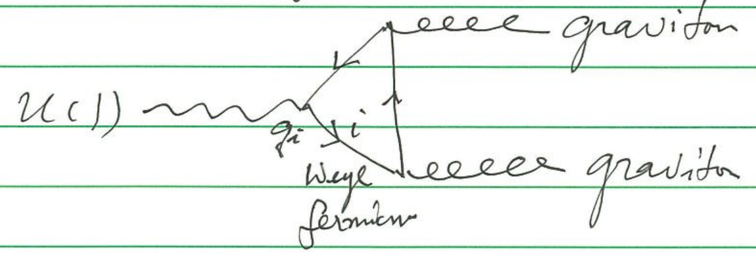
$$\sum_i q_i^3 = 0 \quad \text{or} \quad \sum_{left} q_i^3 = \sum_{right} q_i^3 \qquad (3).$$

* Simple solution for (3) is to have the following setup: For each left-handed Weyl fermion with charge q, we pair with a right-handed Weyl fermion with charge q as well. Those are vector-like theories, like QED & QCD. There are no $[U(1)]^3$ anomaly in vector-like theories.

* In vector-like theories, the left & right-handed Weyl-fermions paired up to form Dirac fermions, as in QED & QCD.
One would end up in chiral gauge theories if the left & right-handed spinors couldn't pair up 'nicely' to form Dirac fermions. Such theories necessarily break parity, like electroweak SM, which has a hypercharge charge $U(1)_Y$. To cancel $[U(1)_Y]^3$ anomaly, we must require

$$\sum_i Y^3[i] = 0 \quad \text{or} \sum_{\text{left}} Y^3 = \sum_{\text{right}} Y^3 . \qquad (4).$$

* The following triangle diagram



+ crossing

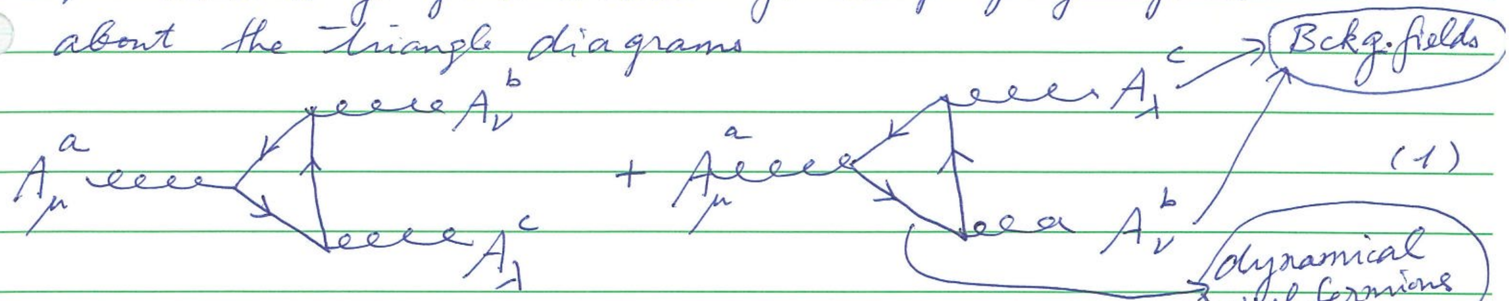
also give rise to the so-called gravitational anomaly. To have a consistent QFT with Weyl fermions on a curved space-time, we must also require

$$\sum_i q_i = 0 \quad \text{or} \quad \sum_{\text{left}} q_i = \sum_{\text{right}} q_i$$

Again vector-like theories fullfill this condition trivially. More interesting case is the chiral gauge theories which can lead to non-trivial constraints on the particle content of the models.

Non-Abelian Gauge Anomalies

Non-abelian gauge anomalies for a gauge group G concerns about the triangle diagrams



(1)

with some Weyl fermions, running inside the loops, are sitting in a representation R, with generators T^a of G.

The group theoretic factor of these 2 diagrams is then given by the so-called 'd-symbol(?)':

$$d^{abc}(R) = \mathrm{Tr}_R\left[T^a\{T^b, T^c\}\right] \qquad (2)$$

* Note that left-handed & right-handed Weyl fermions contributed with opposite sign to the $d^{abc}(R)$.

* Thus anomaly cancellation requires

$$\sum_{\substack{\text{left handed}\\ \text{fermion } i=1}}^{N_L} d^{abc}(R_{Li}) = \sum_{\substack{\text{Right handed}\\ \text{fermion } j=1}}^{N_R} d^{abc}(R_{Rj}) \qquad (3)$$

for N_L left-handed Weyl fermions in R_{Li} $(i=1,\dots,N_L)$ &
N_R right-handed Weyl fermions in R_{Rj} $(j=1,\cdots,N_R)$

* How do one solve (3) for G?

(A) Have equal # of left- and right-handed fermions transforming in the same representations of the gauge group G. This is Vector-like theory. For example, QCD $SU(3)_c$ in SM: $q_L \in 3$, $q_R \in 3$ of $SU(3)$.

Vector-like theory allows us to have Dirac mass term like $\sum_q \bar{q}_L q_R + \bar{q}_R q_L$!

(B) Anomaly vanishes for any representation that is real (e.g. adjoint rep.) or pseudoreal (e.g. 2, the fundamental rep. of SU(2).).

pseudoreal: $\bar{T}^a$ conjugate rep. is related to T^a via a similarity transf. by a unitary matrix U

(4) $$\bar{T}^a = U T^a U^{-1}, \quad U U^\dagger = U^\dagger U = 1.$$

Let $g \in G$. then $g = \exp(i\alpha^a T^a)$ with real α^a.

In the conjugate rep, g, the same group element, is given by $g = \exp(-i\alpha^a T^{*a})$. Thus $\boxed{\bar{T}^a = -T^{a*}}$ (5)

If one take T^a to be hermitian, i.e. $T^a = T^{a\dagger} = (T^{a*})^T$ we would have

$$\bar{T}^a = -T^{a*} = -(T^a)^T. \qquad (5')$$

For real or pseudoreal rep. R, we have

$$\mathrm{Tr}\, T^a\{T^b, T^c\} \underset{(4)}{=} \mathrm{Tr}\, \bar{T}^a\{\bar{T}^b, \bar{T}^c\} \underset{(5)'}{=} (-)^3\, \mathrm{Tr}\,(T^a)^T\{(T^b)^T, (T^c)^T\}$$

$$= -\mathrm{Tr}\left((T^a)^T(T^b)^T(T^c)^T + (T^a)^T(T^c)^T(T^b)^T\right)$$

$$= -\mathrm{Tr}\left(T^c T^b T^a + T^b T^c T^a\right) \qquad (6)$$

$$= -\mathrm{Tr}\left(\{T^b, T^c\} T^a\right) = -\mathrm{Tr}\left(T^a\{T^b, T^c\}\right)$$

Thus for real or pseudoreal rep. $\mathrm{Tr}\, T^a\{T^b, T^c\} = 0$. (6)

Example:

2 of SU(2) is associated with $\{\frac{\sigma^i}{2}\}$, $\bar{2}$ is associated with $\{-\frac{\sigma^{i*}}{2}\}$

Since $\epsilon\left(-\frac{\sigma^{i*}}{2}\right)\epsilon^{-1} = \frac{\sigma^i}{2}$ with $\epsilon = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\bar{2} \cong 2$!

Note that $\epsilon^2 = -1 \Rightarrow \epsilon^{-1} = -\epsilon$. $\epsilon^\dagger = -\epsilon$, so $\epsilon^{-1} = \epsilon^\dagger$, ϵ is unitary!

* In fact, all reps. of SU(2) are either real or pseudo-real!

For fermions in real representation, one can always write down a Majorana mass term. Thus, such fermions do not suffer from gauge anomalies. Gauge anomalies concern about massless fermions.

$\Rightarrow$ Only Lie groups with complex representations are needed to worry about.

For simply laced groups (no $U(1)$ factor), these are

$$G = \begin{cases} SU(N) & \text{with } N \geq 3 \\ SO(4N+2) & \\ E_6 & \end{cases} \tag{7}$$

plus $U(1)$ which is a special case of interest.

Question: $Sp(2N)$, G_2, F_4, E_7, E_8 have no complex representation?

Furthermore, Group theory tells us that

(8). $$d^{abc}(R) = 0 \quad \text{for } E_6 \text{ \& } SO(4N+2) \text{ with } N \geq 2.$$

We left with $SU(N)$ with $N \geq 3$ and $SO(6)$. However, $SO(6) \simeq SU(4)$. We only need to worry about $SU(N)$ with $N \geq 3$ for gauge anomaly cancellation.

For $SU(N)$ with $N \geq 3$, one can show that

$$d^{abc}(R) = A(R)\, d^{abc}(N) \tag{9}$$

with N is the fundamental rep. of $SU(N)$.

$A(R)$ is called the anomaly coefficient of $SU(N)$ for rep. R.

Anomaly Cancellation Condition:

$$\sum_{\substack{\text{left-handed}\\ \text{fermions}}} A(R) = \sum_{\substack{\text{right-handed}\\ \text{fermion}}} A(R) \tag{10}$$

Anomaly arithmetic:

(1) $A(R_1 \oplus R_2) = A(R_1) + A(R_2)$

(2) $R_1 \otimes R_2$ is generated by $\left(1_1 \otimes T_2^a \oplus T_1^a \otimes 1_2\right)$

(11). Thus

$$A(R_1 \otimes R_2) = \dim(R_1) \cdot A(R_2) + \dim(R_2) A(R_1)$$

(3) $A(\bar{R}) = -A(R) \Leftarrow \boxed{\begin{array}{l}\text{Since } \mathrm{Tr}\, \bar{T}^a\{\bar{T}^b, \bar{T}^c\} \\ \quad = -\mathrm{Tr}[T^a\{T^b, T^c\}] \\ \qquad \text{from (6).}\end{array}}$

Example: $SU(3)$. By definition, $A(3) = -A(\bar{3}) = 1$.
(4)

$3 \otimes 3 = \bar{3} \oplus 6$

$$\Rightarrow A(6) = A(3\otimes 3) - A(\bar{3})$$
$$= 3A(3) + 3A(3) + A(3) = 7A(3) = 7.$$

$3 \otimes \bar{3} = 1 \oplus 8$

$$\Rightarrow A(8) = 3A(3) + 3A(\bar{3}) - A(1)$$
$$= 3A(3) - 3A(3) - 0 = 0$$ as expected for the adj. 8. which is real.

* For adj. rep., recall that

$$(T^a)_{bc} = -i f^{abc}, \quad f^{abc} \text{ real}$$

Thus, $(\bar{T}^a)_{bc} \underset{(5)}{=} -(T^a)^*_{bc} = -(+if^{abc}) = -if^{abc}$

$$= (T^a)_{bc}.$$

From the derivation of (6), we have $\mathrm{Tr}\, \bar{T}^a\{\bar{T}^b, \bar{T}^c\} = -\mathrm{Tr}\, T^a\{T^b, T^c\}$.
Therefore, $d^{abc}(\text{adj}) = 0$ for any group G.

* Deep theorem: Adler-Bardeen theorem concerns dynamical gauge fields

$+ \cdots \Rightarrow$ No higher-order corrections to gauge anomalies.

Topological origin! Index theorems.

June 8, 2024

Anomaly Cancellation in SM.

SM is a chiral gauge theory and therefore it can suffer from quantum anomalies. These anomalies must be cancelled to have a consistent QFT.

$$G_{SM} : SU(3)_C \times SU(2)_L \times U(1)_Y .$$

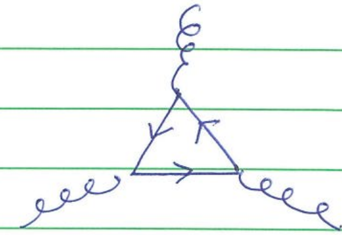
The particle content of SM fermions for a single generation is:

	SU(3)	SU(2)	U(1)
Q_L	3	2	$1/6$
E_L (L_L)	1	2	$-1/2$
u_R	3	1	$2/3$
d_R	3	1	$-1/3$
e_R	1	1	-1
BSM: ν_R	1	1	0
For the Higgs scalar, H	1	2	$1/2$

* The hypercharge assignements are just convention. One could multiply every representation by 6 to make all hypercharges integral, i.e. $Y(Q_L) = 1, \cdots, Y(e_R) = -6$. The above fractional assignements will reproduce our familiar convention for electric charges: $Q(e) = -1$, $Q(u) = 2/3$, $Q(d) = -1/3$, $Q(\nu) = 0$, as one can verify in other lectures.

* See Tong's lectures on 'Gauge Theory' for an alternative hypercharge assignements for the SM.

$[SU(3)]^3$ Anomaly cancellation requires



$$\sum_{\substack{\text{left-handed}\\ \text{quarks}}} A(R) = \sum_{\substack{\text{Right-handed}\\ \text{quarks}}} A(R)$$

Since $A(3) = 1$, we compute for Q_L

$$2 \times A(3) = 2.$$

$\uparrow$ has 2 comp. for

while for u_R & d_R,

$$1 \cdot A(3) + 1 \cdot A(3) = 2$$

$\Rightarrow$ $\sum A(R)$ same for left-handed and right-handed quarks.

$[SU(2)]^3$: $A(R) = 0$ for $SU(2)$.

$SU(3)$ & $SU(2)$ mixed cases:

There's no mixed anomaly between $SU(2)$ & $SU(3)$ because $Tr[T^A] = 0$ for both $SU(2)$ & $SU(3)$.

(Actually for any simple Lie groups.)

So we left with anomalies involve the $U(1)$ factor:

$[U(1)]^3$, $U(1)[SU(2)]^2$, $U(1)[SU(3)]^2$, $[U(1)]^2 SU(2)$, $[U(1)]^2 SU(3)$ & $U(1)SU(2)SU(3)$. The last 3 cases vanish because $Tr[T^A] = 0$ as before for $SU(2)$ & $SU(3)$.

$[U(1)]^3$:

$$\sum_{\substack{\text{left-handed}\\ \text{quarks \& leptons}}} Y^3 = \underbrace{3 \cdot 2 \cdot \left(\tfrac{1}{6}\right)^3}_{Q_L} + \underbrace{2 \cdot \left(-\tfrac{1}{2}\right)^3}_{E_L} = \frac{1}{36} - \frac{1}{4} = -\frac{2}{9}$$

$$\sum_{\substack{\text{Right-handed}\\ \text{quarks \& leptons}}} Y^3 = \underbrace{3 \cdot \left(\tfrac{2}{3}\right)^3}_{u_R} + \underbrace{3\left(-\tfrac{1}{3}\right)^3}_{d_R} + \underbrace{(-1)^3}_{e_R} + \boxed{\underbrace{0^3}_{\nu_R}} = \frac{8}{9} - \frac{1}{9} - 1 = -\frac{2}{9}$$

of $[SU(2)]^2$ & $U(1)$

$[SU(2)]^2 \cdot U(1)$: The anomaly factored into 2 pieces for Q_L & E_L only)

$$\sum_{\substack{\text{left-handed}\\ \text{quarks \& leptons}}} Y = \underset{\substack{\uparrow\\ \text{color factor}}}{3} \cdot \underbrace{\left(\tfrac{1}{6}\right)}_{Q_L} + \underbrace{\left(-\tfrac{1}{2}\right)}_{E_L} = 0 \;!$$

$\boxed{\text{*The } SU(2)^2 \text{ part is trivial: } \mathrm{Tr}\{T^A, T^B\} \propto \delta^{AB}}$

For the right-handed fermions, there's no anomalies by definition because they are $SU(2)$ singlets!

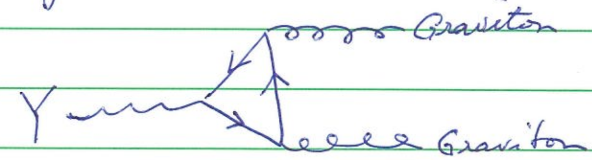
$[SU(3)]^2 \cdot U(1)$: Only Q_L, u_R & d_R are needed to be considered. Again the anomaly factored into $SU(3)$ piece & $U(1)$ piece:

$$\sum_{\substack{\text{left-handed}\\ \text{quarks}}} Y = \underset{\substack{\uparrow\\ \text{from } SU(2)}}{2} \cdot \underbrace{\tfrac{1}{6}}_{Q_L} = \frac{1}{3}$$

$$\sum_{\substack{\text{right-handed}\\ \text{quarks}}} Y = 1 \cdot \underbrace{\left(\tfrac{2}{3}\right)}_{u_R} + 1 \cdot \underbrace{\left(-\tfrac{1}{3}\right)}_{d_R} = \frac{1}{3}$$

Finally, to couple SM consistent with gravity, we have to require

$$\sum_{\substack{\text{All left-handed}\\ \text{fermions}}} Y = \sum_{\substack{\text{All right-handed}\\ \text{fermions}}} Y$$



$$\sum_{\text{Left}} Y = 3 \cdot 2 \cdot \underbrace{\left(\tfrac{1}{6}\right)}_{Q_L} + 2 \cdot \underbrace{\left(-\tfrac{1}{2}\right)}_{E_L} = 0$$

$$\sum_{\text{Right}} Y = 3 \cdot \underbrace{\left(\tfrac{2}{3}\right)}_{u_R} + 3 \cdot \underbrace{\left(-\tfrac{1}{3}\right)}_{d_R} - \underbrace{1}_{e_R} = 0.$$

* Witten 'Global' $SU(2)$ (non-perturbative) Anomaly requires even number of Weyl fermion doublets for cancellation. For one generation, SM has 3 colored Q_L doublets & 1 E_L doublet, a total of 4 $SU(2)$ doublets.

* Thus, SM is anomaly free for one (each) generation.