Restoring source function from correlation function using Richardson-Lucy algorithm with maximum-entropy regularization

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WPCF 2024 - 17th Workshop on Particle Correlations and Femtoscopy

Extract Source function

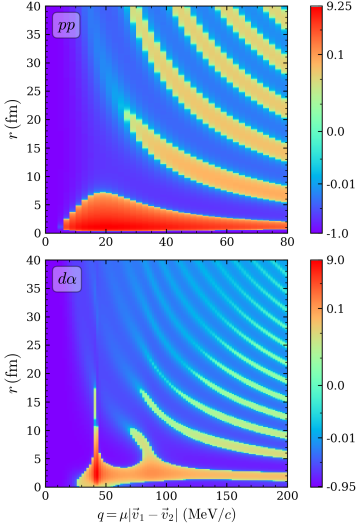
1D Koonin-Pratt equation

$$C(q) = 1 + 4\pi \int r^2 S(r) K(q, r) \mathrm{d}r$$

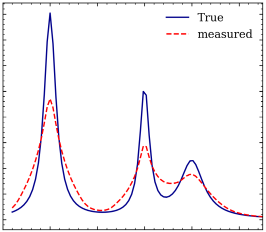
- measure correlation function $C(q) \propto \mathcal{A}/\mathcal{B}$
- derive the kernel $K(q, r) = |\Psi|^2 - 1$
- extract source function $S(r)$
  - fit for a Gaussian-shaped source
  - model-independent method : imaging, deblurring

$\rightarrow$ Deblurring : restoration via *Richardson-Lucy algorithm* (Pierre & Pawel, 846, 138247)

$\rightarrow$ **this work : advancement with max-entropy regularization**



Richardson-Lucy algorithm



- works for the inverse problem $\phi(x) = \int \psi(x')P(x|x')dx'$
  - $\phi(x)$: observed data
  - $\psi(x')$: underlying truth
  - $P(x|x')$: probability of observing x given x'

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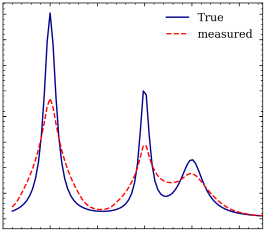
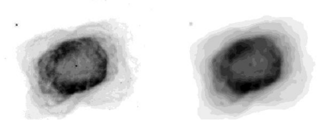
- iterative solution :

$$\psi_j^{(r+1)} = \psi_j^{(r)} \left[\sum_i \frac{\tilde{\phi}_i}{\phi_i^{(r)}} \mathcal{P}_{ij} \right]$$

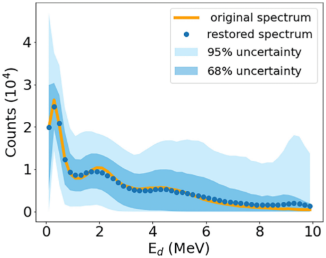
- images, astrophysics, nuclear physics : $M_{\rm inv}$ spectrum



source : scikit-image



nebula, from A&A 539, A133 (2012)



(P. Nzabahimana et.al, Phys. Rev. C 107, 064315)

Richardson-Lucy algorithm *[AJ, Vol. 79, p. 745 (1974), J. Opt. Soc. Am. 62, 55-59 (1972)]*

$$\psi_j^{(r+1)} = \psi_j^{(r)} \left[\sum_i \frac{\tilde{\phi}_i}{\phi_i^{(r)}} \mathcal{P}_{ij} \right]$$

- known data $\tilde{\phi}$ from measurement
- fixed transfer matrix $\mathcal{P}$ from theory or simulation

- starts with an initial guess of source $\psi^{(0)}(x)$, then for each iteration (r)
  - use the forward model to estimate $\phi^{(r)}(x) = \int \psi^{(r)}(x')P(x|x')\mathrm{d}x'$
  - use the data $\tilde{\phi}$ to update the image

- what remains is to recognize the KP equation in the form of forward model

$$C(q) = 1 + 4\pi \int r^2 S(r) K(q,r) \mathrm{d}r \quad \Longleftrightarrow \quad \phi^{(r)}(x) = \int \psi^{(r)}(x')P(x|x')\mathrm{d}x'$$

application to Konnin-Pratt equation

$$\phi^{(r)}(x) = \int \psi^{(r)}(x')P(x|x')\mathrm{d}x'$$

$$C(q) = 1 + 4\pi \int r^2 S(r) K(q,r)\mathrm{d}r$$

$$\lambda_{\text{purity}} = 4\pi \int r^2 S(r)\mathrm{d}r$$

discretize →

$$\mathcal{C} = \mathcal{K}\mathcal{S}$$

$$\mathcal{K}_{ij} = 4\pi r_j^2 \left(K(q_i, r_j) + \frac{1}{\lambda} \right) \Delta r$$

- discretization, absorb the purity factor into the kernel.
- normalize before applying the Richardson-Lucy algorithm, i.e.

$$\sum_i \mathcal{C}_i \Delta q = 1, \quad \sum_j \mathcal{S}_j \Delta r = 1 \quad \text{and} \quad \sum_i \mathcal{K}_{ij} \Delta q = 1$$

- recognize that $\phi \leftrightarrow \mathcal{C}$, $\psi \leftrightarrow \mathcal{S}$ and $\mathcal{K} \leftrightarrow \mathcal{P}$

sanity check - toy model

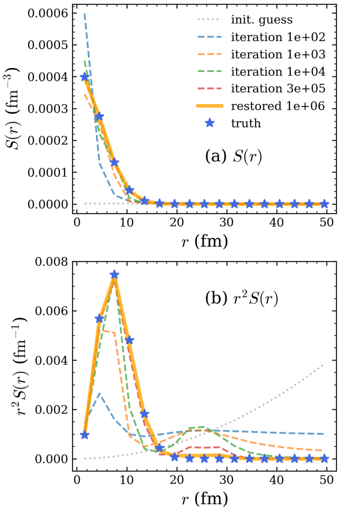
standard RL :

$$\mathcal{S}_j^{(r+1)} = \mathcal{S}_j^{(r)} \left[\sum_i \frac{\tilde{\mathcal{C}}_i}{\mathcal{C}_i^{(r)}} \mathcal{K}_{ij} \right]$$

- pp correlation with Gaussian source function
- $R_G = 3.5$ fm, $\lambda_{\text{purity}} = 0.8$, noise-free
- initial guess is taken as an uniform function
- general shape restored in short iterations

properties:

- positive-definiteness
- integral of $\mathcal{S}$ is perserved, i.e. $\sum \mathcal{S}^{(r+1)} = \sum \mathcal{S}^{(r)}$
- requires regularization if data is noisy
  - $\rightarrow$ total-variation
  - $\rightarrow$ maximum-entropy

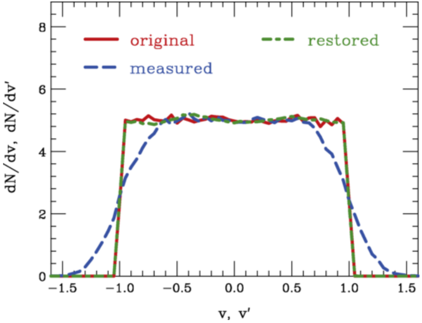


total-variation regularization *[Physica D. (1992) 60, no1-4, 259-268]*

$$I_i^{(r)}(\lambda_{\mathrm{TV}}) = \begin{cases} \dfrac{1}{1-\lambda_{\mathrm{TV}}} & \text{if } \mathcal{S}_i^{(r)} > \mathcal{S}_{i\pm1}^{(r)} \\ \dfrac{1}{1+\lambda_{\mathrm{TV}}} & \text{if } \mathcal{S}_i^{(r)} < \mathcal{S}_{i\pm1}^{(r)} \\ 1 & \text{otherwise} \end{cases}$$

- Full iterative update:

$$\mathcal{S}_j^{(r+1)} = \mathcal{S}_j^{(r)} \left[\sum_i \frac{\tilde{\mathcal{C}}_i}{\mathcal{C}_i^{(r)}} \mathcal{K}_{ij} \right] I^{(r)}(\lambda_{\mathrm{TV}})$$



[*Pawel and Mizuki, PRC 105, 034608*]

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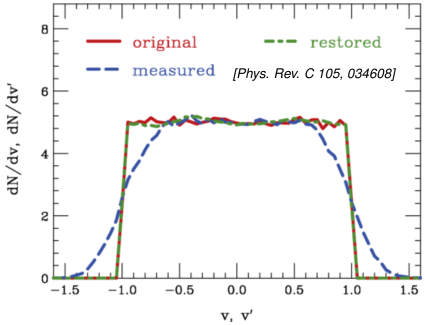
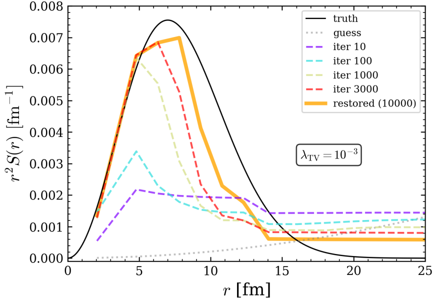
- normalization of the "image" is destroyed, i.e. $\forall r$

$$\sum_i \mathcal{S}_i^{(r+1)} \neq \sum_i \mathcal{S}_i^{(r)}, \text{ in general}$$

$\rightarrow$ solution : *maximum-entropy regularization*





Maximum-entropy regularization

same as before

rewritten as additive update :

$$\Delta\mathcal{S}_j^{(r)} = \Delta^H\mathcal{S}_j^{(r)} + \Delta^{\mathbb{S}}\mathcal{S}_j^{(r)}$$

data term : $\Delta^H\mathcal{S}_j^{(r)} = \mathcal{S}_j^{(r+1)} - \mathcal{S}_j^{(r)} = \mathcal{S}_j^{(r)}\left[\sum_i \frac{\tilde{\mathcal{C}}_i}{\mathcal{C}_i^{(r)}}\mathcal{K}_{ij} - 1\right]$

regularization : $\Delta^{\mathbb{S}}\mathcal{S}_j^{(r)} = \quad ?$

- define entropy $\mathbb{S} = -\sum_j \mathcal{S}_j \ln(\mathcal{S}_j/\chi_j)$
- χ refers to our prior or known solution to $\mathcal{S}$ which, of course, does not exist.
- *floating default* : take $\chi_j = \sum_k \Pi_{jk}\mathcal{S}_k$ as smoothened version $\mathcal{S}$
  - choose gaussian smoothing kernel $\Longrightarrow$ determination of σ_r

Maximum-entropy regularization

$$\text{data term}: \Delta^H \mathcal{S}_j^{(r)} = \mathcal{S}_j^{(r)} \left[\sum_i \frac{\tilde{\mathcal{C}}_i}{\mathcal{C}_i^{(r)}} \mathcal{K}_{ij} - 1 \right]$$

$$\text{regularization}: \Delta^{\mathbb{S}} \mathcal{S}_j^{(r)} = -\alpha \mathcal{S}_j \left[\mathbb{S} + \ln \frac{\mathcal{S}_j}{\chi_j} + 1 - \sum_k \frac{\mathcal{S}_k}{\chi_k} \Pi_{kj} \right]$$

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- *floating default* : take $\chi_j = \sum_k \Pi_{jk} \mathcal{S}_k$ as smoothened version $\mathcal{S}$
  - choose gaussian smoothing kernel $\implies$ determination of σ_r
  - α = regularization strength

Maximum-entropy regularization

$$\text{data term}: \Delta^H \mathcal{S}_j^{(r)} = \mathcal{S}_j^{(r)} \left[\sum_i \frac{\tilde{\mathcal{C}}_i}{\mathcal{C}_i^{(r)}} \mathcal{K}_{ij} - 1 \right]$$

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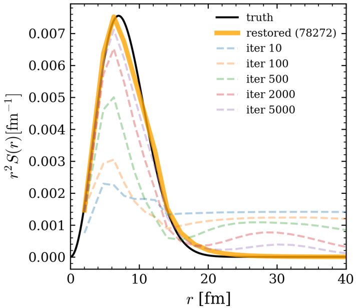
- prepare smoothing matrix Π and kernel $\mathcal{K}$
- starts with uniform guess $\mathcal{S}^{(0)}$, for each iteration (r)
  - compute the update from data (same as before)
  - compute the χ based on previous iteration (r-1)
  - compute entropy and ots update in $\mathcal{S}$
- converges if $|\Delta^H \mathcal{S}_j + \Delta^{\mathbb{S}} \mathcal{S}_j| / (|\Delta^H \mathcal{S}_j| + |\Delta^{\mathbb{S}} \mathcal{S}_j|) \leq 1e-3$



Maximum-entropy regularization

$$\text{data term}: \Delta^H \mathcal{S}_j^{(r)} = \mathcal{S}_j^{(r)} \left[\sum_i \frac{\tilde{\mathcal{C}}_i}{\mathcal{C}_i^{(r)}} \mathcal{K}_{ij} - 1 \right]$$

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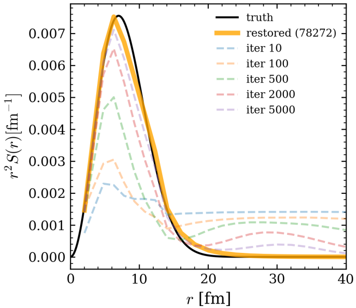
- ✓ clear convergenece criteria
- ✓ conservation of integralby construction,

$$\sum_j \Delta^H \mathcal{S}_j^{(r)} = \sum_j \Delta^{\mathbb{S}} \mathcal{S}_j^{(r)} = 0$$

- ☹ need to optimize extra parameters
  - → regularization strength α
  - → width of smoothing matrix σ_r



optimization of (hyper-)parameters

- for a single correlation function, we need to set
  - purity parameter λ
  - regularization strength α
  - width of smoothing matrix σ_r
- minimizes χ^2 of predicted correlation and data

$$\mathcal{C} = \mathcal{K}\mathcal{S}$$

$$\mathcal{K}_{ij} = 4\pi r_j^2 \left(K(q_i, r_j) + \frac{1}{\lambda} \right) \Delta r$$

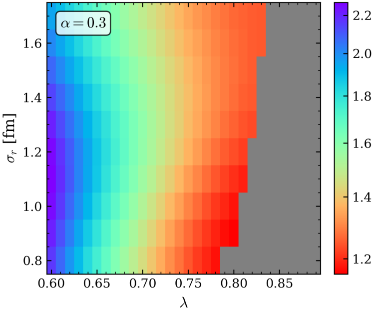
$$\Delta^{\mathbb{S}} \mathcal{S}_j^{(r)} = -\alpha \mathcal{S}_j \left[\mathbb{S} + \ln \frac{\mathcal{S}_j}{\chi_j} + 1 - \sum_k \frac{\mathcal{S}_k}{\chi_k} \Pi_{kj} \right]$$

$$\chi_j = \sum_k \Pi_{jk} \mathcal{S}_k$$

$$\Pi_{jk} \propto \exp\left(-(r_j - r_k)^2 / 2\sigma_r\right)$$

optimization of (hyper-)parameters

- for a single correlation function, we need to set
  - purity parameter λ
  - regularization strength α
  - width of smoothing matrix σ_r
- minimizes χ^2 of predicted correlation and data
- grid search



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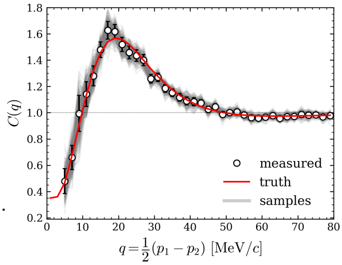
Test calculation

→ prepare data

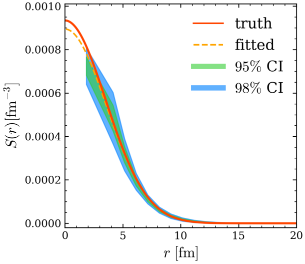
- gaussian correlation with $\lambda_G = 0.65$ and $R_G = 2.5\,\mathrm{fm}$
- add noise and perturb → treated as measured data
- sample according to $C(q) \sim \mathcal{N}(C(q), \delta q)$

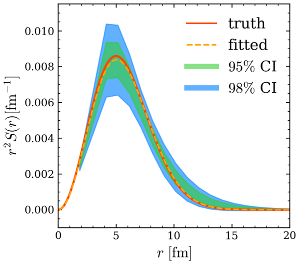
→ for each sample, apply deblurring to restore source function

- deblurring
- bootstrap uncertainty

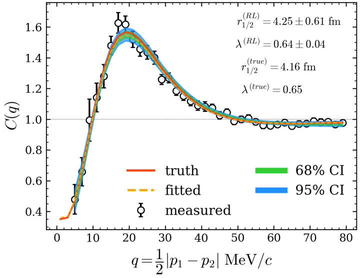


deblur





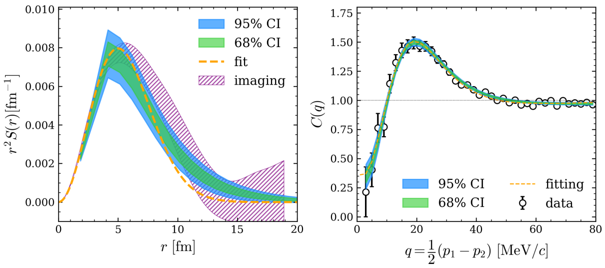
reconstruct

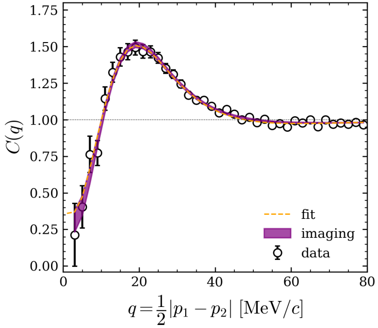


Results in pp HiRA data

- V. Henzl et. al., PRC 85, 014606
- $^{40}\mathrm{Ca} + ^{40}\mathrm{Ca}$ at 80 AMeV
- high $P_{cms} \in (740, 900)$ MeV/c
- forward angle $\theta_{lab} \in (33°, 58°)$

Parameter	MEM-RL	Gaussian fit	Imaging
λ	$0.66^{+0.05}_{-0.05}$	$0.61^{+0.11}_{-0.08}$	$0.69^{+0.19}_{-0.12}$
$r_{1/2}$ [fm]	$4.17^{+0.38}_{-0.38}$	$4.20^{+0.29}_{-0.21}$	$4.06^{+0.23}_{-0.40}$
σ_r [fm]	$1.07^{+0.24}_{-0.24}$	N/A	N/A

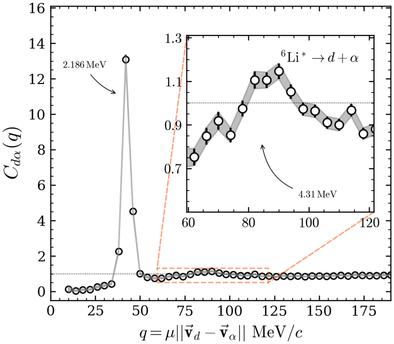




HiRA10 data - $d\alpha$

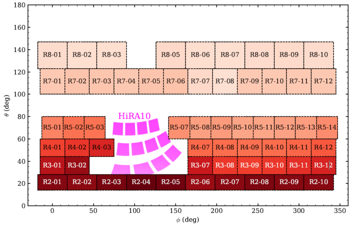
Decay of excited state $^6Li^* \to d + \alpha$

- sharp peak at $q \approx 42\ \mathrm{MeV}/c$
  - $2.186\ \mathrm{MeV}, J^\pi = 3^+$
- broad peak at $q \approx 84\ \mathrm{MeV}/c$
  - $4.312\ \mathrm{MeV}, J^\pi = 2^+$
  - $5.65\ \mathrm{MeV}, J^\pi = 1^+$



High Resolution Array

- 10cm of long CsI crystal, $E_{\mathrm{kin}} \in (15.0, 131.5)\ \mathrm{MeV}/A$
- forawrd angle $\theta_{\mathrm{lab}} \in (30^\circ, 75^\circ)$
- angular resolution better than 0.5° D. DellAquila et. al, 929, 2019, 162-172



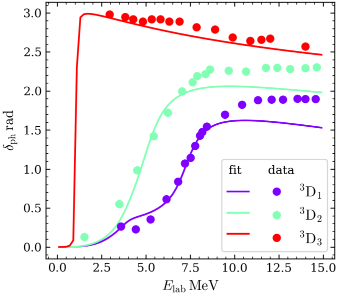
HiRA10 data - $d\alpha$

$\rightarrow$ $^{40}\mathrm{Ca} + ^{112}\mathrm{Sn}$ at 140 AMeV

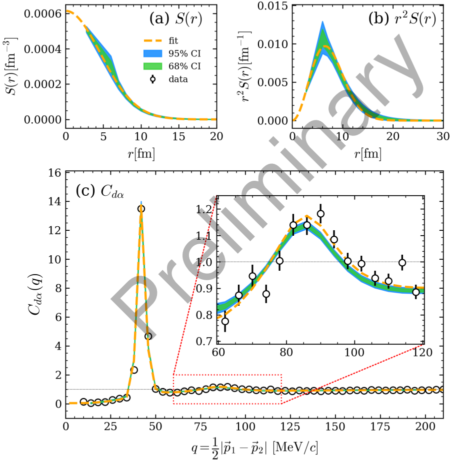
$\rightarrow$ 40% central events, mid-rapidity $|y_{\mathrm{CMS}}| \leq 0.15$

Decay of excited state $^{6}Li^{*} \rightarrow d + \alpha$

- sharp peak at $q \approx 42$ MeV$/c$
  - 2.186 MeV, $J^{\pi} = 3^{+}$
- broad peak at $q \approx 84$ MeV$/c$
  - 4.312 MeV, $J^{\pi} = 2^{+}$
  - 5.65 MeV, $J^{\pi} = 1^{+}$



Pierre & Pawel, 846, 138247



summary and outlook

summary

- ✓ developed deblurring method for source function with maximum-entropy regularization
- ✓ restored source function from pp and $d\alpha$ from data
- ✓ applicable to source from transport models such as AMD and BUU (see Pierre's talk)
- ✓ challenges : discretization, source size characterization, and optimization of parameters

upcoming...

- deblur Ca + Ni and Ca + Sn sources from HiRA10 data
  - $^{40,48}\mathrm{Ca} + {}^{58,64}\mathrm{Ni}$ at $56, 140$ AMeV
  - $^{40,48}\mathrm{Ca} + {}^{112,124}\mathrm{Sn}$ at $56, 140$ AMeV

Thoughts...

- restore 2D, 3D source function ?

Resources...

- simple demo code for restoring source function
- web app for experimenting effects of parameters

Thank you and Q. and A.

backup – Richardson-Lucy algorithm

- recall the forward model : $\phi(x) = \int \psi(x')P(x|x')\mathrm{d}x'$
- let $Q(x'|x)$ be the reciprocal of $P(x|x')$, consider the probability of $x \in (x + \mathrm{d}x)$ and $x' \in (x' + \mathrm{d}x')$,

$$Q(x'|x)\phi(x) = P(x|x')\psi(x')$$

- from the normalization of $P(x|x')$, it follows that $\psi(x') = \int Q(x'|x)\phi(x)\mathrm{d}x$.
- this suggests an iterative update scheme starting from a guess of $\psi^{(0)}(x)$.
  - use the forward model to estimate $\phi^{(r)}(x) = \int \psi^{(r)}(x')P(x|x')\mathrm{d}x'$
  - updates $\psi^{(r+1)}(x') = \int Q^{(r)}(x'|x)\tilde{\phi}(x)\mathrm{d}x$
  - where $\phi(x)$ is approximated with observed data $\tilde{\phi}(x)$. Eliminating $Q(x'|x)$ gives

$$\psi_j^{(r+1)} = \psi_j^{(r)}\left[\sum_i \frac{\tilde{\phi}_i}{\phi_i^{(r)}}\mathcal{P}_{ij}\right]$$

backup – convergence

- convergence criteria : $t_j = \dfrac{|\Delta^H \mathcal{S}_j + \Delta^{\mathbb{S}} \mathcal{S}_j|}{|\Delta^H \mathcal{S}_j| + |\Delta^{\mathbb{S}} \mathcal{S}_j|} \leq 1e-3.$