Symmetries

Standard Model (SM) is constructed using principles of invariance, i.e. symmetries.

Two kinds of symmetries:

(i) External Symmetries

(ii) Internal Symmetries

The symmetries can be discrete or continuous, global or local.

External symmetries — space-time symmetries

e.g. Poincaré symmetries : Space-time translation (P_μ)
$\oplus$ Lorentz translation ($M_{\mu\nu} = -M_{\nu\mu}$)

These are continuous symmetries; basic ingredient of QFT, in particular SM.

e.g. Discrete symmetries : parity ($\vec{x} \to -\vec{x}$), P
time reversal ($t \to -t$) T
Charge Conjugation : particle $\leftrightarrow$ antiparticle C

Internal symmetries — non-space-time.

e.g. Isospin ($p \leftrightarrow n$) $\begin{pmatrix} p \\ n \end{pmatrix} \in$ doublet of $SU(2)$

phase inv. in quantum electrodynamics (QED)
gauge sym. in SM:
$$G : \underbrace{SU(3)_{\text{Color}} \otimes SU(2)_L \otimes U(1)_Y}_{} \quad \text{local sym.}$$
Y: hypercharge

e.g. Z_2, Z_N in general, A_4, etc. are often used in flavor physics.

Spacetime Symmetries

In particle physics, spacetime is Minkowski space $R^{1,3}$, equipped with a flat metric

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \quad \text{: Mostly minus convention.} \tag{1}$$

* Gravity community prefers mostly plus convention. One might get used to both conventions as you grow older (& stronger).

Lorentz transf. Λ makes the Minkowski metric invariant, i.e. $x \to \Lambda x$, $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, $x_\mu \to \Lambda_\mu{}^\nu x_\nu$

$$\Lambda^T \cdot \eta \cdot \Lambda = \eta, \quad \eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta} \quad \text{or} \quad \eta^{\mu\nu} = \Lambda^\mu{}_\alpha \eta^{\alpha\beta} \Lambda^\nu{}_\beta \tag{2}$$

Besides Lorentz boosts & rotations, these Λs include discrete parity P & time reversal T :

$$\Lambda_{parity} = diag(1, -1, -1, -1)$$
$$\Lambda_{Time\text{-}Reversal} = diag(-1, +1, +1, +1) \tag{3}$$

Lorentz inv. scalar product:

$$x^\mu = (x^0, \vec{x}), \quad y^\nu = (y^0, \vec{y})$$
$$x \cdot y = x^\mu y_\mu = x^0 y^0 - \vec{x} \cdot \vec{y} = x^\mu \eta_{\mu\nu} y^\nu = x^T \cdot \eta \cdot y \tag{4}$$

Under Λ, $x \to \Lambda x$, $y \to \Lambda y$,
but $x \cdot y \to x^T \Lambda^T \cdot \eta \cdot \Lambda y = x^T \eta y = x \cdot y$.

* Note

$$x_\mu = \eta_{\mu\nu} x^\nu = (x_0, -\vec{x}) \tag{5}$$

e.g. $p \cdot x = p_\mu x^\mu = x^0 E - \vec{p} \cdot \vec{x}$.

(2) means

$$\eta_{\mu\nu} = (\Lambda^T)_\mu{}^\alpha \eta_{\alpha\beta} \Lambda^\beta{}_\nu = \eta_{\alpha\beta} \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu$$

$$\Rightarrow \eta_{00} = \eta_{\alpha\beta} \Lambda^\alpha{}_0 \Lambda^\beta{}_0 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2 = 1$$

(Einstein's convention!)

$$\Rightarrow \eta^{00} = \eta^{\alpha\beta} \Lambda^0{}_\alpha \Lambda^0{}_\beta = (\Lambda^0{}_0)^2 - (\Lambda^0{}_i)^2 = 1$$

i.e. $(\Lambda^0{}_0)^2 = 1 + (\Lambda^0{}_i)^2 = 1 + (\Lambda^i{}_0)^2$

$\Rightarrow |\Lambda^0{}_0| \geq 1$ since $(\Lambda^i{}_0)^2$ & $(\Lambda^0{}_i)^2$ are positive definite. (6)

Taking the determinant of (2) implies

$$|\mathrm{Det}\,\Lambda| = 1 \qquad (7)$$

All Λs satisfying (2) form the Lorentz group. (6) & (7) imply there are 4-disconnected components of the Lorentz group.

Det Λ	Sgn $\Lambda^0{}_0$	Component / Remarks
+ 1	+ 1	$L_+^\uparrow$ contains unity $\mathbb{I}$
− 1	+ 1	$L_-^\uparrow$ contains space inversion
+ 1	− 1	$L_+^\downarrow$ contains spacetime inversion
− 1	− 1	$L_-^\downarrow$ contains time reversal (inversion)

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Subgroups

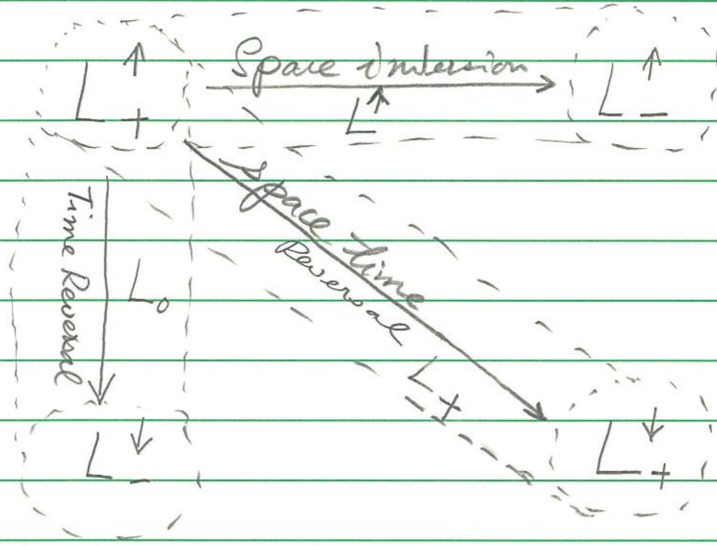
Jargons:

$\mathrm{Det}\,\Lambda = 1$ proper Lorentz group L_+

$\mathrm{Sgn}\,\Lambda^0_0 = 1$ orthochronous Lorentz group $L^\uparrow$

$\mathrm{Det}\,\Lambda \cdot \mathrm{Sgn}\,\Lambda^0_0 = 1$ Orthochorons Lorentz group : L_0

These are the three important subgroups of the Lorentz group, as indicated in the following diagram.



(8)

$L^\uparrow_+$ is the restricted Lorentz group.
One can show that

$$L^\uparrow_+ \approx SL(2,\mathbb{C})/Z_2 \quad , \tag{9}$$

where $SL(2,\mathbb{C})$ is formed by all <u>unimodular</u> 2×2 complex matrices. S stands for special, i.e. $\det = 1$, L stands for linear, 2×2, $\mathbb{C}$ stands for complex. It has $(4-1) = 3$ complex parameters, same as Lorentz group, which has 3 rotations & 3 boost parameters.

$Z_2 = \{ I_{2\times 2}, -I_{2\times 2} \}$ This Z_2 relates to spinors pick up minus sign under 2π rotation.

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In the literature, the Lorentz group is denoted by $SO(1,3)$ as well.

$$SO(3) \longrightarrow SO(4) \longrightarrow SO(1,3) \longrightarrow SO(1,d)$$

3-dim Euclidean	4-dim Euclidean	4-dim Minkowski	$d+1$ space-time

* $SO(1,3)$ doesn't have spinor representation.

$$SO(1,3) \cong Spin(1,3)/\mathbb{Z}_2 \quad \text{(locally isomorphic)} \qquad (10)$$

$Spin(1,3)$ has spinor representation.

* Mapping to previous language, we have

$$\left.\begin{array}{l} L_+^\uparrow \Longrightarrow SO(1,3) \\ \text{with } SL(2,\mathbb{C}) \cong Spin(1,3) \end{array}\right\} \qquad (11)$$

* $\cong$: locally isomorphic in group. share same algebra.

Representation of Lorentz transformation Λ:

It depends on the object that Λ acts upon.

We know under Λ, a 4-vector x^μ transforms as

$$x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu$$

How do we find the diff. representations of Λ? $\to$ Look at infinitesimal transf. of the Lorentz group & study its Lie algebra.

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \theta^\mu{}_\nu \qquad |\theta^\mu{}_\nu| \ll 1.$$

Invariance of η, i.e. $\eta^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta^{\rho\sigma}$ implies

$$\eta^{\mu\nu} = (\delta^\mu{}_\rho + \theta^\mu{}_\rho)(\delta^\nu{}_\sigma + \theta^\nu{}_\sigma)\eta^{\rho\sigma}$$

$$= \eta^{\mu\nu} + (\theta^{\mu\nu} + \theta^{\nu\mu}) + O(\theta^2)$$

i.e. $\boxed{\theta^{\mu\nu} = -\theta^{\nu\mu}}$ anti-sym. in (μ, ν). (12)

In 4-dim, an anti-sym. matrix has $\frac{4\times 3}{2} = 6$ indep. comps, which agrees with the 6 diff. Lorentz transfs: 3 rotations & 3 boosts. One can introduce a basis of these 6 4×4 anti-symmetric matrices

$$(M^A)^{\mu\nu} \qquad A = 1, 2, \cdots, 6 \quad (3 \text{ rots} + 3 \text{ boost})$$

or maps $A \to [\rho\sigma]$: $(M^{\rho\sigma})^{\mu\nu}$ with $(M^{\rho\sigma})^{\mu\nu} = -(M^{\sigma\rho})^{\mu\nu}$

i.e. $(M^{01})^{\mu\nu} = -(M^{10})^{\mu\nu}, \ldots, (M^{12})^{\mu\nu} = -(M^{21})^{\nu\mu}, \ldots$

with M^{0i} $(i = 1, 2, 3)$ corresponds to Lorentz boosts in $\hat{x}^i$ & M^{ij} corresponds to rotation in the 3-diff. orthogonal axis of the (ij) plane.

$$(M^{\rho\sigma})^{\mu\nu} = -(M^{\rho\sigma})^{\nu\mu} \text{ because of } \theta^{\mu\nu} = -\theta^{\nu\mu}.$$

Now, one can expand $\theta^\mu{}_\nu$ in terms of these basis, i.e.

$$\theta^\mu{}_\nu \equiv -\frac{i}{2}\omega_{\rho\sigma}(M^{\rho\sigma})^\mu{}_\nu, \quad \omega_{\rho\sigma} = -\omega_{\sigma\rho} \quad 6 \text{ indep. real parameters.} \qquad (13)$$

A basis of the above 6 4×4 anti-sym matrices can be written down immediately

$$(M^{\mu\nu})^{\rho\sigma} = i(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\nu\rho}\eta^{\mu\sigma})$$

And for $\Lambda^\mu{}_\nu$ we need

$$(M^{\mu\nu})^\rho{}_\sigma = \eta_{\sigma\sigma'}(M^{\mu\nu})^{\rho\sigma'}$$

$$= i(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma)$$

Note that $(M^{\mu\nu})^\rho{}_\sigma \neq -(M^{\mu\nu})^\sigma{}_\rho$

i.e. $(M^{\mu\nu})^\rho{}_\sigma$ is no longer anti-sym. in $(\rho\,\sigma)$.

One can prove <u>in general</u> $\{M^{\mu\nu}\}$ satisfies the following Lorentz algebra:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i[\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho}] \quad (14)$$

$\{M^{\mu\nu}\}$: generators of $SO(1,3)$ (or $Spin(1,3)$ which shares the same algebra)

<u>Exercise</u>: Show that $(M^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu{}_\sigma - \eta^{\nu\rho}\delta^\mu{}_\sigma)$ satisfies the Lorentz algebra of $SO(1,3)$. ✓

* We can promote $\{M^{\mu\nu}\}$ that satisfy the Lorentz algebra as abstract objects and its representation depends on what objects (x^μ, ϕ, A_μ, ψ_α, …) it acts upon. For 4-vector x^μ we know

<u>Exercise</u>: $(M^{01})^\mu{}_\nu = i\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \\ 0 & & 0 \end{pmatrix} \xleftrightarrow[\omega_{01}=-\omega_{10}=\beta]{} \Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\beta & \pm\sinh\beta & 0 \\ \pm\sinh\beta & \cosh\beta & \\ 0 & & 1 \end{pmatrix}$ ✓ boost along x-axis

$$\boxed{\begin{aligned} \cosh\beta &= (1-\beta^2)^{-1/2} \\ \sinh\beta &= \beta(1-\beta^2)^{-1/2} \\ \beta &\equiv v/c = v \\ &= \eta \text{ (rapidity)} \end{aligned}}$$

&

$(M^{12})^\mu{}_\nu = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xleftrightarrow[\omega_{12}=-\omega_{21}=\theta]{} \Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ rotation about (x-y plane) ✓

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For scalar field $\phi(x)$, under Λ, $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$.
So, $M^{\mu\nu} = 0$!

For the vector potential A^μ in E&M, we learnt that
$A^\mu \to A'^\mu = \Lambda^\mu{}_\nu A^\nu$, just like x^μ.

In general, we have a set of field $\{\phi^a\}$, under Lorentz transformation Λ,

$$\phi^a(x) \longrightarrow D[\Lambda]^a{}_b \phi^b(\Lambda^{-1}x) ,$$

$a, b \in$ general Lorentz indices including spinor

where $D[\Lambda]$ is a matrix forming a representation of the Lorentz group, i.e

$$\left.\begin{aligned} & D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2] \\ \& \quad & D[\Lambda^{-1}] = (D[\Lambda])^{-1} \\ \& \quad & D[\mathbb{1}] = \mathbb{I} \end{aligned}\right\}$$

Examples: Scalar ϕ : $\phi(x) \mapsto \phi'(x) = \phi(\Lambda^{-1}x)$. $M = 0$, $D[\Lambda] = 1$.

Dirac spinor ψ : $\psi(x) \to \psi'(x) = D[\Lambda]\psi(\Lambda^{-1}x)$, $M^{\mu\nu} = S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$,
$D[\Lambda] = \exp[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}]$

Vector field A^μ : $A^\mu(x) \to A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$, $(M^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)$
$D[\Lambda] = \Lambda$

Decompositions of 6 Lorentz transf. into 3 rotations J_i & 3 boosts K_i is done as follows:

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} , \quad K_i = M_{0i} \tag{15}$$

$\boxed{\text{Note } M_{ij} = \epsilon_{ijk}J_k}$

Note that
$$J_i^\dagger = J_i \quad \text{Hermitian} \quad \left(\underset{\text{is compact}}{SO(3)} \simeq \frac{SU(2)}{Z_2}\right)$$

while
$$K_i^\dagger = -K_i \quad \text{Anti-hermitian (due to non-compactness of Lorentz boosts)}$$

Finite Lorentz transformation:
Divide Θ into $\frac{\Theta}{N} \longrightarrow$
$$\Lambda = \lim_{N\to\infty}\left(1 + \frac{\Theta}{N}\right)^N = e^\Theta = \exp\left(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right) \tag{16}$$

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From the Lorentz algebra (15), one can derive

$$\left.\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk} J_k \quad &(17a)\\ [J_i, K_j] &= i\epsilon_{ijk} K_k \quad &(17b)\\ [K_i, K_j] &= -i\epsilon_{ijk} J_k \quad &(17c) \end{aligned}\right\}$$

$\Rightarrow$

(1) $\{J_i, K_j\}$ forms closed algebra.

(2) (17a) implies $\{J_i\}$ (rotations) forms a $SU(2)$ subalgebra. Recall that $SO(3) \cong SU(2)/Z_2$. $SO(3)$ & $SU(2)$ share same alg.

(3) (17b) tells us $\{K_i\}$ behave like a vector under rotations.

Define
$$\left.\begin{aligned} A_i &= \frac{1}{2}(J_i + iK_i) \\ B_i &= \frac{1}{2}(J_i - iK_i) \end{aligned}\right\} \quad (18)$$

Note that $A_i^\dagger = A_i$, $B_i^\dagger = B_i$ both A_i, B_i are hermitian. They satisfy

$$\left.\begin{aligned} [A_i, B_j] &= 0 \\ [A_i, A_j] &= i\epsilon_{ijk} A_k \\ [B_i, B_j] &= i\epsilon_{ijk} B_k \end{aligned}\right\} \quad (19)$$

$\Rightarrow$ $\{A_i\}$ & $\{B_i\}$ form <u>two</u> $SU(2)$ sub-algebras.

$\Rightarrow$ Lorentz algebra $SO(1,3) \cong 2SU(2)$ subalgebras.

Thus representations of the Lorentz algebra can be labeled by

$$(j_1, j_2) \quad \text{with } j_1, j_2 \in \frac{1}{2}\mathbb{Z} \quad (20).$$

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(j_1, j_2) with $j_1, j_2 \in \frac{1}{2}\mathbb{Z} = \{0, 1/2, 1, 3/2, 2, \ldots\}$

Dim. of representation $= (2j_1 + 1)(2j_2 + 1)$. (21)

Examples:

(22)
$(0, 0)$: Scalar ϕ complex in general
$(1/2, 0)$: left-handed Weyl spinor ψ_L
$(0, 1/2)$: right-handed Weyl spinor ψ_R
$(1/2, 1/2)$: vector / 1-form A_μ / $A_\mu dx^\mu$
$(1, 0)$: self-dual 2-form $\tilde{F}_{\mu\nu} = F_{\mu\nu}$
$(0, 1)$: anti-self-dual 2-form $\tilde{F}_{\mu\nu} = -F_{\mu\nu}$
⋮

$$\left\{\begin{aligned} *F_{\mu\nu} &\equiv \tilde{F}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \\ F &= \frac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu \end{aligned}\right\} \quad (23)$$

Physical spin j of a particle is specified by the quantum number under rotation $\vec{J} = \vec{A} + \vec{B}$ according to (18), i.e. $\boxed{j = j_1 + j_2}$ (24)

Spin-statistics Theorem:

$$j \in \mathbb{Z} \longrightarrow \text{bosons, canonically quantized using } [\,,\,]$$
$$j \in \mathbb{Z} + 1/2 \longrightarrow \text{fermions, quantized using } \{\,,\,\} \quad (25)$$

A puzzle: (26)

$$SU(2) \otimes SU(2) \cong \text{Spin}(4) \quad \text{with} \quad SO(4) \cong \text{Spin}(4)/\mathbb{Z}_2$$

We know that $SO(4)$ is compact. Nonetheless Lorentz group is non-compact !! Keep boosting your inertial frame, you got farther & farther away from where you started. Under rotations, you will return to original point either by 2π (bosons) or 4π (fermions), hence compact.

Resolution:
Lie algebra of $SO(1,3)$ is not two mutually commuting copies of real Lie algebra $SU(2)$.
Mathematically or rigorously, it should be

$$so(1,3) \cong su(2) \otimes su(2)^* \quad (27)$$

This implies

$$(j_1, j_2)^* = (j_2, j_1) \quad (28)$$

In particular, we have

$$(1/2, 0)^* = (0, 1/2) \quad (29)$$

Which means

$$\psi_L^* \sim \psi_R \quad (30)$$

$$\begin{pmatrix} \text{Left-handed} \\ \text{Weyl Spinor} \end{pmatrix}^* \sim \begin{matrix} \text{Right-handed} \\ \text{Weyl-Spinor} \end{matrix} \quad (31)$$

$$\omega_{\mu\nu} M^{\mu\nu} = 2\omega_{0i} M^{0i} + \omega_{ij} M^{ij}$$
$$= 2\omega_{0i} K^i + \omega_{ij} \epsilon^{ijk} J^k$$

Let $\omega_{0i} = \beta_i$ & $\epsilon^{ijk}\omega_{ij} = 2\theta^k$ (i.e. $\omega_{12} = \theta^3$, $\omega_{23} = \theta^1$, $\omega_{31} = \theta^2$)

$$\Rightarrow \omega_{\mu\nu} M^{\mu\nu} = 2\vec{\beta}\cdot\vec{K} + 2\vec{\theta}\cdot\vec{J}$$

$$\Rightarrow \Lambda = \exp\left(-\tfrac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) = \exp\left(-i\vec{\theta}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K}\right) \qquad (16)'$$

Rewrite $\vec{J}, \vec{K}$ in terms of $\vec{A}, \vec{B}$ of the 2 SU(2)'s generators: $\vec{J} = \vec{A} + \vec{B}$, $i\vec{K} = \vec{A} - \vec{B}$ (18)'

$\Rightarrow$

$$\Lambda = \exp\left(-i\vec{\theta}\cdot\vec{J} - i\vec{\beta}\cdot\vec{K}\right) =$$
$$= \exp\left(-i\vec{\theta}\cdot(\vec{A}+\vec{B}) - i\vec{\beta}\cdot(-i\vec{A} + i\vec{B})\right)$$
$$= \exp\left(-i(\vec{\theta} - i\vec{\beta})\cdot\vec{A} - i(\vec{\theta} + i\vec{\beta})\cdot\vec{B}\right) \qquad (16)''$$

Thus, for Lorentz boosts with $\vec{\theta} = 0$ & $\vec{\beta} \neq 0$, the generators $\vec{A}$ & $\vec{B}$ correspond to imaginary angle rotations. These transf. are thus non-unitary, in-fact anti-unitary, reflecting the fact that Lorentz group is non-compact!

$\Rightarrow$ There are <u>no</u> finite-dim. unitary representations of the Lorentz group.

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<u>Poincaré Symmetry</u> : Lorentz Transformations + Space-time translations
$(M_{\mu\nu})$ (P_μ)

The Lorentz algebra generalizes to

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\nu\sigma}M^{\mu\rho} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\mu\rho}M^{\nu\sigma})$$

$$\left.\begin{aligned} &[P^\mu, P^\nu] = 0 \\ &[M^{\mu\nu}, P^\sigma] = i(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma}) \end{aligned}\right\} (32)$$

Thus $\{P^\sigma, M^{\mu\nu}\}$ forms a closed algebra, called Poincaré algebra, corresponding the Poincaré group $ISO(1,3)$, or $ISO(1,d)$ in $(1+d)$ dim.

<u>Irreducible representation of Poincaré group $ISO(1,3)$:</u>
<u>2 Casimirs:</u>

$$\left.\begin{aligned} C_1 &= P_\mu P^\mu \\ C_2 &= W_\mu W^\mu \quad \text{with } W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu M_{\alpha\beta} \end{aligned}\right\} (33)$$

(Pauli-Lubański Vector)

It's clear that, due to the ϵ-tensor,

$$[W_\mu, P_\nu] = 0 \qquad (34)$$

With some labors, one can also show that (32) implies

$$[W_\mu, M_{\nu\rho}] = i(\eta_{\mu\nu}W_\rho - \eta_{\mu\rho}W_\nu) \qquad (35)$$

$$[W_\mu, W_\nu] = -i\,\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \qquad (36)$$

* quadratic in generators rather than linear!

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Irreducible representations of $ISO(1,3)$ is labeled by

$$(p_\mu, C_1, C_2) \tag{37}$$

which is infinite dimensional due to the continuous spectrum of p_μ.

Eigenvalue of $C_1 = P_\mu P^\mu = m^2 = \begin{cases} > 0 & \text{Massive} \\ = 0 & \text{Massless} \\ < 0 & \text{Tachyon} \end{cases}$ (38)

What is the eigenvalue of C_2? We have to consider $m^2=0$ & $m^2 \neq 0$ separately.

Massive Case: Pick the rest frame $p_0^\mu = (m, \vec{0})$, $m>0$
$(m^2>0)$ Then, from its definition $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}P_\nu M_{\alpha\beta}$, one has

$$\left.\begin{aligned} W^0 &= \frac{1}{2}\epsilon^{00ij}P_0 M_{ij} = 0 \\ W^i &= \frac{1}{2}\epsilon^{i0jk}P_0 M_{jk} = -\frac{1}{2}m\,\epsilon^{ijk}M_{jk} \\ &= -\,m J^i \end{aligned}\right\} \tag{39}$$

$\Rightarrow W^\mu \propto (0, \vec{J})$, this generates the so-called Wigner's little group, that leaves p_0^μ unchanged! (Here, $SU(2)$!)

$$\Rightarrow \quad C_2 = W_\mu W^\mu = -\frac{1}{4}m^2 \vec{J}^2 = -\frac{1}{4}m^2 j(j+1), \quad j \in \frac{1}{2}\mathbb{Z} \tag{40}$$

$\Rightarrow (p_\mu, C_1, C_2) \xleftarrow[\text{is induced by}]{} (p_{\mu 0}, m^2, j)$ labeled the state for massive case!

Induced rep.

$\Rightarrow$ The whole irrep. is filled out by the different values of j_3 with $|j_3| \leq j$. (41)

i.e.

$$\underbrace{-j, -j+1, \dots < j_3 < \dots, j-1, j}_{(2j+1)\text{ degeneracy}}$$

$\Rightarrow$ Massive irrep. of $ISO(1,3)$ is $|p_\mu, j_3\rangle$. (42)

Massless case $m=0$:

Now, pick $p_0^\mu = (E,0,0,E)$ which has $p_0^\mu p_{\mu 0} = 0$.
Now, the Pauli-Lubanski vector in this frame is easily derived as

*Note $M_{0i} = -M^{0i}$, $M_{ij} = M^{ij}$

$$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} P^\nu M^{\alpha\beta} \overset{\text{Exercise}}{=} E(-M_{12}, M_{23}-M_{02}, M_{31}+M_{01}, M_{12})$$

$$= E\left(-J_3, J_1 - K_2, J_2 + K_1, J_3\right) \tag{43}$$

Note that $W_\mu p_0^\mu = 0$, W_μ leaves the massless p_0^μ invariant!
What is the algebra of $\{W_\mu\}$? It's easy to check that

$$\left.\begin{aligned} [W_1, W_2] &= 0 \\ [W_3, W_1] &= iEW_2 \\ [W_2, W_1] &= -iEW_1 \end{aligned}\right\} \tag{44}$$

(Note: $W_0 = -W_3$)

which generates the $ISO(2)$ group acting on the Euclidean 2-plane $\mathbb{R}^2$, with W_1 & W_2 generate the translations while W_3 generates rotation. $\Rightarrow$ $ISO(2)$ is the little group of massless state, it leaves $p_0^\mu = (E,0,0,E)$ unchanged!

* Even though, $ISO(2)$ doesn't act on p_0^μ, it may act on other degrees of freedom that our state carries. In other words, those other degrees of freedom must furnish a representation of the 2d Euclidean group $ISO(2)$.
* Since $[W_1, W_2] = 0$, the maximal # of commuting generators are W_1 & W_2, which we can choose to diagonalize simultaneously!

* We can then label the states by a pair of numbers w_1, w_2 which are eigenvalues of W_1, W_2 respectively. i.e.

$$W_i |w_1, w_2\rangle = w_i |w_1, w_2\rangle \qquad i = 1, 2. \qquad (45)$$

& $$C_2 = W_\mu W^\mu = W_0^2 - W_1^2 - W_2^2 - W_3^2$$ $W_0^2 = W_3^2 = E^2 J_3^2$

$$= -(W_1^2 + W_2^2) \to -(w_1^2 + w_2^2) \qquad (46)$$

Let's consider the special case of $w_1 = w_2 = 0$. Then the ISO(2) reduces to only rotations generated by J_3, according to (43). This is just a U(1), which is labeled by a single eigenvalue h of J_3 such that

$$e^{i\theta J_3} |h\rangle = e^{i\theta h} |h\rangle \qquad (47)$$

This eigenvalue h is identified as the helicity of the state — the spin analogy for massless particle.

In general for any null vector p^μ satisfies $p^2 = 0$, the helicity tells us the eigenvalue of the state under rotation along the direction of motion:

$$\exp(i\vec{J}\cdot\hat{p}\,\theta)|p_\mu; h\rangle = e^{ih\theta} |p_\mu; h\rangle \qquad (48)$$

Since this $U(1) \in SU(2)$, we must have

$$h \in \frac{1}{2}\mathbb{Z} \quad \text{as well} \qquad (49)$$

$\Rightarrow$ Under 2π rotation, the states are either left the same (for $h \in \mathbb{Z}$) or pick up a minus sign (for $h \in \mathbb{Z} + 1/2$). (50)

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Furthmore, there's CPT theorem in QFT tells us that for massless particles, $h \xrightarrow{CPT} -h$. Thus

$$|p_\mu; h\rangle \quad \text{and} \quad |p_\mu; -h\rangle \tag{51}$$

must come in pairs.

In nature, we know photon has 2 polarization states, and graviton also has 2 polarization states. If massless Weyl fermion exists, it should have 2 polarization states, according to CPT theorem.

In the general case of $w_i \neq 0$. Since $C_2 = -(W_1^2 + W_2^2) = \text{const.}$, we can parametrize W_1 & W_2 as

(52) $W_1 = \rho \cos\alpha$, $W_2 = \rho \sin\alpha$, with $\alpha \in [0, 2\pi)$ and the action

$$\left. \begin{aligned} W_1 |\alpha\rangle &= \rho\cos\alpha\,|\alpha\rangle, \\ W_2|\alpha\rangle &= \rho\sin\alpha\,|\alpha\rangle \end{aligned} \right\} \tag{53}$$

and $C_2 = -\rho^2$. i.e. $|w_1, w_2\rangle \longrightarrow |\alpha\rangle$ with $\alpha \in [0, 2\pi)$

What is the action of W_3? It's given by

$$e^{i\theta J_3} |\alpha; h\rangle = e^{ih\theta} |\alpha + \theta; h\rangle \tag{54}$$

$$\Longrightarrow J_3|\alpha; h\rangle = h|\alpha; h\rangle - i\frac{d}{d\alpha}|\alpha; h\rangle \tag{55}$$

Exercise: Show that (53) & (55) furnish a representation of the 2d Euclidean alegbra given in (44).

In summary, for $w_i \neq 0$, we can label the states by

$$|p_\mu, \alpha; h\rangle \quad \text{(continuous spin representation)} \tag{56}$$

This is also infinite dim, even for a fixed p_μ, since $\alpha \in [0, 2\pi)$ is continuous!

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Coleman-Mandula Theorem

In any dimension greater than 1+1 (1 time, 1 space), it's impossible to combine Poincaré spacetime symmetry with any internal symmetry. In other words, any nontrivial (interacting) QFT must factorize as

$$\underset{\text{Sym.}}{\text{Poincaré}} \otimes \text{Internal Sym} \tag{57}$$

* Historic Note: SU(6) relativistic quark model

$$\underbrace{SU(3)}_{\substack{\text{flavor} \\ (u,d,s)}} \otimes \underbrace{SU(2)}_{\substack{\text{Spin} \\ (\uparrow,\downarrow)}} \xrightarrow{\;?\;} \underset{\substack{\text{Relativistic} \\ \text{field}}}{SU(6)} \tag{58}$$

Non-Relativistic

* Two exceptional cases:

(1) Conformal Invariance $SO(1,3) \longrightarrow SO(2,4)$

Coleman-Mandula assumed the theory has a mass gap, i.e. massive theory. $\Rightarrow$ No IR divergencies in the S-matrix. (59)

Scale transformation $x^\mu \rightarrow \lambda x^\mu$ (Dilatation D)

Special conformal transf.
$$x^\mu \rightarrow \frac{x^\mu - a^\mu x^2}{1 - 2a\cdot x + a^2 x^2} \tag{60}$$

where a^μ is a vector associated with a generator K^μ which is also a vector.

The Poincaré algebra (32) has to extend to include

$$\left.\begin{aligned} &[D, K^\mu] = -iK^\mu, \quad [D, P^\mu] = iP^\mu \\ &[K^\mu, P^\nu] = 2i(D\eta^{\mu\nu} - M^{\mu\nu}) \\ &[M^{\mu\nu}, K^\sigma] = i(K^\nu \eta^{\mu\sigma} - K^\mu \eta^{\nu\sigma}) \end{aligned}\right\} \text{(Conformal Algebra)} \tag{61}$$

There are many interacting conformal invariant theories in physics. String theory, e.g., is a 2-d (super-) conformal field theory.

(2) Supersymmetry

Haag-Lopuszański-Sohnius theorem (1975):
If both commutating & anti-commutating generators are considered, the only nontrivial way to mix spacetime & internal symmetries is through supersymmetry

Supersymmetry algebra:

$$\{Q_\alpha^i, Q_{\dot\beta}^{j\dagger}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta_{ij} , \quad i,j = 1,2,\cdots,N \qquad (62)$$

Q_α^i: Weyl spinors, $i = 1, \cdots, N$ = # of supersymmetry
α: spinor index

$\Rightarrow$ Lie Superalgebra

There are many interacting supersymmetric QFT, including $N=4$ Super YM, $N=8$ Supergravity at 4D and 11 D supergravity.

Active viewpoint

Scalar field ϕ : $(0,0)$ of Lorentz group. $\phi(x) \xrightarrow{\Lambda} \phi'(x) = \phi(\Lambda^{-1}x)$

In general we can consider a collection of real scalar fields $\{\phi^a\}$ with $a = 1, \cdots, n$ denotes non-spacetime index.

* $\{\phi^a(t,\vec{x})\}$ can form an irrep. of some internal group G. We'll come to this later.

Free scalar field satisfies Klein-Gordon eq. which can be derived from the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^a\partial_\nu\phi^a - \frac{1}{2}m^2\phi^a\phi^a$$

$$= \frac{1}{2}(\dot{\phi}^a)^2 - \frac{1}{2}(\vec{\nabla}\phi^a)\cdot(\vec{\nabla}\phi^a) - \frac{1}{2}m^2\phi^a\phi^a \qquad (1)$$

Compare with $L = T - V$, we can obtain

$$\underbrace{T = \int \frac{1}{2}(\dot{\phi}^a)^2 d^3\vec{x}}_{\text{Kinetic Energy}} \;\&\; \underbrace{V = \int d^3\vec{x}\left[\frac{1}{2}m^2\phi^a\phi^a + \frac{1}{2}(\vec{\nabla}\phi^a)\cdot(\vec{\nabla}\phi^a)\right]}_{\text{potential}}$$

$$\frac{\partial\mathcal{L}}{\partial\phi^a} = -m^2\phi^a, \quad \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a} = \partial^\mu\phi^a = (\dot{\phi}^a, -\vec{\nabla}\phi^a) \qquad (2)$$

*Note: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

$\Rightarrow \partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)$

Euler-Lagrange Eq:

$$\frac{\partial\mathcal{L}}{\partial\phi^a} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\right) \Longrightarrow \underbrace{\partial_\mu\partial^\mu}_{\equiv\Box}\phi^a + m^2\phi^a = 0 \qquad (KG)$$

$$\text{i.e. } (\Box + m^2)\phi^a = 0 \qquad (3)$$

* In general, $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi^a\cdot\partial_\nu\phi^a - \underbrace{V(\phi)}_{\text{arbitrary potential}}$ (4)

(Not include $(\nabla\phi)^2$ term but include $m^2\phi^2$!)

the eq. of motion is then

$$\Box\phi^a + \frac{\partial V}{\partial\phi^a} = 0, \quad \Box \equiv \partial_\mu\partial^\mu \qquad (5)$$

* For passive viewpoint, $x \xrightarrow{\Lambda} x' = \Lambda x$,

$$\phi \xrightarrow{\Lambda} \phi'(x) = \phi(\Lambda x)$$

Hamiltonian of K.G. theory

$$\pi^a(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}, \quad \mathcal{H} = \pi^a \dot{\phi}_a - \mathcal{L} = \text{Hamiltonian density} \quad (6)$$

For K.G theory, $\pi^a(x) = \dot{\phi}^a \Rightarrow \dot{\phi}^a = \pi^a$ (Solve $\dot{\phi}$ in terms of π ☺)

$$H = \text{Hamiltonian} = \int d^3\vec{x}\, \mathcal{H} = \int \left[\frac{1}{2}\left(\pi^{a2} + (\vec{\nabla}\phi^a)^2\right) + V(\phi)\right] d^3\vec{x} \quad (7)$$

Hamiltonian eqs read

$$\dot{\phi}^a(\vec{x},t) = \frac{\partial H}{\partial \pi_a(\vec{x},t)} \quad \& \quad \dot{\pi}^a(\vec{x},t) = -\frac{\partial H}{\partial \phi(\vec{x},t)} \quad (8)$$

These eqs. lack manifest Lorentz invariance, under the Euler-Lagrangian equation.

Let's look at free KG theory with $V = m^2\phi^2$. Then KG is

$$(\Box + m^2)\phi = (\partial_\mu \partial^\mu + m^2)\phi = 0.$$

Consider the Fourier transf.

$$\phi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \phi(\vec{p},t) e^{i\vec{p}\cdot\vec{x}} \quad \text{Fourier transfs. (only spatial parts!)}$$

$$\Rightarrow \left[\frac{\partial^2}{\partial t^2} + (m^2 + \vec{p}^2)\right] \phi(\vec{p},t) = 0 \quad (9)$$

Compare with the SHO for coordinate q:

$$\left[\frac{d^2}{dt^2} + \omega^2\right] q(t) = 0, \quad \omega = \text{freq.} \quad (10)$$

For each fixed $\vec{p}$, $\phi(\vec{p},t)$ behaves like a SHO with freq. given by

$$\omega_{\vec{p}} \equiv \omega(\vec{p}) = +(m^2 + \vec{p}^2)^{1/2} \quad (11)$$

Simple Harmonic Oscillator (SHO) : (A brief review)

Hamiltonian $H = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{q}^2$ (Drop ^ from now on)

Canonical quantization $[\hat{q}, \hat{p}] = i\hbar = i$ ($\hbar = 1$)

To find the spectrum of H, the easiest way to to define creation & annihilation operators (a & $a^\dagger$)

$$a = \sqrt{\frac{\omega}{2}}\, q + \frac{i}{\sqrt{2\omega}}\, p, \quad a^\dagger = \sqrt{\frac{\omega}{2}}\, q - \frac{i}{\sqrt{2\omega}}\, p \qquad (12)$$

Invert give

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \quad \& \quad p = -i\sqrt{\frac{\omega}{2}}\,(a - a^\dagger) \qquad (13)$$

So $[q, p] = i \implies [a, a^\dagger] = 1$.

Thus, $H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) =$

$$= \frac{1}{2}\omega(1 + a^\dagger a + a^\dagger a)$$

$$= \omega(a^\dagger a + \frac{1}{2}) \to \text{Zero-point energy}$$

Simple exercises lead us to

$$[H, a^\dagger] = \omega a^\dagger, \quad [H, a] = -\omega a$$

These 2 eqs. tell us, $(a, a^\dagger)$ take us traveling between different energy eigenstate. Suppose $H|E\rangle = E|E\rangle$, then

$$Ha^\dagger|E\rangle = (a^\dagger H + \omega a^\dagger)|E\rangle = (E + \omega)\, a^\dagger|E\rangle. \text{ Similarly}$$

$$Ha|E\rangle = (aH - \omega a)|E\rangle = (E - \omega)\, a|E\rangle.$$

$\Rightarrow$ The spectrum looks like

ground state zero-point Energy energy

$\cdots, E - 2\omega, E - \omega, E, E + \omega, E + 2\omega, \cdots$

Ground state (if exists) : $a|0\rangle = 0 \Rightarrow H|0\rangle = \frac{1}{2}\omega|0\rangle$.

Excited states (unnormalized) : $|n\rangle = (a^\dagger)^n|0\rangle$ with $H|n\rangle = (n + \frac{1}{2})\omega|n\rangle$.

Since $\phi(\vec{p},t)$ behaves like SHO for a fixed $\vec{p}$, we can write at a given $t = t_0 = 0$, according to (13),

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a_{-\vec{p}}^{\dagger} \right) e^{+i\vec{p}\cdot\vec{x}} \qquad (14)$$

& Similarly,

$$\pi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\dagger}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} - a_{-\vec{p}}^{\dagger} \right) e^{+i\vec{p}\cdot\vec{x}} \qquad (15)$$

Now (16) turns into $[a_{\vec{p}}, a_{\vec{p}'}^{\dagger}] = (2\pi)^3 \delta^3(\vec{p}-\vec{p}')$

along with $[a_{\vec{p}}, a_{\vec{p}'}] = 0 = [a_{\vec{p}}^{\dagger}, a_{\vec{p}'}^{\dagger}] = 0$

~~One can show easily that~~

$$\left.\begin{aligned} &[a_{\vec{p}}, a_{\vec{q}}] = 0 \\ &[a_{\vec{p}}^{\dagger}, a_{\vec{q}}^{\dagger}] = 0 \\ &[a_{\vec{p}}, a_{\vec{q}}^{\dagger}] = (2\pi)^3 \delta(\vec{p}-\vec{q}) \end{aligned}\right. \iff \begin{aligned} &[\phi(\vec{x}), \phi(\vec{y})] = 0 \\ &[\pi(\vec{x}), \pi(\vec{y})] = 0 \\ &[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^3(\vec{x}-\vec{y}) \end{aligned} \qquad (16)$$

"Equal-time commutation relation"

Furthermore one can show that

$$H = \int d^3\vec{x} \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\vec{\nabla}\phi)^2 + \tfrac{1}{2}m^2\phi^2 \right] = \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} \left[a_{\vec{p}} a_{\vec{p}}^{\dagger} + a_{\vec{p}}^{\dagger} a_{\vec{p}} \right]$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} \left[a_{\vec{p}}^{\dagger} a_{\vec{p}} + \frac{1}{2}\delta^3(0)(2\pi)^3 \right]$$

Vacuum $|0\rangle$:

$$a_{\vec{p}}|0\rangle = 0 \quad \forall\, \vec{p} \qquad (17)$$

Thus the vacuum has ∞ energy! $H|0\rangle = E_0|0\rangle$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2} \omega_{\vec{p}} (2\pi)^3 \delta^3(0) |0\rangle = \infty |0\rangle \; !$$

Recall that

$$\int d^n x \, e^{ik\cdot x} = (2\pi)^n \delta^{(n)}(k)$$

$$\Rightarrow (2\pi)^3 \delta^3(0) = \lim_{L\to\infty} \int_{-L/2}^{+L/2} d^3\vec{x} \, e^{i\vec{k}\cdot\vec{x}} \Big|_{\vec{k}=0} = \text{Volume} = V$$

$$\Rightarrow \mathcal{E}_0 = \underset{\text{Energy density}}{\text{Ground state}} = \frac{E_0}{V} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2} \omega_{\vec{p}} \longrightarrow \text{UV divergence!}$$

Philosophically, one may argue that even a free theory we don't expect it to well-defined at arbitray small scale. ☺ $\longrightarrow$ More fundamental theory is expected to replace QFT which breaks down at $l \le l_{Planck} \simeq 10^{-33}$ cm

Practically, people invented the so called the notion of normal-ordering:

$$\phi_1(\vec{x}_1) \cdots \phi_n(\vec{x}_n) \longrightarrow :\phi_1(\vec{x}_1) \;\; \phi_n(\vec{x}_n): \qquad (18)$$

$:\,:$ put all the annihilation operators $a_{\vec{p}}$ inside the fields' product to the <u>right</u>.

for example,

for $H = \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3} \omega_{\vec{p}} \left[a_{\vec{p}} a^{\dagger}_{\vec{p}} + a^{\dagger}_{\vec{p}} a_{\vec{p}} \right]$,

$$:H: = \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p \left[a^{\dagger}_{\vec{p}} a_p + a^{\dagger}_p a_p \right] = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p a^{\dagger}_p a_p \, .$$

This implies $\langle 0| :H: |0\rangle = 0$.

1-particle state $|p\rangle$ with relativistic normalization:

Since $\delta^3(\vec{p}-\vec{q})$ is not Lorentz inv., $E_p\,\delta^3(\vec{p}-\vec{q})$ is!

One defines

$$|p\rangle = \sqrt{2E_p}\,\underbrace{a^\dagger_{\vec{p}}|0\rangle}_{\equiv |\vec{p}\rangle}, \quad E_p = \sqrt{m^2+\vec{p}^{\,2}} \qquad (19)$$

so that

$$\langle p|q\rangle = 2E_p(2\pi)^3\,\delta^3(\vec{p}-\vec{q}) \text{ while } \langle \vec{p}|\vec{q}\rangle = \frac{\delta^3(\vec{p}-\vec{q})}{(2\pi)^3}$$

* Factor of 2 is just convention!

* The factor of $2E_p$ shows up in many places. e.g.

$$\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4}\, 2\pi\,\delta(p^2-m^2)\Big|_{p^0>0} \qquad (20)$$

* 1-particle completeness relation

$$(\mathbb{1})_{1\text{-particle}} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}|p\rangle\langle p| = \int \frac{d^3\vec{p}}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}| \qquad (21)$$

* For any Hamiltonian H, quantum fields satisfy $\underbrace{i\partial_t\phi(x) = [\phi, H]}_{\text{"Heisenberg EOM"}}$

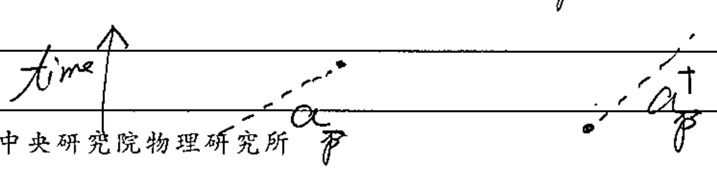
$$\Rightarrow \phi(x) = e^{iHt}\phi(\vec{x})e^{-iHt}$$

$$= e^{+iHt - i\vec{p}\cdot\vec{x}}\,\phi(0)\,e^{-iHt}e^{+i\vec{p}\cdot\vec{x}}$$

$$= e^{iP\cdot x}\,\phi(0)\,e^{-iP\cdot x}$$

Here. $P^\mu = (E, \vec{P})$.

$\Rightarrow$ For free field, $$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\left(a_{\vec{p}}e^{-ip\cdot x} + a^\dagger_{\vec{p}}e^{+ip\cdot x}\right) \qquad (22)$$



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Feynman Propagator for scalar field is given by

$$\Delta_F(x-y) \equiv \langle 0 | T[\phi(x)\phi(y)] | 0 \rangle$$

$$= \begin{cases} \langle 0 | \phi(x)\phi(y) | 0 \rangle & \text{for } x^0 > y^0 \\ \langle 0 | \phi(y)\phi(x) | 0 \rangle & \text{for } y^0 > x^0 \end{cases}$$

Recall time ordering T is defined as for 2 scalar fields

$$T[\phi(x)\phi(y)] = \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & y^0 > x^0 \end{cases}$$

One can show that in QFT,

$$\Delta_F(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)}$$

$$(\epsilon > 0)$$

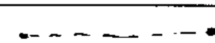
~~It satisfies~~

$$(\Box_x + m^2)\Delta_F(x-y) = -i\,\delta^{(4)}(x-y)$$

Momentum space $\Delta_F(p)$ is defined as $\Delta_F(x) \equiv \int \frac{d^4p}{(2\pi)^4} \Delta_F(p)\, e^{-ip}$

Thus

$$\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \qquad (23)$$



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$(a,b) = (1/2, 0)$ or $(0, 1/2)$ <u>Weyl fermions</u>

<u>$(1/2, 0)$</u> (Left Weyl spinor).

From (4), $B_i = 0 \Rightarrow \vec{J} = i\vec{K} \Rightarrow \vec{A} = \frac{1}{2}(\vec{J} + i\vec{K}) = \vec{J}$

For spin 1/2, we have $\vec{J} = \frac{1}{2}\vec{\sigma}$ and hence $\vec{K} = -\frac{i}{2}\vec{\sigma}$

$\Longrightarrow$ ξ = 2-comp. spinor (Left-handed)

Similarly, for <u>$(0, 1/2)$</u>, $A_i = 0 \Rightarrow \vec{J} = -i\vec{K}$ and

$$\vec{B} = \frac{1}{2}(\vec{J} - i\vec{K}) = \vec{J} = \frac{1}{2}\vec{\sigma} \text{ too!}$$

$\Longrightarrow$ η : 2-comp. spinor (Right-handed)

<u>Summary</u>:

$$\left(\vec{A} = \frac{1}{2}\vec{\sigma}, \vec{B} = 0\right) \quad (1/2, 0) \quad \xi_\alpha \quad \alpha = 1, 2$$

$$\left(\vec{A} = 0, \vec{B} = \frac{1}{2}\vec{\sigma}\right), \quad (0, 1/2) \quad \eta^{\dot\alpha} \quad \dot\alpha = 1, 2 \tag{23}$$

Let $\mathbb{P}$ be the parity operator.

Since $\mathbb{P}\vec{J}\mathbb{P}^{-1} = \vec{J}$, $\mathbb{P}\vec{K}\mathbb{P}^{-1} = -\vec{K}$

$$\mathbb{P}\, i\, \mathbb{P}^{-1} = i \tag{24}$$

$$\Longrightarrow \mathbb{P}\vec{A}\mathbb{P}^{-1} = \vec{B}, \quad \mathbb{P}\vec{B}\mathbb{P}^{-1} = \vec{A} \tag{25}$$

So under parity. $\xi \longleftrightarrow \eta \qquad (1/2, 0) \longleftrightarrow (0, 1/2)$ (26)

* In E&M (QED), parity is a <u>good</u> quantum number, we need both

$$\xi \;\&\; \eta \Longrightarrow \psi_{Dirac} = \text{Dirac Spinor} = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot\alpha} \end{pmatrix} \tag{27}$$

* In weak interaction, parity is violated (e.g. β-decay)

We need only ξ or η for (for ν_L, $\overline{\nu_R}$) (ν_R, $\overline{\nu_L}$) in weak theory.

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Define two sets of Pauli-Matrices

$$\sigma^\mu = (\mathbb{1}, \vec{\sigma})_{\alpha\dot{\beta}} \qquad \vec{\sigma} = \text{Pauli matrices} \tag{28}$$

$$\bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma})^{\dot{\alpha}\beta} \qquad = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Then we can construct two bilinears

$$\xi^\dagger \bar{\sigma}^\mu \xi \sim (0, 1/2) \otimes (1/2, 0) = (1/2, 1/2) + \text{singlet}$$

$$\eta^\dagger \sigma^\mu \eta \sim (1/2, 0) \otimes (0, 1/2) = (1/2, 1/2) + \text{singlet} \tag{29}$$

Up to an overall sign, we can construct two Lorentz invariances (for Lagrangian densities for ξ & η)

$$\mathcal{L}_L = i\xi^\dagger \bar{\sigma}^\mu \partial_\mu \xi = i\xi^\dagger (\partial_t - \vec{\sigma} \cdot \vec{\nabla}) \xi$$

$$\mathcal{L}_R = i\eta^\dagger \sigma^\mu \partial_\mu \eta = i\eta^\dagger (\partial_t + \vec{\sigma} \cdot \vec{\nabla}) \eta \tag{30}$$

Recall $\partial_\mu = (\partial_t, \vec{\nabla})$ while $x^\mu = (t, \vec{x})$.

E.O.M.

$$\left.\begin{aligned} \frac{\delta \mathcal{L}_L}{\delta \xi^\dagger} = 0 &\Rightarrow (\partial_t - \vec{\sigma} \cdot \vec{\nabla}) \xi = 0 \\ \frac{\partial \mathcal{L}_R}{\partial \eta^\dagger} = 0 &\Rightarrow (\partial_t + \vec{\sigma} \cdot \vec{\nabla}) \eta = 0 \end{aligned}\right\} \tag{31}$$

* Treating $\xi, \xi^\dagger$ as <u>indep.</u> objects.
* " $(\eta, \eta^\dagger)$ "

Note that $\mathcal{L}_L$ & $\mathcal{L}_R$ are inv. under global $U(1)$ phase transformations.

$$\left.\begin{aligned} \xi \to e^{i\theta} \xi &\Rightarrow \mathcal{L}_L \to \mathcal{L}_L' = \mathcal{L}_L \\ \eta \to e^{i\theta'} \eta &\Rightarrow \mathcal{L}_R \to \mathcal{L}_R' = \mathcal{L}_R \end{aligned}\right\} \tag{32}$$

Now $(\partial_t \mp \vec{\sigma}\cdot\vec{\nabla})(\partial_t \pm \vec{\sigma}\cdot\vec{\nabla}) = \partial_t^2 - \sigma_i\partial_i\sigma_j\partial_j$

$= \partial_t^2 - \frac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i)\partial_i\partial_j = \partial_t^2 - \frac{1}{2}\{\sigma_i, \sigma_j\}\partial_i\partial_j$

$= \partial_t^2 - \frac{1}{2}\cdot 2\cdot\delta_{ij}\partial_i\partial_j = \partial_t^2 - \vec{\nabla}^2 = \Box$

$$\Rightarrow \quad \Box\xi = 0 \quad \& \quad \Box\eta = 0. \qquad (33)$$

both ξ & η satisfy K.G. equations. $\Rightarrow$ plane wave solutions

$$\left.\begin{aligned} \xi(x) &= \xi(k)\, e^{-ik\cdot x} & k_0 &= |\vec{k}| \\ \eta(x) &= \eta(k)\, e^{-ik\cdot x} & k_0 &= |\vec{k}| \end{aligned}\right\} \qquad (34)$$

Plugging these sols. into (31), we have (using $(\partial_t, \vec{\nabla})\, e^{-ik\cdot x} = (-ik_0, i\vec{k})\, e^{-ik\cdot x}$)

$$(-ik_0 - i\vec{k}\cdot\vec{\sigma})\xi = 0 \quad \text{and} \quad (-ik_0 + i\vec{k}\cdot\vec{\sigma})\eta = 0$$

i.e. $\frac{\vec{\sigma}\cdot\vec{k}}{|\vec{k}|}\xi = -\xi \Rightarrow \xi$ has helicity $-1/2$

& $\frac{\vec{\sigma}\cdot\vec{k}}{|\vec{k}|}\eta = +\eta \Rightarrow \eta$ has helicity $+1/2$ (35)

$\Rightarrow$ (left / Right) $\xrightarrow[\text{massless case}]{}$ helicity $-1/2$ / $+1/2$

$\quad -\quad +1$

* Helicity Operator $h \equiv \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|}$ (36) $\quad \vec{S} = \frac{1}{2}\vec{\sigma}$ for spin 1/2 case.

(*Not an Lorentz inv. in general, unless for massless case like the Weyl fermions here).

~~i.e. for the Weyl spinors, $2h$ = chirality~~

* Introduce chirality later in 4-comp. spinors. Not here because chirality has something to do with γ_5.

In terms of dimensi.

The billinear quantities transforms like $(1/2, 1/2) \sim (2, 2)$
What about the singlet piece of $(0, 0)$. This is the follow object

$$\left.\begin{aligned} \xi^T \epsilon \xi &= \epsilon^{\alpha\beta} \xi_\alpha \xi_\beta \\ \eta^T \epsilon \eta &= \epsilon_{\dot\alpha\dot\beta} \bar\eta^{\dot\alpha} \bar\eta^{\dot\beta} \end{aligned}\right\} \neq 0 \qquad (37)$$

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \epsilon_{\dot\alpha\dot\beta}$$

Note: ξ & η are <u>Grassmanian Variables</u>!

These two billinears can provide a <u>mass</u> term for the Weyl spinors ξ & η.

$$\left.\begin{aligned} \Rightarrow \mathcal{L}_L &= i \xi^\dagger \bar\sigma^\mu \partial_\mu \xi + \frac{1}{2} m (\xi^T \epsilon \xi + h.c.) \\ \mathcal{L}_R &= i \eta^\dagger \sigma^\mu \partial_\mu \eta + \frac{1}{2} m (\eta^T \epsilon \eta + h.c.) \end{aligned}\right\} (38)$$

<u>Ex</u> Show that the two h.c. terms in (38) are

$$-\frac{1}{2} m \xi^\dagger \epsilon \xi^* \quad \& \quad -\frac{1}{2} m \eta^\dagger \epsilon \eta^* \qquad (39)$$

where m is real.

<u>Ex</u> Treating $\xi, \xi^\dagger$ as indep. objects, show that the E.O.M. for ξ is

$$i \bar\sigma^\mu \partial_\mu \xi - m \epsilon \xi^* = 0 \qquad (40)$$

Similarly, for η & $\eta^\dagger$ we have

$$i \sigma^\mu \partial_\mu \eta - m \epsilon \eta^* = 0 \qquad (41)$$

Let M be a Lorentz transformation acting on the Weyl spinor

$$M = \exp\Big(-\frac{i}{2}\underbrace{\vec{\sigma}\cdot\vec{\theta}}_{\text{rotation angle}}\Big) \quad \text{or } \exp\Big(-\frac{\vec{\sigma}}{2}\cdot\vec{\beta}\Big) \qquad (42)$$

boost or rapidity?

Under M, $\xi^T\epsilon\xi$ transforms as

$$\xi^T\epsilon\xi \longrightarrow \xi^T \underbrace{M^T\epsilon M}\,\xi$$

Displacing indices for $M^T\epsilon M$:

$$(M^T)^{\delta\alpha}\epsilon^{\alpha\beta}M^{\beta\gamma} = M^{\alpha\delta}\epsilon^{\alpha\beta}M^{\beta\gamma}$$

$$= \epsilon^{\alpha\beta}M^{\alpha\delta}M^{\beta\gamma}$$

$$= \epsilon^{\delta\gamma}\underbrace{\operatorname{Det}M}_{1} = \epsilon^{\delta\gamma} \qquad (43)$$

i.e. ϵ is an inv. tensor under Lorentz transf.

$$M^T\epsilon M = \epsilon. \qquad (43)$$

* $\xi^T\epsilon\xi$ & $\eta^T\epsilon\eta$ are indeed Lorentz invariants

* $SO(1,3) \simeq SL(2,\mathbb{C})$: 2x2 complex matrices with det = 1.

* In general, M is given by

$$M = \exp\left[-i\vec{J}\cdot\vec{\theta} - i\vec{K}\cdot\vec{\beta}\right] = \exp\left[-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right]$$

$$(1/2, 0) \Rightarrow i\vec{K} = \vec{J} = \frac{\vec{\sigma}}{2} \quad M = \exp\left[-i\frac{\vec{\sigma}}{2}\cdot\vec{\theta} - \frac{\vec{\sigma}}{2}\cdot\vec{\beta}\right]$$

$$(0, 1/2) \Rightarrow i\vec{K} = -\vec{J} = -\frac{\vec{\sigma}}{2} \quad M = \exp\left[-i\frac{\vec{\sigma}}{2}\cdot\vec{\theta} + \frac{\vec{\sigma}}{2}\cdot\vec{\beta}\right]$$

$(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ Dirac Spinor is a 4-comp. object.

$\psi_D = \begin{pmatrix} \xi_\alpha \\ \eta^{\dot\alpha} \end{pmatrix}$. The EOM of ξ & η are modified as

$$\left.\begin{aligned} i\bar\sigma^\mu\partial_\mu\xi &= m\eta \\ i\sigma^\mu\partial_\mu\eta &= m\xi \end{aligned}\right\} \Rightarrow i\begin{pmatrix} \bar\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}\partial_\mu\psi = m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi \qquad (44)$$

Eq (44) can be derived from the following Lagrangian

$$\mathcal{L}_D = i\psi^\dagger\begin{pmatrix} \bar\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix}\partial_\mu\psi - m\psi^\dagger\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi \qquad (45)$$

Define $\gamma^0 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $(\gamma^0)^2 = \mathbb{I}$ (46)

& $\gamma^\mu = \gamma^0\begin{pmatrix} \bar\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$ Weyl representation

& $\bar\psi \equiv \psi^\dagger\gamma^0 = (\xi^\dagger, \eta^\dagger)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \eta^\dagger \\ \xi^\dagger \end{pmatrix}$ Dirac adjoint. (47)

$$\Rightarrow \mathcal{L}_D = i\bar\psi\gamma^\mu\partial_\mu\psi - m\bar\psi\psi \qquad (48)$$

EOM. $0 = \frac{\delta\mathcal{L}_D}{\delta\bar\psi} \Rightarrow i\gamma^\mu\partial_\mu\psi - m\psi = 0$ (49)

$\Rightarrow (\Box + m^2)\psi = 0 \Rightarrow \psi$ satisfies KG eq.

$\Rightarrow$ plane wave sol.

Aside:

$$(i\gamma^\mu\partial_\mu + m)(i\gamma^\mu\partial_\mu - m)$$
$$= -\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu - m^2 = -\frac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2g^{\mu\nu}}\partial_\mu\partial_\nu - m^2 = -(\Box + m^2)$$

Dirac algebra: $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ (Clifford Algebra) (50)

Indep. of representation

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Chirality:

$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (Weyl), $(\gamma^5)^2 = 1$

$\Longrightarrow$ eigenvalues $= \pm 1$

$\{\gamma^5, \gamma^\mu\} = 0 \quad \forall \mu = 0, 1, 2, 3.$

$P_L = \frac{1}{2}(1-\gamma_5)$, $P_R = \frac{1}{2}(1+\gamma_5)$ } Project. op.

$P_L^2 = P_L$, $P_R^2 = P_R$ $P_L P_R = 0$. $P_L + P_R = 1$

$$P_L = \frac{1}{2}\left(\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} - \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{51}$$

$$P_R = \frac{1}{2}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow \quad P_L \psi_D = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$. $P_R \psi_D = \begin{pmatrix} 0 \\ \eta \end{pmatrix}$.

*Recall ξ has $-1/2$ helicity while η has $+1/2$ helicity.
* Massless case: helicity $\Leftrightarrow$ chirality

*P_L, P_R are chirality projection operators, project out left & right Weyl fermions from ψ_D.

Construct 6 4×4 matrices $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$ (antisym in μ, ν) as

$$\sigma^{\mu\nu} \equiv \frac{1}{2} i [\gamma^\mu, \gamma^\nu] \tag{52}$$

$\{\sigma^{\mu\nu}\}$ furnishes a representation (reducible) of the Lorentz Algebra; namely $\simeq (1/2, 0) \oplus (0, 1/2)$

(53) $\mathcal{M}^{\mu\nu} \equiv \frac{1}{2}\sigma^{\mu\nu}$, $D(\Lambda) = \exp(-\frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu})$

$\boxed{\text{Change } \mathcal{M}^{\mu\nu} \to S^{\mu\nu}}$

$$= 1 - \frac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} + \cdots \tag{54}$$

Ex: $[\mathcal{M}^{\mu\nu}, \mathcal{M}^{\rho\sigma}]$

$$= i\left(\eta^{\nu\rho}\mathcal{M}^{\mu\sigma} - \eta^{\mu\rho}\mathcal{M}^{\nu\sigma} - \eta^{\nu\sigma}\mathcal{M}^{\mu\rho} + \eta^{\mu\sigma}\mathcal{M}^{\nu\rho}\right) \tag{55}$$

Ex: $[\gamma^5, \sigma^{\mu\nu}] = 0 \quad \forall \mu, \nu$ (56) $\Longrightarrow$ Reducible rep $(1/2, 0) \oplus (0, 1/2)$ (Schur's Lemma)

Schur's Lemma: A representation of a group is irreducible iff all matrices commuting with every element of the representation are proportional to identity.

* Since γ_5 is non-trivial, the rep. must be reducible.

Indeed, we know, by construction, the rep. is $(1/2, 0) \oplus (0, 1/2)$. Since the eigenvalues of γ_5 are ± 1, the rep. is classified by + & − chirality.

* $[\gamma_5, \sigma^{\mu\nu}] = 0 \;\; \forall \mu, \nu$ also implies chirality is a Lorentz invariant concept!

$$\psi_L = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \quad \psi_R = \begin{pmatrix} 0 \\ \eta \end{pmatrix}, \quad \gamma_5 \psi_L = \underset{\text{Chirality}}{-}\, \psi_L, \quad \gamma_5 \psi_R = \underset{\text{Chirality}}{+}\, \psi_R \qquad (57)$$

* Often in the literature, the Dirac spinor is written as

$$\psi_D = \begin{pmatrix} \xi \\ \epsilon \eta^* \end{pmatrix}.$$

*Note Majorana spinor $\eta = \xi$: $\psi_M = \begin{pmatrix} \xi \\ \epsilon \xi^* \end{pmatrix}$, $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv i\sigma^2$ (58)

Then the Dirac Mass term can be written as

$$-m\bar{\psi}\psi = -m \left(\xi^\dagger \;\; -\eta^T \epsilon \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \epsilon \eta^* \end{pmatrix}$$

$$= -m \left(-\eta^T \epsilon \;\; \xi^\dagger \right) \begin{pmatrix} \xi \\ \epsilon \eta^* \end{pmatrix}$$

$$= -m \left(\xi^\dagger \epsilon \eta^* - \eta^T \epsilon \xi \right) \qquad (59)$$

i.e. One can use two different Weyl spinors to rewrite a Dirac mass term as (59)

In this case, $\psi_L = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$, $\psi_R = \begin{pmatrix} 0 \\ \epsilon \eta^* \end{pmatrix}$ (60)

$\psi_L \sim (1/2, 0)$, $\psi_R \sim (0, 1/2)$

* For Majorana spinor $\psi_M = (\xi, \epsilon\xi^*)^T$.

The mass term (59) becomes

$$-\frac{1}{2}m\bar{\psi}_M\psi_M = -\frac{1}{2}m\left(\xi^\dagger\epsilon\xi^* - \xi^T\epsilon\xi\right) \qquad (59)'$$

which is known as Majorana mass term.

One can rewrite this Majorana mass term in terms of ψ_L as follows:

$$-\frac{1}{2}m\psi_L^T C\psi_L + h.c. \qquad \text{with } C = \begin{pmatrix} -\epsilon & 0 \\ 0 & +\epsilon \end{pmatrix} \ \& \ \psi_L = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$$

$$\psi_L = \begin{pmatrix} \xi \\ 0 \end{pmatrix}, \Rightarrow \psi_L^T = (\xi^T, 0) \Rightarrow \psi_L^T C\psi_L = (\xi^T\ 0)\begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}\begin{pmatrix} \xi \\ 0 \end{pmatrix}$$

$$= (\xi^T\ 0)\begin{pmatrix} -\epsilon\xi \\ 0 \end{pmatrix} = -\xi^T\epsilon\xi$$

$$\& \ h.c. = (\psi_L^T C\psi_L)^\dagger = (-\xi^T\epsilon\xi)^\dagger = \xi^\dagger(-\epsilon)^\dagger\xi^* = +\xi^\dagger\epsilon\xi^*$$

Thus $$-\frac{1}{2}m\psi_L^T C\psi_L + h.c. = -\frac{1}{2}m\left(\xi^\dagger\epsilon\xi^* - \xi^T\epsilon\xi\right)$$

$$= -\frac{1}{2}m\bar{\psi}_M\psi_M \qquad (59)''$$

Note that here in Weyl basis,

$$C = \pm i\gamma^2\gamma^0 = \pm\begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \pm\begin{pmatrix} 0 & +\epsilon \\ -\epsilon & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \pm\begin{pmatrix} +\epsilon & 0 \\ 0 & -\epsilon \end{pmatrix}$$

$$i\sigma^2 = i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \epsilon$$

* Peskin-Schroeder used $C = -i\gamma^2\gamma^0 \underset{\text{Weyl}}{=} \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$

* $C = \pm i\gamma^2\gamma^0$: Charge conjugation matrix (See Discrete symmetries notes.)

* Similarly, one can write down Majorana mass term for ψ_R as

$$-\frac{1}{2}m\psi_R^T C\psi_R + h.c. \qquad (59)'''$$

This is useful for sterile neutrinos N_R's in BSM.

Plane wave solution of Dirac's equation. (49)

$$(49)' \quad (i\partial\!\!\!/ - m)\psi(x) = 0 \tag{61}$$

positive freq. sol. $\psi_s(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} u(p,s)\, e^{-ip\cdot x}$ (62)

$(s = \pm 1/2)$ $\quad (p_0 = \sqrt{\vec{p}^2+m^2} > 0)$
$\equiv \omega_p$

Negative freq. sol.

$$\chi_s(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} v(p,s)\, e^{+ip\cdot x} \tag{63}$$

($p_0 > 0$ too using anti-particle interpretation)

Go to mom. space & Weyl. rep.

$$i\partial\!\!\!/ \to p\!\!\!/ = \gamma^\mu p_\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} p_\mu = \begin{pmatrix} 0 & \sigma\cdot p \\ \bar\sigma\cdot p & 0 \end{pmatrix} \tag{64}$$

$$\left.\begin{aligned} (i\partial\!\!\!/ - m)\psi(x) \Rightarrow & \begin{pmatrix} -m & \sigma\cdot p \\ \bar\sigma\cdot p & -m \end{pmatrix} u(p,s) = 0 \\ \& & \begin{pmatrix} -m & -\sigma\cdot p \\ -\bar\sigma\cdot p & -m \end{pmatrix} v(p,s) = 0 \end{aligned}\right\} \tag{65}$$

At rest, $p_0 = (m, \vec{0})$, we have

check if o.k with (74)

$$(66) \quad \left.\begin{aligned} +m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p_0,s) = 0 \\ -m\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} v(p_0,s) = 0 \end{aligned}\right\} \Rightarrow u(p_0,s) \propto \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix},\ v(p_0,s) \propto \begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix} \tag{67}$$

where ξ^s, η^s are const. 2-comp. spinors. For example the following 4-indep. solutions:

$$u(p_0,\tfrac12) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ u(p_0,-\tfrac12) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix};\ v(p_0,\tfrac12) = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ v(p_0,-\tfrac12) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

e.g. $e^-\uparrow$ $\quad e^-\downarrow$ $\quad e^+\uparrow$ $\quad e^+\downarrow$

(68)

For arbitrary p, the solutions are

$$u(p,s) = \begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi^s \\ \sqrt{\bar\sigma\cdot p}\,\xi^s \end{pmatrix}, \quad v(p,s) = \begin{pmatrix} \sqrt{\sigma\cdot p}\,\eta^s \\ -\sqrt{\bar\sigma\cdot p}\,\eta^s \end{pmatrix} \tag{69}$$

One can easily check that the above expressions of u & v satisfy the Dirac eq.

$$\begin{pmatrix} -m & \sigma\cdot p \\ \bar\sigma\cdot p & -m \end{pmatrix} u(p,s) = \begin{pmatrix} -m & \sigma\cdot p \\ \bar\sigma\cdot p & -m \end{pmatrix}\begin{pmatrix} \sqrt{\sigma\cdot p}\,\xi^s \\ \sqrt{\bar\sigma\cdot p}\,\xi^s \end{pmatrix} = \begin{pmatrix} (-m\sqrt{\sigma\cdot p} + \sigma\cdot p\sqrt{\bar\sigma\cdot p})\,\xi^s \\ (\bar\sigma\cdot p\sqrt{\sigma\cdot p} - m\sqrt{\bar\sigma\cdot p})\,\xi^s \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\sigma\cdot p}\,(-m + \sqrt{\sigma\cdot p}\sqrt{\bar\sigma\cdot p})\,\xi^s \\ \sqrt{\bar\sigma\cdot p}\,(\sqrt{\sigma\cdot p}\sqrt{\bar\sigma\cdot p} - m)\,\xi^s \end{pmatrix} = 0 \tag{70}$$

where we have used

$$\sigma\cdot p\;\bar\sigma\cdot p = p^2 = m^2 \tag{71}$$

Similarly

$$\begin{pmatrix} -m & -\sigma\cdot p \\ -\bar\sigma\cdot p & -m \end{pmatrix} v(p,s) = \begin{pmatrix} -m & -\sigma\cdot p \\ -\bar\sigma\cdot p & -m \end{pmatrix}\begin{pmatrix} \sqrt{\sigma\cdot p}\,\eta^s \\ -\sqrt{\bar\sigma\cdot p}\,\eta^s \end{pmatrix} = 0 \tag{72}$$

Here ξ^s, η^s are just any 2-comp spinors, satisfying the normalization conditions:

(73) $$\xi^{s\dagger}\xi^{s'} = \delta^{ss'}, \quad \eta^{s\dagger}\eta^{s'} = \delta^{ss'}$$ e.g. $\xi^{1/2} = \eta^{1/2} \doteq \begin{pmatrix}1\\0\end{pmatrix}$, $\xi^{-1/2} = \eta^{-1/2} \doteq \begin{pmatrix}0\\1\end{pmatrix}$

Thus.

$$\left.\begin{aligned} u^\dagger(p,s)u(p,s') &= 2E\,\xi^{s\dagger}\xi^{s'} = 2E\,\delta_{ss'} \\ v^\dagger(p,s)\,v(p,s') &= 2E\,\eta^{s\dagger}\eta^{s'} = 2E\,\delta_{ss'} \end{aligned}\right\} \tag{74}$$

Spin Sum (spinor outer product)

$$\sum_s u(p,s)\bar u(p,s) = \not p + m, \quad \sum_s v(p,s)\bar v(p,s) = \not p - m \tag{75}$$

* Quantizing Dirac field using commutation relations will lead negative-energy excitations from the vacuum. ⇒ Unstable ground state.

* Pauli's Spin-statistics Theorem:
Lorentz (Poincaré) invariance, positive energies, positive norms and Causality
⇒ Integer spin particles obey Bose-Einstein statistics.
But Half-integer spin particles obey Fermi-Dirac statistics

From QFT, we have, for Dirac field, with $\omega_p = \sqrt{m^2+\vec{p}^2} > 0$

$$\psi(x) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[b_{\vec{p}}^{s} u^s(\vec{p}) e^{-ip\cdot x} + c_{\vec{p}}^{s\dagger} v^s(\vec{p}) e^{+ip\cdot x} \right]$$

$$\psi^\dagger(x) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[b_{\vec{p}}^{s\dagger} u^s(\vec{p})^\dagger e^{+ip\cdot x} + c_{\vec{p}}^{s} v^s(\vec{p})^\dagger e^{-ip\cdot x} \right] \tag{76}$$

And we have the equal-time anti-commutation relations

$$\left. \begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\} = 0 \\ \{\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\} &= \delta_{\alpha\beta}\, \delta^{(3)}(\vec{x}-\vec{y}) \end{aligned} \right\} \tag{77}$$

α, β Dirac spinor indices.

which is equivalent to

$$\begin{aligned} 0 &= \{b_{\vec{p}}^r, b_{\vec{q}}^s\} = \{c_{\vec{p}}^r, c_{\vec{q}}^s\} = \{b_p^r, c_q^s\} = \{b_p^r, c_q^{s\dagger}\} \\ &= \{b_{\vec{p}}^{r\dagger}, b_{\vec{q}}^{s\dagger}\} = \{c_{\vec{p}}^{r\dagger}, c_{\vec{q}}^{s\dagger}\} = \{b_p^{r\dagger}, c_q^{s\dagger}\} = \{b_p^{r\dagger}, c_q^{s}\} \end{aligned} \tag{78}$$

&

$$\{b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^3(\vec{p}-\vec{q}), \quad \{c_{\vec{p}}^r, c_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^3(\vec{p}-\vec{q})$$

Feynman Propagator for Dirac field

$$S_F(x,y) = \langle 0 | T[\psi(x)\bar{\psi}(y)] | 0 \rangle$$

with

$$T[\psi(x)\bar{\psi}(y)] = \begin{cases} \psi(x)\bar{\psi}(y) & x^0 > y^0 \\ (-)\bar{\psi}(y)\psi(x) & y^0 > x^0 \end{cases} \qquad (79)$$

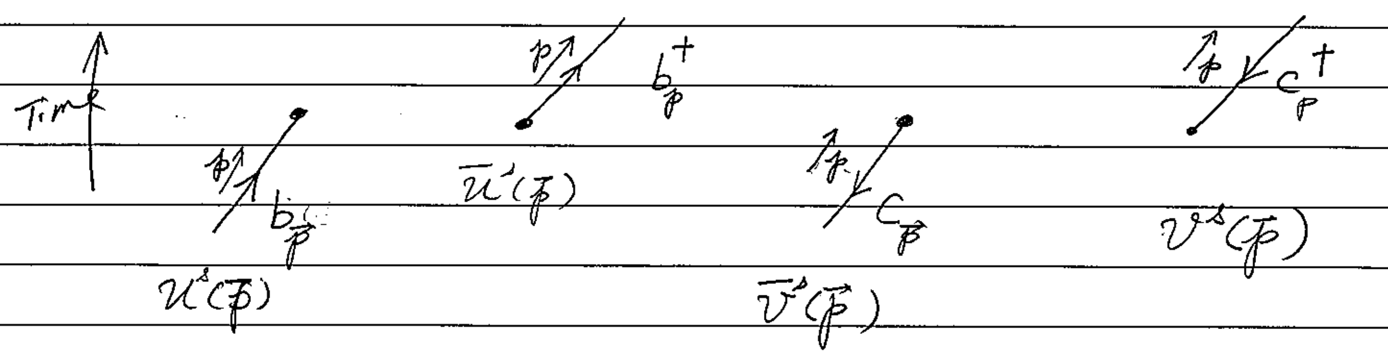
Fermion

One can show that

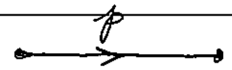
$$S_F(x,y) = \int \frac{d^4p}{(2\pi)^4} S_F(p)\, e^{-ip\cdot(x-y)}$$

$$= i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{\not{p} - m + i\epsilon} \qquad (80)$$

$$\Rightarrow S_F(p) = \frac{i}{\not{p} - m + i\epsilon} = \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} \qquad \epsilon = 0^+$$



(81)



$$S_F(p) = \frac{i}{\not{p} - m}$$

Euler-Lagrange Equation

Action Principle:

$\mathcal{L}[\phi, \partial_\mu \phi]$ Lorentz Scalar — up to 1st derivative! Lagrangian density

$$S = \int d^4x \, \mathcal{L} = \int dt \, L, \quad L = \int d^3\vec{x} \, \mathcal{L}$$

$L \to$ Lagrangian

* Principle of least action says

$$\frac{\delta S}{\delta \phi} = 0 \iff \text{Equation of Motion}$$

$$\delta S = \int d^4x \, \delta\mathcal{L} = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \partial_\mu \phi \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \cdot \delta\phi \right]$$

$$= \int d^4x \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)}_{\text{Total derivative}} \right] \quad (0)$$

$\to$ Surface term $\to$ Can be dropped if fields vanish on asymptotic boundaries at infinities

Thus $\delta S / \delta\phi = 0$ leads to

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0} \quad (1)$$

This is the Euler-Lagrange eq. which holds in both Classical & quantum field theories. (Schwinger Action Principle)

* Generalize to $\{\phi_a\}$ a collection of fields in trivial way.

* Since S is a Lorentz scalar, the resulting equation is invariant under Lorentz transformation, which is a required property in high energy physics.

* Eq. (0) implies $\boxed{\frac{\delta S}{\delta \phi_a(x)} = \frac{\partial \mathcal{L}}{\partial \phi_a(x)} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a(x))} \right)}$

(external or internal)

Noether's Theorem:

If a Lagrangian admit a continuous symmetry, then there exists an associated current that is conserved if the eq. of motion is satisfied.

Suppose the symmetry is parameterized by some $\delta\phi_a(x) = F_a(\phi)$ that can be taken to be small. Then the Lagrangian is a symmetry if $\delta\mathcal{L} = \partial_\mu K^\mu$ for some set of function of K^μ. Consider arbitrary $\delta\phi_a$. Then

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) = \frac{\partial\mathcal{L}}{\partial\phi_a} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\delta\phi_a)$$

$$= \underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right)\right]}\delta\phi_a + \partial_\mu\left[\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_a}\right)\delta\phi_a\right]$$

$\equiv$ 0 if Euler-Lagrangian eq. is satisfied (On-shell!)

Then, since $\delta\phi_a = F_a(\phi)$ for the symmetry, $\delta\mathcal{L} = \partial_\mu K^\mu$.

Define

$$J^\mu \equiv \left(\sum_a \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\cdot F_a(\phi) - K^\mu\right) \quad (2)$$

the so-called Noether current.

Then it satisfies

$$\partial_\mu J^\mu = 0 \quad \text{(Continuity eq.)} \quad (3)$$

J^μ is conserved because the total charge Q, defined as

$$Q \equiv \int_V d^3\vec{x}\, J^0 \quad (4)$$

satisfies

$$\partial_0 Q = \int_V d^3\vec{x}\,\partial_0 J^0 = -\int_V d^3\vec{x}\,\vec{\nabla}\cdot\vec{J} = \begin{cases} 0 & \text{if } V = \mathbb{R}^3 \text{ \& } \vec{J} \text{ falls off rapidly at } |\vec{x}|\to\infty \\ -\int_{\partial V}\vec{J}\cdot\hat{n}\,dS & \text{for finite Volume, (local conservation law)!} \end{cases} \quad (5)$$

by Stokes' theorem.

$$* \; \partial_\mu J^\mu = \partial_t J^0 + \vec{\nabla}\cdot\vec{J} \text{ with } J^\mu = (J^0, \vec{J}),\ \partial_\mu = (\partial_t, \vec{\nabla}) \quad (6)$$

✱ Note that Noether's theorem concerns $\delta\mathcal{L}$, while Euler-Lagrange eq. concerns about δS!

$* \longrightarrow \dagger$
classical field QFT

Examples:

(1) Scalar fields, $\phi = \phi_1 + i\phi_2$ Complex scalars

$$\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - m^2 \phi^* \phi \qquad (7)$$

is invariant under (α real const.)

$$\phi \to e^{-i\alpha}\phi, \quad \phi^* \to e^{+i\alpha}\phi^*$$

for small α, $\delta\phi = -i\alpha\phi$, $\delta\phi^* = +i\alpha\phi^*$ } Global $U(1)$ phase transf. (8)

i.e. $\delta\phi/\delta\alpha = -i\phi$, $\delta\phi^*/\delta\alpha = +i\phi^*$ & $\delta\mathcal{L} = \partial_\mu K^\mu = 0$!

Thus the Noether's current (2) is

$$J^\mu = \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\frac{\delta\phi}{\delta\alpha} + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\frac{\delta\phi^*}{\delta\alpha}\right)\delta\alpha$$

$$= \left[(\partial^\mu\phi)^*(-i\phi) + (\partial^\mu\phi)(i\phi^*)\right]\delta\alpha$$

$$= -i\left(\partial^\mu\phi^*\cdot\phi - \partial^\mu\phi\cdot\phi^*\right) \quad \text{(Drop arbitrary } \delta\alpha\text{)} \quad (9)$$

$$\Rightarrow \partial_\mu J^\mu = -i\Big(\underbrace{\Box\phi^*}_{-m^2\phi^*\ \text{(EOM)}}\cdot\phi + \cancel{\partial^\mu\phi^*\partial_\mu\phi} - \underbrace{\Box\phi}_{-m^2\phi\ \text{(EOM)}}\cdot\phi^* - \cancel{\partial^\mu\phi\cdot\partial_\mu\phi^*}\Big)$$

$$= -i\left(-m^2\phi^*\phi + m^2\phi\phi^*\right) = 0 \ !$$

$$\Rightarrow N_\phi = \int d^3\vec{x}\, J^0 = i\int d^3\vec{x}\,(\phi^*\cdot\partial^0\phi - \partial^0\phi^*\cdot\phi) \qquad \boxed{\partial^0 \equiv \partial_t}$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\,(a_p^\dagger a_p - b_p^\dagger b_p) \qquad (10)$$

$$= N - \bar{N}$$

$$= (\#\text{ of particle}) - (\#\text{ of antiparticle})$$

* Not negative probability interpretation in 1st quantized theory

N_ϕ is conserved, according to Noether's theorem.

Thus, Invariance of $\mathcal{L}$ under the Global $U(1)$ phase transf. gives rise to particle number Conservation law.

$$\boxed{\text{Symmetry (Global, Continuous)} \Longrightarrow \text{Conservation Law}}$$

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(2) Dirac field ψ

$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ has a global $U(1)$ phase transf. invariance.

(11) $\psi \to e^{-i\alpha}\psi$, $\bar{\psi} \to e^{+i\alpha}\bar{\psi}$. For infinitesimal α,

$$\delta\psi = -i\alpha\psi, \quad \delta\bar{\psi} = +i\alpha\bar{\psi} \qquad \alpha : \text{const. real parameter}$$

Rewrite $\mathcal{L} = \frac{1}{2}\bar{\psi}i\overrightarrow{\partial\!\!\!/}\psi - \frac{1}{2}\bar{\psi}i\overleftarrow{\partial\!\!\!/}\psi + \underbrace{\frac{1}{2}i\partial_\mu(\bar{\psi}\gamma^\mu\psi)}_{\text{total derivative!}} - m\bar{\psi}\psi$

$$\Rightarrow J^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi}\delta\psi + \delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}}$$

$$= \frac{1}{2}\bar{\psi}i\gamma^\mu\delta\psi - \frac{1}{2}\delta\bar{\psi}i\gamma^\mu\psi = \frac{1}{2}\bar{\psi}i\gamma^\mu(-i\alpha\psi) - \frac{1}{2}i\alpha\bar{\psi}i\gamma^\mu\psi$$

$$= \alpha\bar{\psi}\gamma^\mu\psi \quad \Rightarrow \text{Vector current. } J^\mu \equiv \bar{\psi}\gamma^\mu\psi \qquad (12)$$

$$\partial_\mu J^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$$

$$= (+im\bar{\psi})\gamma^\mu\psi + \bar{\psi}(-im\psi) = 0 \ !$$

* Note: $(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow \partial\!\!\!/\psi = -im\psi \Rightarrow \psi^\dagger\overleftarrow{\partial\!\!\!/}^\dagger = +im\psi^\dagger$

$\Rightarrow \psi^\dagger\gamma_0^2\overleftarrow{\partial\!\!\!/}^\dagger\gamma_0 = im\psi^\dagger\gamma_0 \quad (13)$

$\boxed{\gamma_0(\gamma^\mu)^\dagger\gamma_0 = \gamma^\mu}$ $\Rightarrow \bar{\psi}\gamma_0\overleftarrow{\partial\!\!\!/}^\dagger\gamma_0 = im\bar{\psi}$

$\Rightarrow \bar{\psi}\overleftarrow{\partial\!\!\!/} = im\bar{\psi}$

* $J^\mu = (n, \vec{J})$

$$0 = \partial_\mu J^\mu \Rightarrow \frac{\partial n}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \quad \text{(continuity eq.)} \qquad (14)$$

* $J^\mu = \bar{\psi}\gamma^\mu\psi = \bar{\psi}_L\gamma^\mu\psi_L + \bar{\psi}_R\gamma^\mu\psi_R = J^\mu_L + J^\mu_R$

where $J^\mu_{L,R} \equiv \bar{\psi}_{L,R}\gamma^\mu\psi_{L,R}$, $\psi_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5)\psi \qquad (15)$

Exercise: Show that

$$\left.\begin{aligned} \partial_\mu J^\mu_L &= -im\bar{\psi}\gamma_5\psi \\ \& \quad \partial_\mu J^\mu_R &= +im\bar{\psi}\gamma_5\psi \end{aligned}\right\} \qquad (16)$$

* Global $U(1)$ phase inv. in Dirac eq $\Longrightarrow$ Conserved Vector Current J^μ

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The Dirac Lagrangian can be rewritten as

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$
$$= \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi) \quad (17)$$

The Kinetic term $\bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R$ is invariant under the global chiral transformation

$$\psi \to e^{-i\alpha\gamma_5}\psi, \quad \alpha \text{ is real const.} \quad (18)$$

Since $\gamma_5\psi_L = -\psi_L$ & $\gamma_5\psi_R = +\psi_R$

we have
$$\left.\begin{aligned} \psi_R &\to e^{-i\alpha\gamma_5}\psi_R = e^{-i\alpha}\psi_R \\ \psi_L &\to e^{-i\alpha\gamma_5}\psi_L = e^{+i\alpha}\psi_L \end{aligned}\right\} \quad (19).$$

& the Kinetic term is invariant under chiral rotation

However the mass term

$$-m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \longrightarrow -m\, e^{-2i\alpha}(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \quad (20)$$

is not!

For infinitesimal α, (18) is $\delta\psi = -i\alpha\gamma_5\psi$, $\frac{\delta\psi}{\delta\alpha} = -i\gamma_5\psi$.

The Noether current from the Kinetic term is then (Drop α!)

$$J_5^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\cdot\frac{\delta\psi}{\delta\alpha} = \bar{\psi} i\gamma^\mu(-i\gamma_5)\psi = \bar{\psi}\gamma^\mu\gamma_5\psi \quad (21)$$

This is called the axial $U(1)$ current associated with the chiral rotation (18). $\Longrightarrow$ Chiral Symmetry.

Decomposing $\psi = \psi_L + \psi_R$, (21) becomes

$$J_5^\mu = (\bar{\psi}_L + \bar{\psi}_R)\gamma^\mu\gamma_5(\psi_L + \psi_R) = -\bar{\psi}_L\gamma^\mu\gamma_5\psi_L + \bar{\psi}_R\gamma^\mu\gamma_5\psi_R$$
$$= -J_L^\mu + J_R^\mu \quad (22)$$

Thus, from (16), we have

$$\partial_\mu J_5^\mu = -\partial_\mu J_L^\mu + \partial_\mu J_R^\mu = +2im\bar{\psi}\gamma_5\psi \neq 0 \qquad (23)$$

* The axial current J_5^μ is not conserved unless $m=0$.

* Massless Dirac's Lagrangian has two global continuous symmetries:

$$\left.\begin{array}{ll} \text{Vector } U(1) & J^\mu = \bar{\psi}\gamma^\mu\psi \\ \text{Axial } U_A(1) & J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \end{array}\right\}$$

$$\partial_\mu J^\mu = 0 \quad , \quad \partial_\mu J_5^\mu = 0 \qquad (\text{massless})$$

This is classical result. At quantum (loop) level, Classical symmetry is not necessarily hold. It turns out that the axial U(1) sym. is not preserved at quantum level. $\longrightarrow$ Anomaly (ABJ)

* Chiral Sym. is broken by mass term at tree level, and broken at quantum level in massless case.

$\Longrightarrow$ Anomaly cancellation is required if chiral sym. is a gauge (local) symmetry.

$\Longrightarrow$ Global Chiral anomaly is responsible to $\pi^0 \to \gamma\gamma$ in low energy physics.

$$\partial_\mu J_5^\mu = -\frac{e^2}{8\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} \Rightarrow$$

+ crossing diagram (24).

Here $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$

$J_5^\mu \propto f_\pi \partial^\mu \pi^0$, f_π: Pion decay constant

Chiral Symmetry (Global)

$$SU(2)_L \otimes SU(2)_R \longrightarrow SU(2)_V \qquad V = L+R$$

* Noether's theorem applies to non-abelian case too, where $\alpha \to \{\alpha_a T_a\}$ $\{T_a\}$: generators

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(3) Poincaré Sym.

(3a) Translation For infinitesimal translation,

$$x^\mu \to x^\mu - \epsilon^\mu \qquad |\epsilon^\nu| \ll 1, \qquad (25)$$

the scalar field ϕ_a, transforms as

$$\phi_a(x) \to \phi_a(\Lambda^{-1}x) = \phi_a(x) + \epsilon^\nu \partial_\nu \phi_a(x) \Rightarrow \boxed{\delta\phi_a = \epsilon^\nu \partial_\nu \phi_a} \qquad (26)$$

The Lagrangian is also a scalar, so,

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}(x). \qquad (27)$$

i.e. $\delta\mathcal{L} = \partial_\mu K^\mu$ with $K^\mu = \epsilon^\mu \mathcal{L}$

Thus the Noether current

$$J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a - K^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\epsilon^\nu\partial_\nu\phi_a - \epsilon^\mu\mathcal{L} \qquad (28)$$

$$\equiv (J^\mu)_\nu \epsilon^\nu \quad \text{with} \quad (J^\mu)_\nu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\partial_\nu\phi_a - \delta^\mu{}_\nu \mathcal{L} \qquad (29)$$

i.e. For each ν, there's a conserved current.

The energy-momentum tensor is defined as

$$T^\mu{}_\nu \equiv (J^\mu)_\nu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}\partial_\nu\phi_a - \delta^\mu{}_\nu\mathcal{L} \qquad (30)$$

And the 4-conserved charges are ($dE/dt = 0$, $d\vec{P}/dt = 0 \Leftarrow$ Noether's theorem)

$$(31) \quad \left.\begin{aligned} E &= \int d^3\vec{x}\, T^{00} &&= \text{total energy} \\ P^i &= \int d^3\vec{x}\, T^{0i} &&= \text{total momentum} \end{aligned}\right\} \text{of field configuration.}$$

For a scalar field ϕ with Lagrangian $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$,

one can derive $T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L} = T^{\nu\mu}$. (32)

hence

$$\left.\begin{aligned} E &= \int d^3\vec{x}\left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right] \\ \& \quad \vec{P} &= \int d^3\vec{x}\, \dot{\phi}\nabla\phi \end{aligned}\right\} \qquad (33).$$

(3b) Under infinitesimal Lorentz transformation $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \cong x^\nu + \theta^\mu{}_\nu x^\nu$

Thus
$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x) \cong \phi(x^\mu - \theta^\mu{}_\nu x^\nu) = \phi(x) - \theta^\mu{}_\nu x^\nu \partial_\mu \phi$$

i.e.
$$\delta\phi = -\theta^\mu{}_\nu x^\nu \partial_\mu \phi = -\partial_\mu(\theta^\mu{}_\nu x^\nu \phi) \qquad (34)$$

Thus the Lagrangian, being a scalar, also transfs like ϕ,
$$\delta\mathcal{L} = -\theta^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} \underset{\boxed{\text{use } \theta^\mu{}_\mu = 0}}{=} -\partial_\mu\big(\underbrace{\theta^\mu{}_\nu x^\nu \mathcal{L}}_{(-K^\mu)}\big) \qquad (35)$$

Once again the Lagrangian changed by a total derivative, so we have Lorentz symmetry and we can apply Noether's theorem.

$$\Rightarrow J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi - K^\mu$$
$$= \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\left(-\theta^\rho{}_\sigma x^\sigma \partial_\rho\phi\right) + \theta^\mu{}_\sigma x^\sigma \mathcal{L}$$
$$= -\theta^\rho{}_\sigma \underbrace{\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\rho\phi - \eta^\mu{}_\rho \mathcal{L}\right]}_{\text{Energy-Momentum tensor}} x^\sigma$$
$$= -\theta^\rho{}_\sigma \, T^\mu{}_\rho \, x^\sigma$$
$$= -\theta_{\rho\sigma} T^{\mu\rho} x^\sigma = -\frac{1}{2}\theta_{\rho\sigma}\left(T^{\mu\rho}x^\sigma - T^{\mu\sigma}x^\rho\right) \qquad (36)$$

Define
$$(\mathcal{J}^\mu)^{\rho\sigma} \equiv x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} \qquad (37)$$

Then $\partial_\mu J^\mu = 0$ implies $\partial_\mu (\mathcal{J}^\mu)^{\rho\sigma} = 0$ since $\theta_{\rho\sigma}$ are arbitrary parameters. So there are 3 conserved charges associated with total angular momentum for $\rho, \sigma = 1, 2, 3$:
$$Q^{ij} = \int d^3\vec{x}\, (\mathcal{J}^0)^{ij} = \int d^3\vec{x}\,(x^i T^{0j} - x^j T^{0i}) \qquad (38)$$

For $\rho, \sigma = 0, i$, we have 3 more conserved charges
$$Q^{0i} = \int d^3\vec{x}\,(\mathcal{J}^0)^{0i} = \int d^3\vec{x}\,(x^0 T^{0i} - x^i T^{00}) \qquad (39)$$

associated with the 3 boosts.

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Conserved charge Q^{0i} means $dQ^{0i}/dt = 0$.

$$\Rightarrow 0 = \frac{dQ^{0i}}{dt} = \int d^3\vec{x}\,(T^{0i}) + t\int d^3\vec{x}\,\frac{\partial T^{0i}}{\partial t} - \frac{d}{dt}\int d^3\vec{x}\,x^i T^{00}$$

$$= P^i + t\frac{dP^i}{dt} - \frac{d}{dt}\int d^3\vec{x}\,x^i T^{00}$$

Since P^i, the total momentum is a conserved charge too, $dP^i/dt = 0$, $P^i = $ const.

$$\Rightarrow \quad \frac{d}{dt}\underbrace{\int d^3\vec{x}\,x^i T^{00}}_{\text{center of field energy}} = \text{const.} \qquad (40)$$

i.e. The center of field energy moves with const. velocity.

$\Rightarrow$ Inertia law for field theory.

* These examples are for scalar fields. For other nonzero spin fields, We have to include the non-trivial transformation matrix D

$$\phi_a(x) \longrightarrow D_{ab}[\Lambda]\,\phi_b(\Lambda^{-1}x) \quad (\text{Active})$$

In this situation, $(J^\mu)^{\rho\sigma}$ contains two parts

$$(J^\mu)^{\rho\sigma} = (L^\mu)^{\rho\sigma} + (S^\mu)^{\rho\sigma}$$

with $(L^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}$ = orbital angular mom. current as before & $(S^\mu)^{\rho\sigma}$ = Spin current.

For example, for Dirac spinor. $(S^\mu)^{\rho\sigma} = \bar{\psi}\gamma^\mu S^{\rho\sigma}\psi$

with

$$S^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma] \quad , \{\gamma^\mu\} = \text{Dirac matrices.}$$

&

$$D[\Lambda] = \exp[-i\,\omega_{\rho\sigma}S^{\rho\sigma}] \quad , \omega_{\rho\sigma} = -\omega_{\sigma\rho}$$