Discrete Spacetime Symmetries P. C. T.

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Parity P: Inversion of spatial coordinates

$$P : (t, \vec{x}) \longrightarrow (t, -\vec{x}) \qquad (1)$$

* P can be viewed as a Lorentz transf. that's not connected to identity
* P is better viewed as a reflection

$$R : (x, y, z) \longrightarrow (x, y, -z)$$

followed by a rotation of 180° in the (x, y) plane
Advantage: R works in all dim. of spacetime
Whereas, P only works in odd-dim of spatial dim.
But P treats all spatial coordinates symmetrically
* P isn't a symmetry in weak processes, but it is a perfect sym in QED and almost perfect in QCD.

Consider QED: $D_\mu = \partial_\mu - ieQ A_\mu \qquad (2)$

Let $Q = -1$ for simplicity. $\Rightarrow D_\mu = \partial_\mu + ie A_\mu \quad (2')$

As is well known, under

$$\left.\begin{aligned} \psi(x) &\rightarrow e^{i\alpha(x)} \psi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \end{aligned}\right\} \qquad (3)$$

we have

$$D_\mu \psi \rightarrow e^{i\alpha(x)} D_\mu \psi . \qquad (4)$$

Under parity, ∂_0 unchanged while $\partial_i \rightarrow -\partial_i$. This tells us that parity must act as

$$P : A_0(t, \vec{x}) \rightarrow + A_0(t, -\vec{x}) , \; \vec{A}(t, \vec{x}) \rightarrow - \vec{A}(t, -\vec{x}) \qquad (5)$$

electric field   Magnetic field

Recall that $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$ & $\vec{B} = \nabla \times \vec{A}$, we have

$$P: \vec{E}(t,\vec{x}) \longrightarrow -\vec{E}(t,-\vec{x}), \quad \vec{B}(t,\vec{x}) \to +\vec{B}(t,-\vec{x}) \tag{6}$$

$\Rightarrow$ $\vec{E}$ is a vector while $\vec{B}$ is a pseudovector.

* Another example of pseudovector is the angular momentum $\vec{L} = \vec{r} \times \vec{p}$. Under parity, $\vec{L} \to \vec{L}$ doesn't change sign ~~too~~!

(Wigner's theorem)

In quantum theory, P is implemented as an unitary operator acting on the infinite dim. Hilbert space: i.e.

$$\left.\begin{aligned} \hat{P} A_0(t,\vec{x}) \hat{P}^\dagger &= A_0(t,-\vec{x}) \\ \hat{P} \vec{A}(t,\vec{x}) \hat{P}^\dagger &= -\vec{A}(t,-\vec{x}) \end{aligned}\right\} \tag{7}$$

$$\underbrace{P^\dagger = P^{-1}}_{\text{unitary}} \tag{8}$$

where $\hat{P}$, A_0, $\vec{A}$ are now <u>operators</u>!

Next, spinors. Consider a left-handed Weyl spinor ψ_L which satisfies the massless Weyl eq

Weyl proposed this eq. in 1929. But Pauli rejected it due to P

$$\bar{\sigma}^\mu \partial_\mu \psi_L = 0, \quad \bar{\sigma}^\mu = (\mathbb{1}, -\vec{\sigma}) \tag{9}$$

Under Parity P, $\partial_0 \to \partial_0$, $\partial_i \to -\partial_i$. So, Weyl eq. is <u>not</u> invariant under parity. Similarly for ψ_R

$$\sigma^\mu \partial_\mu \psi_R = 0, \quad \sigma^\mu = (\mathbb{1}, +\vec{\sigma}) \tag{10}$$

* The different signs in the spatial comp. of σ^μ & $\bar{\sigma}^\mu$ suggests that one can compensate for a parity transf. if one also exchange ψ_L & ψ_R!

Thus, we want under parity P,

$$\left.\begin{aligned} \hat{P}\,\psi_L(t,\vec{x})\,\hat{P}^\dagger &= \psi_R(t,-\vec{x}) \\ \hat{P}\,\psi_R(t,\vec{x})\,\hat{P}^\dagger &= \psi_L(t,-\vec{x}) \end{aligned}\right\} \qquad (11)$$

Recall the Dirac spinor ψ is defined as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad (12)$$

(11) implies that

$$\hat{P}\,\psi(t,\vec{x})\,\hat{P}^\dagger = \begin{pmatrix} P\psi_L(t,\vec{x})P^\dagger \\ P\psi_R(t,\vec{x})P^\dagger \end{pmatrix} = \begin{pmatrix} \psi_R(t,-\vec{x}) \\ \psi_L(t,-\vec{x}) \end{pmatrix}$$

$$= \gamma^0 \begin{pmatrix} \psi_L(t,-\vec{x}) \\ \psi_R(t,-\vec{x}) \end{pmatrix} = \gamma^0\,\psi(t,-\vec{x}) \qquad (13)$$

* Note = up to ± 1 due to the fact that fermions always come in bilinear! (14)

See Peskin-Schroeder chp 3

where $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

Recall that for a stationary Dirac fermion $u \propto \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}$ whereas for a stationary anti-fermion $v \propto \begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix}$ where ξ^s, η^s are 2-comp. spinors describing the orientation of the spin of the particle. $(s=1,2)$

$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\eta^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \eta^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\left.\begin{aligned} &\gamma^0 \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} = \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \Rightarrow \text{particle's intrinsic parity} = +1 \\ &\gamma^0 \begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix} = -\begin{pmatrix} \eta^s \\ -\eta^s \end{pmatrix} \Rightarrow \text{antiparticle's intrinsic parity} = -1 \end{aligned}\right\} \qquad (15)$$

If $\hat{P}\psi(t,\vec{x})\hat{P}^{\dagger} = \gamma^0\psi(t,-\vec{x})$, then

$$\hat{P}\bar{\psi}(t,\vec{x})\hat{P}^{\dagger} = \bar{\psi}(t,-\vec{x})\gamma^0 \qquad (16)$$

One can show that

This means $\bar{\psi}(t,-\vec{x})\psi(t,-\vec{x})$. Arguments are often omitted!

scalar $P: \bar{\psi}\psi \to +\bar{\psi}\psi$

pseudo-scalar $P: \bar{\psi}\gamma^5\psi \to -\bar{\psi}\gamma^5\psi$

vector $P: \bar{\psi}\gamma^\mu\psi \to (\bar{\psi}\gamma^0\psi, -\bar{\psi}\gamma^i\psi)$ (17)

pseudo-vector $P: \bar{\psi}\gamma^\mu\gamma^5\psi \to (-\bar{\psi}\gamma^0\gamma^5\psi, \bar{\psi}\gamma^0\gamma^i\psi)$

$P: \quad A^\mu, \partial^\mu \to (A^0, -\vec{A}), (\partial^0, -\vec{\nabla})$

(See Peskin - Schroeder Chp. 3.)

$$* \quad \int_{-\infty}^{+\infty} d^4x \to \int_{-\infty}^{+\infty} dt \int_{+\infty}^{-\infty} d^3\vec{x}'(-1)^3 = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d^3\vec{x}' \qquad (18)$$

$(\vec{x} \to \vec{x}' \equiv -\vec{x})$

$$* \quad S = \int d^4x \left(i\bar{\psi}\gamma^\mu D_\mu\psi - M\bar{\psi}\psi\right) \qquad (19)$$

with $D_\mu = \partial_\mu - ieQA_\mu$

is invariant under parity ((13) & (16)).

(Exercise)

$$* \quad S = \int d^4x \left(i\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\gamma_5\psi\right) \qquad (20)$$

is not invariant under parity defined by (13) & (16). However, define a new parity operator P' as

$$P'\psi(t,\vec{x})P'^{\dagger} = \gamma^5\gamma^0\psi(t,-\vec{x}) \qquad (21)$$

(20) is invariant under this modified parity P'.

In (21), $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ & $\gamma^5 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ in (22)

Weyl representation (or chiral basis): $\gamma^5\gamma^0 = \begin{pmatrix} 0 & +\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = -\gamma^0\gamma^5$

中央研究院物理研究所 Thus, $P'\psi_L(t,\vec{x})P'^{\dagger} = \psi_R(t,-\vec{x})$, $P'\psi_R(t,\vec{x})P'^{\dagger} = -\psi_L(t,-\vec{x})$ # (23)

Finally, for scalar fields we can define parity as

$$(23) \quad \hat{P}\,\phi(t,\vec{x})\,\hat{P}^{+} = \pm\,\phi(t,-\vec{x}) \quad \begin{matrix} + \text{ scalar} \\ - \text{ pseudoscalar} \end{matrix}$$

Obviously, $(\partial_\mu \phi)^2$ doesn't distinguish scalar or pseudoscalar fields. One needs Yukawa interactions to tell the differences. For example $\bar{\psi}\psi\phi$ versus $\bar{\psi}\gamma^5\psi\phi$.

If parity is conserved in all the interactions in nature, like all conservation laws, this put constraints (selection rules) on the elementary processes, like in QED & QCD.

However, parity can be broken <u>explicitly</u> or due to <u>quantum</u> <u>anomaly</u>!

Charge Conjugation C: particle $\xleftrightarrow{C}$ antiparticle

We expect, for scalar field (complex) ϕ

$$C: \phi \longrightarrow \pm \phi^\dagger \qquad (24)$$

$+$ or $-$ will be fixed by the interactions of ϕ with other fields.

How about gauge fields?

$$\left.\begin{aligned} \because\ D_\mu \phi &= (\partial_\mu - ieQA_\mu)\phi \\ \Rightarrow (D_\mu\phi)^\dagger &= (\partial_\mu + ieQA_\mu)\phi^\dagger \end{aligned}\right\} (25)$$

(24) implies that under C (regardless the $\pm$ ambiguities in (24))

$$C: A_\mu \longrightarrow -A_\mu \qquad (26)$$

so that $|D_\mu\phi|^2$ is invariant under C.

In quantum theory, C is promoted to an unitary operator:

$$\left.\begin{aligned} &\hat{C}\,\phi\,\hat{C}^\dagger = \pm\phi^\dagger \\ &\hat{C}\,A_\mu\,\hat{C}^\dagger = -A_\mu \text{ (Abelian theories)} \\ \text{or } &\hat{C}\,A_\mu\,\hat{C}^\dagger = -A_\mu^\dagger\ ? \text{ (for Nonabelian theories)} \end{aligned}\right\} (27)$$

* Check $F_{\mu\nu}F^{\mu\nu}$ & $\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu})$ are C-invariant!

* $\hat{C}\,W_\mu^+\,\hat{C}^\dagger = -W_\mu^-$, $\hat{C}\,Z_\mu\,\hat{C}^\dagger = -Z_\mu$, $\hat{C}\,A_\mu\,\hat{C}^\dagger = -A_\mu$ in Standard model.

* What about gluon fields $G_\mu = G_\mu^a T^a$?

Finally, spinor fields.

Dirac eq. reads

$$i\gamma^\mu(\partial_\mu - ieQA_\mu)\psi - M\psi = 0 \quad (28)$$

Under charge conjugation, we look for a 4×4 matrix $\tilde{C}$ such that

$$C : \psi \longrightarrow \tilde{C}\psi^* \quad (29)$$

ψ^* → column spinor

$\tilde{C}$ → 4×4 matrix (Same notation as the operator!)

Taking the complex conjugation of (28)

$$-i(\gamma^\mu)^*(\partial_\mu + ieQA_\mu)\psi^* - M\psi^* = 0 \quad (30)$$

Taking the charge conjugation of (28)

$$i\gamma^\mu(\partial_\mu - ieQ(-A_\mu))\tilde{C}\psi^* - M\tilde{C}\psi^* = 0$$

$$\Rightarrow i\gamma^\mu\tilde{C}(\partial_\mu + ieQA_\mu)\psi^* - \tilde{C}M\psi^* = 0$$

$$\Rightarrow i\tilde{C}^{-1}\gamma^\mu\tilde{C}(\partial_\mu + ieQA_\mu)\psi^* - M\psi^* = 0 \quad (31)$$

Comparing (31) with (30), we need

$$\tilde{C}^{-1}\gamma^\mu\tilde{C} = -(\gamma^\mu)^* \quad (32)$$

* $\tilde{C}$ depends on your choices of basis for the γ^μ! In the Chiral (Weyl) basis, all Γ's are real except γ^2 which is purely imaginary. This leads to the possibility of

$\boxed{\text{Note: } \tilde{C}^2 = i^2\gamma^2\gamma^2 = (-1)(-1) = 1}$

$\boxed{\text{Note: } \pm \text{ because fermions always come in pairs!}}$

$$\tilde{C} = \pm i\gamma^2 = \pm\begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (33)$$

such that $C : \psi \to \psi^c = \pm i\gamma^2\psi^*$, with $\gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}$ (34)

Decompose ψ into ψ_L & ψ_R, we have

$$\pm i\gamma_2 \psi^* = \pm i \gamma_2 \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} = \pm i \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \psi_L^* \\ \psi_R^* \end{pmatrix} = \begin{pmatrix} \pm i \sigma^2 \psi_R^* \\ \mp i \sigma^2 \psi_L^* \end{pmatrix}$$

Thus, like parity, under charge-conjugation, one exchanges the two Weyl spinors in ψ. i.e.

$$C: \left.\begin{aligned} \psi_L &\longrightarrow \pm i \sigma^2 \psi_R^* \\ \psi_R &\longrightarrow \mp i \sigma^2 \psi_L^* \end{aligned}\right\} \qquad (35)$$

In QFT, this becomes

$$\left.\begin{aligned} \hat{C} \psi_L \hat{C}^\dagger &\longrightarrow \pm i \sigma^2 \psi_R^* \\ \hat{C} \psi_R \hat{C}^\dagger &\longrightarrow \mp i \sigma^2 \psi_L^* \end{aligned}\right\} \qquad (36)$$

(11) & (36) can be combined to form a CP operation as:

$$\begin{aligned} CP: \psi_L(t,\vec{x}) &\longrightarrow (\widehat{CP}) \psi_L (\widehat{CP})^\dagger \\ &= \hat{C}\hat{P} \psi_L(t,\vec{x}) \hat{P}^\dagger \hat{C}^\dagger \\ &= \hat{C} \psi_R(t,-\vec{x}) \hat{C}^\dagger = \mp i \sigma^2 \psi_L^*(t,-\vec{x}) \end{aligned} \qquad (37)$$

$$\begin{aligned} \& \quad \psi_R(t,\vec{x}) &\longrightarrow (\hat{C}\hat{P}) \psi_R (\hat{C}\hat{P})^\dagger \\ &= \hat{C}\hat{P} \psi_R(t,\vec{x}) \hat{P}^\dagger \hat{C}^\dagger \\ &= \hat{C} \psi_L(t,-\vec{x}) \hat{C}^\dagger = \pm i \sigma^2 \psi_R^*(t,-\vec{x}) \end{aligned} \qquad (38)$$

* While a single Weyl fermion is invariant under neither parity nor charge conjugation, it is possible to couple a single Weyl fermion to a gauge field in a CP invariant way! But, quantum anomalies ......?

For theories that are invariant under charge conjugation, one can assign $C = \pm 1$ to each particle, usually referred to as C-parity in the literature.
For example in QED & QCD, we have

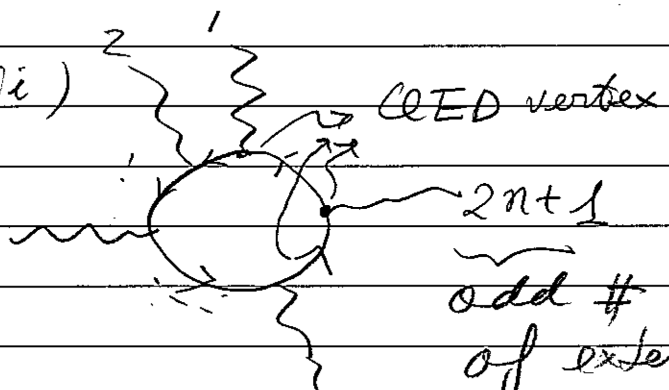
$\pi\pi \to \pi\pi\pi$ $\hookrightarrow$
$$C(\pi^0) = +1 \quad \& \quad C(A_\mu) = -1 \tag{39}$$

Charge conjugation invariant implies selection rules in elementary processes. For example,

(i) $\pi^0 \longrightarrow \gamma\gamma$ (40)

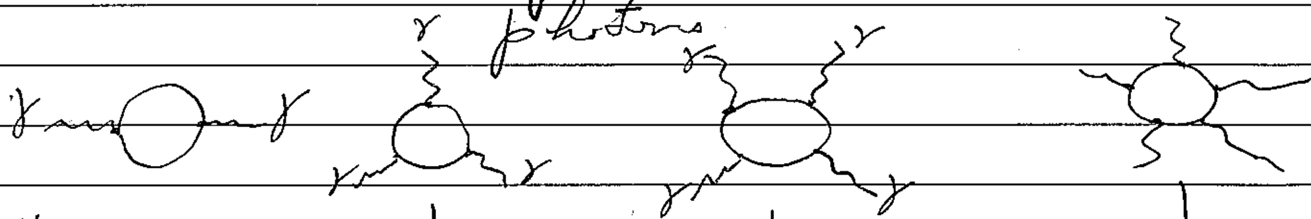
$\not\longrightarrow \gamma\gamma\gamma$ (forbidden by C-parity)

(ii)



$= 0$ (Furry's theorem)

odd # of external photons



Vacuum polarization $\downarrow$ QED β-function

$\downarrow$ 0

$\downarrow$ Euler-Heisenberg Lagrangian (Nonlinear optics)

$\downarrow$ 0

* In the literature, we also see ψ^c is defined as $\psi^c = C\bar{\psi}^T$.
Compare with (34) we have
$$\psi^c = C\bar{\psi}^T = C(\psi^\dagger\gamma^0)^T = C\gamma^{0T}\psi^* = C\gamma^0\psi^*$$
$$\Rightarrow C\gamma^0 = \pm i\gamma^2 \quad \text{i.e.} \quad C = \pm i\gamma^2\gamma^0 \quad \boxed{\text{Note: } (C)^2 = -1} \tag{34'}$$

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Time Reversal T (Wigner's T)

By definition, time reversal means

$$T : (t, \vec{x}) \longrightarrow (-t, \vec{x}) \qquad (41)$$

In QFT, according to Wigner's theorem, T has to be implemented as anti-unitary operator, rather than the more familiar unitary operator.

Anti-unitary means (anti-linear)

$$T(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha^* T|\psi_1\rangle + \beta^* T|\psi_2\rangle \qquad (42)$$

&

$$\langle T\psi_2 | T\psi_1\rangle = \langle \psi_2|\psi_1\rangle^* \qquad (43)$$

* Note that (43) preserves transition probability $|\langle\psi_2|\psi_1\rangle|^2$.

* (42) implies $i \to -i$!

To appreciate the physics of anti-unitary operator, we compare the Schrödinger eq.

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi \qquad (44)$$

with the heat equation that describes how the temperature field $T(t,\vec{x})$ diffuses in a system,

$$\frac{\partial T}{\partial t} = \kappa\nabla^2 T \qquad (45).$$

Obviously, the heat eq. violates time-reversal and it is expected physically since diffusion process increase entropy and hence it provides a preferred time direction!

On the other hand, we don't expect Schrödinger eq. to violate time reversal since it is just a qunatum system for a single particle. However, if $\psi(t)$ is a solution, $\psi(-t)$ is certainly not a solution! But there's a 'i' in Schrödinger eq. while there's no 'i' in the diffusion heat eq. Thus, in QM, we expect under time reversal,

$$T: \psi(t, \vec{x}) \longrightarrow \psi^*(-t, \vec{x}) \qquad (46)$$

for the wave function.

How about the $U(1)$ gauge field? Again, let's look at the covariant derivative ($Q = -1$)

$$D_\mu = \partial_\mu + ieA_\mu \qquad (47)$$

Under time reversal, $\partial_0 \to -\partial_0$, $i \to -i$. So we would like to have

$$T: A_0(t,\vec{x}) \longrightarrow +A_0(-t,\vec{x}), \quad \vec{A}(t,\vec{x}) \longrightarrow -\vec{A}(-t,\vec{x}) \qquad (48)$$

to preserve the form of covariant derivative.

To check if (48) makes sense physically, we can see how the electric & magnetic fields transform under T:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A} \qquad (49)$$

So, under T,

$$T: \vec{E}(t,\vec{x}) \longrightarrow +\vec{E}(-t,\vec{x}), \quad \vec{B}(t,\vec{x}) \longrightarrow -\vec{B}(-t,\vec{x}) \qquad (50)$$

i.e. $\vec{E}$, $\vec{B}$ behave like a pseudovector & vector respectively under time-reversal. This is consistent with the Lorentz force $m\ddot{\vec{x}} = q(\vec{E} + \dot{\vec{x}} \times \vec{B})$.

What about Dirac fermions? Dirac eq. reads

$$i\gamma^\mu(\partial_\mu + ieA_\mu)\psi(t,\vec{x}) - M\psi(t,\vec{x}) = 0 \quad (51)$$

Under time reversal, we seek for an anti-linear anti-unitary operator T of the form

$$\hat{T}:\ \psi(t,\vec{x}) \longrightarrow T\psi(-t,\vec{x}) \quad (52)$$

(consistent with Schwartz's QFT)

Not to be confused with the simple $\hat{T}$ in 11.6.1 in Schwartz!

where

T is a 4×4 matrix in the Dirac spinor space.

* Note that there's no complex conjugation in (52) compared with the NR Schrödinger eq case in (46)

Under T, (51) goes to

$$(-i)\left[(\gamma^0)^*(-D_0) + (\gamma^i)^*(D_i)\right]T\psi(-t) - MT\psi(-t) = 0$$

$$\Rightarrow iT^{-1}\left[(\gamma^0)^* D_0 - (\gamma^i)^* D_i\right]T\psi(-t) - M\psi(-t) = 0 \quad (53)$$

If
$$\left.\begin{aligned} \cancel{\hat{T}}^{-1}(\gamma^0)^*\cancel{\hat{T}} &= \gamma^0 \\ \cancel{\hat{T}}^{-1}(\gamma^i)^*\cancel{\hat{T}} &= -\gamma^i \end{aligned}\right\} \qquad \boxed{T^{-1}iT = +i}$$

(53) will go back to the original ((51) but for $\psi(-t)$ & $A_\mu(-t,\vec{x})$)

* In the Weyl basis, $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$,

$\gamma^0, \gamma^1, \gamma^3$ are real, γ^2 is pure imaginary, one can check

$$T = \gamma^1\gamma^3 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{pmatrix} \quad (55)$$

does the magic!
$$= \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix} \quad (\text{up to } \pm 1)$$

Splitting $\psi = (\psi_L, \psi_R)^T$ in (52), we obtain Wigner's T

$$T: \psi_L(t,\vec{x}) \longrightarrow -i\sigma^2 \psi_L(-t,\vec{x})$$
$$\psi_R(t,\vec{x}) \longrightarrow -i\sigma^2 \psi_R(-t,\vec{x})$$

* Don't flip particle into antiparticle ($\leftarrow 6$)!
* Only flip spin and 3-momenta

* Like CP, Wigner's T doesn't exchange the two Weyl spinors! T only flips the spins of particles (by $-i\sigma^2$)!

* Dipole moments (MDM & EDM)

	C	P	T	Θ
$\vec{S} = \vec{\sigma}/2$	$+\vec{S}$	$+\vec{S}$	$-\vec{S}$	$-\vec{S}$
A^μ	$(-A^0(t,\vec{x}), -\vec{A}(t,\vec{x}))$	$(A^0(t,-\vec{x}), -\vec{A}(t,-\vec{x}))$	$(A^0(-t,\vec{x}), -\vec{A}(-t,\vec{x}))$	$-A^\mu(-x)$
$\vec{E}$	$-\vec{E}(t,\vec{x})$	$-\vec{E}(t,-\vec{x})$	$+\vec{E}(-t,\vec{x})$	$+\vec{E}$
$\vec{B}$	$-\vec{B}$	$\vec{B}$	$-\vec{B}$	$+\vec{B}$
EDM $\vec{S}\cdot\vec{E}$	$-\vec{S}\cdot\vec{E}$	$-\vec{S}\cdot\vec{E}$	$-\vec{S}\cdot\vec{E}$	$-\vec{S}\cdot\vec{E}$
MDM $\vec{S}\cdot\vec{B}$	$-\vec{S}\cdot\vec{B}$	$+\vec{S}\cdot\vec{B}$	$+\vec{S}\cdot\vec{B}$	$-\vec{S}\cdot\vec{B}$
$\vec{p} = i\vec{\nabla}$	$+\vec{p}$	$-\vec{p}$	$-\vec{p}$	$+\vec{p}$

(57)

* Thus an electric dipole moment of an elementary particle would imply both $\not{P}$ & $\not{T}$

* SM particles have very tiny EDMs: $|d_e/e| < 10^{-34}$ cm.

* Experimental limits:

$$\boxed{\begin{aligned} \mathcal{H}_{int} &= -d\,\vec{\sigma}\cdot\vec{E} \\ &= -2d\,\vec{S}\cdot\vec{E} \end{aligned}}$$

	Present limit	Future Sensitivity
$\lvert d_n e\rvert$ [cm]	$(0.0 \pm 1.1)\times 10^{-26}$	
$\lvert d_e/e\rvert$ [cm]	$< 1.1\times 10^{-29}$	(Advanced ACME)
$\lvert d_\mu/e\rvert$ [cm]	$< 1.8\times 10^{-19}$	$\sim 6\times 10^{-23}$ (PSI)

* SM contribution to quark EDM starts at 3 loops.

Classics paper: (1) Ellis, Gaillard, Nanopoulos, NPB109, 213 (1976)
(2) Shabalin, Sov. J. Nucl. Phys. 32, 228 (1980)

Wigner's Version of $\tilde{T}$

CPT invariant $\quad Ⓗ \equiv CPT$

For U(1) gauge field, we have

$$A_\mu(t,\vec{x}) \xrightarrow[(15)]{P} (A_0(t,-\vec{x}), -\vec{A}(t,-\vec{x}))$$
$$\xrightarrow[(48)]{T} (A_0(-t,-\vec{x}), +\vec{A}(-t,-\vec{x}))$$
$$\xrightarrow[(27)]{C} (-A_0(-t,-\vec{x}), -\vec{A}(-t,-\vec{x}))$$

i.e.

$$Ⓗ : A_\mu(x) \longrightarrow -A_\mu(-x) \qquad (59)$$

For Dirac fermion $\psi(t,x)$, we have

$$\psi(t,x) \xrightarrow[(13)]{P} \gamma^0\psi(t,-\vec{x}) \xrightarrow[(52)]{T} \gamma^1\gamma^3\gamma^0\psi(-t,-\vec{x})$$
$$\xrightarrow[(34)]{C} (\pm i\gamma^2)\gamma^1\gamma^3\gamma^0\psi^*(-t,-\vec{x})$$
$$= \pm i\gamma^0\gamma^1\gamma^2\gamma^3\psi^*(-t,-\vec{x})$$
$$= \pm\gamma^5\psi^*(-t,-\vec{x})$$

$\boxed{*\ \gamma^0, \gamma^1, \gamma^3 \text{ are real in Weyl basis}}$

i.e.

$$Ⓗ : \psi(t,\vec{x}) \longrightarrow \pm\gamma^5\psi^*(-t,-\vec{x}) \qquad (60)$$

with $\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & +\mathbb{1} \end{pmatrix}$ in the Weyl representation

Exercise: Check $\bar{\psi}\psi$, $i\bar{\psi}\gamma_5\psi$, $i\bar{\psi}\not{\partial}\psi$, $\bar{\psi}\not{A}\psi$, $\bar{\psi}\not{A}\gamma_5\psi$, $\bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu}$ & $i\bar{\psi}\sigma_{\mu\nu}\gamma_5\psi F^{\mu\nu}$ are all invariant under $Ⓗ \equiv CPT$.

For example: $\bar{\psi}(x)\psi(x) \xrightarrow{Ⓗ} \bar{\psi}(-x)\psi(-x)$

$\bar{\psi}(x)\gamma_5\psi(x) \xrightarrow{Ⓗ} \bar{\psi}(-x)\gamma_5\psi(-x)$ etc.

(ψ: Dirac)

* note $\int_{-\infty}^{+\infty} d^4x \rightarrow \int_{-\infty}^{+\infty} d^4x' \quad (x'_\mu = -x_\mu)$

CPT theorem = All relativistic local Lagrangian QFTs must be invariant under the combined action of CPT. [Consequence of Lorentz inv. & unitarity] (Streater & Wightman, 1989)

Consequences of CPT invariance:

① $m_{particle} = m_{antiparticle}$

$\tau_{particle} = \tau_{anti\text{-}particle}$

$Q(particle) = -Q(antiparticle)$

$\mu(particle) = \mu(anti\text{-}particle)$

$d(particle) = d(anti\text{-}particle)$

e.g. 1

$$\left|\frac{m(K^0) - m(\bar{K}^0)}{m(K^0)}\right| < 10^{-18}$$

$K^0(d\bar{s})$

$\bar{K}^0(s\bar{d})$ (61)

e.g. 2

$$\begin{array}{ccccc} \bar{\nu}_\mu & \longleftarrow & \pi^- & \longrightarrow & \mu^- \\ & & \downarrow C & & \\ \nu_\mu & \longleftarrow & \pi^+ & \longrightarrow & \mu^+ \\ & & \downarrow P & & \\ \mu^+ & \longleftarrow & \pi^+ & \longrightarrow & \nu_\mu \\ & & \downarrow T & & \\ \nu_\mu & \longleftarrow & \pi^+ & \longrightarrow & \mu^+ \end{array}$$

② Neutrino oscillations

$$\begin{tikzcd} \left|Amp(\nu^\mu \to \nu^e)\right|^2 \arrow[r, leftrightarrow, "T"] \arrow[d, leftrightarrow, "CP"] & \left|Amp(\nu^e \to \nu^\mu)\right|^2 \arrow[dl, leftrightarrow, "CPT \equiv \Theta"] \\ \left|Amp(\bar{\nu}^\mu \to \bar{\nu}^e)\right|^2 & \end{tikzcd}$$

CPT inv. implies

$$\left|Amp(\bar{\nu}^\mu \to \bar{\nu}^e)\right|^2 = \left|Amp(\nu^e \to e^\mu)\right|^2$$

③ CPT theorem implies a massless particle has 2 helicities h & $-h$! (62)

* Thus far, all experimental tests have failed to find any violations of CPT.