Combinants and correlation functions in nuclear collisions: some intriguing properties of multiplicity distributions

Maciej Rybczyński
(in collaboration with **Zbigniew Włodarczyk**)

Jan Kochanowski University, Kielce, Poland

17$^{\rm th}$ Workshop on Particle Correlations and Femtoscopy
4-8 November 2024
Toulouse, France

Motivation

- Multiplicity distributions of charged particles produced in $e^+ + e^-$ and proton-proton collisions exhibit, after closer inspection, peculiarly enhanced void probability and oscillatory behavior of the modified combinants.

- The set of combinants, C_j provides a similar measure of fluctuations as the set of cumulant factorial moments, K_q, which are very sensitive to the details of the multiplicity distribution and were frequently used in phenomenological analyses of data.

- While cumulants are best suited to the study of the densely populated region of phase space, combinants are better suited for the study of sparsely populated regions because calculation of C_j requires only a finite number of probabilities $P(N < j)$.

M. Rybczyński *et al.*, Phys. Rev. D **99** 094045
H. Ang *et al.*, Eur. Phys. J. A **56** 117
H. Ang *et al.*, Mod. Phys. Lett. A **34** 1950324
M. Rybczyński, G. Wilk, Z. Włodarczyk, Phys. Rev. D **103** 114026
V. Z. Reyna Ortiz, M. Rybczyński, Z. Włodarczyk, Phys. Rev. D **108** 074009

Combinants, modified combinants, and correlation functions

- The dynamics of multiparticle production process is hidden in the way in which the consecutive measured multiplicities N are connected.
- In the simplest case one assumes that the multiplicity N is directly influenced only by its neighboring multiplicities $(N \pm 1)$ in the way dictated by the simple recurrence relation:

$$(N+1)\,P(N+1) = g(N)\,P(N), \quad g(N) = \alpha + \beta N, \tag{1}$$

where $\beta > 0$ for negative binomial distribution (NBD), $\beta < 0$ for binomial distribution (BD) and $\beta = 0$ for Poisson distribution (PD).

Combinants, modified combinants, and correlation functions

- We propose a more general form of the recurrence relation, used in counting statistics when dealing with multiplication effects in point processes.
- Contrary to Eq. (1), it now connects all multiplicities by means of some coefficients C_j, which define the corresponding $P(N)$ in the following way:

$$(N+1)\,P\,(N+1) = \langle N\rangle \sum_{j=0}^{N} C_j P\,(N-j)\,. \tag{2}$$

- The coefficients C_j contain the memory of particle $N+1$ about all the $N-j$ previously produced particles. They can be directly calculated from the experimentally measured $P(N)$ by reversing Eq. (2) and putting it in the form of the following recurrence formula for C_j:

$$\langle N\rangle C_j = (j+1)\left[\frac{P\,(j+1)}{P\,(0)}\right] - \langle N\rangle \sum_{i=0}^{j-1} C_i \left[\frac{P\,(j-i)}{P\,(0)}\right]. \tag{3}$$

Combinants, modified combinants, and correlation functions

The modified combinants C_j defined by the recurrence relation (3) are closely related to the *combinants* $C_j^\star$ introduced long a time ago by means of the generating function, $G(z) = \sum_{N=0}^{\infty} P(N)z^N$, as

$$C_j^\star = \frac{1}{j!}\frac{d^j \ln G(z)}{dz^j}\bigg|_{z=0} \qquad (4)$$

or

$$\ln G(z) = \ln P(0) + \sum_{j=1}^{\infty} C_j^\star z^j. \qquad (5)$$

Namely,

$$C_j = \frac{j+1}{\langle N\rangle} C_{j+1}^\star. \qquad (6)$$

Note that, although the combinants, C_j^* were already known for a long time, and their possible oscillatory behavior was also known, they have so far scarcely been used and were not directly extracted from the experimental data.

S.K. Kauffmann and M. Gyulassy, Phys. Rev. Lett. **40** 298
S.K. Kauffmann and M. Gyulassy, J. Phys. A **11** 1715
J. Bartke, Phys. Scrip. **27** 226
A.B. Balantekin and J.E. Seger, Phys. Lett. B **266** 231
Ding-wei Huang, J. Phys. G **23** 895

Combinants, modified combinants, and correlation functions

The set of modified combinants, C_j, provides a similar measure of fluctuations as the set of cumulant factorial moments, K_q, which are very sensitive to the details of the multiplicity distribution and were frequently used in phenomenological analyses of data,

$$K_q = F_q - \sum_{i=1}^{q-1} \binom{q-1}{i-1} K_{q-i} F_i, \tag{7}$$

where

$$F_q = \sum_{N=q}^{\infty} N(N-1)(N-2)\dots(N-q+1)\,P(N) \tag{8}$$

are the factorial moments. The K_q can be expressed as an infinite series of the C_j,

$$K_q = \sum_{j=q}^{\infty} \frac{(j-1)!}{(j-q)!} \langle N \rangle C_{j-1}, \tag{9}$$

and, conversely, the C_j can be expressed in terms of the K_q,

$$C_j = \frac{1}{\langle N \rangle} \frac{1}{(j-1)!} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} K_{p+j}. \tag{10}$$

Combinants, modified combinants, and correlation functions

The modified combinants C_j defined by the recurrence relation (2) can be expressed by the generating function of multiplicity distribution, $G(z) = \sum_{N=0}^{\infty} P(N) z^N$ as:

$$\langle N \rangle C_j = \frac{1}{j!} \frac{d^{j+1} \ln G(z)}{dz^{j+1}} \bigg|_{z=0}. \tag{11}$$

The generating function can be shown to be a sum over the 'averaged' connected correlation function of all orders ($\xi_{(N)}$):

$$\ln G(z) = \sum_{N=1}^{\infty} \frac{(z-1)^N}{N!} \langle N \rangle^N \bar{\xi}_{(N)}, \tag{12}$$

where the correlation function

$$\bar{\xi}_{(N)} = \frac{1}{V^N} \int \cdots \int \xi_{(N)} \left(r_{ij} \right) dV_1 \cdots dV_N. \tag{13}$$

Combinants, modified combinants, and correlation functions

Glauber predicted that the maximal value of the N-body correlation function for thermal light is related to the order of the function by relationship $N!$. This $N!$ dependence is a consequence of Wick's theorem. Recently its validity is confirmed for massive particles. $\bar{\xi}_{(N)}$ can be expressed in terms of C_j as:

$$\bar{\xi}_{(N)} = \frac{1}{\langle N\rangle^{N-1}} \sum_{j=N}^{\infty} \frac{(j-1)!}{(j-N)!} C_{j-1}. \quad (14)$$

- For NBD:
  - $\bar{\xi}_{(N)} = (N-1)!/k^{N-1}$,
  - $C_j = \frac{k}{\langle N\rangle} p^{j+1}$.
- For BD:
  - $\bar{\xi}_{(N)} = (-1)^{N+1} (N-1)!/K^{N-1}$,
  - $C_j = (-1)^j \frac{K}{\langle N\rangle} \left(\frac{p}{1-p}\right)^{j+1}$.
- For PD:
  - $\bar{\xi}_{(1)} = 1$ and $\bar{\xi}_{(N>1)} = 0$,
  - $C_0 = \frac{1}{\langle N\rangle}$ and $C_{j>0} = 0$.

S.D.M. White, MNRAS **186** 145
R.G. Dall *et al.*, Nature Physics **9** 341

Compound distributions

- Because a single distribution of the NBD or BD type cannot describe data we shall check the idea of *compound distributions* (CD). They are applicable when the production process consists of a number M of some objects (clusters/fireballs/etc.) produced according to some distribution $f(M)$ (defined by a generating function $F(z)$), which subsequently decay independently into a number of secondaries, $n_{i=1,\dots,M}$, following some other (always the same for all M) distribution, $g(n)$ (defined by a generating function $G(z)$).
- The resultant multiplicity distribution,

$$h\left(N = \sum_{i=0}^{M} n_i\right) = f(M) \otimes g(n), \tag{15}$$

is a compound distribution of f and g with generating function

$$H(z) = F[G(z)]. \tag{16}$$

Compound distributions

The immediate consequence of Eq. (16) is that in the case where $f(M)$ is a Poisson distribution, P_{PD}, with generating function

$$F(z) = \exp[\lambda(z-1)], \tag{17}$$

then, for any other distribution $g(n)$ with generating function $G(z)$, the combinants obtained from the compound distribution $h(N) = P_{PD} \otimes g(n)$ and calculated using Eq. (11), do not oscillate and are equal to

$$C_j = \frac{\lambda(j+1)}{\langle N \rangle} g(j+1). \tag{18}$$

Compound distributions

- The modified combinants C_j for the BD with generating function

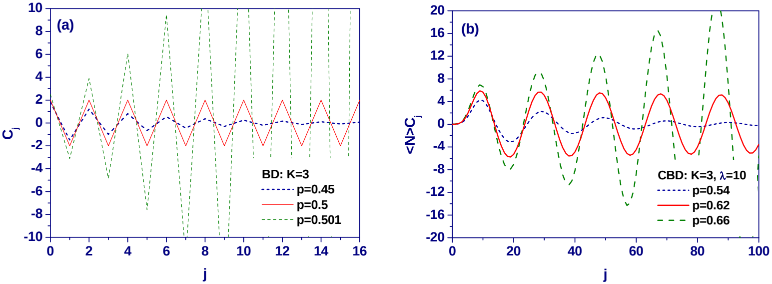
$$F(z) = (pz + 1 - p)^K \tag{19}$$

  oscillate with a period of 2, whereas the amplitudes of these oscillations depend on the probability emission p. To control the period of the oscillations one has to compound this BD with some other distribution.

- We show an example of using for this purpose a Poisson distribution with a generating function given by Eq. (17) (for which $C_0 = 2$ and $C_{j>0} = 0$). The generating function of the resulting Compound Binomial Distribution (CBD) is

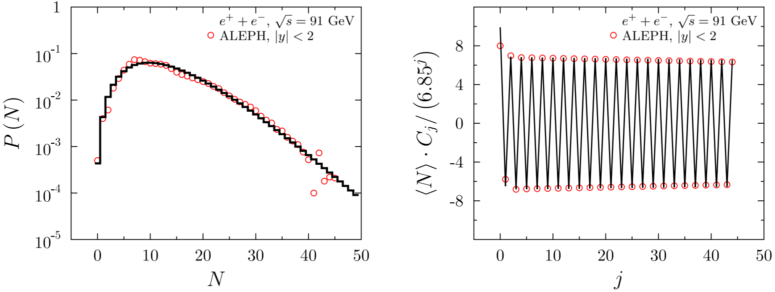
$$H(z) = \{p \exp[\lambda(z-1)] + 1 - p\}^K . \tag{20}$$

Compound distributions



- (a) C_j for a single BD for different probabilities of particle emission.
- (b) The same BD compounded with a Poison distribution with $\lambda = 10$.

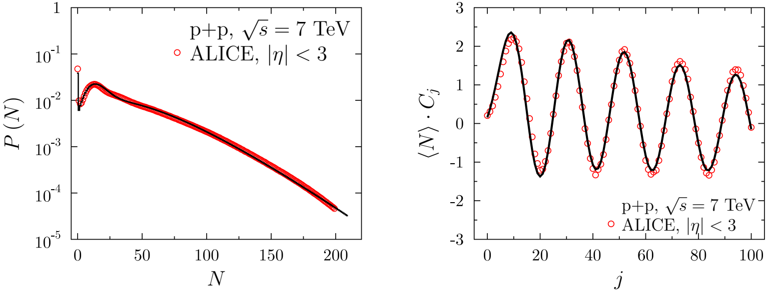
Results



- Left panel: data on charged particles multiplicity distribution $P(N)$ measured in $e^+ + e^-$ collisions by the ALEPH experiment at $\sqrt{s} = 91$ GeV.
- Right panel: the modified combinants C_j derived from these data (note the significant dependence of the amplitude on rank j).
- The oscillation amplitude of the plot has been scaled accordingly making it possible to plot the results. Otherwise the amplitudes would grow in a power-law fashion.

Data from:
ALEPH Coll., Z. Phys. C **69** 15

Results



- Left panel: Charged particle multiplicity distribution $P(N)$ measured in proton-proton collisions by ALICE at $\sqrt{s} = 7$ TeV.
- Right panel: The corresponding modified combinants C_j emerging from it fitted using a two-compound distribution.

Data from:
ALICE Coll., Eur. Phys. J. C **77** 852

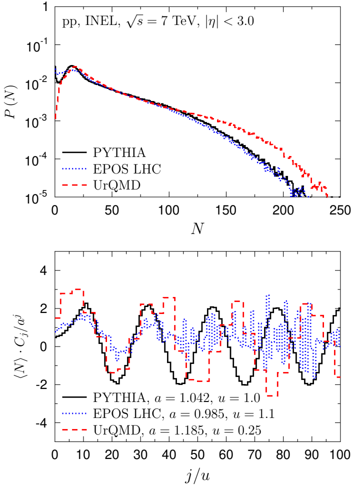
Comparison with models

- The aim of this section is to show that the observed oscillations have a physical origin and are not the result of experimental procedures. We focus on the analysis of the Monte Carlo simulated events and comparison with existing experimental data.
- In this study we have used PYTHIA 8.308 (Bierlich *et al.*, SciPost Phys.Codeb. **2022** 8), EPOS LHC (Pierog *et al.*, Phys. Rev. C **92** 034906) and UrQMD 3.4 set to LHC mode (Bass *et al.* Prog. Part. Nucl. Phys. **41** 255) to generate proton-proton interactions at $\sqrt{s} = 7$ TeV in accordance to data on charged particles multiplicity distributions obtained by the ALICE experiment at CERN LHC (ALICE Coll., Eur. Phys. J. C **77** 852).

Comparison with models

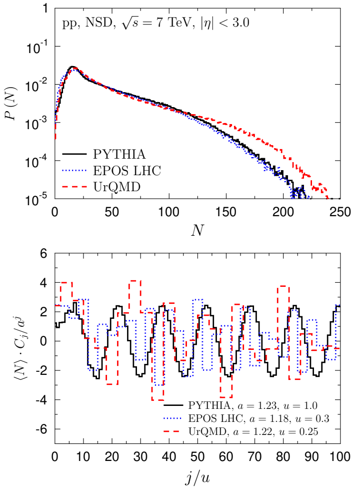
- In PYTHIA simulation we have implemented the inelastic component of the total cross-section for soft-QCD processes with the parameter *SoftQCD:inelastic=on*. The remaining set of PYTHIA parameters we left with its default values. In the case of EPOS LHC and UrQMD we used default values of the parameters.
- To match with the experimental conditions, charged particle multiplicities have been chosen in the trigger conditions and acceptance of the ALICE detector, defined in (ALICE Coll., Eur. Phys. J. C **77** 852). Namely, the generated events of collisions (EOCs) were divided into two classes: inelastic (*INEL*) class and non-single diffractive (*NSD*) class.
- The generated EOC belongs to INEL class if there is at least one charged particle in either the $-3.7 < \eta < -1.7$, $|\eta| < 1.98$, or $2.8 < \eta < 5.1$ pseudorapidity interval corresponding to the acceptances of the V0-C, SPD, and V0-A ALICE sub-detectors, respectively.
- The NSD class requires charged particles to the detected in both $-3.7 < \eta < -1.7$ and $2.8 < \eta < 5.1$ pseudorapidity intervals (ALICE Coll., Eur. Phys. J. C **77** 852).

Comparison with models



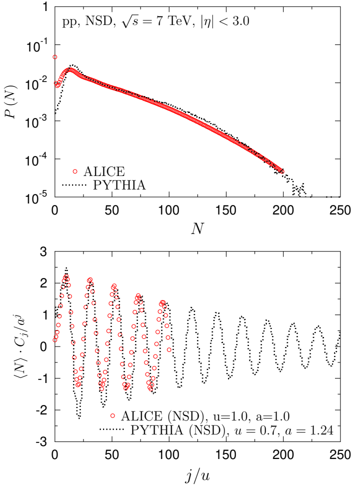
- (upper panel) Multiplicity distributions $P(N)$ of charged particles generated in proton-proton interactions at $\sqrt{s} = 7$ TeV.
- (lower panel) The corresponding modified combinants C_j emerging from them. PYTHIA 8 with *SoftQCD:inelastic* processes (solid lines), EPOS LHC (dotted lines) and UrQMD 3.4 set to LHC mode (dashed lines).
- For all models the applied kinematic cuts as described in the ALICE paper (*INEL* class).

Comparison with models



- Same as on previous plot, but for *NSD* class.

Comparison with models



- (upper panel) Multiplicity distribution $P(N)$ of charged particles produced in proton-proton *non-single diffractive* interactions at $\sqrt{s} = 7$ TeV as measured by ALICE experiment (ALICE Coll., Eur. Phys. J. C **77** 852) (NSD class).
- (lower panel) The corresponding modified combinants C_j emerging from them. PYTHIA 8 (dotted lines) with *SoftQCD:inelastic* processes and all kinematic cuts as described in the ALICE paper.

Conclusions

- Modified combinants, C_j, deduced from the measured multiplicity distributions of charged particles, $P(N)$, together with the already measured void probabilities, could provide additional information on the dynamics of the particle production.

- A detailed analysis of the modified combinants derived from the experimental $P(N)$'s reveals differences between the various processes.
    - In $e^+ + e^-$ annihilation, the C_j's oscillate with a period of 2 with amplitudes increasing as a power-law.
    - On the other hand, proton-proton collisions produce C_j's oscillating with approximately 10 times the period of their $e^+ + e^-$ counterparts, with decaying amplitudes.

- Modified combinants evaluated from models exhibit oscillatory behavior, though the oscillation period differs from experimental data.

Additional slides

Results

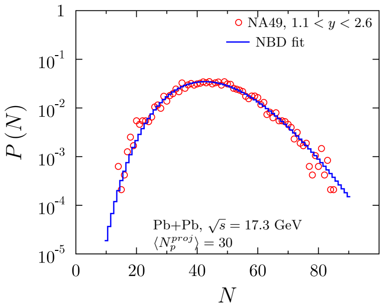


Fig. 3. Charged particle multiplicity distribution $P(N)$ measured in semi-peripheral Pb+Pb collisions by NA49 experiment at $\sqrt{s_{NN}} = 17.3$ GeV.

Data from:
NA49 Coll., Phys. Rev. C **75** 064904

Results

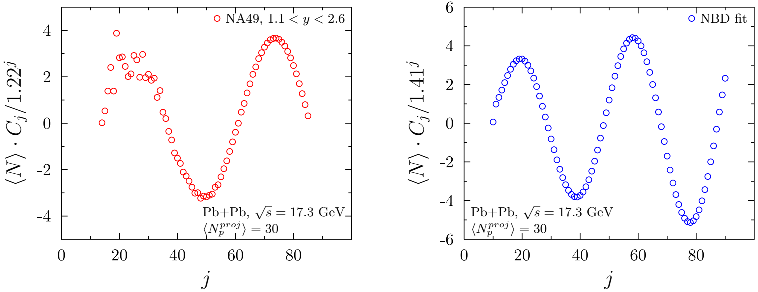


Fig. 4. Left panel: the modified combinants C_j derived from the NA49 data. Right panel: the modified combinants C_j derived from the NBD fit to the NA49 data.

Data from:
NA49 Coll., Phys. Rev. C **75** 064904

Combinants, modified combinants, and correlation functions