Covariant Derivatives

Again consider a complex scalar $\phi = \phi_1 + i\phi_2$.
Suppose we let

$$\phi \to e^{-i\alpha(x)}\phi \,, \quad \alpha(x) \text{ is } x\text{-dep. instead of a const.} \qquad (1)$$

$\Rightarrow \quad m^2\phi^*\phi \to m^2\phi^*\phi$, but $(\partial_\mu\phi^*\partial^\mu\phi) \not\to (\partial_\mu\phi^*)(\partial^\mu\phi)$

Indeed,

$$\partial_\mu\phi \to e^{-i\alpha(x)}(\partial_\mu\phi - i(\partial_\mu\alpha)\phi)$$

&

$$\partial_\mu\phi^* \to e^{+i\alpha(x)}(\partial_\mu\phi^* + i(\partial_\mu\alpha)\phi^*)$$

To make $|\partial_\mu\phi|^2$ invariant, we can introduce a vector field A_μ that transforms

$$A_\mu \longrightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha \qquad (2)$$

so that

$$(\partial_\mu + ieA_\mu)\phi \to (\partial_\mu + ieA_\mu + i\partial_\mu\alpha)\, e^{-i\alpha(x)}\phi$$
$$= e^{-i\alpha(x)}(\partial_\mu + ieA_\mu)\phi \qquad (3)$$

This leads us to the concept of covariant derivative D_μ:

$$D_\mu \equiv \partial_\mu + ieA_\mu \qquad (4)$$

Under a local phase transf., $D_\mu\phi$ transf. covariantly like

$$\left.\begin{aligned} D_\mu\phi &\to e^{-i\alpha(x)}D_\mu\phi \\ \& \quad D_\mu\phi^* &\to e^{+i\alpha(x)}D_\mu\phi^* \end{aligned}\right\} \qquad (5)$$

$$\Rightarrow \quad |D_\mu\phi|^2 \longrightarrow |D_\mu\phi|^2 .$$

The local phase transf. is called gauge transformation of $U(1)$. The following Lagrangian (Scalar QED)

$$\mathcal{L} = (D_\mu\phi^*)(D^\mu\phi) - m^2\phi^*\phi \qquad (6)$$

is invariant under this local $U(1)$ gauge transf.

And A_μ is called the $U(1)$ gauge field.

More generally, we have

$$\phi_n \rightarrow e^{iQ_n \alpha(x)} \phi_n \ , \quad Q_n : \text{charge of } \phi_n \quad (7)$$

&

$$D_\mu \phi_n = \left(\partial_\mu - iQ_n e A_\mu\right) \phi_n . \quad (8)$$

In SM, $Q(e^-) = -1$, $Q(\nu_e) = 0$, $Q(u) = +\frac{2}{3}$, $Q(d) = -1/3$.
Of course, the matter fields of SM are fermions, not scalars. But the idea of covariant derivative applies to Dirac eq. as well.

$\bar{\psi} i \not\partial \psi$ is not invariant under $\psi \rightarrow e^{-i\alpha(x)} \psi$.

But

$$D_\mu \psi = (\partial_\mu - ieQA_\mu)\psi \longrightarrow e^{-i\alpha(x)} D_\mu \psi \quad (9)$$

if

$$A_\mu \longrightarrow A_\mu - \frac{1}{eQ}\partial_\mu \alpha$$

i.e. $D_\mu \psi$ transf. covariantly under local $U(1)$ gauge transf.

Thus, $\bar{\psi} \not{D} \psi \longrightarrow \bar{\psi} \not{D} \psi$ is inv. as well.

So is the mass term. $m\bar{\psi}\psi$.

Thus the following Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \ , \quad D_\mu \equiv \partial_\mu - ieQA_\mu \quad (10)$$

is invariant under local $U(1)$. Q is the charge of ψ in unit of e. (Our convention is $e > 0$.)

* (6) & (10) provide the interactions of scalar & fermions couple to the gauge field A_μ.

$U(1)$ Gauge field

What's the dynamics of A_μ?

$$\partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow \partial_\mu A_\nu - \frac{1}{eQ}\partial_\mu\partial_\nu\alpha - \left(\partial_\nu A_\mu - \frac{1}{eQ}\partial_\nu\partial_\mu\alpha\right)$$

$$\longrightarrow \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{since } \partial_\mu\partial_\nu\alpha = \partial_\nu\partial_\mu\alpha$$

Thus $(\partial_\mu A_\nu - \partial_\nu A_\mu)$ is invariant under $U(1)$ gauge transf.
To find a Lagrangian for A_μ, we need second order in A_μ & Lorentz invariant + gauge inv. $\Rightarrow (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$
Indeed,

(11). $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ (field strength)

is the familiar kinetic term for E & M.
Thus, we have two interacting theories for complex scalar & Dirac fermion with electromagnetic field.

$$\mathcal{L}_{\text{scalar QED}} = \sum_n (D_\mu\phi_n)^\dagger (D^\mu\phi_n) - m^2\phi_n^\dagger\phi_n - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (12)$$

&

$$\mathcal{L}_{QED} = \sum_n \bar\psi_n (i\not{D} - m)\psi_n - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (13)$$

with
$$\left.\begin{aligned} D_\mu\phi_n &= (\partial_\mu - ieQ_n A_\mu)\phi_n \\ \& \quad D_\mu\psi_n &= (\partial_\mu - ieQ_n A_\mu)\psi_n \end{aligned}\right\} \quad (14).$$

* Note a mass term $A_\mu A^\mu$ is <u>not</u> allowed by gauge invariance. But, ...

* <u>Stueckelberg Mechanism</u>: Introduce the Stueckelberg field σ (a scalar) transforms under the $U(1)$ gauge transformation as

$$\left.\begin{aligned} \sigma &\to \sigma - \frac{m_\gamma}{e}\alpha \\ \text{along with } A_\mu &\to A_\mu + \frac{1}{e}\partial_\mu\alpha \end{aligned}\right\} \quad (15)$$

The following mass term for the photon

(16). $+\frac{m_\gamma^2}{2}\left(A_\mu + \frac{1}{m_\gamma e}\partial_\mu\sigma\right)^2$ is invariant under (15).

The experimental limit of photon's mass is

$$m_\gamma < 1 \times 10^{-18}\ eV$$

In fact, from PDG (2022), Prog. Theor. Exp. Phys. 2022, 083C01, we have the profile of photon:

γ (Photon)	$J^{PC} = 1^{--}$	
Mass m_γ	$< 1 \times 10^{-18}\ eV$	
Charge q_γ	$< 1\times 10^{-46}\ e$	(Mixed Charge) ?
	$< 1\times 10^{-35}\ e$	(single charge)
Mean life τ	$=$ Stable	

(17)

$\Rightarrow$ Forget the Stueckelberg mass term for the photon!
* (17) q_γ is vanishing small, photon doesn't interact among themselves!

But it is quite popular to use Stueckelberg Mechanism for dark photon from a hidden $U(1)_D$ sector in BSM.

Now we have the kinetic term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ for the photon.
What's the propagator for the photon?
Recall that the propagator is the inverse of the kinetic operator. For example, for the real scalar ϕ

$$\langle 0 | T[\phi(x)\phi(y)] | 0\rangle = \langle x | \frac{i}{-(\Box + m^2)} | y \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2} \tag{18}$$

for the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2 = -\frac{1}{2}\phi(\Box + m^2)\phi$ + total derivative

And for Dirac fermion with $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$,

$$\langle 0 | T[\psi(x)\bar{\psi}(y)] | 0\rangle = \langle x | \frac{i}{i\partial\!\!\!/ - m} | y \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p\!\!\!/ - m}$$

i.e.

$$\langle 0 | T[\psi_\alpha(x)\bar{\psi}_\beta(y)] | 0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i(p\!\!\!/ + m)_{\alpha\beta}}{p^2 - m^2} \tag{19}$$

Now for the photon field, we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{4}\partial_\mu A_\nu \partial^\mu A^\nu - \frac{1}{4}\partial_\nu A_\mu \partial^\nu A^\mu + \frac{1}{4}\partial_\mu A_\nu \partial^\nu A^\mu + \frac{1}{4}\partial_\nu A_\mu \partial^\mu A^\nu$$

$$= -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu$$

$$= -\frac{1}{2}\Big(\underbrace{\partial_\mu(A_\nu \partial^\mu A^\nu)}_{\text{total derivative}} - A_\nu \Box A^\nu\Big) + \frac{1}{2}\Big(\underbrace{\partial_\mu(A_\nu \partial^\nu A^\mu)}_{\text{total derivative}} - A_\nu \partial_\mu \partial^\nu A^\mu\Big)$$

total derivative ⟵ Drop ⟶ total derivative
without impact on EOM.

$$= +\frac{1}{2}A^\mu \Box g_{\mu\nu} A^\nu - \frac{1}{2}A^\mu(\partial_\mu \partial_\nu)A^\nu$$

$$= \frac{1}{2}A^\mu \underbrace{\left(g_{\mu\nu}\Box - \partial_\mu \partial_\nu\right)}_{\text{Kinetic operator for } A^\mu} A^\nu \qquad (20)$$

The trouble is this kinetic operator is <u>not</u> invertible.
How to see this? In momentum space,

$$g_{\mu\nu}\Box - \partial_\mu\partial_\nu \longrightarrow (-g_{\mu\nu}q^2 + q_\mu q_\nu) . \qquad (21)$$

Since $(-g_{\mu\nu}q^2 + q_\mu q_\nu)q^\nu = -q^2 q_\mu + q_\mu q^2 = 0$, that means

q^ν is an eigenvector of $(-g_{\mu\nu}q^2 + q_\mu q_\nu)$ with 0 eigenvalue!

$\Rightarrow$ $\mathrm{Det}(-g_{\mu\nu}q^2 + q_\mu q_\nu)$ must be vanishing.

$\Rightarrow$ $(-g_{\mu\nu}q^2 + q_\mu q_\nu)$ has <u>no</u> inverse! (Linear algebra)

The solution is <u>gauge fixing</u> because the non-invertible is a manifestation of gauge invariance. One has to break the gauge symmetry somehow!

<u>Exercise</u>: Show that $\mathrm{Det}(-g_{\mu\nu}q^2 + q_\mu q_\nu) = 0$ &

$$(-g_{\mu\nu}q^2 + q_\mu q_\nu)q^\nu = 0 . \qquad (22)$$

Gauge fixings: In E&M, we learnt about the Coulomb gauge $\nabla\cdot\vec{A}=0$ & Lorenz gauge $\partial_\mu A^\mu=0$. In QED, or general gauge theories, covariant gauge conditions are more useful in practical computations. To implement gauge fixings in a more systematic way in Lagrangian formulation, one introduces auxiliary field (non-propagating) ξ as Lagrange Multiplier to enforce the constraint through EOM. Thus

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{23}$$

* Euler-Lagrangian eq. for ξ is simply $\frac{\partial\mathcal{L}}{\partial\xi}=0 \Rightarrow (\partial_\mu A^\mu)^2=0$
$\Rightarrow \partial_\mu A^\mu = 0$.
* We need $(\partial_\mu A^\mu)^2$ instead of $(\partial_\mu A^\mu)$ because we want to modify the kinetic operator

From (20), we have

$$\mathcal{L} = \frac{1}{2}A^\mu(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)A^\nu + \frac{1}{2\xi}A^\mu\partial_\mu\partial_\nu A^\nu + \text{total derivative}$$

$$= \frac{1}{2}A^\mu\left(g_{\mu\nu}\Box - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu\right)A^\nu + \text{total derivative}$$

$$\xrightarrow[\text{space}]{\text{Momentum}} \frac{1}{2}\tilde{A}^\mu(q)\left(-g_{\mu\nu}q^2 + \left(1-\frac{1}{\xi}\right)q_\mu q_\nu\right)\tilde{A}^\nu(-q) \tag{24}$$

Now $-g_{\mu\nu}q^2 + (1-\frac{1}{\xi})q_\mu q_\nu$ has an inverse, since its determinant is nonvanishing in general.

Check:

$$\left.\begin{aligned} &\mathrm{Det}\left(-g_{\mu\nu}q^2 + \left(1-\frac{1}{\xi}\right)q_\mu q_\nu\right) = -\frac{1}{\xi}(q^2)^4 \neq 0 \\ \& \quad &\left(-g_{\mu\nu}q^2 + \left(1-\frac{1}{\xi}\,q_\mu q_\nu\right)\right)q^\nu = \frac{-1}{\xi}q^2\cdot q_\mu \neq 0 \end{aligned}\right\} \tag{25}$$

The inverse of $(-q^2 g_{\mu\nu} + (1-\frac{1}{\xi}) q_\mu q_\nu)$ is

$$\Pi_{\mu\nu} = -\left(\frac{g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}}{q^2} \right) \qquad (R_\xi \text{ gauge}) \qquad (26)$$

R_ξ: Renormalization gauge

One can check easily that

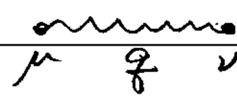
$$\left(-q^2 g_{\mu\nu} + (1-\tfrac{1}{\xi}) q_\mu q_\nu\right) \Pi^{\nu\alpha} = g_\mu{}^\alpha = \delta_\mu{}^\alpha \qquad (27)$$

This implies the photon propagator is given by

$$\langle 0 | T[A_\mu(x) A_\nu(y)] | 0 \rangle = \langle x | \frac{i}{g_{\mu\nu}\Box - (1-\frac{1}{\xi})\partial_\mu \partial_\nu} | y \rangle$$

$$= \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)} (-i) \frac{1}{q^2+i0^+} \left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2} \right)$$

$$\equiv \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot(x-y)}\, i\,\Pi_{\mu\nu}(q) \qquad (28)$$



Covariant Gauges

(1) 't Hooft-Feynman gauge $\xi = 1$

$$i\Pi_{\mu\nu}(q) = \frac{-i\, g_{\mu\nu}}{q^2 + i0^+} \qquad (29)$$

(2) Lorenz gauge $\xi = 0$ (Invert first and set $\xi \to 0$ afterward!)

$$(30) \qquad i\Pi_{\mu\nu}(q) = -\frac{i}{q^2+i0^+}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)$$

(* $\xi \to 0$ forces $\partial_\mu A^\mu = 0$ in the path integral!)

(3) Unitary gauge $\xi \to 0$

Non-usefull for photon. But important for EW gauge theory. (later)

Noncovariant gauges:

(1) Light cone gauge $n_\mu A^\mu = 0$, $n_\mu n^\mu = 0$ (light-like)

(2) Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$

(3) Radial (Fock-Schwinger) gauge $x_\mu A^\mu(x) = 0$.

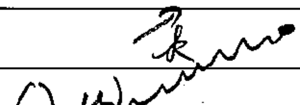
* For QED, most calculations are performed in covariant gauges.
* Lorentz-invariant quantities must be gauge-independent, i.e. independent of the ξ parameter. i.e. the $q^\mu q^\nu$ term in $i\Pi^{\mu\nu}(q)$ should not have physical effects!
See proof in QFT textbook, e.g. M. Schwartz

Quantization: $\frac{1}{\sqrt{2\omega_k}}|k, \epsilon(k,\lambda)\rangle = a_k^\dagger(\lambda)|0\rangle$, $\langle 0|A_\mu(x)|k, \epsilon\rangle = \epsilon_\mu(k,\lambda)e^{-ik\cdot x}$

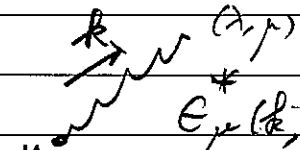
$$A_\mu(x) = \sum_\lambda \int \frac{d^4k}{(2\pi)^4}\frac{1}{2\omega_k}\Big[\underbrace{a_k(\lambda)}_{\text{annihilates a photon}}\epsilon_\mu(k,\lambda)e^{-ik\cdot x} + \underbrace{a_k^\dagger(\lambda)}_{\text{creates a photon}}\epsilon_\mu^*(k,\lambda)e^{+ik\cdot x}\Big]$$

$\boxed{\epsilon_\mu \text{ is transverse}: p^\mu\epsilon_\mu = 0}$

$[a_k(\lambda), a_{k'}^\dagger(\lambda')] = (2\pi)^3\delta(\vec{k}-\vec{k}')\delta_{\lambda\lambda'}$



$\epsilon_\mu(k,\lambda)$



$\epsilon_\mu^*(k,\lambda)$

$\epsilon_\mu(k,\lambda)$ = polarization wave function of helicity λ and momentum k.

Linear polarization:

$$\epsilon_1^\mu = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad \epsilon_2^\mu = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

Wave travels in $\hat{z}$ direction
$k = (E, 0, 0, E)$

$$\boxed{\text{Polarization Sum: } \sum_\lambda \epsilon_\mu^*\epsilon_\nu \to -g_{\mu\nu}}$$

Circular polarization:

$$\epsilon_L^\mu = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\\0\end{pmatrix}, \quad \epsilon_R^\mu = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\\0\end{pmatrix}$$

$$\epsilon_L^\mu = \frac{1}{\sqrt{2}}(\epsilon_1^\mu - i\epsilon_2^\mu), \quad \epsilon_R^\mu = \frac{1}{\sqrt{2}}(\epsilon_1^\mu + i\epsilon_2^\mu)$$

Now back to Scalar QED in spinor QED in (12) & (13).

$$\mathcal{L}_{\text{Scalar QED}} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{gauge fixing} \quad (31)$$
$$= \mathcal{L}_0 + \mathcal{L}_{int}$$

$$\mathcal{L}_0 = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - m^2\phi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (32)$$

$$\mathcal{L}_{int} = +ieQ\phi^\dagger A_\mu\partial^\mu\phi - ieQ(\partial_\mu\phi^\dagger)A^\mu\phi + e^2Q^2A_\mu A^\mu\phi^\dagger\phi \quad (33)$$
$$= eQJ_\mu A^\mu$$

with $J_\mu \equiv -i\left((\partial_\mu\phi^\dagger)\phi - \phi^\dagger\partial_\mu\phi\right) + eQA_\mu\phi^\dagger\phi \quad (34)$

Now this J_μ is the electromagnetic current. In fact, one can show this is a conserved current.

<u>Exercise:</u> Show that $\partial_\mu J^\mu = 0$ using EOM.

[Hint: Derive the eq. of motion for ϕ & ϕ^* first from the Lagrangian (31).]

Next $\mathcal{L}_{QED} = \bar\psi(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \text{gauge fixing} \quad (35)$

$$= \mathcal{L}_0 + \mathcal{L}_{int}$$

where $\mathcal{L}_{int} = \bar\psi i(-ieQ\gamma_\mu)A^\mu\psi = eQJ_\mu A^\mu \quad (36)$

with $J_\mu \equiv \bar\psi\gamma_\mu\psi = \bar\psi_L\gamma_\mu\psi_L + \bar\psi_R\gamma_\mu\psi_R \quad (37)$

= Noëther's current

One can also show that $\partial_\mu J^\mu = 0$ using EOM for ψ & $\bar\psi$.

* Gauge field couples to conserved charge current!

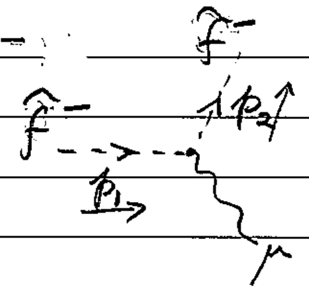
$\tilde{f}^-$ = particle, $\tilde{f}^+$ = anti-particle

Feynman Rules for Scalar & Spinor QED:

Scalar QED

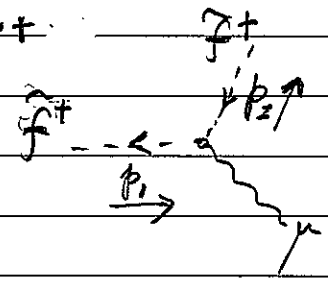
$\longrightarrow$ time flow direction

$\tilde{f}^- \to \tilde{f}^-$



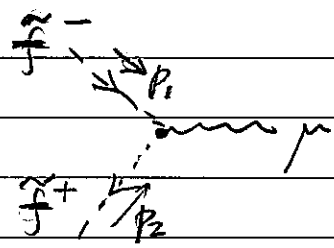
$$-ieQ(-p_1 - p_2)_\mu$$ ✓

$\tilde{f}^+ \to \tilde{f}^+$



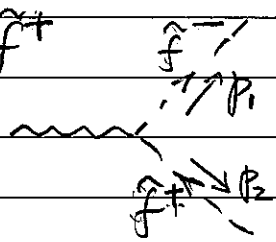
$$-ieQ(p_1 + p_2)_\mu$$ ✓

$\tilde{f}^- \tilde{f}^+ \to \gamma$

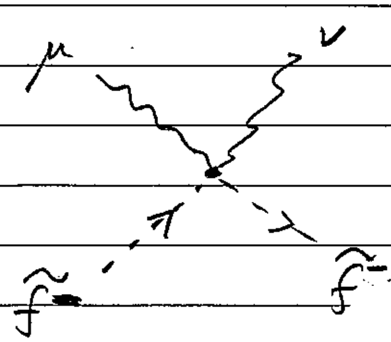


$$-ieQ(-p_1 + p_2)_\mu$$ ✓

$\gamma \to \tilde{f}^- \tilde{f}^+$



$$-ieQ(-p_1 + p_2)_\mu$$ ✓

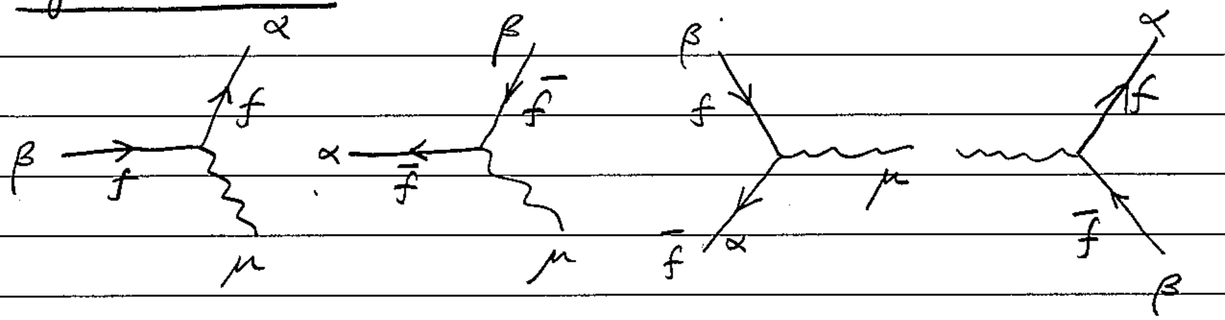


$$2 i e^2 Q^2 g_{\mu\nu}$$ ✓

due to 2 possible Wick's contractions!

Symmetry factor, if any

Spinor QED

(i)  $= ieQ_f(\gamma^\mu)_{\alpha\beta}$

(ii) Figure out the overall sign of the diagram. (Fermi statistics)

(iii) Closed fermion loop: (-1) and trace over Dirac matrices.

(iv) Symmetry factor.

Exercise: (Scalar QED = SQED)

Use the SQED Feynman rules to draw the Feynman diagrams and write down the Lorentz invariant amplitudes for each of the following processes to leading order:

(1) $\tilde{e}^- \tilde{e}^- \to \tilde{e}^- \tilde{e}^-$ (Selectron Scattering)

(2) $\tilde{e}^- \tilde{e}^+ \to \tilde{e}^- \tilde{e}^+$ (Selectron - Positron Scattering)

(3) $\tilde{e}^- \gamma \to \tilde{e}^- \gamma$ (Compton Scattering)

(4) $\tilde{e}^- \tilde{e}^+ \to \gamma \gamma$ (Selectron-Spositron annihilation)

Exercise: Repeat the above exercise for Spinor QED.

Non-Abelian Gauge Theories

Consider N complex Dirac fields ψ^a, $a = 1, 2, \ldots, N$.

The free Lagrangian

$$\mathcal{L}_0 = \sum_{a=1}^{N} \bar{\psi}_a (i\not{\partial} - m)\psi^a \quad , \quad m_a = m \;\; \forall a = 1, \ldots, N$$

is invariant under the <u>global</u> transf.

$$\psi(x) \to \psi'(x) = \mathcal{U}\,\psi(x), \qquad \mathcal{U} \in U(N)$$

any unitary $N\times N$ matrix

* If not all the masses are equal, we will have a smaller subgroup symmetry of $U(N)$. In general, $\{\psi_a\}$ will belong to some reducible representation $\underset{\frown}{R}$ of Lie group (algebra) G.

* Classical groups G:

$\boxed{S \to \text{Special}, \; \text{Det} = 1}$

(Orthogonal, Unitary, Symplectic) $SO(N)$, $SU(N)$, $Sp(2N)$

5 Exceptional Groups: G_2, F_4, E_6, E_7, E_8

* We will focus on G = unitary group: $SU(N)$ in these lectures.

Let $\mathcal{U} \in G$, a generic element, then

$$\mathcal{U} = \exp(i\alpha_A T^A) \quad , \{T^A\},\; A = 1, 2, \ldots, \text{dim. of } G$$

$\alpha_A \to$ real constants. $\{T^A\}$ are the generators of G, i.e. Lie's algebra of G.

Lie algebra of G

$$[T^A, T^B] = i f^{ABC} T^C \quad , \quad f^{ABC}$$

: structure const. of Lie Algebra of G. They can be chosen to be totally antisym. in A, B & C.

* For real constants α_A, T^A are traceless & Hermitian $N\times N$ matrices. They can be chosen to satisfy the normalization for fundamental irrep. F

$$* \quad \text{Tr}[T^A T^B] = \frac{1}{2}\delta^{AB}$$

$$* \quad T^A = (T^A)^\dagger \; , \quad \text{Tr}\, T^A = 0$$

$\to$ fix f^{ABC} normalization

$\Rightarrow$ Simple (No $U(1)$ factor)

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We note that $\{T^A\}$ are abstract objects in general. The Lie algebra $[T^A, T^B] = i f^{ABC} T^C$

implies the Jacobi identity

$$[T^A, [T^B, T^C]] + [T^B, [T^C, T^A]] + [T^C, [T^A, T^B]] = 0$$

which is satisfied by matrices too.

~~Jacobi-identity implies constraints on structure constants~~

$$f^{ADE} f^{BCD} + f^{BDE} f^{CAD} + f^{CDE} f^{ABD} = 0.$$

<u>Adjoint irrep:</u> $\dim(\text{adj}) = \dim(G)$

$$(T^A(\text{adj.}))_{BC} = -i f^{ABC}.$$

<u>Fundamental irrep:</u> $\dim(F)$ smallest dim.

with $\mathrm{Tr}(T^A(F)\, T^B(F)) = \frac{1}{2}\delta^{AB}$

In general, $\mathrm{Tr}(T^A(R)\, T^B(R)) = I(R)\,\delta^{AB}$

$I(R)$ is called the Dynkin index. $I(F) = 1/2$.

Quadratic Casimir operator: $\quad \rightarrow$ Casimir

$$T^A(R)\, T^A(R) = C_2(R)\,\mathbb{1}$$

<u>Exercise:</u> Show that

$$I(R)\cdot \mathrm{Dim}(G) = C_2(R)\cdot \mathrm{Dim}(R)$$

No $U(1)$s floating around! Otherwise

Classical Lie Algebras G (Compact, simple) ; semi-simple

G	$SU(N)$	$SO(N)$	$Sp(2N)$	G_2	F_4	E_6	E_7	E_8
Dim G	N^2-1	$\frac{1}{2}N(N-1)$	$N(2N+1)$	14	52	78	183	248
Dim F	N	N	$2N$	7	6	27	56	248
Rank	$N-1$	n for $N=2n+1$ n for $N=2n$	N	2	4	6	7	8

* F : Fundamental irrep. (smallest irrep.)

* $Sp(2) = SU(2)$.

Other notations for $Sp(2N)$ = $Sp(N)$ or $USp(2N)$

* Cartan's Notations:

$$A_n = SU(n+1)$$
$$B_n = SO(2n+1)$$
$$C_n = Sp(2n)$$
$$D_n = SO(2n)$$
$$G_2,\ F_4,\ E_6,\ E_7,\ E_8$$

* Nonabelian feature of G implies in general that

$$\exp(i\alpha_A T^A)\exp(i\beta_A T^A) \neq \exp(i\beta_A T^A)\exp(i\alpha_A T^A).$$

* The conserved global noether current can be easily generalized to the non-abelian case with multiple fields $\{\phi^a\}$

$\delta\mathcal{L}=0$
$\Rightarrow K^\mu=0$

$$J^\mu = + \sum_a \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}\delta\phi^a$$

← This form of J^μ contains infinitesimal parameters α's

where $\delta\phi^a$ is the infinitesimal transf. undergone by the field ϕ^a.

Conserved Charge is

$$Q = \int d^3\vec{x}\, J^0 = + \int d^3\vec{x} \underbrace{\frac{\partial\mathcal{L}}{\partial\partial_0\phi^a}}_{\equiv \Pi_a \text{ Generalized Momentum}} \delta\phi^a = + \int d^3\vec{x}\, \Pi_a \delta\phi^a$$

[Recall $\Pi_a \equiv \frac{\partial\mathcal{L}}{\partial\partial_0\phi^a} = \frac{\partial\mathcal{L}}{\partial\dot\phi^a}$ (conjugate Momentum of ϕ^a)]

For infinitesimal small αs, we have

$$\mathcal{U} = \exp(i\alpha_A T^A) = 1 + i\alpha_A T^A + \cdots$$

$$\delta\phi^a = i\alpha_A (T^A)^a{}_b \phi^b$$

Then, $J^\mu = +i\alpha_A \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(T^A)^a{}_b\phi^b \equiv +\alpha_A j^{\mu A}$

where

$$j^{\mu A} \equiv +i \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi^a}(T^A)^a{}_b\phi^b \quad \text{for each } A=1,\cdots,\dim. \text{ of } G.$$

[e.g. N^2-1 for $SU(N)$]

So the corresponding conserved charge is

$$Q^A = \int d^3\vec{x}\, j^{0A} = +i\int d^3\vec{x}\, \Pi_a (T^A)^a{}_b \phi^b \quad \text{for each } A.$$

* In CM, (or the Hamiltonian formulation of classical physics), given conserved charges, we can reconstruct the associated symmetry: For any (non-spinor) operator O,

*See Georgi's Group Theory Book: Section 2.6 ← $\delta_A O = i[Q^A, O]$ ✓ $\Rightarrow \delta_A\phi^a = i[Q^A,\phi^a] = i(T^A)^a{}_b\phi^b$ ✓!

(Jacobi-identity &)

In QFT, one can show that using canonical quantization (Commutation or anti-commutation relations depending on bosons or fermions), the conserved charge Q^A satisfies the Lie algebra of the sym. group G. i.e.

$$[Q^A, Q^B] = i f^{ABC} Q^C \qquad \text{"Inverse Noëther theorem"}$$

So far, it is quite general. We can apply to N Dirac fermions $\{\phi_a\} \to \{\psi_a\}$. Then the conserved current is

$$j^{\mu A} = +i \frac{\partial \mathcal{L}_0}{\partial \partial_\mu \psi^a} (T^A)^a{}_b \psi^b = +i \left(-i\bar\psi_a \gamma^\mu (T^A)^a{}_b\right)\psi^b$$

Grassmanian - !

where

$$\mathcal{L}_0 = \sum_{a=1}^{N} \bar\psi_a (i\gamma^\mu \partial_\mu - m)\psi^a$$

i.e. for each generator T^A, we have a conserved current

$$j^{\mu A} = \bar\psi T^A \gamma^\mu \psi, \quad \psi^T = (\psi^1, \psi^2, \cdots, \psi^N).$$

corresponding to the $SU(N)$. i.e $\mathcal{L}_0$ has a $SU(N)$ symmetry.

Exercise: What is the maximal internal symmetry of a system of N free massless Dirac fields ψ^a $(a=1,\cdots,N)$ specified by the following Lagrangian

$$\mathcal{L}_0 = \sum_{a=1}^{N} \bar\psi_a (i\gamma^\mu \partial_\mu)\psi^a.$$

Hint: $\psi = \psi_L + \psi_R$.

If one promotes $\alpha_A \to \alpha_A(x)$ for local transformation, then

$$\partial_\mu \psi \not\to \exp(i\alpha_A(x) T^A)\, \partial_\mu \psi .$$

As in the abelian $U(1)$ case, we introduce covariant derivative D_μ such that $D_\mu \psi(x) \longrightarrow \exp(i\alpha_A(x) T^A)\, D_\mu \psi(x)$.

i.e. $D_\mu \psi$ & ψ transform the same way, which implies $(D_\mu \cdots D_\nu)\psi$ transf. like ψ as well.

Let $D_\mu \equiv \partial_\mu - ig B_\mu$. g: gauge coupling, real.

↑ (Peskin - Schroeder) $B_\mu = N \times N$ matrix acting on $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^N \end{pmatrix}$

* Convention dependent. Some authors used + !

B_μ → Vector fields (gauge fields)

Denote $\mathcal{U} \equiv \exp(i\alpha_A(x) T^A)$.

$$
\begin{aligned}
D_\mu \psi \to (\partial_\mu - ig B_\mu')\psi' &= (\partial_\mu - ig B_\mu')\, \mathcal{U}(x) \psi(x) \\
&= \mathcal{U}(x)\partial_\mu \psi + (\partial_\mu \mathcal{U}(x))\psi - ig B_\mu' \mathcal{U}(x) \psi \\
&= \mathcal{U}(x)\partial_\mu \psi + \underbrace{\mathcal{U}\mathcal{U}^{-1}}_{\mathbb{1}}(\partial_\mu \mathcal{U})\psi - ig \underbrace{\mathcal{U}\mathcal{U}^{-1}}_{\mathbb{1}} B_\mu' \mathcal{U} \psi \\
&= \mathcal{U}\left[\partial_\mu \psi + \mathcal{U}^{-1}(\partial_\mu \mathcal{U})\psi - ig \mathcal{U}^{-1} B_\mu' \mathcal{U} \psi\right] \\
&= \mathcal{U}\left[\partial_\mu + \mathcal{U}^{-1}\partial_\mu \mathcal{U} - ig \mathcal{U}^{-1} B_\mu' \mathcal{U}\right]\psi .
\end{aligned}
$$

Wanted!

$$= \mathcal{U} D_\mu \psi$$

This implies $\mathcal{U}^{-1}\partial_\mu \mathcal{U} - ig\,\mathcal{U}^{-1} B_\mu' \mathcal{U} = -ig B_\mu$

i.e. B_μ must transf. as

$$\boxed{B_\mu' = \mathcal{U} B_\mu \mathcal{U}^{-1} - \frac{i}{g}(\partial_\mu \mathcal{U})\mathcal{U}^{-1}}$$

To proceed further, let $\mathcal{U}(x) = 1 + i\alpha_A(x)\, T^A + \cdots$

Then

$$\mathcal{U} B_\mu \mathcal{U}^{-1} = (1 + i\alpha_A T^A + \cdots) B_\mu (1 - i\alpha_B T^B - \cdots)$$

$$= B_\mu + i\alpha_A(x) [T^A, B_\mu] + O(\alpha^2)$$

&

$$(\partial_\mu \mathcal{U}) \mathcal{U}^{-1} = i(\partial_\mu \alpha_A) T^A + O(\alpha^2)$$

$$\Rightarrow \delta B_\mu = B'_\mu - B_\mu = i\alpha_A [T^A, B_\mu] + \frac{1}{g}(\partial_\mu \alpha_A) T^A + O(\alpha^2)$$

Let $B_\mu \equiv A^A_\mu T^A$, $A^A_\mu : (A = 1, \cdots, N^2 - 1)$
$N^2 - 1$ gauge fields.

In terms of components, we have

$$(B_\mu)^a{}_b = (A_\mu)(T^A)^a{}_b \Longrightarrow (\delta B_\mu)_{ab} = \delta A^A_\mu (T^A)_{ab}$$

$(T^A)^a{}_b \longrightarrow (T^A)_{ab}$ * from now on, we will be sloppy about up/down indices in the generators. They are usually summed over, anyway

Thus

$$(\delta A^C_\mu)(T^C)_{ab} = i\alpha_A [T^A, T^B]_{ab} A^B_\mu + \frac{1}{g}(\partial_\mu \alpha_C)(T^C)_{ab}$$

$$= i\alpha_A i f^{ABC} (T^C)_{ab} A^B_\mu + \frac{1}{g}(\partial_\mu \alpha_C)(T^C)_{ab}$$

we thus obtain for infinitesimal local transformation,

A^C_μ transforms like

$$\delta A^C_\mu = \underbrace{- \alpha_A f^{ABC} A^B_\mu}_{\text{Extra piece due to non-abelian feature.}} + \underbrace{\frac{1}{g}\partial_\mu \alpha_C}_{\mathcal{U}(1) \text{ abelian case}} \checkmark$$

Extra piece due to non-abelian feature.

This is homogeneous term, i.e. no derivative!

$\mathcal{U}(1)$ abelian case

Convention dep. (Peskin - Schroeder)
This correlates with
$D_\mu \equiv \partial_\mu \ominus ig A_\mu$

Field Strength Curvature is defined as

$$[D_\mu, D_\nu]\psi = \ominus ig F_{\mu\nu}\psi = -ig F^A_{\mu\nu} T^A \psi$$

Under gauge transformation, Definition of covariant derivatives

$$
\begin{aligned}
[D_\mu, D_\nu]\psi \longrightarrow ([D_\mu, D_\nu]\psi)' &= \mathcal{U}([D_\mu, D_\nu]\psi] \\
&= \mathcal{U}(-ig F_{\mu\nu}\psi) \\
&\xrightarrow{\text{Definition}} = -ig F'_{\mu\nu}\psi' \\
&= -ig F'_{\mu\nu}\mathcal{U}\psi
\end{aligned}
$$

Since ψ is arbitrary, we must have

$$\mathcal{U} F_{\mu\nu} = F'_{\mu\nu}\mathcal{U} \Rightarrow \boxed{F'_{\mu\nu} = \mathcal{U} F_{\mu\nu}\mathcal{U}^{-1}}$$

* For non-abelian theories, the field strength is no longer invariant by itself. It transforms homogenously instead.

* However $\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$ is invariant under $F'_{\mu\nu} = \mathcal{U} F_{\mu\nu}\mathcal{U}^{-1}$!

$$\Rightarrow \mathcal{L}_{K.E.} = -\frac{1}{2}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$ is a gauge invariant kinetic term for the gauge field A_μ.

$$= -\frac{1}{2} F^A_{\mu\nu} F^{B\mu\nu} \underbrace{\mathrm{Tr}[T^A T^B]}_{\frac{1}{2}\delta^{AB}\ (\text{fundamental})}$$

$$= -\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu A}$$ is analogous to Maxwell's theory.

What is $F^A_{\mu\nu}$? From definition, $-ig F_{\mu\nu}\psi = [D_\mu, D_\nu]\psi = [\partial_\mu - igB_\mu, \partial_\nu - igB_\nu]\psi = [\partial_\mu, \partial_\nu]\psi - ig[B_\mu, \partial_\nu]\psi - ig[\partial_\mu, B_\nu]\psi - g^2[B_\mu, B_\nu]\psi$

$$
\begin{aligned}
&= 0 - ig[B_\mu\partial_\nu\psi - \partial_\nu(B_\mu\psi)] - ig[\partial_\mu(B_\nu\psi) - B_\nu\partial_\mu\psi] - g^2[B_\mu, B_\nu]\psi \\
&= -ig[B_\mu\partial_\nu\psi - (\partial_\nu B_\mu)\psi - B_\mu\partial_\nu\psi] - ig[(\partial_\mu B_\nu)\psi + B_\nu\partial_\mu\psi - B_\nu\partial_\mu\psi] \\
&\quad - g^2[B_\mu, B_\nu]\psi = -ig[-(\partial_\nu B_\mu)\psi + (\partial_\mu B_\nu)\psi] - g^2[B_\mu, B_\nu]\psi
\end{aligned}
$$

$$\Rightarrow \quad -ig F_{\mu\nu} \psi = -ig [(\partial_\mu B_\nu) - (\partial_\nu B_\mu)] \psi - g^2 [B_\mu, B_\nu] \psi$$

Since ψ is arbitrary we derive

$$\Rightarrow F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu - ig [B_\mu, B_\nu]$$

$$\Rightarrow F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f^{ABC} A_\mu^C$$

Crucial features:
* ~~Renormalizable~~
* C&P invariance
* gauge invariant

<u>SU(N) gauge theory</u> (Yang Mills)

$$\left.\begin{array}{l} \mathcal{L} = -\frac{1}{2} \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i \not{D} - m) \psi \\ D_\mu = \partial_\mu - ig T^A A^A_\mu , \quad T^A : \text{Generators of } SU(N) \text{ (or any Lie group } G) \end{array}\right\}$$

$$\psi = (\psi_1, \psi_2, \cdots, \psi_N)^T$$

In terms of components,

$$\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu A} + \bar{\psi}_a (i\not{\partial} - m_a) \psi_a + g A^A_\mu J^{\mu A}$$

with $J^{\mu A} \equiv \bar{\psi}_a \gamma^\mu (T^A)_{ab} \psi_b$

* <u>QCD</u>: $N=3$. $\psi^a \to$ 6 quark fields $(u, d, s, c, b, t)^a$ each comes with 3 colors!
* $J^{\mu A}$ is a global current of the fermion fields. i.e. $\partial_\mu J^{\mu A} = 0 \quad \forall A = 1, \cdots, N^2-1$. $\quad \dim(SU(N)) = N^2 - 1$.
* The above Lagrangian holds for <u>any</u> Lie group G.
* The mass term $A^A_\mu A^{\mu A}$ is not allowed since it is <u>not</u> invariant under gauge transformation.
* There's <u>no</u> Stueckelberg Mechanism to give gauge-inv. mass to non-abelian gauge bosons like the abelian case!

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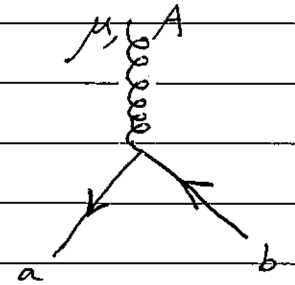
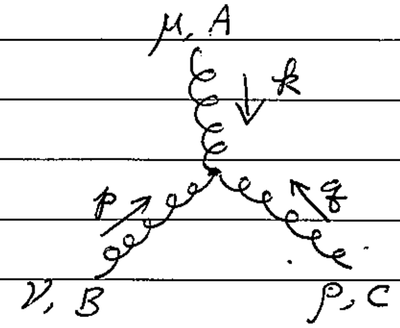
The most elegant approach to quantize a YM gauge theory is using Feynman's Path integral quantization à la Faddeev-Popov.

Details can be found in any decent QFT textbooks.
Partial lists of the Feynman rules are (Peskin-Schroeder, Fig. 16.1, page 507):

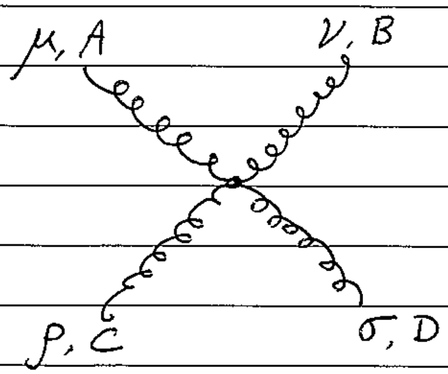


$$= ig\gamma^\mu (T^A)_{ab}$$



$$= gf^{ABC}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$

$(p+q+k=0)$



$$= -ig^2\left[f^{ABE}f^{CDE}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ACE}f^{BDE}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ADE}f^{BCE}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\right]$$

Fermion, Ghost loop (-1)

+ Ghost vertices.

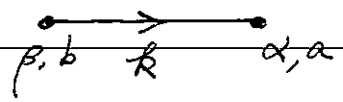
Fermion Propagator

$$\left(S_F(x,y)\right)_{\alpha\beta,ab} = \langle 0 | T\{\psi_{\alpha a}(x)\,\bar{\psi}_{\beta b}(y)\} | 0 \rangle$$

$\alpha, \beta \in$ Dirac indices

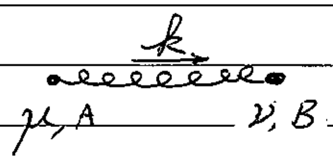
$a, b = 1, \cdots, \dim(\underset{\sim}{R})$

$$= \delta_{ab} \int \frac{d^4k}{(2\pi)^4} \left(\frac{i}{\not{k} - m}\right)_{\alpha\beta} e^{-ik\cdot(x-y)}$$



Gauge Boson propagator

$$\Delta_F(x,y) = \langle 0 | T\{A^A_\mu(x)\, A^B_\nu(y)\} | 0 \rangle$$



$$= \delta_{AB} \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\mu\nu}}{k^2 + i0^+} e^{-ik\cdot(x-y)}$$

$A, B = 1, \cdots, \dim(G)$

(~~'t Hooft-Feynman gauge~~)

* To apply these Feynman rules to perform perturbative calculations in QCD, one needs to have justifications since QCD is for strong interacting quarks. The coupling constant g is not small at low energy. In fact, we only see hadrons (mesons & baryons) at low energy. PQCD is only applicable at high energies where the coupling constant g can become small.

$\Longrightarrow$ Asymptotic Freedom for Non-Abelian theories $\Big\} \longrightarrow$ $\boxed{\text{QCD Lectures}}$

Spontaneous Symmetry Breaking (SSB)

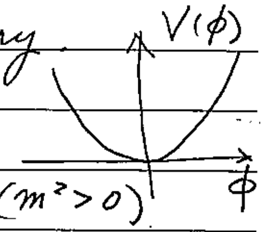
Consider a simple system of a *single* hermitian scalar ϕ:

$$\mathcal{L}(\phi) = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \equiv \frac{1}{2}(\partial^\mu\phi)^2 - V(\phi) \quad (1)$$

Eq. (1) is invariant under $\quad \phi \to -\phi \quad$ (Discrete) $\quad (2)$

$\Rightarrow$ No cubic ϕ^3 term in the original theory!

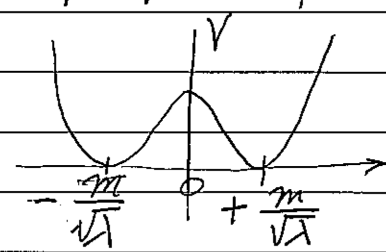
Suppose λ is small, so we use perturbation theory.
For $m^2 > 0$, we have an unique vacuum $\phi = 0$:



Suppose we flip the sign of m^2 term, such that, V is now

$$V = -\frac{m^2}{2}\phi^2 + \frac{1}{4}\lambda\phi^4 = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 \underbrace{- \frac{1}{4}\frac{m^4}{\lambda}}_{\text{irrelevant const. to be dropped}} \quad (3)$$

V is now look like



$\langle\phi\rangle = 0$ is no longer a stable minimum for V, as compared to the other 2 degenerate minima at $\langle\phi\rangle = \pm\frac{m}{\sqrt{\lambda}}$, where the system prefers to spend

One can pick either one of the minima, but the physics should be the same. Suppose we pick $\langle\phi\rangle = m/\sqrt{\lambda}$

and expand the theory around this vacuum; by shifting

$$\phi \to \phi' : \quad \phi' = \phi - \langle\phi\rangle = \phi - \frac{m}{\sqrt{\lambda}} \quad (4)$$

$$\Rightarrow \quad \langle\phi'\rangle = 0 \quad (5)$$

$$\Rightarrow \quad V(\phi') = \frac{1}{4}\lambda\left(\left(\phi' + \frac{m}{\sqrt{\lambda}}\right)^2 - \frac{m^2}{\lambda}\right)^2 = \frac{1}{4}\lambda\left(\phi'^2 + 2\frac{m}{\sqrt{\lambda}}\phi'\right)^2$$

$$= \frac{1}{4}\lambda\left(\phi'^4 + 4\frac{m}{\sqrt{\lambda}}\phi'^3 + 4\frac{m^2}{\lambda}\phi'^2\right)$$

$$= m^2\phi'^2 + \sqrt{\lambda}\,m\,\phi'^3 + \frac{1}{4}\lambda\phi'^4 \quad (6)$$

$$\Rightarrow \quad V(\phi') = m^2\phi'^2 + \sqrt{\lambda}\, m\phi'^3 + \frac{1}{4}\lambda\phi'^4 \qquad (6)$$

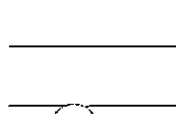
Several observations for the ϕ'-theory:

(1) the discrete symmetry is now <u>hidden</u> since there's the cubic term ϕ'^3 that breaks the symmetry (SSB)

(2) ϕ' has a mass $\sqrt{2}m$.

(3) Despite generating a new cubic term, the theory is still describing by 2 parameters: m & λ! The relations among ϕ'^2, ϕ'^3, & ϕ'^4 are expected to be preserved by quantum effects, as the theory is still renormalizable by power countings and only two counterterms are needed for m^2 & λ! This is an important aspect of SSB, and this idea had been the rationale behind using SSB in gauge theories to give masses to gauge bosons.

(4) The other vacuum $\langle\phi\rangle = -m/\sqrt{\lambda}$ or $\langle\phi'\rangle = \langle\phi\rangle - \frac{m}{\sqrt{\lambda}} = -2m/\sqrt{\lambda}$ <u>does not</u> show up in perturbation theory, because the field must be changed everywhere in spacetime to get there and hence required infinite energy to do so in perturbation theory. However this doesn't exclude non-perturbative effects to transit from one false vacuum to the true one in QFT with a multiple vacua. (Bubble formation, which is not a tunneling effect!)



QM tunneling effects!
⇓
lift the degeneracy
⇓
unique ground state in QM

(5) In finite system where we can take ϕ as q, the position of a particle in QM. Then $|L\rangle = |\langle q\rangle \equiv -m/\sqrt{\lambda}\rangle$, $|R\rangle \equiv |\langle q\rangle = +m/\sqrt{\lambda}\rangle$ are the two degenerate states. According to the general principle of superposition in QM, we should form linear combinations $|\pm\rangle = \frac{1}{\sqrt{2}}(|L\rangle \mp |R\rangle)$. The parity-even $|+\rangle$ will be the true unique ground state in <u>QM</u>. In QFT with infinite spacetime, such state is forbidden by superselection rules. Nonperturbative effects like instantons in QCD, is a different story. There $|\theta\rangle = \sum_n |e^{in\theta}\rangle$!!

Goldstone Theorem

For a set of scalar fields ϕ, we can talk about internal continuous symmetry.

(1) $$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$$, ϕ : A set (Vector) Hermitian fields $\{\phi_i\}$

~~The Lagrangian is invariant under a O(n) continuous sym. group (Global).~~ $\phi = \begin{pmatrix}\phi_1\\ \phi_2\\ \vdots\\ \phi_n\end{pmatrix}$

(2) $$\delta\phi = i\epsilon_a T^a \phi$$, $\{T^a$: generators of $O(n)$, imaginary, antisym. $\}$

* This implies $V(\phi)$ must be a function of $\phi^2 = \phi^T\cdot\phi$

Notation:
$$V_{j_1 j_2\cdots j_n}(\phi) \equiv \frac{\partial^n}{\partial\phi_{j_1}\partial\phi_{j_2}\cdots\partial\phi_{j_n}} V(\phi) \qquad (3)$$

As in the discrete case, we want to find a minimum of V such that one can do perturbative calculations. Let v be an extremum of $V(\phi)$ defined by

$$V_j(v) = 0 \qquad (7)$$

v will be a minimum if

$$V_{ij}(v) \geq 0 \qquad (8)$$

* $V_{ij}(v)$ is the mass matrix of the scalar fields ϕ.

Proof:
$$V(\phi) = V(\phi' + v)$$
$$= V(v) + \phi'_j V_j(v) + \frac{1}{2}\phi'_i\phi'_j V_{ij}(v) + \frac{1}{3!}\phi'_i\phi'_j\phi'_k V_{ijk}(v)$$
$$+ \frac{1}{4!}\phi'_i\phi'_j\phi'_k\phi'_l V_{ijkl}(v) + \cdots \qquad (9)$$
$$= \text{irrelevant const.} + \underset{(\text{eq.}(7))}{0} + \frac{1}{2}V_{ij}(v)\phi'_i\phi'_j + \text{interaction terms.}$$

eq(8)

$$\Rightarrow (M^2)_{ij} = V_{ij}(v) \geq 0 \Rightarrow \text{No tachyons!} \qquad (10)$$

Now what's the effect of SSB to the transformation $\delta\phi = i\epsilon_a T^a \phi$?

$$\langle \delta\phi \rangle = i\epsilon_a T^a \langle \phi \rangle = i\epsilon_a T^a v \qquad (11)$$

One has two possibilities:

(i) $$T^a v = 0 \quad \forall a \qquad (12).$$

$\Rightarrow$ All symmetry is not broken by v.
(Of course, $v = 0$ is a solution but it is unstable solution.)

The important point of (12) is that the vacuum v doesn't carry the charge T^a, so the charge doesn't disappear into the vacuum.

(ii) $$T^a v \neq 0 \quad \text{for some } a \qquad (13).$$

Physically, (13) means there's charge disappearing into the vacuum, even though there's a conserved current in the theory! $\Longrightarrow$ SSB

* The set of $T^a v = 0$ form a subgroup of unbroken symmetry of the theory. This set is closed because $T^a v = 0$, $T^b v = 0 \Rightarrow [T^a, T^b] v = 0$!

Now, since $V(\phi)$ is invariant under the transformation (2), we have $0 = V(\phi + \delta\phi) - V(\phi) = V_j(\phi)\,\delta\phi_j = i V_j(\phi)\,\epsilon_a (T^a)_{jk}\,\phi_k$ (14)

Differentiate (14) again, we have

$$0 = i V_{ij}(\phi)\,\epsilon_a (T^a)_{jk}\,\phi_k + i V_j(\phi)\,\epsilon_a (T^a)_{jk}\,\delta_{ki} \qquad (15).$$

Set $\phi = v$ in (15), and since $V_j(v) = 0$, we deduce

$$M^2 T^a v = 0 \qquad (16)$$

$\Rightarrow \forall$ broken generator T^a where $T^a v \neq 0$, M^2 has a massless eigenvalue!

$\Rightarrow$ for every broken generator in continuous symmetry, there's a massless Goldstone boson!

The precise statement of Goldstone theorem is:

"For any local Lorentz invariant field theory with a positive definite norm Hilbert Space, which has a SSB from a continuous global internal sym. group G to a subgroup H, there must be a massless particle associated with each broken generator."

A rigorous proof can be found in the classic paper by Goldstone, Salam & Weinberg, Phy. Rev. 127, 965 (1962). A nice discussion of the theorem can be found at Tong's lectures on SM.

So far, we haven't specified $O(n)$ explicitly. Let's look at some examples.

(i) $O(2)$: $\quad \vec{\phi} = (\phi_1, \phi_2)^T \quad \phi_{1,2}$ real scalars

$$\mathcal{L}(\vec{\phi}) = \frac{1}{2}(\partial_\mu \vec{\phi})^T \cdot (\partial^\mu \vec{\phi}) - V(\vec{\phi}^T \cdot \vec{\phi}) \tag{17}$$

$$V = -\frac{\mu^2}{2} \vec{\phi}^T \cdot \vec{\phi} + \frac{\lambda}{4} (\vec{\phi}^T \cdot \vec{\phi})^2 = -\frac{\mu^2}{2} \phi_i \phi_i + \frac{\lambda}{4} \phi_i \phi_i \phi_j \phi_j \tag{18}$$

$$\frac{\partial V}{\partial \phi_i} = -\mu^2 \phi_i + \lambda \phi_i \vec{\phi}^T \cdot \vec{\phi} = \phi_i (-\mu^2 + \lambda \vec{\phi}^2) \tag{19}$$

Let $\langle \phi_i \rangle = (v, 0)^T \Rightarrow \left.\frac{\partial V}{\partial \phi_i}\right|_v = 0$ has two solutions:

$$v = 0 \quad \text{or} \quad v = \mu/\sqrt{\lambda} \quad (\mu^2 > 0) \tag{20}$$

Shifting the fields $\phi_1 = \chi + v$ such that $\langle \chi \rangle = 0$ and ϕ_2 doesn't change.

$$\Rightarrow V(\chi, \phi_2) = +\mu^2 \chi^2 + \mu\sqrt{\lambda}\, \chi(\chi + \phi_2^2) + \frac{\lambda}{4}(\chi^2 + \phi_2^2)^2 + \text{const.} \tag{21}$$

$$\Rightarrow m_\chi = \sqrt{2}\mu, \quad m_{\phi_2} = 0 \tag{22}$$

i.e. ϕ_2 is the massless Goldstone boson.

$$\Rightarrow O(2) \longrightarrow \text{Nothing} \tag{23}$$

Under infinitesimal $O(2)$ rotation,

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \rightarrow \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1 + \alpha\phi_2\\ -\alpha\phi_1 + \phi_2\end{pmatrix} + O(\alpha^2) \qquad (24)$$

i.e. $\left.\begin{array}{l}\delta\phi_1 = \alpha\phi_2\\ \delta\phi_2 = -\alpha\phi_1\end{array}\right\} \Rightarrow \left.\begin{array}{l}\delta\chi = \alpha\phi_2\\ \delta\phi_2 = -\alpha(\chi + v)\end{array}\right\}$ (25)

* $\delta\phi_2$ is a rotation plus a translation, tangent to the circle S^1 (degenerate vacuum). This is the main reason of masslessness of the Goldstone mode ϕ_2.

* Suppose α is local, i.e. $\alpha(x)$ has x-dependence. Then $\delta\phi_2 = -\alpha(x)(\chi(x) + v)$. One may wonder if $\alpha(x)$ can be chosen to remove (eliminate) ϕ_2 completely in this local case.

To see this let parameterize $\vec{\phi}$ in polar coordinate system as

$$\vec{\phi} = \rho(\cos\theta, \sin\theta)^T \text{ with } \rho = \sqrt{\phi_1^2 + \phi_2^2} \ \& \ \sin\theta = \phi_2/\rho. \qquad (26)$$

Under infinitesimal $O(2)$, $\rho \to \rho$ & $\theta \to \theta - \alpha$.

Indeed, near the vacuum v, the two coordinate systems coincide each other:

$$\rho = (\phi_1^2 + \phi_2^2)^{1/2} = (\chi^2 + 2v\chi + v^2 + \phi_2^2)^{1/2} \simeq \chi + v \qquad (27)$$

& $\sin\theta = \dfrac{\phi_2}{\rho} \Rightarrow \theta \simeq \dfrac{\phi_2}{v + \chi} \simeq \dfrac{\phi_2}{v}$ which implies

$$\delta\theta \simeq \frac{1}{v}\delta\phi_2 \simeq \frac{1}{v}(-\alpha(\chi + v)) \simeq -\alpha. \qquad (28)$$

Now, promoting $O(2)$ to local, $\rho(x) \to \rho(x)$ unchanged, but $\theta(x) \to \theta(x) - \alpha(x)$. (29)

$\Rightarrow$ By choosing $\alpha(x) = \theta(x)$, one can eliminate the polar angle d.o.f. completely in $\phi(x) = \rho(x)\, e^{i\theta(x)}$.

$\Rightarrow$ Only physical d.o.f. left over is $\rho(x)$!

The above simple calculations lead us to consider in more detail a local $O(2)$ (or $U(1)$) gauge theory.

(30) $\mathcal{L}_\phi = |D_\mu \phi|^2 - V(\phi)$, $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, $\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$

$\Rightarrow O(2) \leftrightarrow U(1)$, $\phi \rightarrow e^{-i\alpha}\phi$. (31)

* Check: $e^{-i\alpha}\phi = (\cos\alpha - i\sin\alpha)\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

$$= \underbrace{\frac{1}{\sqrt{2}}(\cos\alpha\,\phi_1 + \sin\alpha\,\phi_2)}_{} + i\frac{1}{\sqrt{2}}(-\sin\alpha\,\phi_1 + \cos\alpha\,\phi_2)$$

$$= \phi_1' + i\,\phi_2' \quad \#.$$

The complete $U(1)$ gauge theory is described by

[Wrong sign, tachyonic!]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\phi|^2 + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \quad (32)$$

$(\mu^2 > 0)$

* This is the original Higgs model.

* $D_\mu\phi = (\partial_\mu - igA_\mu)\phi$, $(D_\mu\phi)^\dagger = (\partial_\mu + igA_\mu)\phi^\dagger$

* In terms of the polar coordinates, $\phi = \frac{1}{\sqrt{2}}\rho e^{i\theta}$, $\phi^\dagger = \frac{1}{\sqrt{2}}\rho e^{-i\theta}$.

So

$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) = \frac{1}{\sqrt{2}}\rho(\cos\theta + i\sin\theta) \Rightarrow \phi_2 = \rho\sin\theta$ same as before.

* In terms of the shifted fields after SSB,

(33) $\frac{1}{\sqrt{2}}\phi = \frac{1}{\sqrt{2}}\rho e^{i\theta} \underset{\uparrow}{\simeq} \frac{1}{\sqrt{2}}(\chi + v)e^{i\xi/v} \simeq \frac{1}{\sqrt{2}}(\chi + v + i\xi + \cdots)$

'near the vacuum' where we have defined

$\theta \equiv \xi/v$, $v = \mu/\sqrt{\lambda}$.

$\mathcal{L}(\rho \simeq \chi + v,\ \theta \equiv \xi/v)$ reads (Exercise)

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi + \frac{1}{2}\partial_\mu\xi\partial^\mu\xi + \frac{1}{2}g^2v^2A_\mu A^\mu - \mu^2\chi^2$$

$$+\sqrt{2}\,g\,vA_\mu\partial^\mu\xi + \text{'cubic/quartic terms'} \quad (34)$$

The term $A_\mu \partial^\mu \xi$ mixes A_μ with $\partial^\mu \xi$. It's a billinear in fields, so it affects propagators not interactions.
Now, the $O(2)$ transformation ($U(1)$) is local!

$$\phi(x) \to \phi'(x) = e^{-i\alpha(x)}\phi(x) \quad (35)$$

Suppose we pick

$$\alpha(x) = \theta(x),$$

$$\Rightarrow \quad \phi'(x) = e^{-i\alpha(x)}\phi(x) = e^{-i\theta(x)}\frac{1}{\sqrt{2}}\rho(x)e^{+i\theta(x)}$$

$$= \frac{1}{\sqrt{2}}\rho(x), \quad \theta(x) \text{ disappears!} \quad (36)$$

For the gauge field, we have, from previous note;

$$\left.\begin{array}{l} A_\mu \to A'_\mu = A_\mu + \frac{1}{g}\partial_\mu \alpha = A_\mu + \frac{1}{g}\partial_\mu \theta \\ F_{\mu\nu} \to F'_{\mu\nu} = F_{\mu\nu} \end{array}\right\} \quad (37)$$

$$\& \quad (D_\mu \phi) \to (D_\mu \phi)' = (\partial_\mu - igA'_\mu)\phi' = (\partial_\mu - igA'_\mu)\rho/\sqrt{2}$$

$$(D_\mu \phi)^\dagger \to (\partial_\mu + igA'_\mu)\rho/\sqrt{2} \quad (38)$$

In terms of ϕ', the full Lagrangian in (32) becomes

$$\mathcal{L} = -\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{1}{2}(\partial_\mu + igA'_\mu)\rho(\partial^\mu - igA'^\mu)\rho + \mu^2\frac{1}{2}\rho^2 - \frac{1}{4}\lambda\rho^4 \quad (39)$$

Now, compare with (34) we don't have the mixed $A_\mu \partial^\mu \theta$ term in (39).
The potential is $V(\rho) = -\frac{1}{2}\mu^2\rho^2 + \frac{1}{4}\lambda\rho^4$ (40)
$\ast$ Drop $'$ in (39)
Now (40) can lead to SSB for $\mu^2 > 0$. Same story as before! $\Rightarrow \langle\rho\rangle = \pm v$, $v = \mu/\sqrt{\lambda}$ $\Rightarrow$ Field shift to $\rho = \chi + v$.

$$(\partial_\mu + igA_\mu)\rho(\partial^\mu - igA^\mu)\rho = (\partial_\mu\chi + igA_\mu\chi + igA_\mu v)(\partial^\mu\chi - igA^\mu\chi - igA^\mu v)$$

$$= (\partial_\mu\chi)^2 + (gA_\mu\chi + gA_\mu v)^2$$

$$= (\partial_\mu\chi)^2 + g^2(v^2A_\mu A^\mu + 2vA_\mu A^\mu\chi + A_\mu A^\mu\chi^2) \quad (41)$$

$$\Rightarrow A_\mu \text{ obtains a mass } \boxed{m_A = gv} \quad (42)$$

And $V(\chi+v) = -\frac{1}{2}\mu^2(\chi+v)^2 + \frac{1}{4}\lambda(\chi+v)^4$

$$= -\frac{1}{2}\mu^2(\chi^2+2v\chi+v^2) + \frac{1}{4}\lambda(\chi^2+2v\chi+v^2)$$

$$= -\frac{1}{2}\mu^2\chi^2 + \cdots + \frac{1}{4}\lambda(2\chi^2v^2+4v^2\chi^2) + \cdots$$

$$= -\frac{1}{2}\mu^2\chi^2 + \frac{3}{2}\lambda v^2\chi^2 + \cdots$$

$$= -\frac{1}{2}\mu^2\chi^2 + \frac{3}{2}\lambda\frac{\mu^2}{\lambda}\chi^2 + \cdots$$

$$= +\mu^2\chi^2 + \text{cubic/quartic terms} + \text{Const} \quad (43)$$

$\Rightarrow$ χ has a mass $\sqrt{2}\mu$! $\boxed{m_\chi = \sqrt{2}\mu}$ $\boxed{\text{* No tachyonic mode!}}$ (44)

$\Rightarrow$ The massless θ has been <u>eliminated</u> from the theory.

It has been <u>absorbed</u> by the longitudinal component of A_μ : $(A_\mu)_L \sim \partial_\mu\theta$ which gains a mass gv, according to (42).

Thus we find an exception of the Goldstone theorem in a local U(1) gauge theory.

The gauge choice $\alpha(x) = \theta(x)$ such that

$$\phi'(x) = \frac{1}{\sqrt{2}}\rho(x) \quad (45)$$

is called unitary (or unitarity) gauge.

* Unitary gauge is useful to obtain the tree level mass spectra. But it is difficult to use in loop calculation because it is not a renormalizable gauge. In fact, in the unitary gauge, the Vector boson propagator is

$$-i\frac{1}{q^2-m_A^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{m_A^2}\right) \quad (46)$$

The second term (longitudinal) give rise to bad divergencies in loop diagrams! But, ---

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(ii) $O(3)$: A non-abelian case

Exercise: Generalize the $O(2)$ to $O(3)$ by considering the following Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(D_\mu\vec{\phi})^T\cdot(D^\mu\vec{\phi}) - V(\vec{\phi})$$

where $\vec{\phi} = (\phi_1, \phi_2, \phi_3)^T$, $\phi_i\ (i=1,2,3)$ are real fields.

and the potential

$$V(\vec{\phi}) = -\frac{\mu^2}{2}(\vec{\phi}^T\cdot\vec{\phi}) + \frac{\lambda}{4}(\vec{\phi}^T\cdot\vec{\phi})^2 \quad \text{with } \mu^2 > 0$$

The covariant derivative is

$$D_\mu\vec{\phi} = (\partial_\mu - i g W_\mu^a T^a)\vec{\phi}$$

with $(T^a)_{ij} = -i\epsilon_{aij}$, $a, i, j = 1, 2, 3$.

(i) Show that W_μ^1 & W_μ^2 acquire a mass gv, where v is the VEV: $\langle\vec{\phi}\rangle = (0, 0, v)^T$, while W_μ^3 remains massless.

(ii) Work out the properties of the physical Higgs that left over after SSB: (Its mass, self-couplings, couplings with the gauge bosons, ...)

* Thus, we have established the fact that there are exceptions for the Goldstone theorem in local gauge theories in which the Hilbert spaces have indefinite norm, such that the massless Goldstone bosons can be absorbed by the longitudinal components of the gauge bosons. $\boxed{W_L^\mu \sim \partial^\mu G}$