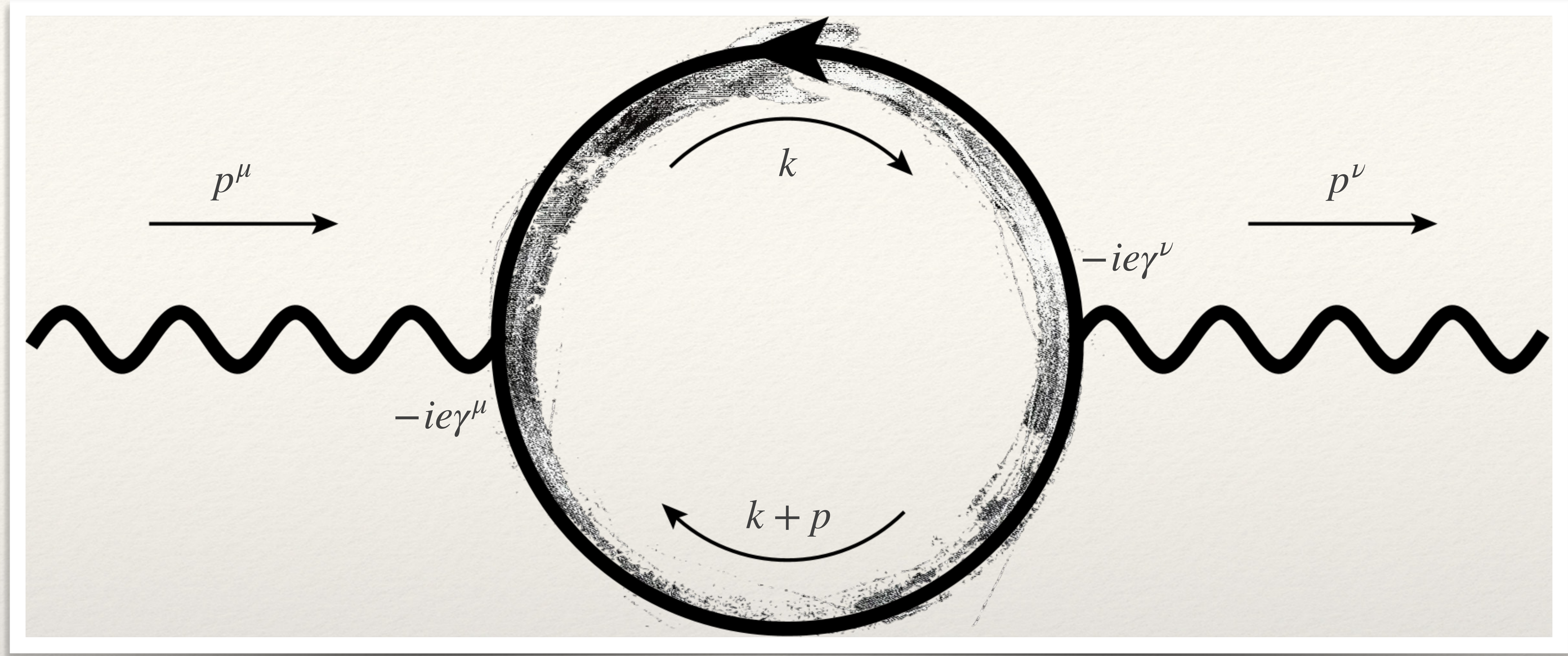


Jonathan Ronca

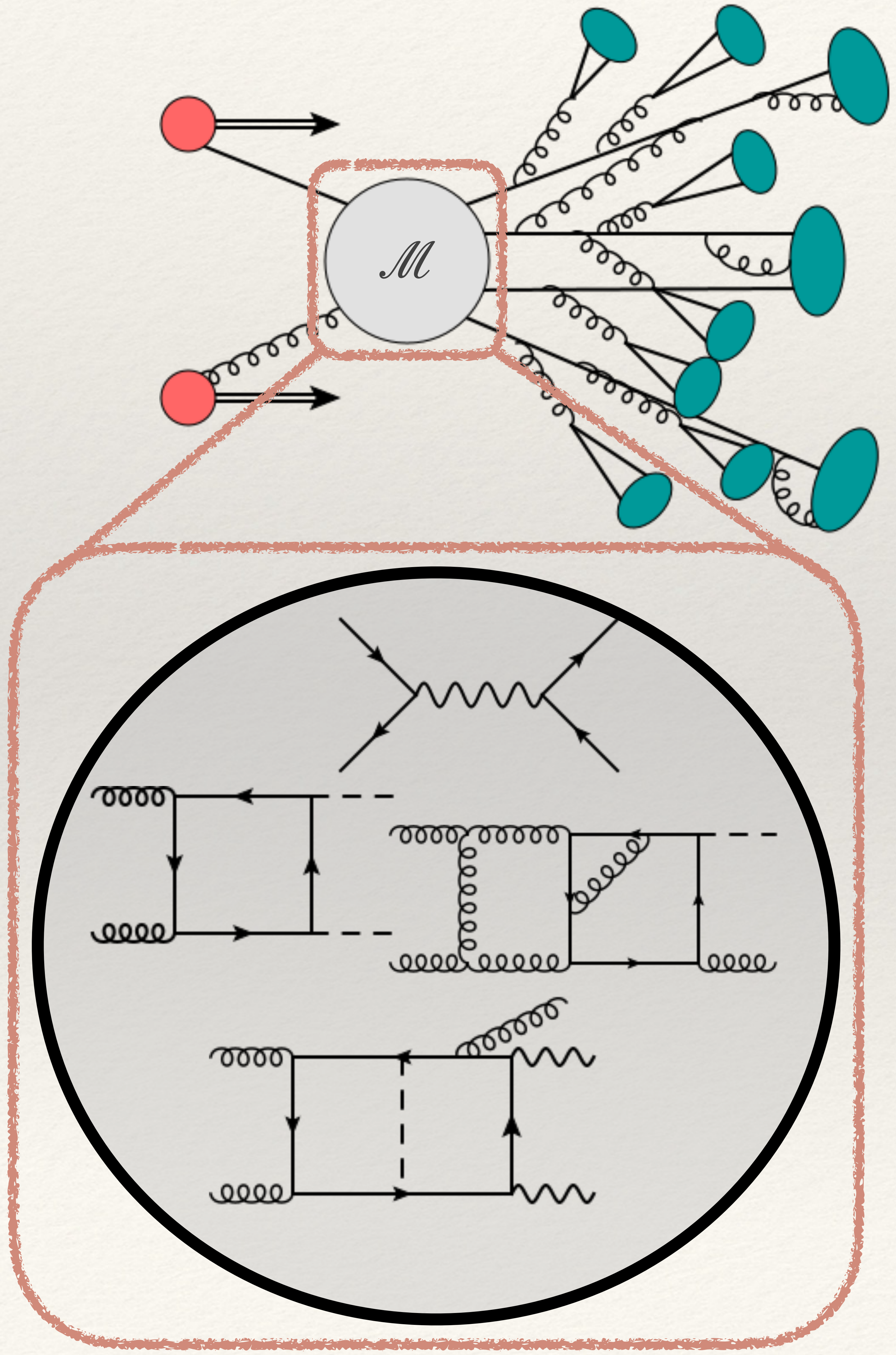
Feynman Integrals in QFT

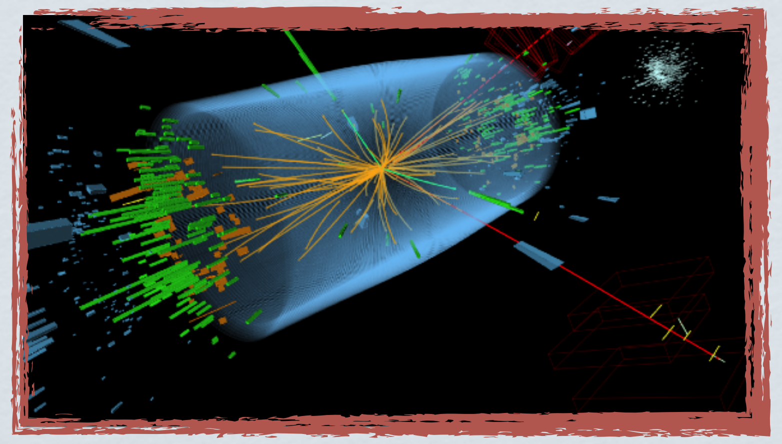
DMF
ROMA TRE UNIVERSITÀ DEGLI STUDI
INFN Istituto Nazionale di Fisica Nucleare
SEZIONE DI ROMA TRE

17 Apr 2024

Why Amplitudes? Why Loops?

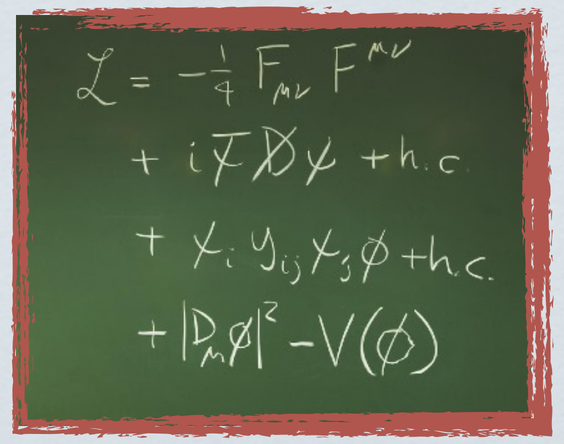
Feynman Integral are the fundamental object of any pQFT prediction to physical observables





HL-LHC will be able to measure Observables with sub-percent precision

Increasing precision

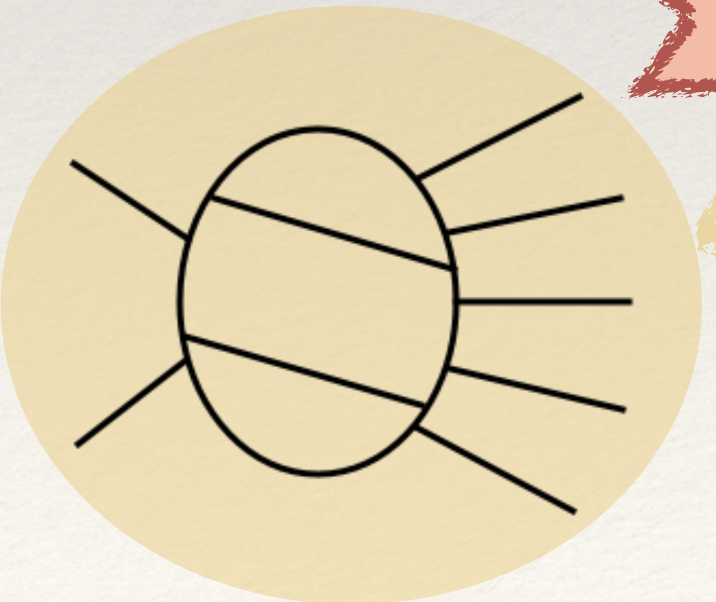


N2LO and N3LO accuracy is demanded

High multiplicity and high loop order

Multi-loop amplitudes are **crucial**

Feynman Integrals are their **core**

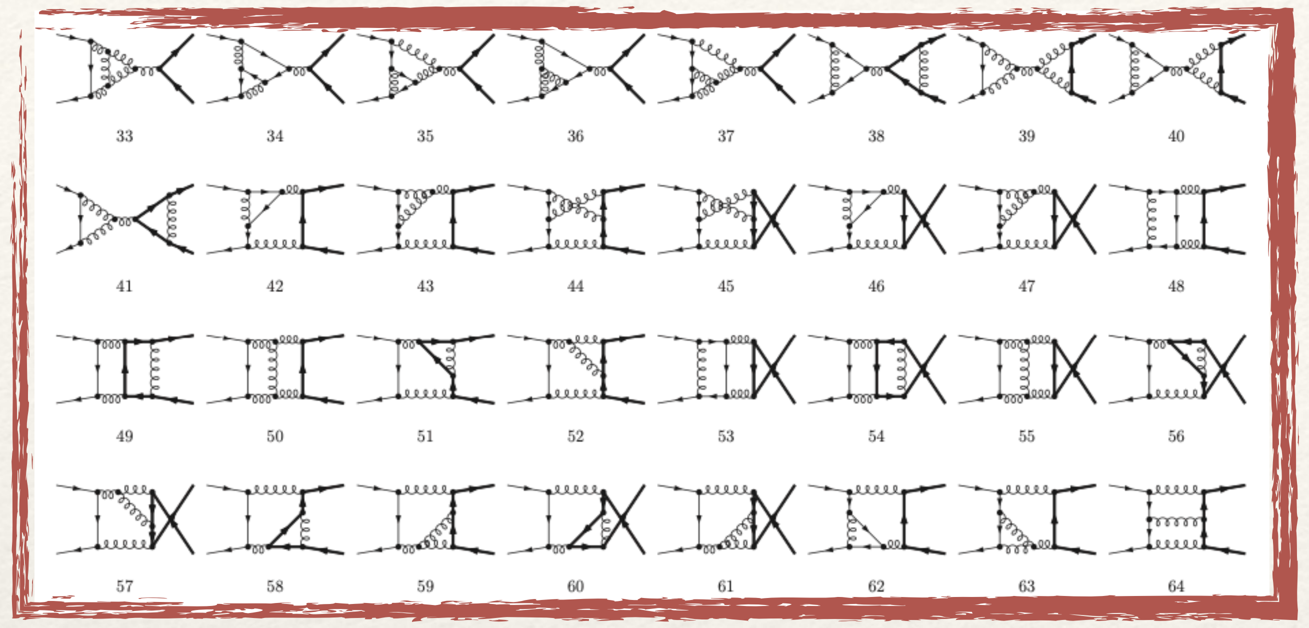


The integrals should have a "nice"/"standard" form

$$I = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{1}{D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} \ldots}, \qquad \alpha_j \lesseqgtr 0$$

[Peraro(2019)]

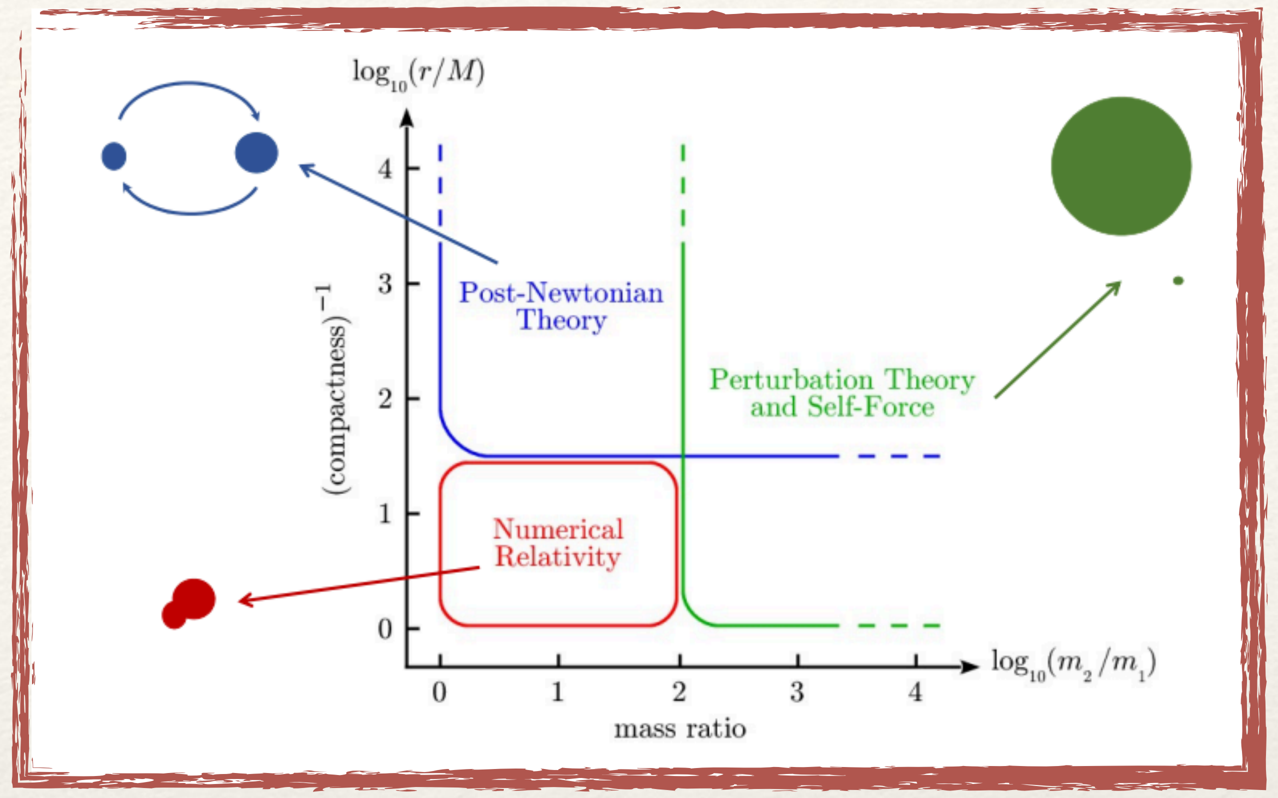
Where do Feynman Integral appear?



[Mandal,Mastrolia,Ronca,Torres:2204.03466]

Collider physics

...but also in



[Le Tiec,2014]

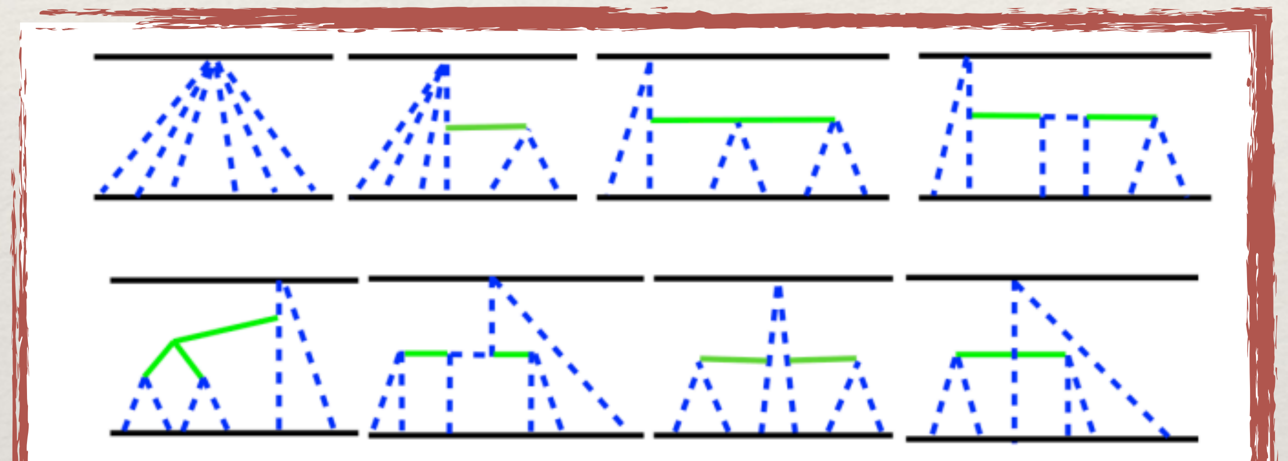


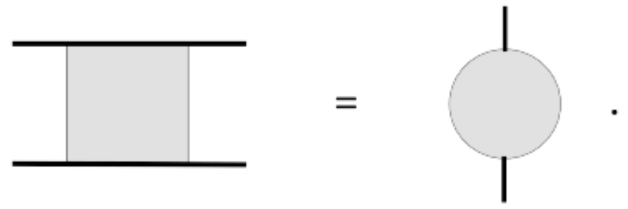
Figure 2. 5PN graphs contributing to the test-particle limit. The last graph (bottom-right) does not contribute to the 5PN potential, because its 4PN subdiagram vanishes.

[Foffa,Mastrolia,Sturanic,Sturm,Torres:1902.1057]

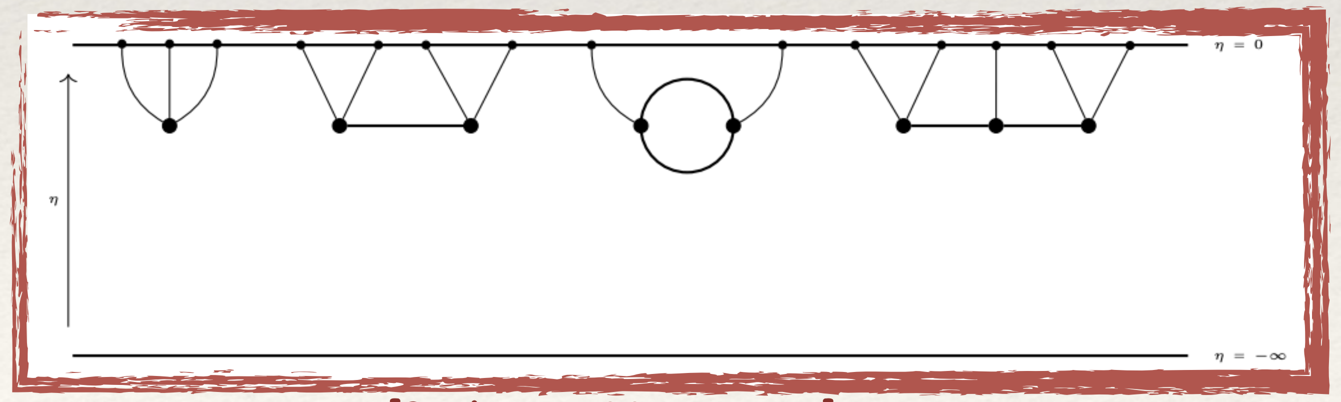
General relativity:
-> Post-Newtonian framework

3 Amplitudes and Feynman Integrals

In general, within the EFT approach, since the sources (black lines) are static and do not propagate, any gravity-amplitude of order G_N^ℓ can be mapped into an $(\ell-1)$-loop 2-point function with massless internal lines and external momentum p, where $p^2 \equiv s \neq 0$,


(3.1)

[Foffa,Mastrolia,Sturanic,Sturm:1612.00482]



[Benincasa:2203.15330]

Cosmology:
-> Cosmological wavefunction and correlators

However, there is a much more interesting way in which this integral representation opens the door to a "timeless" description of cosmological correlators. Our cosmological integrals are special cases of a wide class of integrals of the form

$$I(C,D;n;\varepsilon) = \int_0^\infty \mathrm{d}x_1 \cdots \mathrm{d}x_m \, P(x) \prod_I (C_{Ij}x_j + D_I)^{-n_I+\varepsilon_I}\,, \qquad (1.2)$$

where $P(x)$ is some polynomial in the x variables, and the singularities of the integrand are powers of linear factors, raised to integer powers n_I possibly "twisted" by fractional parameters ε_I.

When applied to the cosmological wavefunction (1.1), we learn a first important fact:

The cosmological wavefunction satisfies a differential equation, which governs how it changes as the external kinematics are varied.

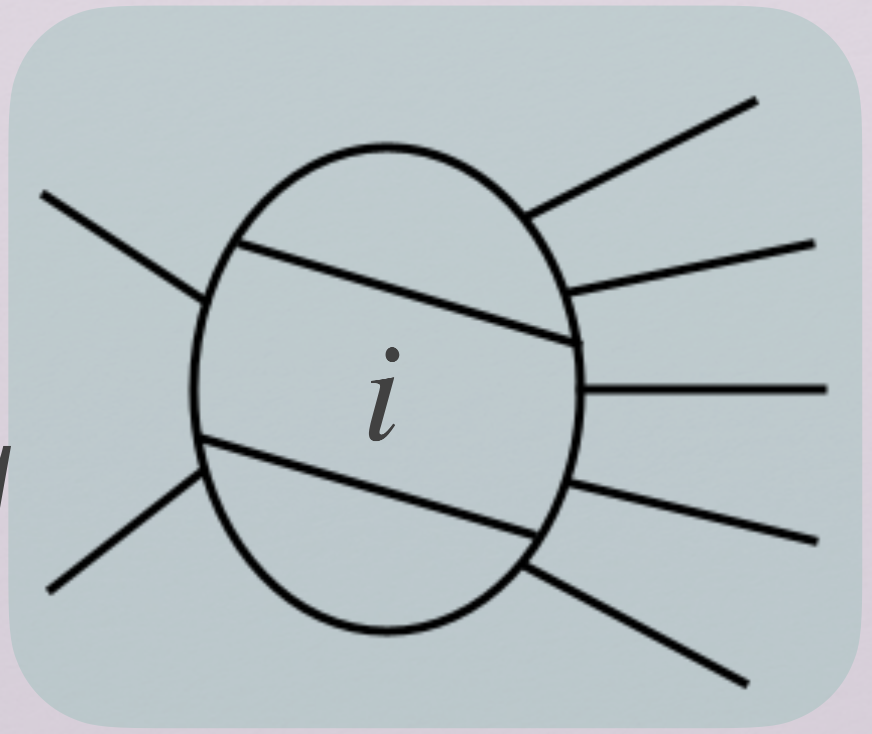
[Arkani-Hamed,Baumann,Hillman,Joyce,Lee,Pimentel:2312.05303]

...And many other fields, where a Lagrangian can be defined...

Calculating Amplitudes: Decomposition

Scattering Amplitude

$$\mathscr{M} = \sum_{i}^{N_d}$$



$N_d \sim O(10^1 - 10^3)$ *

Drawing Feynman Diagrams

Direct Interferences

Helicity Amplitudes

Form Factors

Numerator Algebra

$$\mathscr{M} = \sum_{j}^{N_I} c_j I_j$$

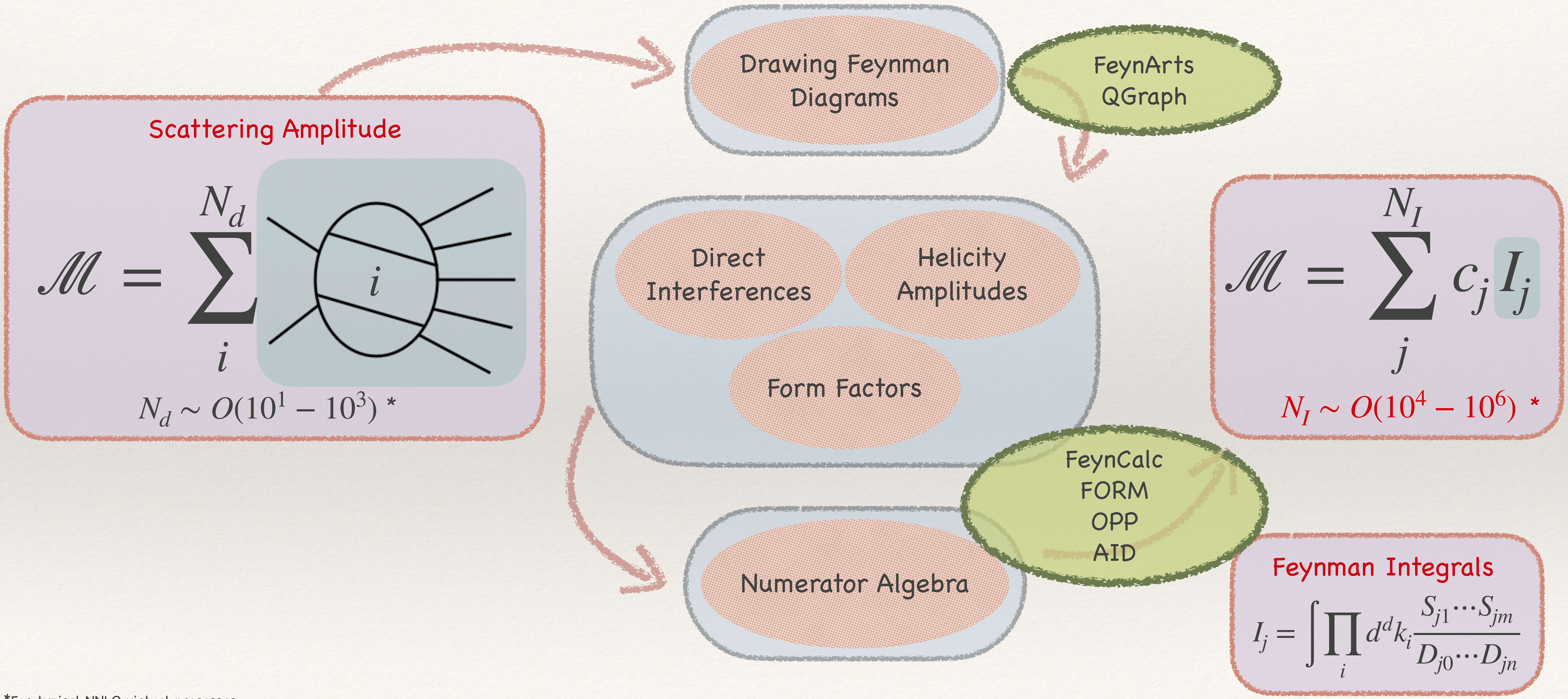
$N_I \sim O(10^4 - 10^6)$ *

Feynman Integrals

$$I_j = \int \prod_i d^d k_i \frac{S_{j1} \cdots S_{jm}}{D_{j0} \cdots D_{jn}}$$

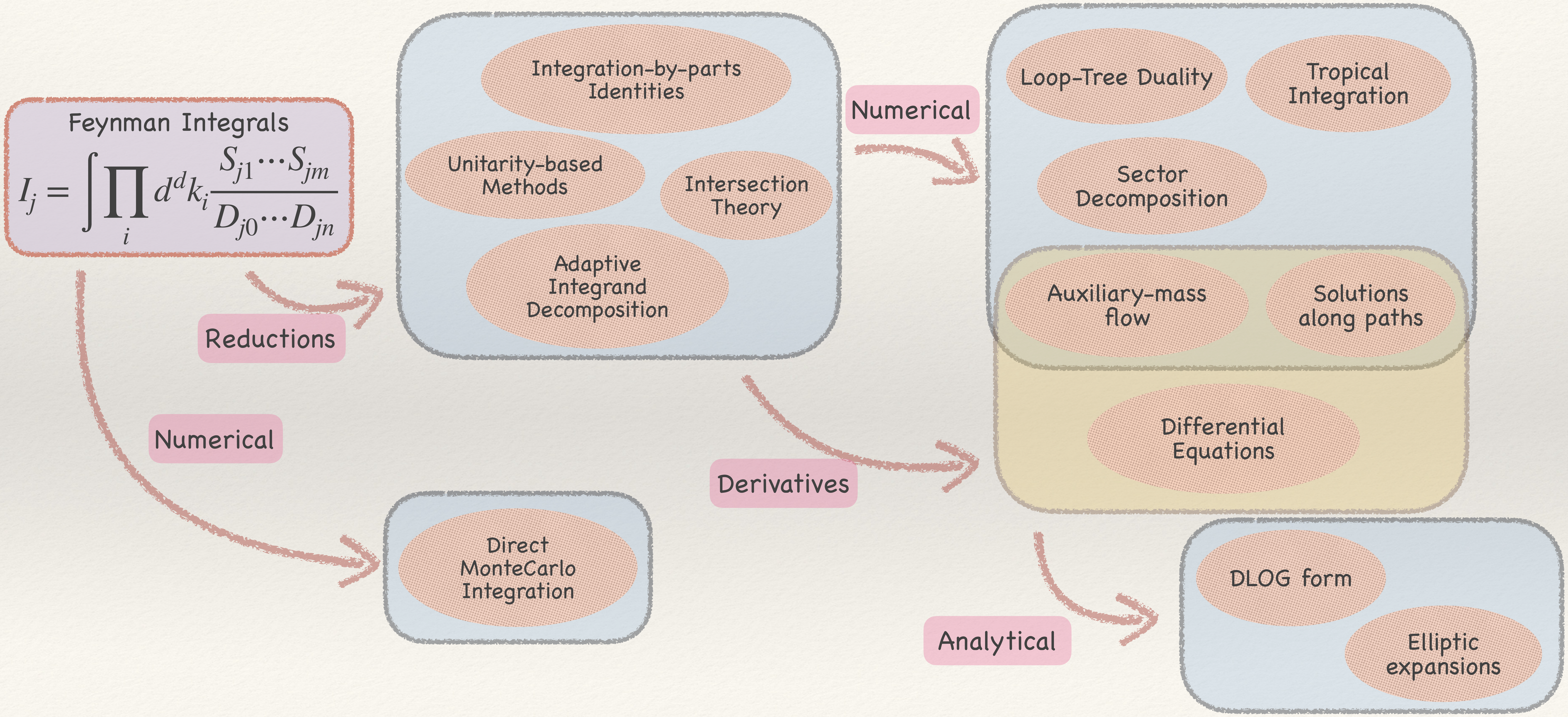
*For typical NNLO virtual processes

Calculating Amplitudes: Decomposition



*For typical NNLO virtual processes

Calculating Amplitudes: Feynman Integrals



Direct Monte Carlo Integration: Applications

$gg \to HH$ @ NLO QCD
$gg \to hH$ @ NLO 2HDM
$gg \to AA$ @ NLO 2HDM

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Higgs-pair production via gluon fusion at hadron colliders: NLO QCD corrections

Julien Baglio,[a] Francisco Campanario,[b,c] Seraina Glaus,[c,d] Margarete Mühlleitner,[c] Jonathan Ronca,[b] Michael Spira[e] and Juraj Streicher[f]

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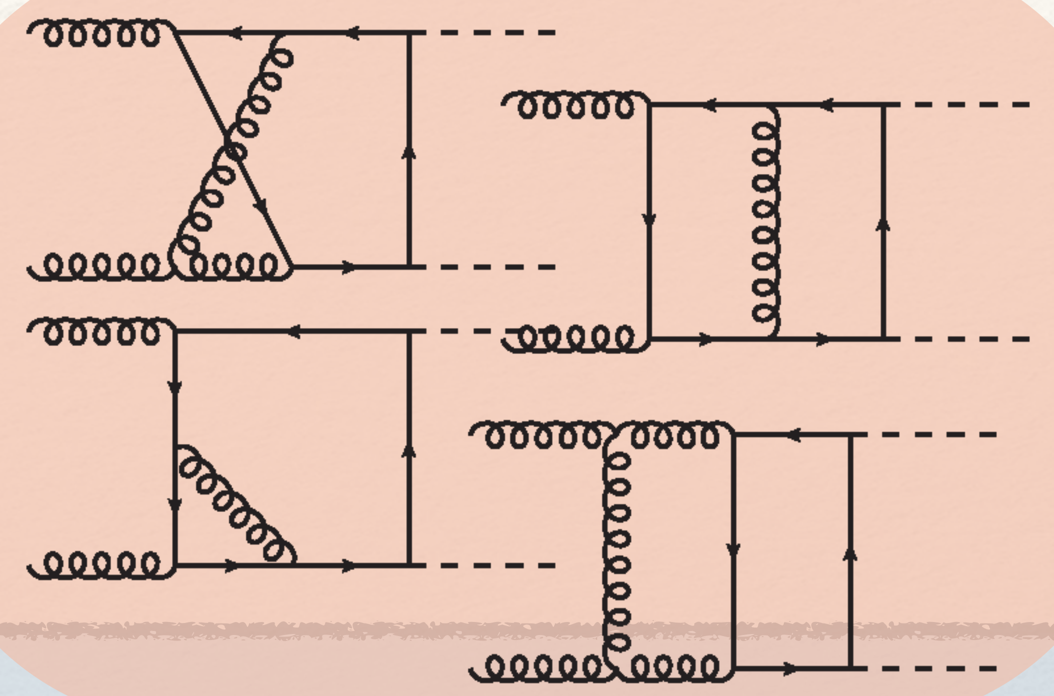
https://doi.org/10.1140/epjc/s10052-023-11957-2
The European Physical Journal C
Check for updates
Regular Article - Theoretical Physics

Full NLO QCD predictions for Higgs-pair production in the 2-Higgs-doublet model

J. Baglio[1,2,a], F. Campanario[3,b], S. Glaus[4,5], M. Mühlleitner[4,c], J. Ronca[6,d], M. Spira[7,e]

Process:
- SM QCD/2HDM
- Four-point two-Loop scattering
- Four mass scales
- IR and UV divergent

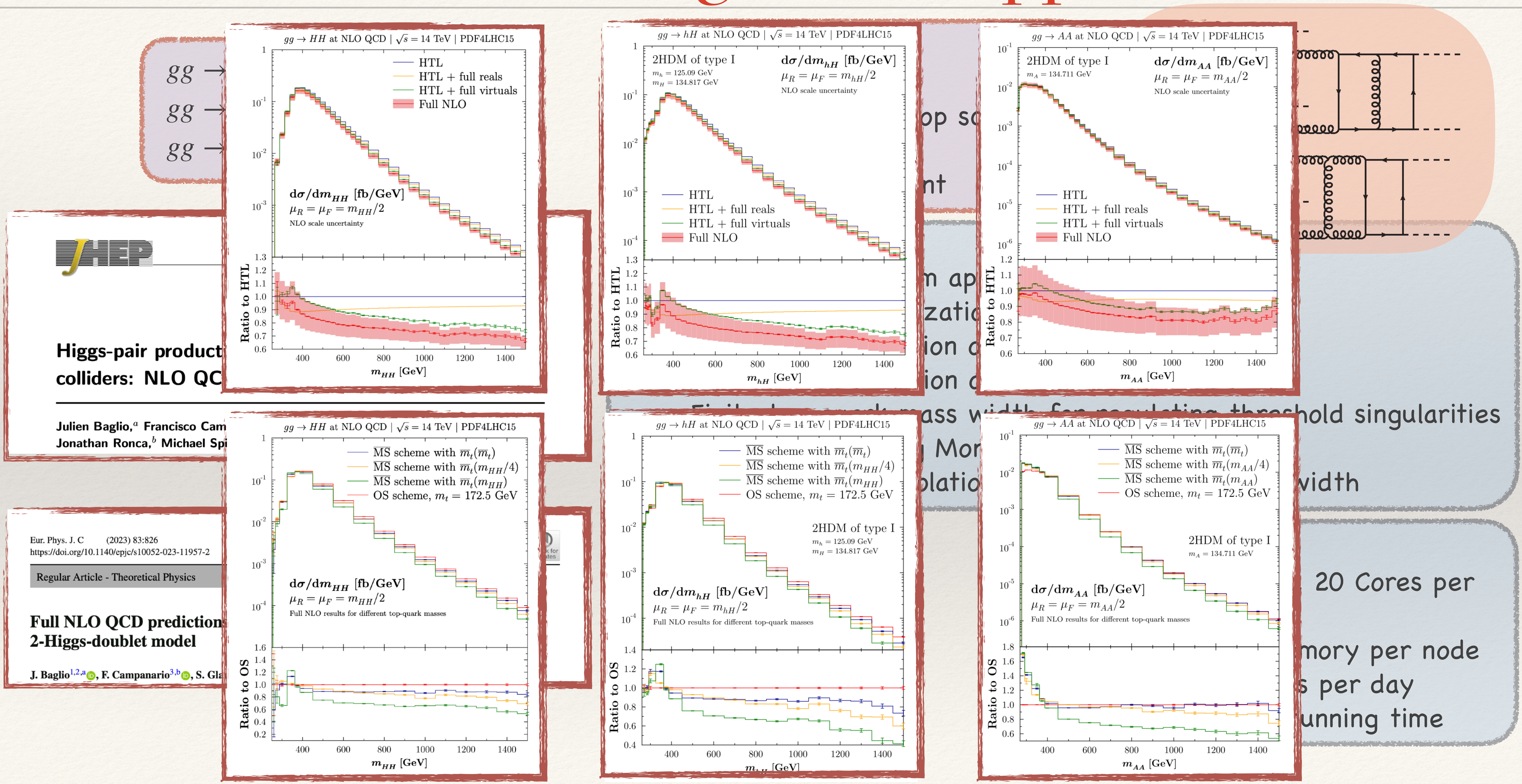


Method:
- Diagram-by-diagram approach
- Feynman Parametrization
- End-point subtraction of UV divergences
- Dedicated subtraction of IR divergences
- Finite top-quark mass width for regulating threshold singularities
- IBPs for stabilizing Monte Carlo integrals
- Richardson Extrapolation to recover top narrow width

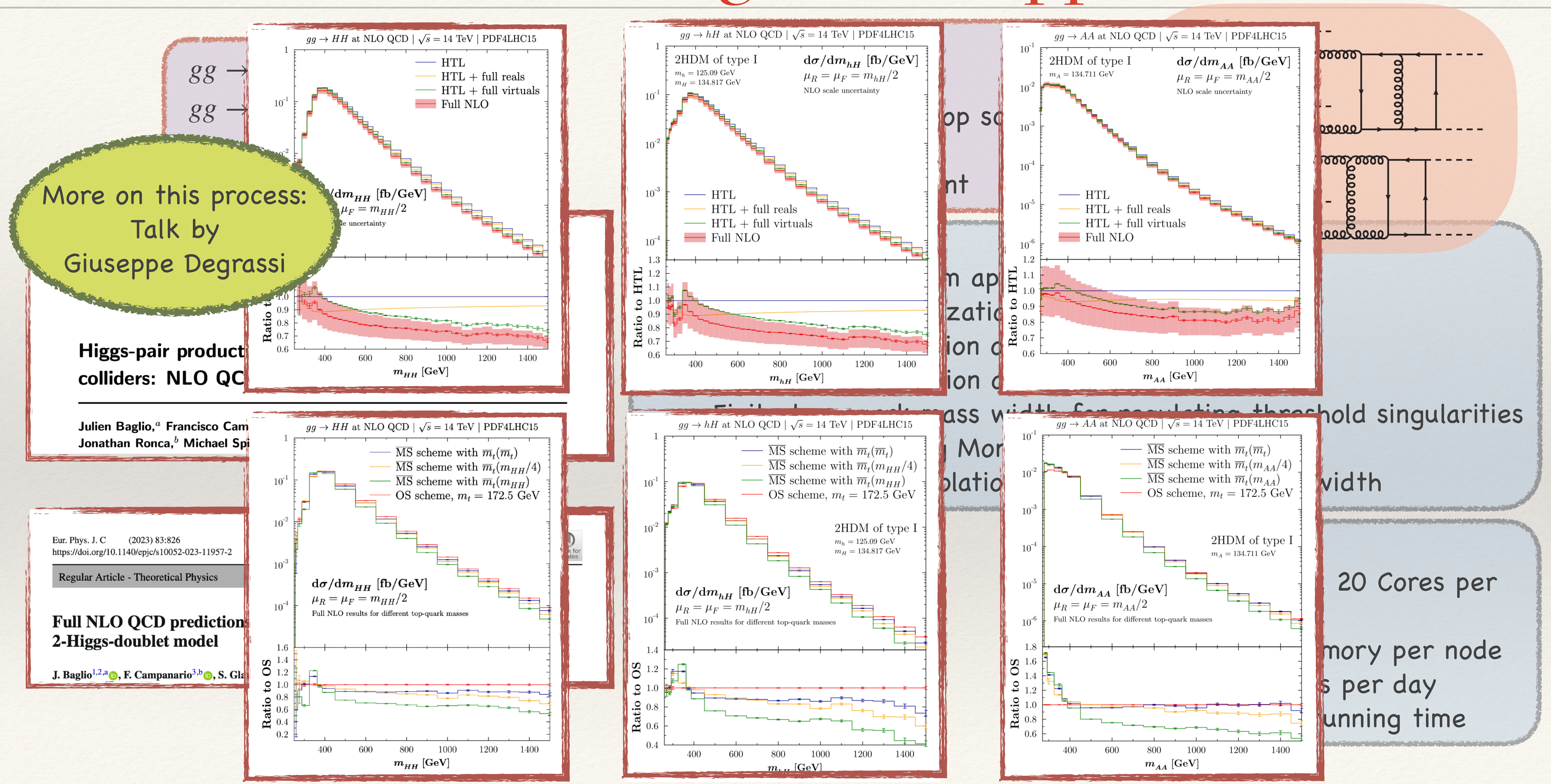
Tools:
- Mathematica
- FeynCalc
- PackageX
- Fortran
- XVegas

Running:
- 900 Nodes, 20 Cores per node
- 128 GB Memory per node
- ~2000 jobs per day
- 1 year of running time

Direct Monte Carlo Integration: Applications



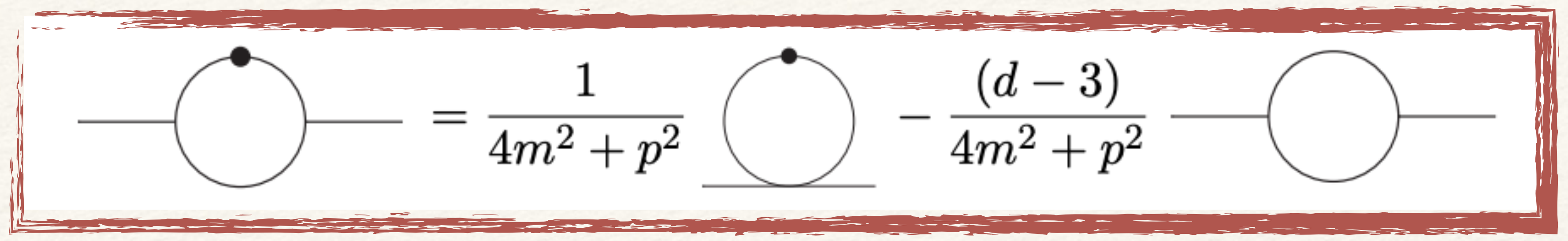
Direct Monte Carlo Integration: Applications



Integration-by-parts identities

Is there a way to **reduce** the number of integrals we need to evaluate?

$$J_n^r(\{s_{ij}\},\{m_1,\ldots,m_{n+s}\};d) = \int \prod_{l=1}^{L} d^d k_l \frac{D_{n+1}^{\nu_{n+1}}\cdots D_{n+r}^{\nu_{n+r}}}{D_1^{\nu_1}\cdots D_n^{\nu_n}}$$




```
{
diag3[1,0,1,0] →
 + 0
,
diag3[-1,1,1,1] →
 + diag3[0,1,1,1]*(((s-3*m2)*t+m2^2)/(s-4*m2))
 + diag3[0,0,1,1]*((-t-m2)/(s-4*m2))
 + diag3[0,1,1,0]*((2*t+s-2*m2)/(s-4*m2))
 + diag3[0,1,0,0]*(((-d+2)*t+(-d+2)*m2)/(((2*d-6)*m2)*s+(-8*d+24)*m2^2))
,
diag3[0,1,0,1] →
 + diag3[0,1,0,0]*((d-2)/((2*d-6)*m2))
,
diag3[1,-1,1,1] →
 + diag3[0,0,1,1]*((((d-2)*s+(-2*d+6)*m2)*t+((-d+4)*m2)*s+(2*d-6)*m2^2)/((d-4)*t^2+((-2*d
 + diag3[1,0,0,1]*(((d-4)*t^2+(-2*s+2*m2)*t+(-d+2)*m2^2)/((d-4)*t^2+((-2*d+8)*m2)*t+(d-4)
,
diag3[1,0,1,1] →
 + diag3[0,0,1,1]*((2*d-6)/((d-4)*t+(-d+4)*m2))
 + diag3[1,0,0,1]*((-2*d+6)/((d-4)*t+(-d+4)*m2))
,
diag3[1,1,-1,1] →
 + diag3[1,1,0,1]*(((s+m2)*t^2+((m2)*s-2*m2^2)*t+m2^3)/(t^2+(-2*m2)*t+m2^2))
 + diag3[1,0,0,1]*((t^2+(2*s-2*m2)*t+m2^2)/(t^2+(-2*m2)*t+m2^2))
 + diag3[0,1,0,0]*((((-3*d+8)*s)*t+((d-4)*m2)*s)/(((2*d-6)*m2)*t^2+((-4*d+12)*m2^2)*t+(2*
,
diag3[1,1,0,0] →
 + diag3[0,1,0,0]*(1/m2)
```

Feynman integrals are **invariant over loop momenta shifts**

$$k_l \to \alpha_{li} k_i + \beta_{lj} p_j \quad \Longrightarrow \quad \int \prod_{l=1}^{L} d^d k_l \frac{d}{dk_j^\mu}\left(v_i^\mu \frac{D_{n+1}^{\nu_{n+1}}\cdots D_{n+r}^{\nu_{n+r}}}{D_1^{\nu_1}\cdots D_n^{\nu_n}} \right) = 0$$

Indices ν_i are integers

Integration-by-part Identities

[Chetyrkin,Tkachov:1981]
[Laporta:hep-ph/0102033]

Using ν_i as seeds: **GIGANTIC** system of equation

[Laporta:hep-ph/0102033]

$$\mathcal{M} = \sum_j^{N_{\rm MI}} c_j J_j$$

Not all rows are independent -> Gauss Elimination
Minimal set of independent integrals: **Master Integrals**

It looks like the space where Feynman integrals live is a **vector space**
How a **scalar product** on such space can be defined?

Promising new method for finding integral identities

Recent mathematical understanding: **Intersection theory**
Scalar product -> **Intersection Number**

[Mastrolia,Mizera:1810.03818]

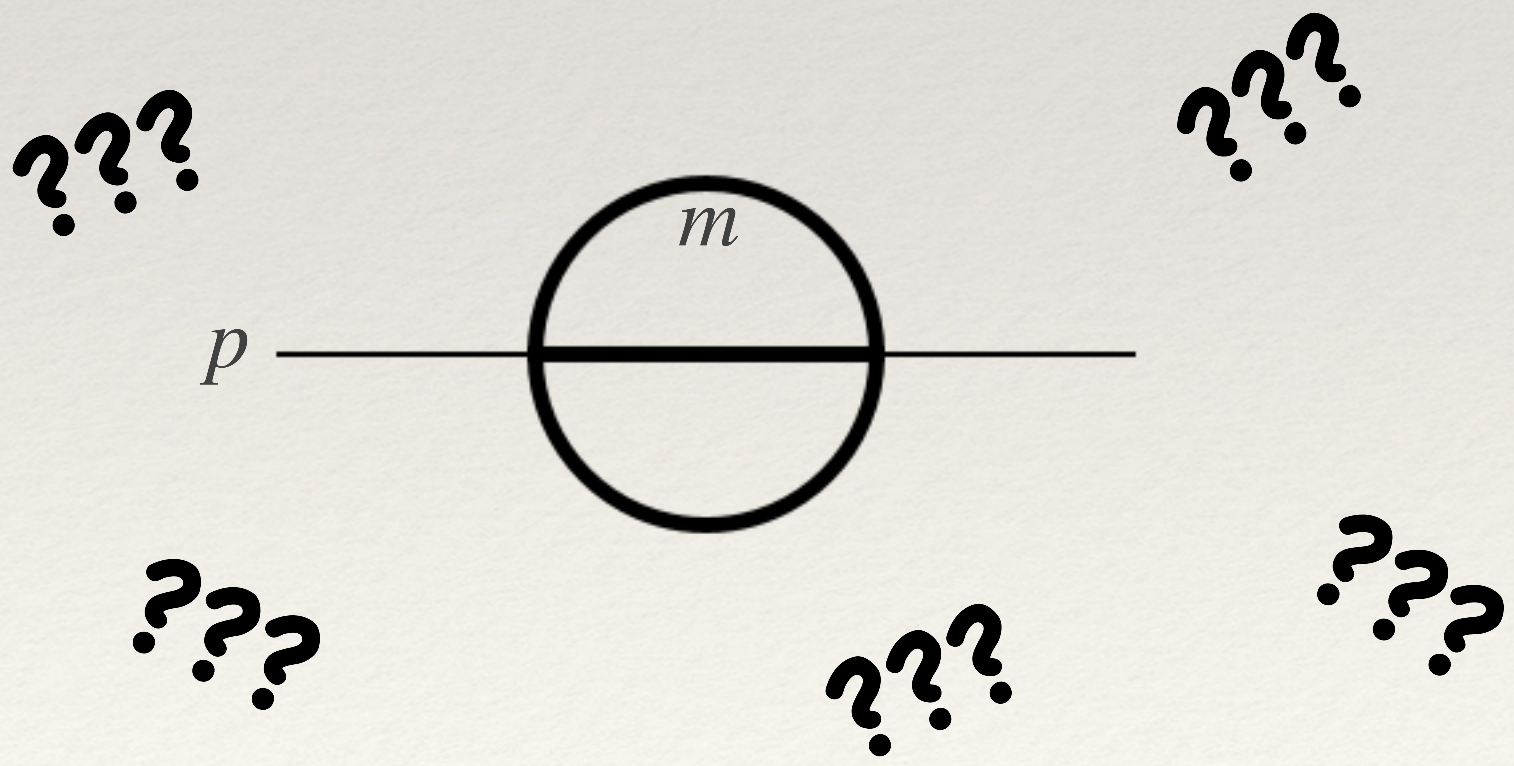
$$I = \int_{\mathcal{C}_R} u(\mathbf{z})\, \varphi_L(\mathbf{z})$$

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{(2\pi i)^n} \int_X \varphi_L \wedge \varphi_R$$

[Frellesvig,Gasparotto,Laporta,Mandal,Mastrolia,Mattiazzi,Mizera:2008.04823]

So, we have found the Master Integrals...

How do we calculate them?

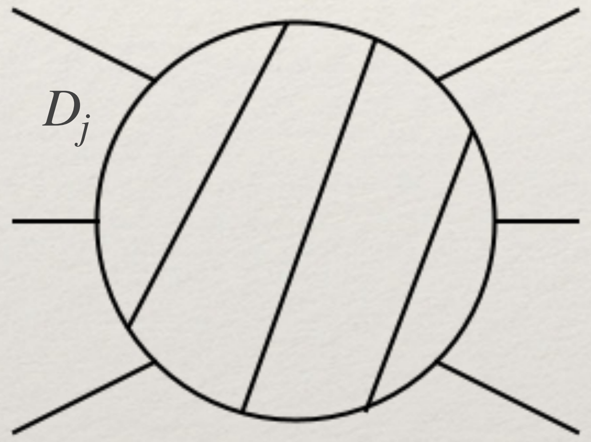


Analytical Integration: Basics

Direct analytical integration is not possible for many practical cases

$$J_n^r(\{s_{ij}\},\{m_1,\ldots,m_{n+r}\};d)=\int\prod_{l=1}^{L}d^dk_l\frac{D_{n+1}^{\nu_{n+1}}\cdots D_{n+r}^{\nu_{n+r}}}{D_1^{\nu_1}\cdots D_n^{\nu_n}}$$

$$D_k=q_k-m_k+i0$$



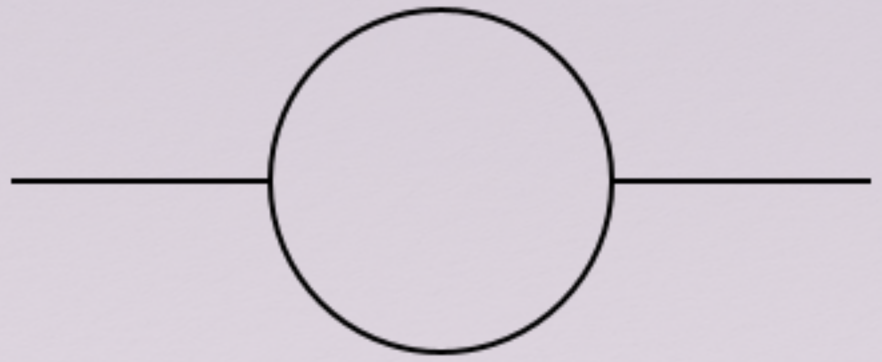
$$(-1)^{N_\nu}\frac{\Gamma(N_\nu-Ld/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_0^\infty x_j^{\nu_j-1}dx_j\,\delta\left(1-\sum_{j=1}^{n}x_j\right)\frac{\mathcal{U}(\bar{x})^{N_\nu-(L+1)d/2}}{\mathcal{F}(\bar{x})^{N_\nu-Ld/2}}$$ (Feynman)

$$(i)^{L-N_\nu}\frac{\Gamma(N_\nu-Ld/2)}{\Gamma(\nu_1)\cdots\Gamma(\nu_n)}\int_0^\infty\alpha_j^{\nu_j-1}d\alpha_j\,e^{-i\frac{\mathcal{U}(\bar{\alpha})}{\mathcal{F}(\bar{\alpha})}}$$ (Alpha)

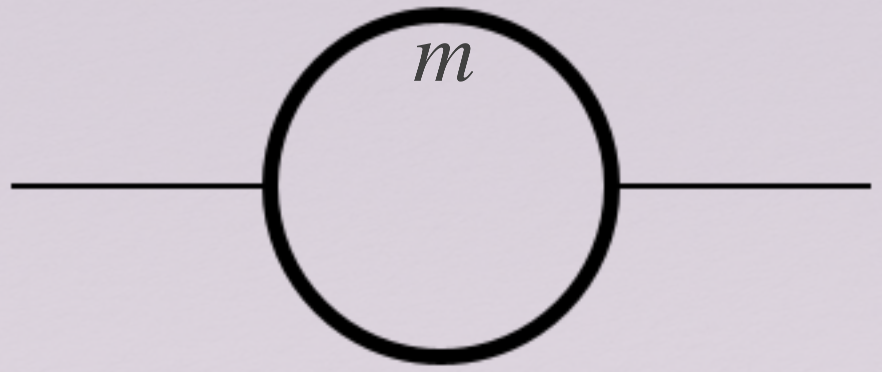
$$\frac{1}{2\pi i}^{n}\int_{-\infty}^{\infty}\prod_j^n dz_j\mathbf{F}(\bar{z},s_{ij};d)\frac{\prod_p^n\Gamma(\Lambda_p)}{\prod_q^n\Gamma(\Lambda_q)}$$ (Mellin-Barnes)

$$C^{(L)}\det(G(\bar{p}))\int_{-\infty}^{\infty}\mathcal{B}(\{s_{ij}\},z_1,\cdots,z_n)\frac{dz_1\cdots dz_n}{z_1^{\nu_1}\cdots z_n^{\nu_n}}$$ (Baikov)

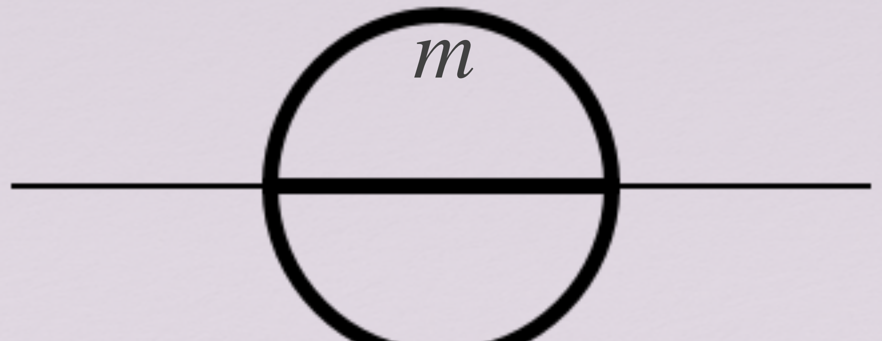
Complexity increases with number of loops and legs



$$=\frac{1}{\epsilon}(-p^2)^{-(1+\epsilon)}\left(-2+\frac{\pi^2}{6}\epsilon^2+\frac{14}{3}\zeta_3\epsilon^3+O(\epsilon^4)\right)$$



$$=\frac{2}{\sqrt{(-p^2)(4m^2-p^2)}}\log\left(\frac{\sqrt{1-4m^2/p^2}+1}{\sqrt{1-4m^2/p^2}-1}\right)+O(\epsilon)$$



$$=-\frac{4\mathrm{K}(\lambda)}{(p^2+m^2)\sqrt{a_{13}a_{24}}}\left[2\mathcal{E}\left(\begin{matrix}0&-1\\0&\infty\end{matrix};1,\bar{a}\right)+\mathcal{E}\left(\begin{matrix}0&-1\\0&0\end{matrix};1,\bar{a}\right)+\mathcal{E}\left(\begin{matrix}0&-1\\0&1\end{matrix};1,\bar{a}\right)\right]+O(\epsilon)$$

Special functions easily appear

Analytical Integration: Differential Equations

Master Integrals satisfy a **system of differential equation**

[Barucchi,Ponzano(1973)]
[Kotikov(1991)]
[Remiddi:hep-th/9711188]
[Gehrmann,Remiddi:hep-th/9912329]

$$\frac{d}{dx_{ij}} J(\{x_{ij}\};\epsilon) = A(x_{ij};\epsilon) J(\{x_{ij}\};\epsilon)$$

$x_{ij} =$ Dimensionless kinematic variables
$d = 4 - 2\epsilon$

A convenient change of basis allows to **factor out the epsilon dependence** from the matrix A

Canonical Form

$$\frac{d}{dx_{ij}} \mathcal{J}(\{x_{ij}\};\epsilon) = \epsilon \tilde{A}(x_{ij}) \mathcal{J}(\{x_{ij}\};\epsilon)$$

[Henn:1304.1806]

DEs in Canonical form admit **analytical solution** in epsilon

$$\mathcal{J}(\{x_{ij}\};\epsilon) = \frac{1}{\epsilon^{2L}} \sum_n \mathcal{J}^{(n)}(\{x_{ij}\})\epsilon^n$$

Iterated integrals (HPL,GPL) enter in the game

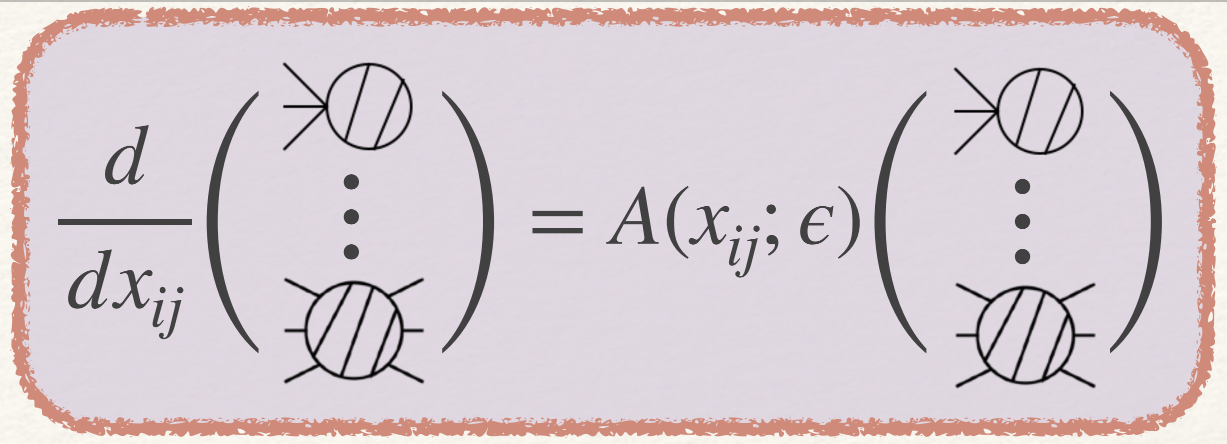
$$\mathcal{G}(a_1,\cdots,a_n;t) = \int_0^x \frac{dt}{t-a_n} \mathcal{G}(a_1,\cdots,a_{n-1};t)$$

[Goncharov(1995)]
[Remiddi,Vermaseren:hep-ph/9905237]

Leading Log singularity allows to find automatically the canonical form

$$\prod_{l=1}^{L} d^d k_l \frac{1}{D_1^{\nu_1}\cdots D_n^{\nu_n}} \overset{?}{=} d\log\tau_1 \cdots d\log\tau_n$$

Alternative method: **Magnus Exponential**
[Argeri,di Vita,Mastrolia,Mirabella,Schlenk,Schubert,Tancredi:1401.2979]



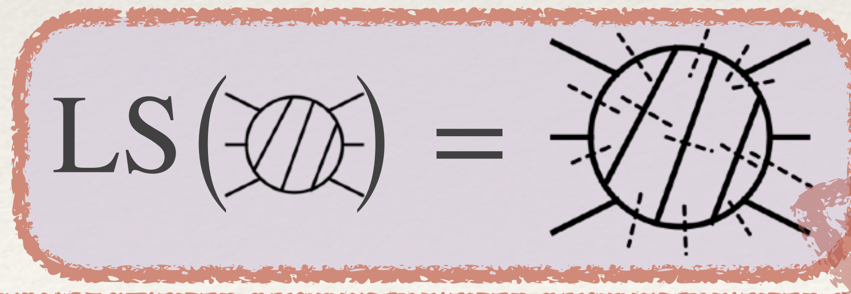
$$d\mathcal{J}(\{x_{ij}\};\epsilon) = \epsilon\, d\tilde{A}(x_{ij})\, \mathcal{J}(\{x_{ij}\};\epsilon)$$

$$\mathcal{J}^{(n)}(\{x_{ij}\};\epsilon) = \sum_{k=0}^{n} \int \underbrace{d\tilde{A}(x_{ij})\cdots d\tilde{A}(x_{ij})}_{k} b^{(n-k)}(\bar{x}_{ij})$$

Boundary

Evaluation with **handyG, PolyLogTools...**

[Naterop,Signer,Ulrich:1909.01656]
[Duhr,Dulat:1904.07279]



Maximal cut

Change of basis

$$\mathcal{J}(\{x_{ij}\};\epsilon) = \text{LS}(J)^{-1} J(\{x_{ij}\};\epsilon)$$

[Henn:1304.1806]

Numerical Integration

Numerical solution of Differential equations

Auxiliary mass flow
(AMFlow)
[Liu,Ma:2201.11669]

- Introducing a mass parameter η into propagators
- Numerical IBPs + DE system depending on η only
- Automatic Boundary condition at $\eta \to \infty$
- Propagating boundaries to $\eta \to 0$

Series expansion methods
(DiffExp, SeaSyde)
[Armadillo,Bonciani,Devoto,Rana,Vicini:2205.03345]
[Hidding:2006.05510]

- Analytical IBPs + Differential equation system
- Boundary condition as input in Euclidean region
- Propagating boundary to physical region

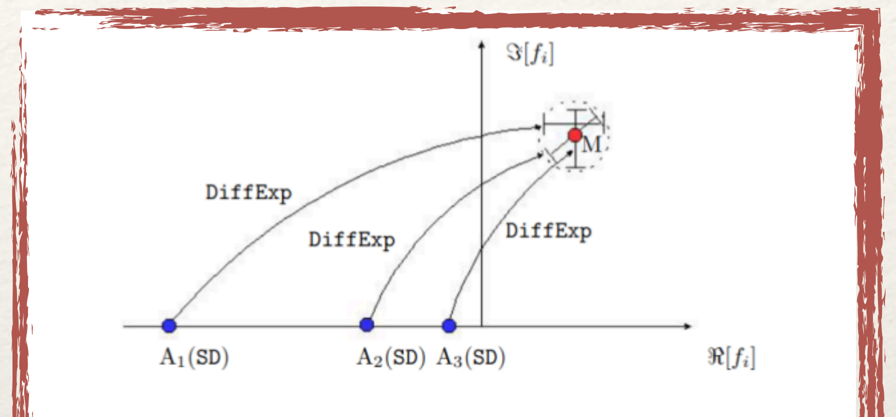


FIG. 1. Illustration of the DE transport method. The bound-

[Dubovyk,Freitas,Gluza,Grzanka,Hidding,Usovitsch:2201.0257]

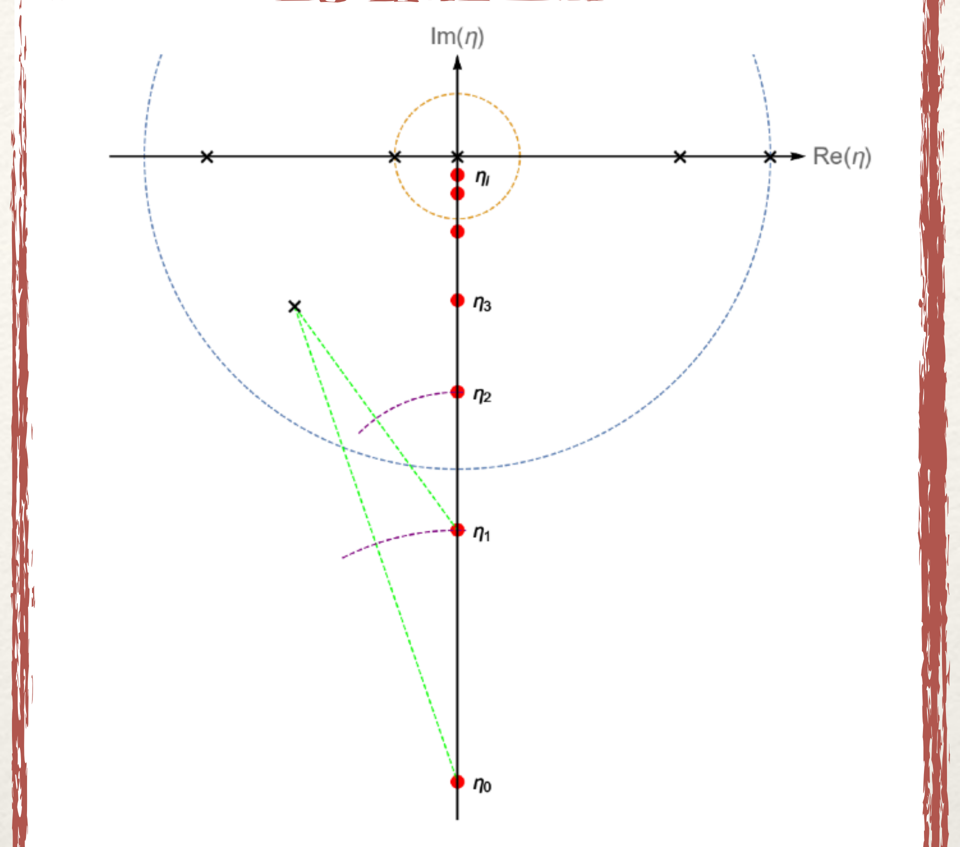


Figure 1: Singularities and analytic continuations. Singularities are labeled as crosses. Solid dots are points where to perform series expansions.

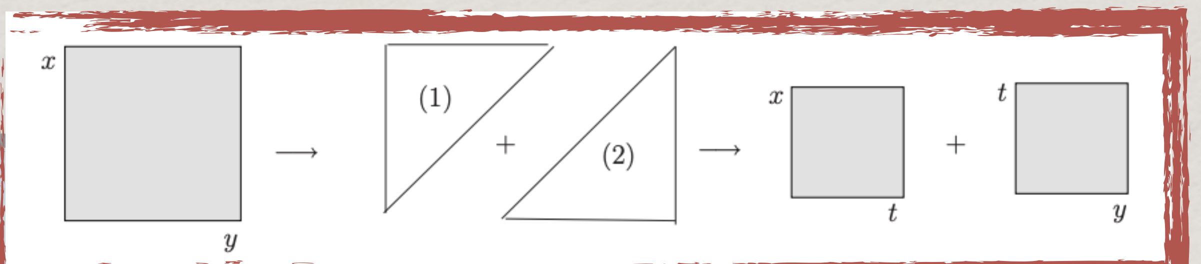
[Liu,Ma:2201.11669]

Recently: **Neural networks** application to numerical solution of DEs
[Calisto,Moodie,Zoia:2312.02067]

MonteCarlo Integration methods

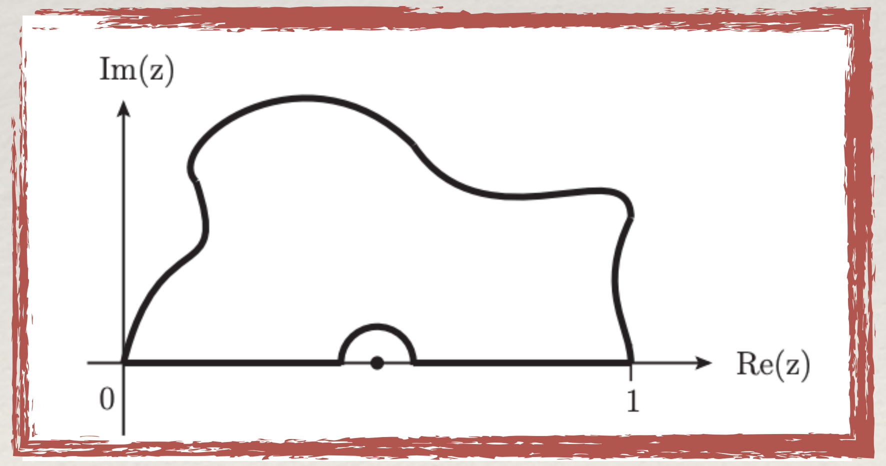
Sector Decomposition
(SecDec, pySecDec)
[Heinrich,Jones,Kerner,Magerya,Olsson,Schlenk:2305.19768]

- Feynman parametrization
- Splitting integration domain
- End-point subtraction of singularities and ϵ expansion
- contour deformation + expansion-by-region
- MonteCarlo integration of finite integrals



[Heinrich:0803.4177]

Recently: Neural network for contour deformation



[Jones:GGI talk,13 Sept 2023]

Tropical integration
(FeynTrop)
[Borinski,Munch,Tellander:2302.08955]

- Feynman parameters + contour deformation $x_i \to x_i e^{-i\lambda \frac{d}{dx_i}\left(\frac{\mathcal{U}(\bar{x})}{\mathcal{F}(\bar{x})}\right)}$
- Tropical approximation of Symanzik Polynomial $\left(\mathrm{Trop}(2x_1x_2 + 4x_3^2) = \max_{\mathrm{supp}}(x_1x_2, x_3^2)\right)$
- MonteCarlo integration improved with tropical sampling
- Improving sampling by geometrical insights (Newton polytopes, generalized permutahedra…)

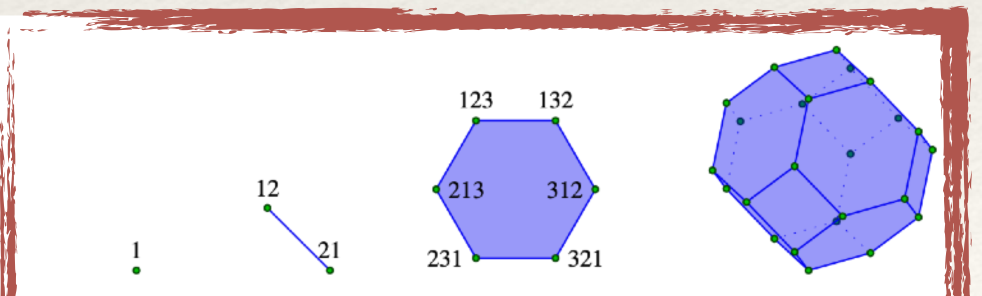


Figure 1: The first four permutohedra where it is easily seen from the explicit vertex coordinates (permutations) that all edges are parallel to $e_i - e_j$ for $i \neq j$.

[Borinsky,Munch,Tellander:2310.19890]

Differential Equation method: Applications

$e^+e^- \to \mu^+\mu^-$ virtuals @ NNLO QED
$e^+\mu^- \to e^+\mu^-$ virtuals @ NNLO QED
$q\bar{q} \to t\bar{t}$ virtuals @ NNLO QCD

PHYSICAL REVIEW LETTERS **128**, 022002 (2022)

Two-Loop Four-Fermion Scattering Amplitude in QED

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Two-loop scattering amplitude for heavy-quark pair production through light-quark annihilation in QCD

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Muon-electron scattering at NNLO

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Process:
- NNLO SM QCD
- Four-point two-loop scattering
- Three mass scales
- IR and UV divergent

Method:
- Constructing double-virtual interferences
- Adaptive integrand decomposition
- Integration-by-parts identities
- Differential equation method for Master Integrals
- Magnus Exponential to expose the canonical basis
- dlog form of the differential matrix
- Analytical expansion in terms of GPLs

Tools:
- Mathematica
- FeynCalc
- AIDA
- handyG
- PolyLogTools

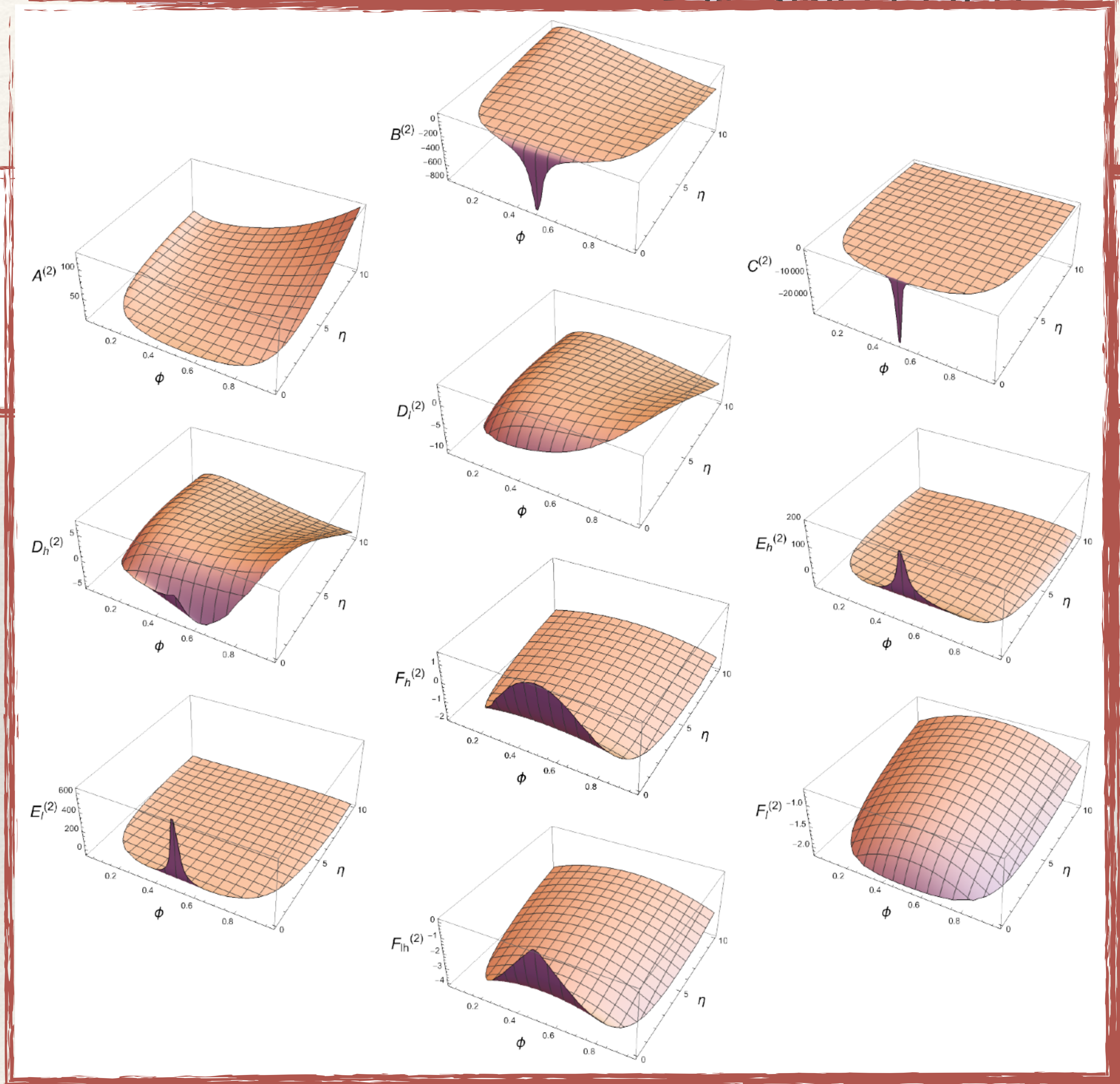
Running:
- Mathematica package: O(10) sec/pt
- Within McMule: O(0.1) sec/pt

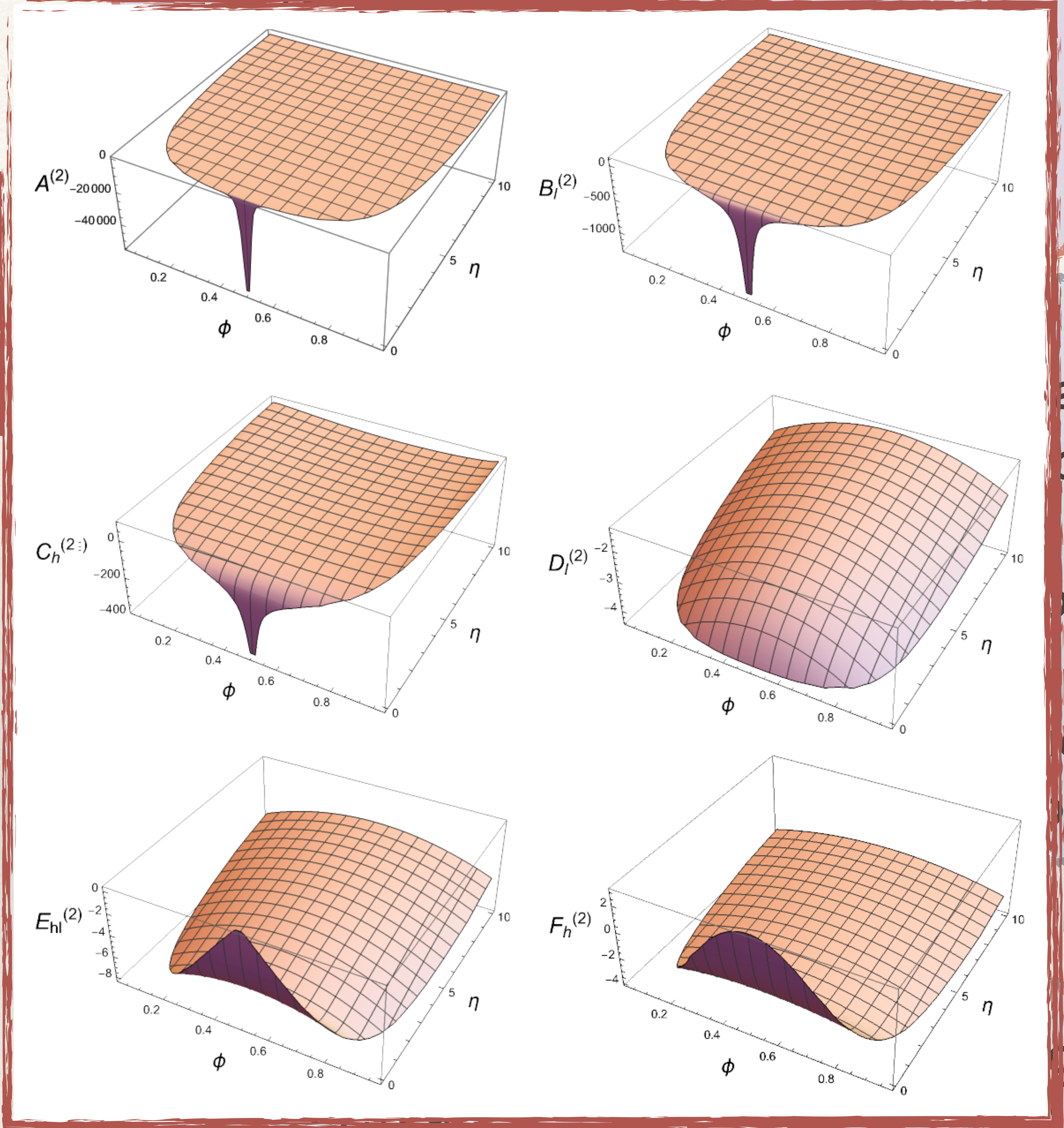
Differential Equation method: Applications

$e^+e^- \to \mu^+\mu^-$ virtuals @ NNLO QED

Process:

- NNLO SM QCD





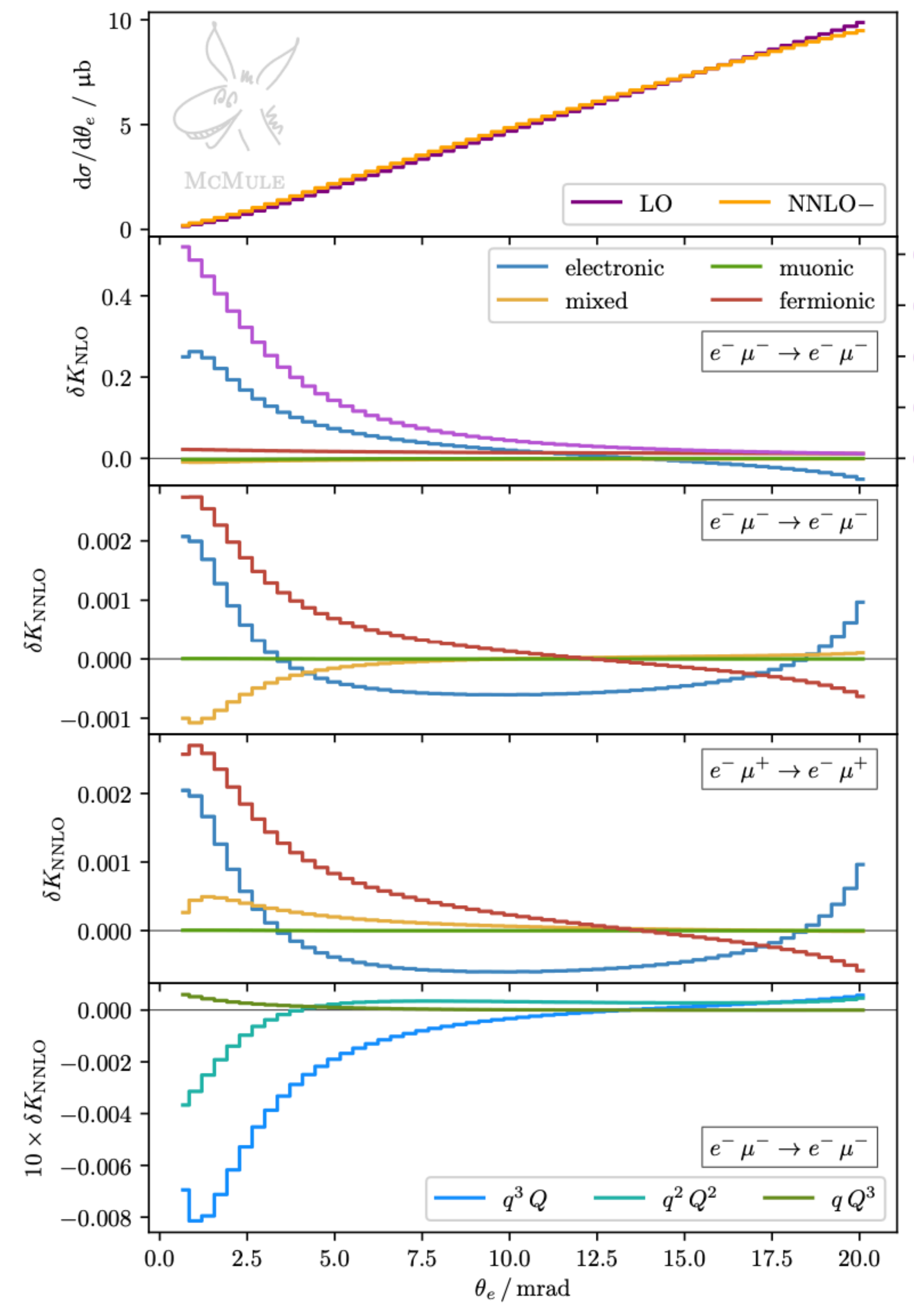


Figure 3: From top to bottom: (i) differential cross section w.r.t. θ_e for S1 at LO (violet), and NNLO (orange) for negative muons; (ii) NLO K factor for negative muons (positive muons have a sign flip for the mixed photonic correction); (iii) NNLO K factor for negative muons; (iv) NNLO K factor for positive muons; (v) NNLO K factor for disentangled mixed photonic corrections, for negative muons. In panels (ii)–(iv) the correction is split into photonic, i.e. electronic, mixed and muonic, and fermionic, including leptonic and hadronic. The hadronic correction at NLO corresponds to the signal of the experiment and is shown separately in purple in panel (ii).

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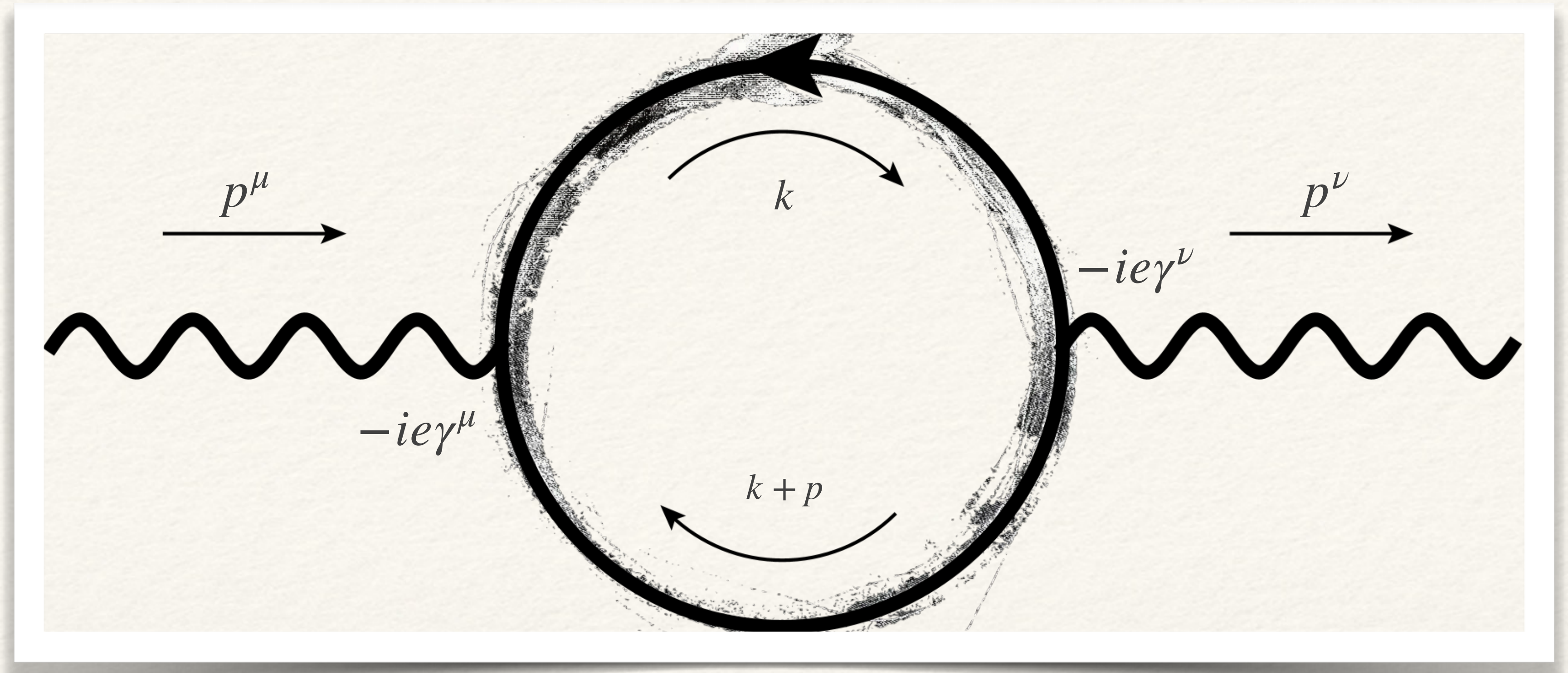
- AIDA
- handyG
- PolyLogTools

- Wi



"The very advanced counting system used by elementary particle theorists for counting the loops is: 'One, two, many'."

— Ettore Remiddi

Thank you!