Femtoscopic Coulomb and strong final state interactions in Fourier space

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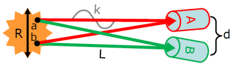
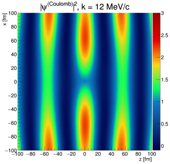
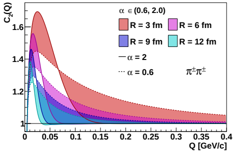
Outline

- Introduction (short; for sake of completeness. . .)
  - HBT correlations, final state interactions (FSI), wave functions
  - Femtoscopy vs. "interaction femtoscopy"
  - Need for new method for FSI treatment
    precision investigations, non-Gaussian sources. . .
- Methods for FSI calculation
  - Direct calculation vs. "Fourier space" calculation
  - Motivation (numerical & "philosophical")
  - Integral kernel for Coulomb w.f., *s*-wave strong interacting w.f.
    (some details still work in progress. . .)
  - Implementation: spherically symmetric case vs. "3D" case
  - (Seemingly) complicated formulas
    $\rightarrow$ Simple and reliable numerical integrals
- Outlook: status report, next steps. . .







Variables

- Momentum variables: $\mathbf{K} := \frac{1}{2}(\mathbf{p}_1+\mathbf{p}_2)$, $\mathbf{k}\equiv\mathbf{k}^* = \frac{m_2\mathbf{k}_1 - m_1\mathbf{k}_2}{m_1+m_2}$
- Coordinate variables: $\mathbf{r} := \mathbf{r}_1-\mathbf{r}_2$, $\boldsymbol{\rho} = \frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2}$

  Working in 3D; assume temporal extent of source scaled into 3D sizes

  e.g. $R^2_{\rm out} \to R^2_{\rm out}+\beta_t^2(\Delta\tau)^2$
- source function: $S(\mathbf{r}_1,\mathbf{p}_1)$; meaning: $N_1(\mathbf{p}_1)=\int d\mathbf{r}_1\, S(\mathbf{r}_1,\mathbf{p}_1)$
- pair wave function: $\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2)$; assume $\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2)=e^{2i\mathbf{K}\boldsymbol{\rho}}\psi_{\mathbf{k}}(\mathbf{r})$

  Working in NR setting as of now
- pair mom. distribution: $N_2(\mathbf{p}_1,\mathbf{p}_2)=\int d\mathbf{r}_1 d\mathbf{r}_2\, S(\mathbf{r}_1,\mathbf{p}_1)S(\mathbf{r}_2,\mathbf{p}_2)\left|\psi^{(2)}(\mathbf{r}_1,\mathbf{r}_2)\right|^2$
- correlation function: $C(\mathbf{p}_1,\mathbf{p}_2)=\dfrac{N_2(\mathbf{p}_1,\mathbf{p}_2)}{N_1(\mathbf{p}_1)N_1(\mathbf{p}_2)}$
- pair source: $D(\mathbf{r},\mathbf{K})=\int d\boldsymbol{\rho}\, S\left(\boldsymbol{\rho}+\frac{\mathbf{r}}{2},\mathbf{K}\right)S\left(\boldsymbol{\rho}-\frac{\mathbf{r}}{2},\mathbf{K}\right)$, $\underline{D(\mathbf{r},\mathbf{K})=D(-\mathbf{r},\mathbf{K})}$
- normalization (for corr. functions): $\int d\mathbf{r}_1\, S(\mathbf{r}_1,\mathbf{p}_1)=1 \quad\Rightarrow\quad \int d\mathbf{r}\, D(\mathbf{r})=1$

  (from now on, $\mathbf{K}$ suppressed)

Basic formulas

- Put together,

$$\underline{\underline{C(\mathbf{k})=\int d\mathbf{r}\, D(\mathbf{r})|\psi_{\mathbf{k}}(\mathbf{r})|^2;}} \quad \overset{Bowler-Sinyukov}{\Rightarrow} \quad C(\mathbf{k})=1-\lambda+\lambda\int d\mathbf{r}\, D(\mathbf{r})|\psi_{\mathbf{k}}(\mathbf{r})|^2.$$

- λ: "intercept"; core-halo parameter *when $D(r)$ has no self-similar part; see e.g. a core-halo paper, Csörgő, Lörstad, Zimányi, Z. Phys. C 71, 491 (1996)*
- Knowing $|\psi_{\mathbf{k}}(\mathbf{r})|^2 \Rightarrow D(r)$ can be reconstructed
  - Imaging (Brown, Danielewicz, *PLB* 398, 252 (1997))
    de-blurring (Nzabahimana, Danielewicz, *PLB* 846, 138247 (2023))
    $\rightarrow$ No assumption ("model") for $D(\mathbf{r})$ functional form (in principle)
  - Assuming a source, parameters from fitting to $C(\mathbf{k})$
- Knowing ($\equiv$ assuming) $D(\mathbf{r}) \Rightarrow |\psi_{\mathbf{k}}(\mathbf{r})|^2$, especially $\delta_{l=0}$ can be reconstructed
- Need for reliably calculate $\int \mathrm{d}\mathbf{r}\, D(\mathbf{r})|\psi_{\mathbf{k}}(\mathbf{r})|^2$ integral; $|\psi_{\mathbf{k}}(\mathbf{r})|^2$ distinctions:
  - No symmetrization vs. symmetrization/antisymmetrization (spin-weighted)
  - No FSI: plane wave (trivial 1 & Fourier transfrom)
  - Coulomb FSI: well-known form (sol. to Schrödinger eq.); repulsive vs. attractive
  - Strong interaction: s-wave (valid for zero range $\Rightarrow$ singular but integrable $|\psi|^2$)
  - Coulomb + strong: s phase shif changes (Ledniczky, *Phys. Part. Nucl.* 40, 307 (2009))

Wave functions — Coulomb interaction

- Basic quantities: $|\mathcal{N}_r|^2$ & $|\mathcal{N}_a|^2$ repulsive & attractive Gamow factors, η Sommerfeld parameter, $\delta_l^c \equiv \arg\Gamma(l{+}1{+}i\eta)$ Coulomb phase shift

$$\eta \equiv \alpha_{\rm em}\frac{mc}{\hbar k}, \qquad \begin{aligned}\mathcal{N}_r &:= e^{-\pi\eta/2}\Gamma(1{+}i\eta),\\ \mathcal{N}_a &:= e^{\pi\eta/2}\Gamma(1{-}i\eta)\end{aligned} \qquad \Rightarrow \qquad \begin{aligned}|\mathcal{N}_r|^2 &= \frac{2\pi\eta}{e^{2\pi\eta}-1},\\ |\mathcal{N}_a|^2 &= \frac{2\pi\eta}{1-e^{-2\pi\eta}}.\end{aligned}$$

- Coulomb wave function $\psi_{\mathbf{k}}(\mathbf{r})$:

Attractive ($\Rightarrow$non-identical): $\mathcal{N}_a^* e^{-ikr}\mathbf{M}\big(1{+}i\eta, 1, i(kr{+}\mathbf{kr})\big)$,

Repulsive, non-identical: $\mathcal{N}_r^* e^{-ikr}\mathbf{M}\big(1{-}i\eta, 1, i(kr{+}\mathbf{kr})\big)$,

Repulsive, symm.: $\frac{\mathcal{N}_r^*}{\sqrt{2}} e^{-ikr}\Big[\mathbf{M}\big(1{-}i\eta, 1, i(kr{+}\mathbf{kr})\big) + \mathbf{M}\big(1{-}i\eta, 1, i(kr{-}\mathbf{kr})\big)\Big]$,

Repulsive, anti.: $\frac{\mathcal{N}_r^*}{\sqrt{2}} e^{-ikr}\Big[\mathbf{M}\big(1{-}i\eta, 1, i(kr{+}\mathbf{kr})\big) - \mathbf{M}\big(1{-}i\eta, 1, i(kr{-}\mathbf{kr})\big)\Big]$,

where $\mathbf{M}(a,b,z)$: (reduced) confluent hypergeometric function

Wave functions — strong interaction (s-wave)

- $\Delta_0^s(k)$ phase shift $\Leftrightarrow$ $f_s(k)$ scattering amplitude; $\sin\Delta_0^s e^{i\Delta_0^s} = kf_s(k)$:
- $\psi_{\mathbf{k}}(\mathbf{r})$ in case of strong interaction but no Coulomb:

  Non-identical: $e^{i\mathbf{kr}} + \frac{e^{-ikr}}{r} f_s^*(k)$,

  Symmetrized: $\frac{1}{\sqrt{2}}\left[e^{i\mathbf{kr}} + e^{-i\mathbf{kr}} + 2\frac{e^{-ikr}}{r} f_s^*(k)\right]$,

  Antisymm'd: $\frac{1}{\sqrt{2}}\left[e^{i\mathbf{kr}} - e^{-i\mathbf{kr}}\right]$.

  Error estimation from assumed zero range instead of r_0:

  $\sim\int_0^{r_0} \mathrm{d}\mathbf{r}\,|f_s|^2\left(\frac{1}{r^2}-\frac{1}{r_0^2}\right) = \frac{8}{3}\pi|f_s|^2 r_0$, vs. $(2\pi)^{3/2}R^3$ for simple Gaussian $D(r)$
- $\psi_{\mathbf{k}}(\mathbf{r})$ in case of strong + Coulomb ($\mathcal{S}_r \equiv 2i\sin\Delta_0^s e^{-i\Delta_0^s} e^{-2i\delta_0^c} e^{\pi\eta/2}$;
  for attractive $\mathcal{S}_a$, replace η with $-\eta$ in repulsive $\mathcal{S}_r$):

  Attractive (non-id.): $\mathcal{N}_a^* e^{-ikr}\left[\mathbf{M}\left(1+i\eta, 1, i(kr+\mathbf{kr})\right) + \mathcal{S}_a \cdot U(1-i\eta, 2, 2ikr)\right]$,

  Repulsive, non-id.: $\mathcal{N}_r^* e^{-ikr}\left[\mathbf{M}\left(1-i\eta, 1, i(kr+\mathbf{kr})\right) + \mathcal{S}_r \cdot U(1-i\eta, 2, 2ikr)\right]$,

  Repulsive, symm. or antisymm.: add or subtract $\mathbf{r} \leftrightarrow -\mathbf{r}$ and divide by $\sqrt{2}$.

  $U(a,b,z)$: Tricomi's function

„Roadmap"

- Possible cases (11; cases of no Coulomb & no strong interaction omitted):

	No Coulomb	Repulsive Coulomb	Attractive Coulomb
Non-identical	– Strong Yes	– Strong Yes – Strong No	Strong Yes Strong No
Symmetrized (Bose-Einstein)	– Strong Yes	– Strong Yes – Strong No	X
Antisymmetrized (Fermi-Dirac)	– Strong Yes	– Strong Yes – Strong No	X

- Altogether, for a given $D(\mathbf{r})$, four integrals may be needed

  similarly to Kincses, Nagy, Csanád, PRC 102, 064912 (2020)

$$\mathcal{I}^{(1)}(\mathbf{k}) := \int \mathrm{d}\mathbf{r}\, D(\mathbf{r}) \mathbf{M}\big(1-i\eta, 1, i(kr+\mathbf{kr})\big) \mathbf{M}\big(1+i\eta, 1, -i(kr+\mathbf{kr})\big),$$

$$\mathcal{I}^{(2)}(\mathbf{k}) := \int \mathrm{d}\mathbf{r}\, D(\mathbf{r}) \mathbf{M}\big(1-i\eta, 1, i(kr+\mathbf{kr})\big) \mathbf{M}\big(1+i\eta, 1, -i(kr-\mathbf{kr})\big),$$

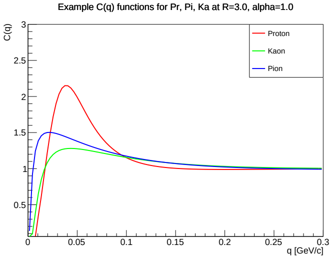
$$\mathcal{I}^{(3)}(\mathbf{k}) := \int \mathrm{d}\mathbf{r}\, D(\mathbf{r})\, U\big(1-i\eta, 2, 2ikr\big)\, U\big(1+i\eta, 2, -2ikr\big),$$

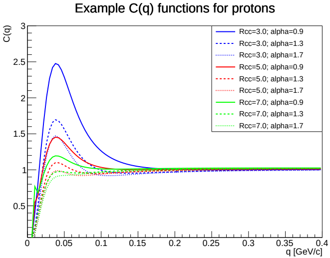
$$\mathcal{I}^{(4)}(\mathbf{k}) := \int \mathrm{d}\mathbf{r}\, D(\mathbf{r}) U\big(1-i\eta, 2, 2ikr\big) \mathbf{M}\big(1+i\eta, 1, -i(kr+\mathbf{kr})\big)$$

- From these & $f_s(k)$ parametrization: result can be assembled for all cases

Calculational schemes: direct integration

- Another distinction: $D(\mathbf{r})$ spherically symmetric vs. non-spherical (besides functional form of $D(\mathbf{r})$, of course)
- Monte Carlo integration: unreliable for slowly decreasing $D(r)$
- Calculating $C(\mathbf{k}) = \int d\mathbf{r}\, D(\mathbf{r}) |\psi_{\mathbf{k}}(\mathbf{r})|^2$, direct calculation suffers from oscillations in $\psi_{\mathbf{k}}(\mathbf{r})$; nevertheless: possible
  - Coulomb & spherical Lévy source: Csanád, Lökös, Nagy, *Universe* 5, 133 (2019)
  - Coulomb+strong, spher. Lévy src: Kincses, Nagy, Csanád, *PRC* 102, 064912 (2020)
- Example for protons & spherically symmetric Lévy source function:





`github.com/csanadm/StrongCoulombLevyBEC`

Calculational schemes: Fourier transform method

- Nagy, Csanád, Purzsa, Kincses, Eur. Phys. J. C **83**, 1015 (2023)
  `github.com/csanadm/CoulCorrLevyIntegral`
- Assume that source is easily expressed as Fourier transform

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

  E.g. Gaussian: $f(\mathbf{q}) = \exp\left(-\frac{1}{2}\mathbf{q}\mathbf{R}^2\mathbf{q}\right)$; if spherically symmetric: $f_s(q) = e^{-q^2R^2/2}$,
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- Mathematical idea: „interchange integrals".

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$$C_2(\mathbf{k}) = \int d^3\mathbf{r}\, |\psi_{\mathbf{k}}^{(2)}(\mathbf{r})|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}}$$

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  Instead, by means of regularization ($\lambda \in \mathbb{R}^+$, $\lambda \to 0$):

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- Step 1: Calculate this last **r**-integral, independently of $f(\mathbf{q})$, i.e. of $D(\mathbf{r})$,

Calculational schemes: Fourier transform method

- Nagy, Csanád, Purzsa, Kincses, Eur. Phys. J. C **83**, 1015 (2023)
  `github.com/csanadm/CoulCorrLevyIntegral`
- Assume that source is easily expressed as Fourier transform

$$D(\mathbf{r}) := \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \quad \Leftrightarrow \quad f(\mathbf{q}) = \int d^3\mathbf{r}\, D(\mathbf{r}) e^{-i\mathbf{q}\mathbf{r}}$$

  E.g. Gaussian: $f(\mathbf{q}) = \exp\left(-\frac{1}{2}\mathbf{q}\mathbf{R}^2\mathbf{q}\right)$; if spherically symmetric: $f_s(q) = e^{-q^2R^2/2}$,
  Lévy source: $f(\mathbf{q}) = \exp\left(-\frac{1}{2}|\mathbf{q}\mathbf{R}^2\mathbf{q}|^{\alpha/2}\right)$; spherically symmetric: $f_s(q) = e^{-(qR)^\alpha/2}$,
- Mathematical idea: „interchange integrals". Not trivial since

$$C_2(\mathbf{k}) = \int d^3\mathbf{r}\, |\psi_\mathbf{k}^{(2)}(\mathbf{r})|^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) e^{i\mathbf{q}\mathbf{r}} \neq \int \frac{d^3\mathbf{q}}{(2\pi)^3} f(\mathbf{q}) \int d^3\mathbf{r}\, e^{i\mathbf{q}\mathbf{r}} |\psi_\mathbf{k}^{(2)}(\mathbf{r})|^2$$

  Instead, by means of regularization ($\lambda \in \mathbb{R}^+$, $\lambda \to 0$):

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- Step 1: Calculate this last **r**-integral, independently of $f(\mathbf{q})$, i.e. of $D(\mathbf{r})$,
- Step 2: *Then* simplify the result of the $\lambda \to 0$ limit of the **q**-integral.

Calculational schemes: Fourier transform method

- Nagy, Csanád, Purzsa, Kincses, Eur. Phys. J. C **83**, 1015 (2023)
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$$D(\mathbf{r}):=\int \frac{d^3\mathbf{q}}{(2\pi)^3}\, f(\mathbf{q})e^{i\mathbf{qr}} \quad \Leftrightarrow \quad f(\mathbf{q})=\int d^3\mathbf{r}\, D(\mathbf{r})e^{-i\mathbf{qr}}$$

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$$C_2(\mathbf{k})=\int d^3\mathbf{r}\,|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2\int \frac{d^3\mathbf{q}}{(2\pi)^3}\, f(\mathbf{q})e^{i\mathbf{qr}} \neq \int \frac{d^3\mathbf{q}}{(2\pi)^3}\, f(\mathbf{q})\int d^3\mathbf{r}\, e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2$$

Instead, by means of regularization ($\lambda\in\mathbb{R}^+$, $\lambda\to 0$):

$$\underline{\underline{C_2(\mathbf{k})}}=\underline{\underline{\lim_{\lambda\to 0}\int \frac{d^3\mathbf{q}}{(2\pi)^3}\, f(\mathbf{q})\int d^3\mathbf{r}\, e^{-\lambda r}e^{i\mathbf{qr}}|\psi^{(2)}_{\mathbf{k}}(\mathbf{r})|^2}} \quad \checkmark$$

- Step 1: Calculate this last **r**-integral, independently of $f(\mathbf{q})$, i.e. of $D(\mathbf{r})$,
- Step 2: *Then* simplify the result of the $\lambda\to 0$ limit of the **q**-integral.
- Result: *functional* of $f(\mathbf{q})$; not simple integral transform.

Spherically symmetric case, Coulomb FSI, symmetrized

- Glimpses shown last year; since then, using extensively in measurements
- If $f(\mathbf{q}) \equiv f_s(q)$, solid angle integrals come first, resulting in

$$C_2(Q) = \frac{|\mathcal{N}|^2}{2\pi^2} \lim_{\lambda\to 0} \int_0^\infty \mathrm{d}q\, q^2 f_s(q) \Big[\mathcal{D}_{1\lambda s}(q) + \mathcal{D}_{2\lambda s}(q)\Big].$$

Using method of A. Nordsieck, *Phys. Rev.* 93, 785 (1954), with $\mathcal{F}_+(x) \equiv {}_2F_1(i\eta, 1+i\eta, 1, x)$

$$\mathcal{D}_{1\lambda s}(q) = \frac{4\pi}{q} \mathrm{Im}\Big[\frac{1}{(\lambda - iq)^2}\Big(1 + \frac{2k}{q+i\lambda}\Big)^{2i\eta} \mathcal{F}_+\Big(\frac{4k^2}{(q+i\lambda)^2}\Big)\Big],$$
$$\mathcal{D}_{2\lambda s}(q) = \frac{4\pi}{q} \mathrm{Im}\Big[\frac{(\lambda - iq - 2ik)^{i\eta}(\lambda - iq + 2ik)^{-i\eta}}{(\lambda - iq)^2 + 4k^2}\Big];$$

$\Rightarrow$ **Step 1 completed.**

- **Step 2:** for $\lambda\to 0$, integral transforms with $\mathcal{D}_{1\lambda s, 2\lambda s}$ become $\mathcal{A}_{1s,2s}$ functionals:

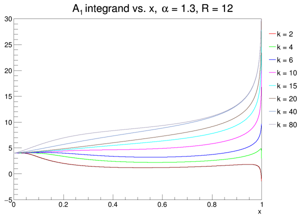
$$C_2(Q) = |\mathcal{N}|^2 \left(1 + f_s(2k) + \frac{\eta}{\pi}\left[\mathcal{A}_{1s} + \mathcal{A}_{2s}\right]\right), \quad \text{where}$$

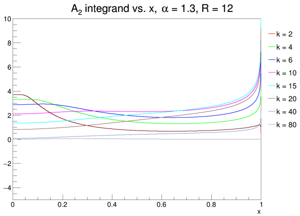
$$\mathcal{A}_{1s} = -\frac{2}{\eta} \int_0^\infty \mathrm{d}q\, \frac{f_s(q) - f_s(0)}{q} \mathrm{Im}\left[\left(1 + \frac{2k}{q}\right)^{2i\eta} \mathcal{F}_+\left(\frac{4k^2}{q^2} - i0\right)\right],$$

$$\mathcal{A}_{2s} = -\frac{2}{\eta} \int_0^\infty \mathrm{d}q\, \frac{f_s(q) - f_s(2k)}{q - 2k} \frac{q}{q+2k} \mathrm{Im} \frac{(q+2k)^{i\eta}}{(q - 2k + i0)^{i\eta}}.$$

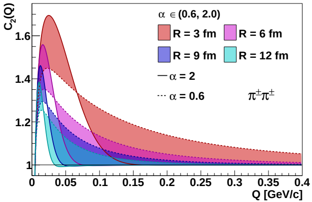
Spherically symmetric case, Coulomb FSI, symmetrized

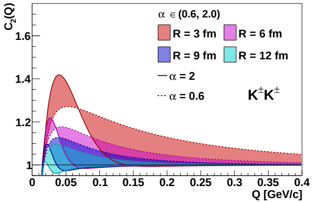
- Numerically: 1 integral for $\mathcal{A}_{1s}$ & $\mathcal{A}_{2s}$; transform from $q\in\mathbb{R}_0^+$ to $x\in[0,1]$
  Gauss-Krohnrod algorithm from `C++ boost` library
- Integrands: well-behaved





- Example result (for Lévy sources, various α and R, $\pi^{\pm}\pi^{\pm}$ & $K^{\pm}K^{\pm}$):





Non-spherically symmetric, Coulomb FSI, symmetrized

- *First 3D measurement using Lévy sources was shown at this conference! (STAR preliminary; Sneha Bhosale)*
- Notation: $\mathbf{q}_\pm \equiv \mathbf{q} \pm 2\mathbf{k}$. NB: free $C^{(0)}(\mathbf{k}) = 1 + f(2\mathbf{k})$.
- Three-dimensional integrals remain at all stages:

$$C_2(\mathbf{k}) = \frac{|\mathcal{N}|^2}{(2\pi)^3} \lim_{\lambda\to 0} \int d^3\mathbf{q}\, f(\mathbf{q}) \Big[\mathcal{D}_{1\lambda}(\mathbf{q}) + \mathcal{D}_{2\lambda}(\mathbf{q}) \Big].$$

$$\mathcal{D}_{2\lambda s}(\mathbf{q}) = \int \mathrm{d}^3\mathbf{r}\, e^{i\mathbf{q}\mathbf{r}} e^{-\lambda r} M\big(1+i\eta, 1, -i(kr - \mathbf{kr})\big) M\big(1-i\eta, 1, i(kr+\mathbf{kr})\big).$$

Again with Nordsieck's method, Step 1 can be completed:

$$\mathcal{D}_{1\lambda}(\mathbf{q}) = -\frac{d}{d\lambda} \frac{4\pi}{\lambda^2+q^2} \mathcal{U}_{1-} \mathcal{F}_+(x_1), \qquad \mathcal{D}_{2\lambda}(\mathbf{q}) = -\frac{d}{d\lambda} \frac{4\pi}{\lambda^2+q_+^2} \mathcal{U}_{2-} \mathcal{F}_+(x_2),$$

with

$$\mathcal{U}_{1\pm} := \frac{(\lambda^2+q^2)^{\pm 2i\eta}}{(\lambda^2+\mathbf{q}\mathbf{q}_\mp \pm 2ik\lambda)^{\pm 2i\eta}}, \qquad \mathcal{U}_{2\pm} := \frac{((\lambda^2+q^2)(\lambda^2+q_+^2))^{\pm i\eta}}{(\lambda^2+\mathbf{q}\mathbf{q}_+ \pm 2ik\lambda)^{\pm 2i\eta}},$$

and $x_1 := \frac{4(\mathbf{qk} - ik\lambda)^2}{(\lambda^2+q^2)^2} \in \mathbb{C} \setminus \mathbb{R}_0^+$, $\quad \lim_{\lambda\to 0} x_1 := X_1 = 1 - \frac{\mathbf{q}\mathbf{q}_+}{q^2}\frac{\mathbf{q}\mathbf{q}_-}{q^2}$,

with $x_2 := \frac{4[k^2q^2 - (\mathbf{qk})^2]}{(\lambda^2+q^2)(\lambda^2+q_+^2)} \in [0, 1[$, $\quad \lim_{\lambda\to 0} x_2 := X_2 = 1 - \left(\frac{\mathbf{q}\mathbf{q}_+}{qq_+}\right)^2$.

In the following, $\mathcal{F}_-(x) \equiv (1+i\eta)\cdot {}_2F_1(-i\eta, 1-i\eta, 2, x)$, $g_+ \equiv \frac{\Gamma(-2i\eta)}{\Gamma(-i\eta)\Gamma(1-i\eta)}$

Non-spherically symmetric case: result

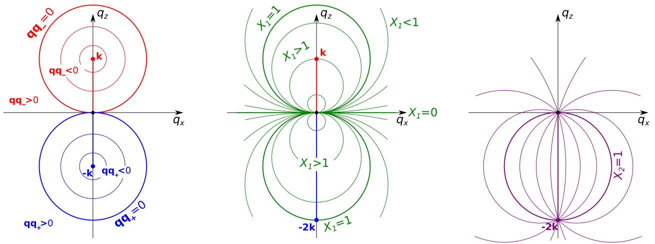
- Step 2 (performing $\lambda\to 0$): possible in exotic curved coordinate system (a, β, φ) and (b, y, φ) fit to $\mathbf{k}$ vector. Using $f(\mathbf{q})=f(-\mathbf{q})$,

$$C_2(Q)=|\mathcal{N}|^2\left(1+f_s(2\mathbf{k})+\frac{\eta}{\pi}\left[\mathcal{A}_1+\mathcal{A}_2\right]\right),\quad \text{where}$$

$$\mathcal{A}_1=\frac{1}{\pi}\int_{\mathbb{R}}\mathrm{d}a\int_{\mathbb{R}^+}\mathrm{d}\beta\int_{-\pi}^{\pi}\mathrm{d}\varphi\,\frac{aY(-a)}{\beta(a+1)}\mathrm{Re}\left\{\frac{|a|^{2i\eta}}{|a+1|^{2i\eta}}\left[g_+^*\frac{4i\,f_1(-1,\beta,\varphi)}{(\beta^2+1)(a-i)^2}-\mathcal{F}_-\left(\tfrac{1+a\cdot i0}{a^2}\right)\frac{f_1(a,\beta,\varphi)}{(a^2+\beta^2)}\right]\right\},$$

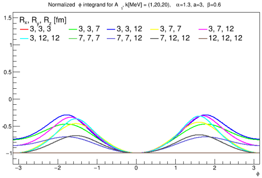
$$\mathcal{A}_2=-\frac{1}{\pi}\int_0^\infty\mathrm{d}b\int_{-1}^1\mathrm{d}y\int_{-\pi}^{\pi}\mathrm{d}\varphi\,\frac{1-y}{1+y}\,\frac{Y(b)}{b-1}\mathrm{Re}\left\{\frac{(b+1)^{2i\eta}}{|b-1|^{2i\eta}}\left[\mathcal{F}_-\left(\tfrac{4b}{(1+b)^2}\right)\frac{f_2(b,y,\varphi)}{1+by^2}-\right.\right.$$

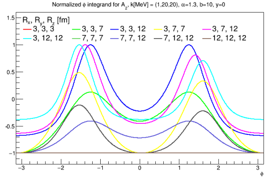
$$\left.\left.-2g_+^*\frac{2f_2(1,y,\varphi)}{(1+b)(1+y^2)}\right]\right\}+\Im(g_+)\frac{e^{2\pi\eta}-1}{2\pi\eta}\int_\Omega\mathrm{d}\mathbf{n}\,\frac{f(\mathbf{k}+k\mathbf{n})-f(2\mathbf{k})}{1-\hat{\mathbf{k}}\mathbf{n}}.$$



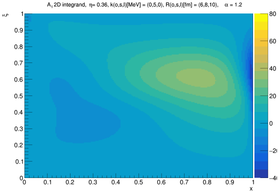
Non-spherically symmetric case: result

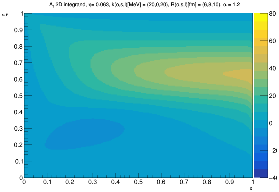
- Rightfully *seems* complicated; gain: tame, well-manaegable 3D integrands!
  Example integrands in φ variable:

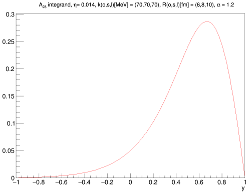




- Example integrands in remaining 2 & 1 variables (after φ done):







- Programmed, working, coming soon in measurements! ☺

Status, summary and outlook

- Fouier-space method: efficient for calculation of FSI for a wide range of $D(\mathbf{r})$
- Coulomb interaction (no strong):
  - Spher. symmetric: Eur. Phys. J. C **83**, 1015 (2023); talked about at WPCF 2023
  - Non-spherically symmetric case: *in the barrel*
  - Generalization (in fact, simplification) for non-identical particles: only $\mathcal{A}_1$ needed
- Strong interaction (s-wave):
  - Calculation done, formulas (even more) complicated at first sight
    $\Rightarrow$ did not include in these slides; programming is being done
- Result (shortly summarized): efficient means of calculating valid 3D Coulomb-interacting & strong interacting & Coulomb+strong interacting correlation functions for various source types (Gauss, Lévy...)

***

New exact analytic formulas in NR quantum mechanics! ☺

Thank you for your attention!