Bootstrapping form factors through six loops and beyond

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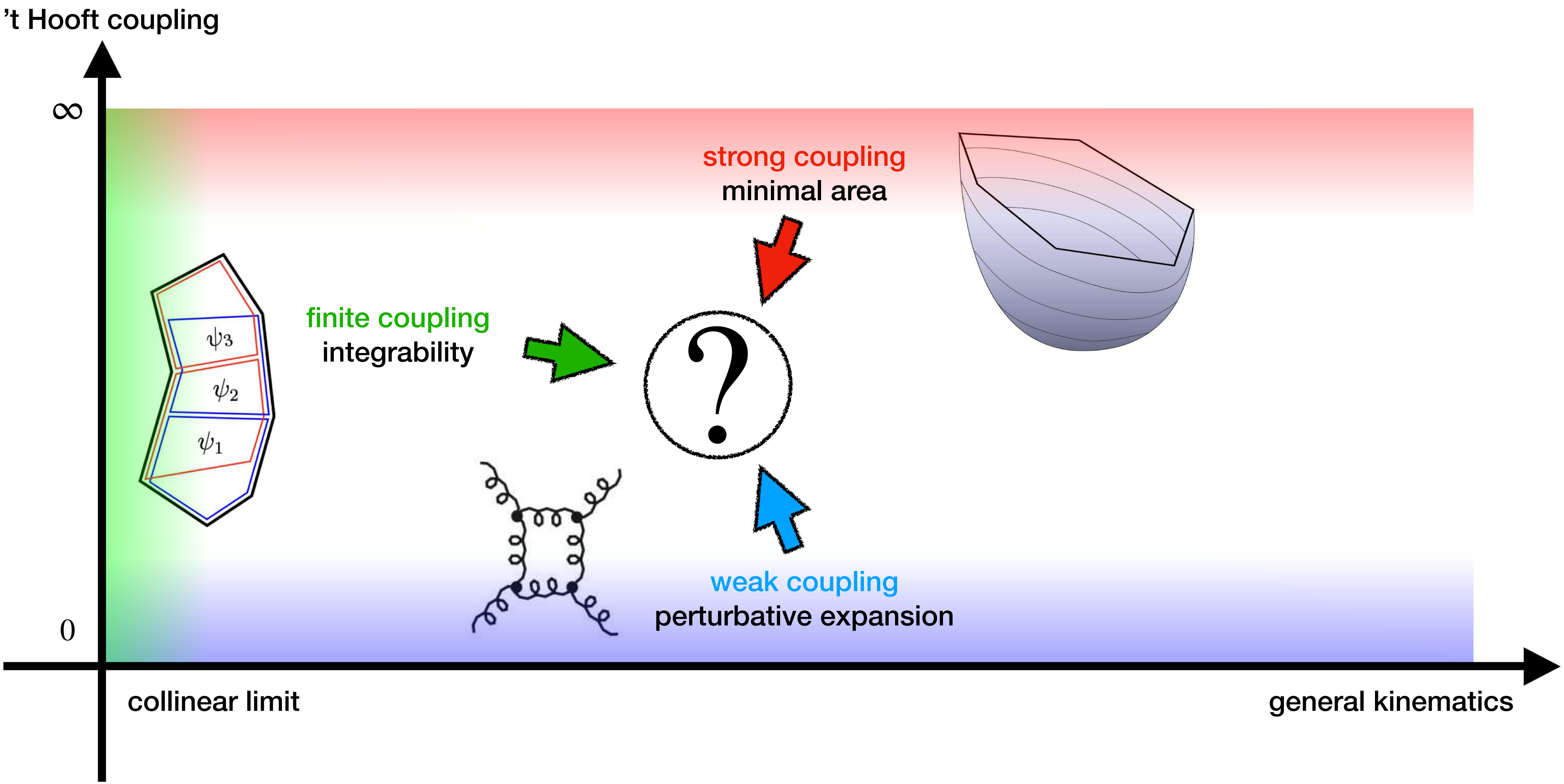
Based on [2308.08432, upcoming] with **Benjamin Basso** and **Lance Dixon**

LAPTh, 15.02.24

Can we “solve” scattering in $\mathcal{N} = 4$ SYM?

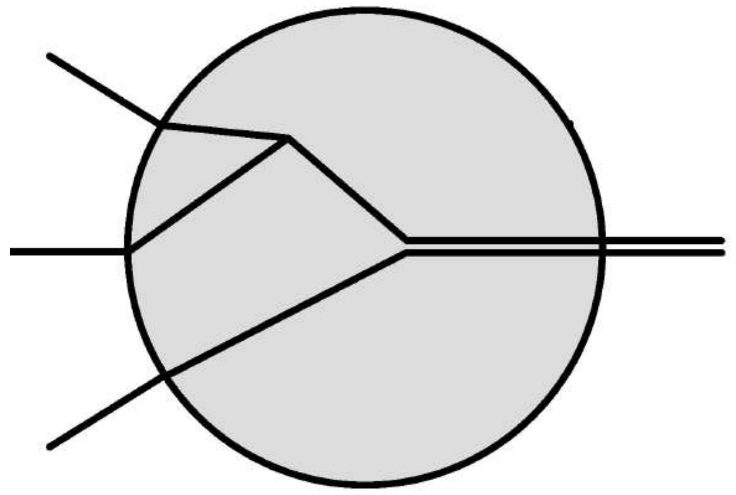


How do we compute amplitudes perturbatively?

Perturbative bootstrap

1. Choosing an observable (amplitudes, form factors, ...)

   Form factor : $F_{\mathcal{O}}(k_1, \ldots, k_n) = \langle k_1, \ldots, k_n | \mathcal{O}(q) | 0 \rangle$



2. Class of functions (classical polylogs, generalized polylogs, elliptic functions, ...)

   form factors of protected operators: generalized polylogs, uniform transcendentality

3. Symbol (replaces polylogarithms with a large vector space that captures their branch cut structure)

$$(\log z)^n \to z \otimes \ldots \otimes z$$

$$\mathrm{Li}_n(z) \to -(1-z) \otimes z \otimes \ldots \otimes z$$

$$\ldots$$

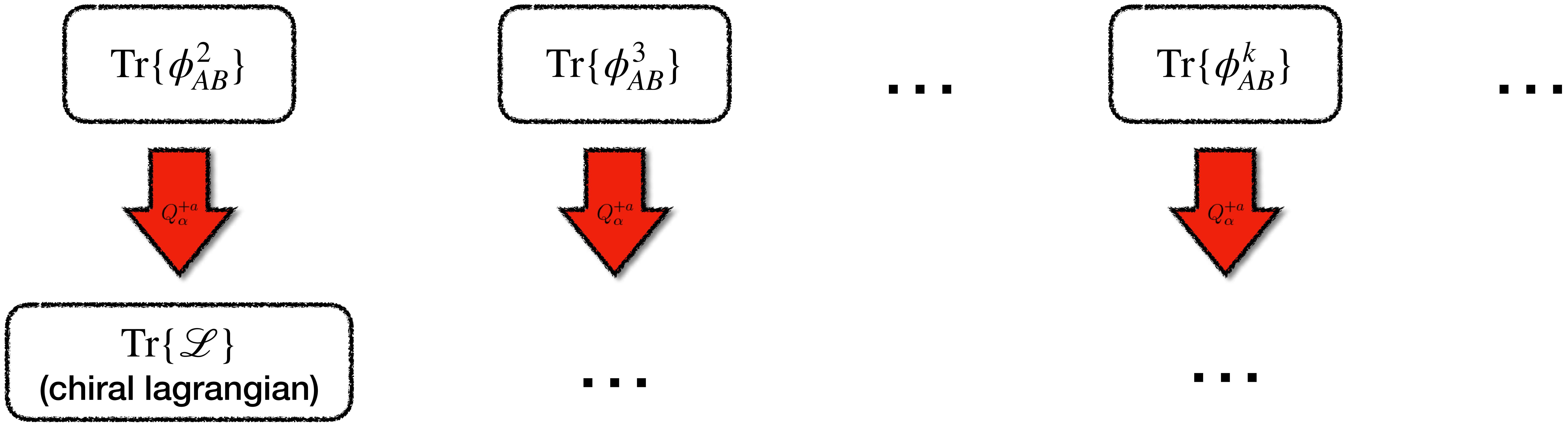
   Length of the symbol = transcendentality

4. Alphabet (entries of the symbol = "letters") $\{a, b, c, d, e, f, \ldots\}$

   Typical ansatz of transcendentality 4: $\sum \alpha_{ijkl}\, a_i \otimes a_j \otimes a_k \otimes a_l$

Choosing an observable

The simplest class of local operators $\mathcal{N}=4$ SYM are the [protected] $1/2$-BPS operators.



$$W_{k,n} = \frac{\mathcal{F}_{k,n}}{\mathcal{F}_{2,n}^{\text{MHV}(0)}}$$

$\text{Tr}\{\phi^k\}$

number of particles

Choosing an observable

Divergent parts of $\mathcal{N} = 4$ amplitudes/form factors are fixed by symmetry. [Bern, Dixon, Smirnov '05]

The first non-trivial finite part appears at $n = 6$ for the amplitude and at $n = 3$ for form factors.

Three-point form factor of $\mathrm{Tr}\,\phi^2$ $W_{2,3}$	Three-point form factor of $\mathrm{Tr}\,\phi^3$ $W_{3,3}$
[Dixon, McLeod, Wilhelm '20] [Dixon, Gurdogan, McLeod, Wilhelm '22]	[Dixon, Basso, AT '24]

The function space (conjecturally) is the same: generalized polylogarithms, six letter symbol alphabet:

$$\mathcal{L}_a = \{a, b, c, d, e, f\} = \{\frac{u}{vw}, \frac{v}{wu}, \frac{w}{uv}, \frac{1-u}{u}, \frac{1-v}{v}, \frac{1-w}{w}\}$$

where $u = \frac{s_{12}}{q^2}, \quad v = \frac{s_{23}}{q^2}, \quad w = \frac{s_{31}}{q^2}, \quad u + v + w = 1,$ q is the momentum carried by the operator

$n = 3$ form factors have two kinematic degrees of freedom.

Reducing the function space size

$$\mathcal{L}_a = \{\frac{u}{vw}, \frac{v}{wu}, \frac{w}{uv}, \frac{1-u}{u}, \frac{1-v}{v}, \frac{1-w}{w}\}$$

At weight w the naive ansatz has 6^w terms. Thankfully, there are some some unexpected constraints that limit it significantly, called the Steinmann-like relations. They forbid certain combinations of letters from appearing:

$$\ldots \cancel{a \otimes d} \ldots \qquad \ldots \cancel{d \otimes a} \ldots \qquad \ldots \cancel{d \otimes e} \ldots \qquad \ldots a \otimes \cancel{abc} \otimes b \ldots$$ + (dihedral images)

Additionally, we also have the so-called first entry conditions, that forbid branch cuts at non-physical thresholds:

$$\cancel{d \otimes} \ldots \qquad \cancel{e \otimes} \ldots \qquad \cancel{f \otimes} \ldots$$

The resulting function space has the following size:

w	1	2	3	4	5	6	7	8
$\mathcal{C}$ symbols	3	9	21	48	108	249	567	1290
$\mathcal{C}$ functions	3	9	22	52	122	284	654	1495

Imposing the constraints

There is a natural action of the dihedral group D_3 generated by two transformations:

cycle: $u \to v \to w \to u \qquad a \to b \to c \to a, \quad d \to e \to f \to d$

flip: $u \leftrightarrow v \qquad a \leftrightarrow b, \quad d \leftrightarrow e$

Both the form factors are invariant under it. Another type of constraints that we are imposing are the multiple-final-entry conditions.

$\mathrm{Tr}\,\phi^2$: $\cancel{\ldots \otimes a} \quad \cancel{\ldots \otimes b} \quad \cancel{\ldots \otimes c}$ $+$ $\ldots \cancel{\otimes a \otimes d} \quad \ldots \cancel{\otimes e \otimes d} \quad \ldots \otimes (b \cancel{\otimes f - b} \otimes d)$ $+ \quad \ldots$

single-final-entry

+ (dihedral images)

double-final-entry

$\mathrm{Tr}\,\phi^3$: $\cancel{\ldots \otimes abc}$ $+ \quad \ldots$

These constraints can be loosely associated with certain supersymmetric Ward identities ($\bar{Q}$ equation), but are much stronger than what one would expect.

Lastly, at L loops, we expect L^{th} discontinuity at $u \to 0$ of the $\mathrm{Tr}\,\phi^2$ form factor to vanish. For the $\mathrm{Tr}\,\phi^3$ form factor, we expect $(L+1)^{\text{st}}$ discontinuity to vanish instead.

Intermediate results

$\mathrm{Tr}\,\phi^2$

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	191	191
L^{th} discontinuity	2	5	17	38	75	171	164

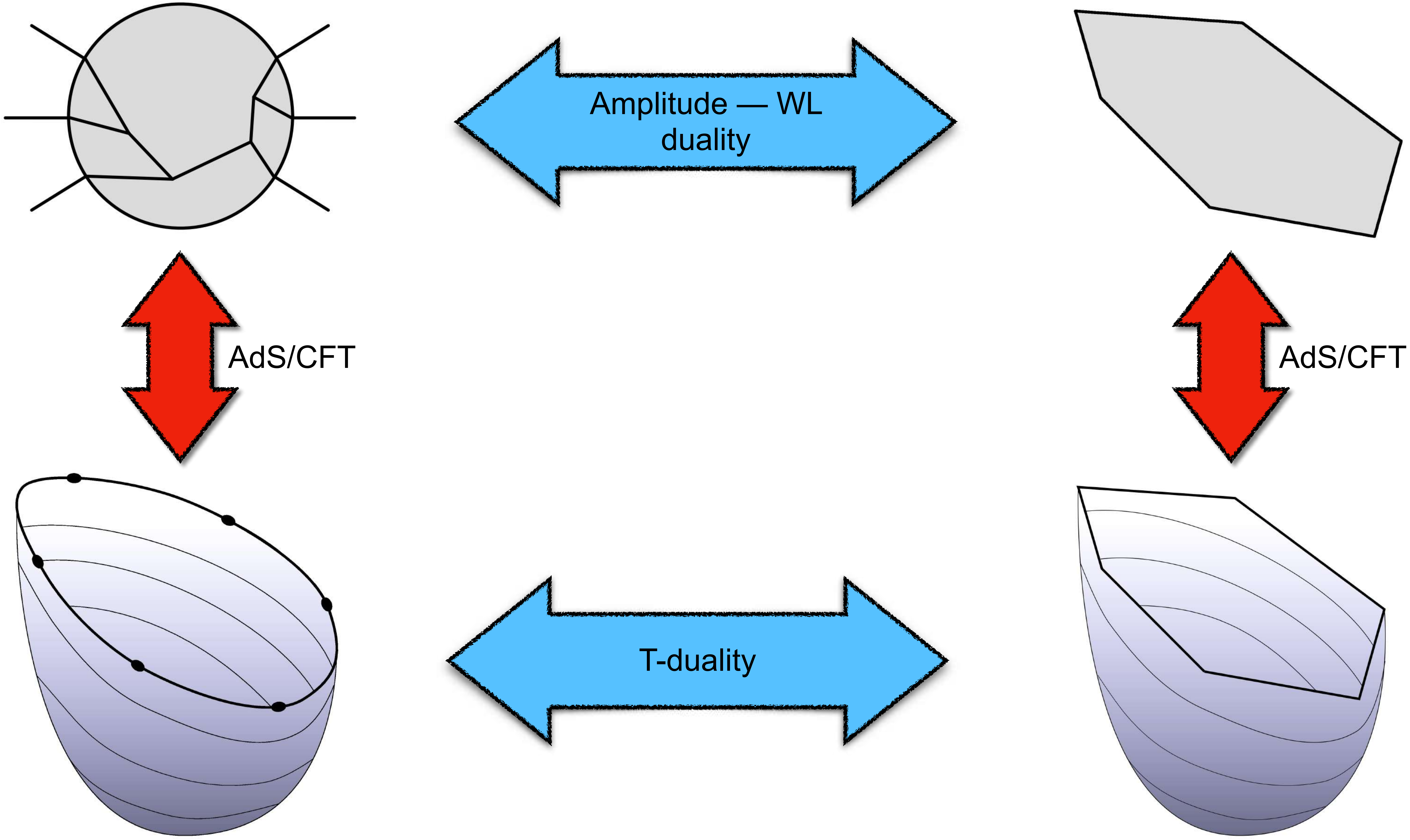
$\mathrm{Tr}\,\phi^3$

L	2	3	4	5	6
functions in $\mathcal{C}$	52	284	1495	$\sim$8000	?????
dihedral symmetry	13	63	302	$\sim$1400	????
$(L-1)$ final entries	4	15	47	190	407
$(L+1)^{\mathrm{st}}$ discontinuity	3	13	43	182	394

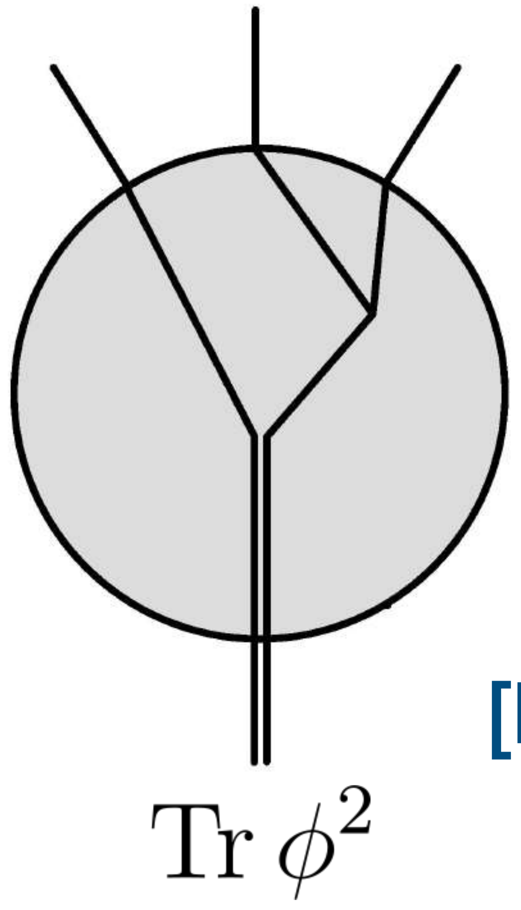
Integrability

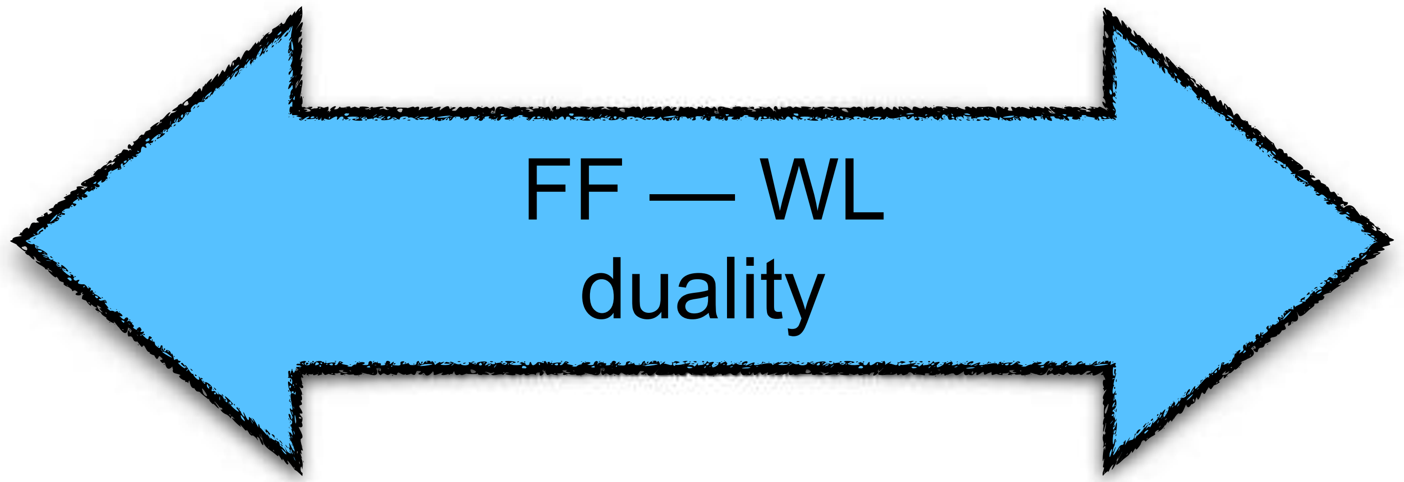
Amplitude — Wilson Loop duality

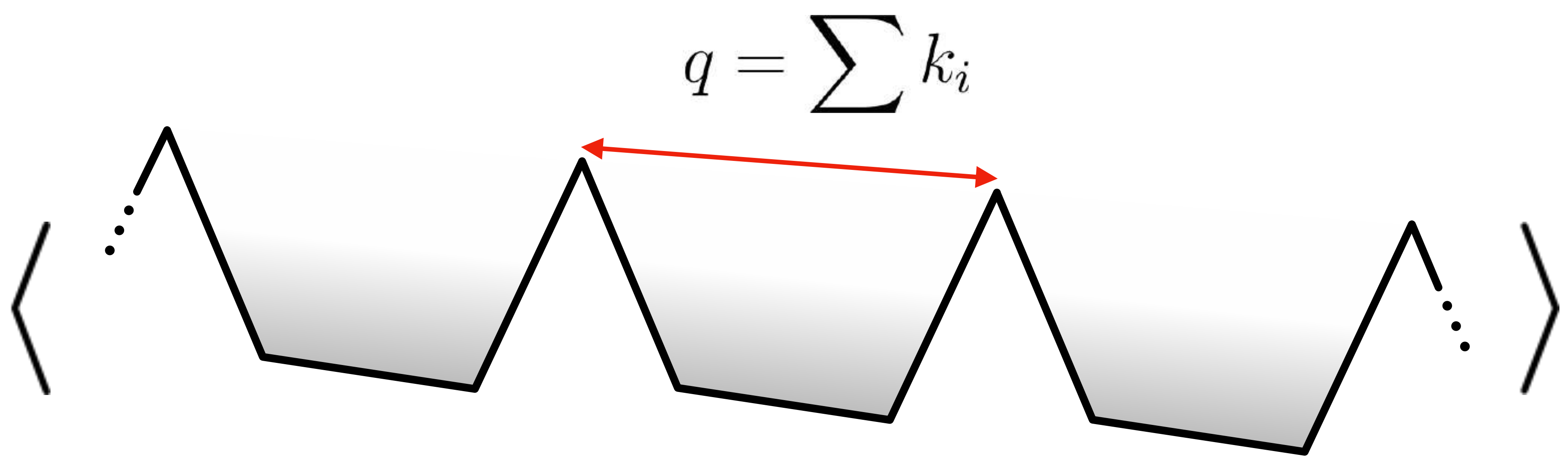
[Alday, Maldacena ’07]



Form Factor — Wilson loop duality for $\mathrm{Tr}\,\phi^2$



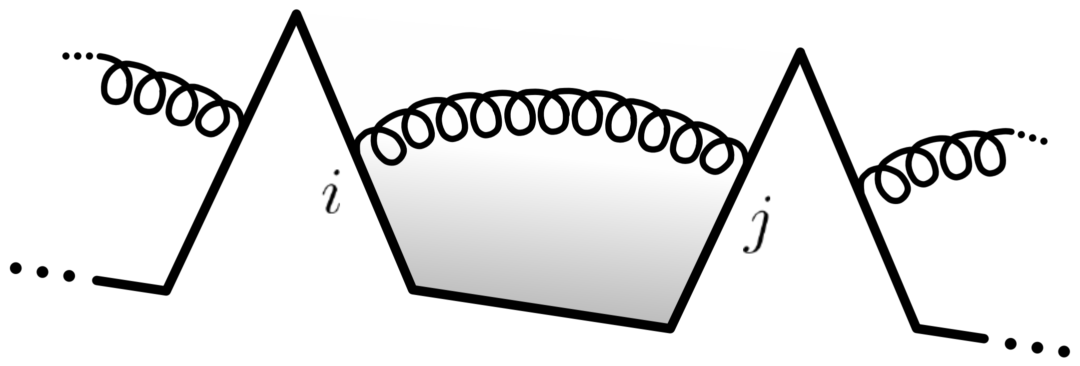




[Brandhuber, Spence, Travaglini, Yang ’10]

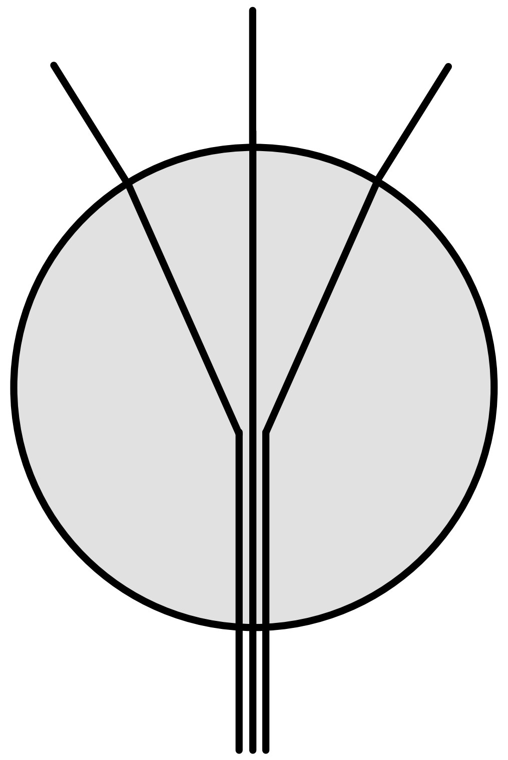
At one loop, corrections arise from dressing the Wilson loop with gluon exchanges between edges:

$$W_{2,n} = 1 + g^2 \sum_{i<j}$$



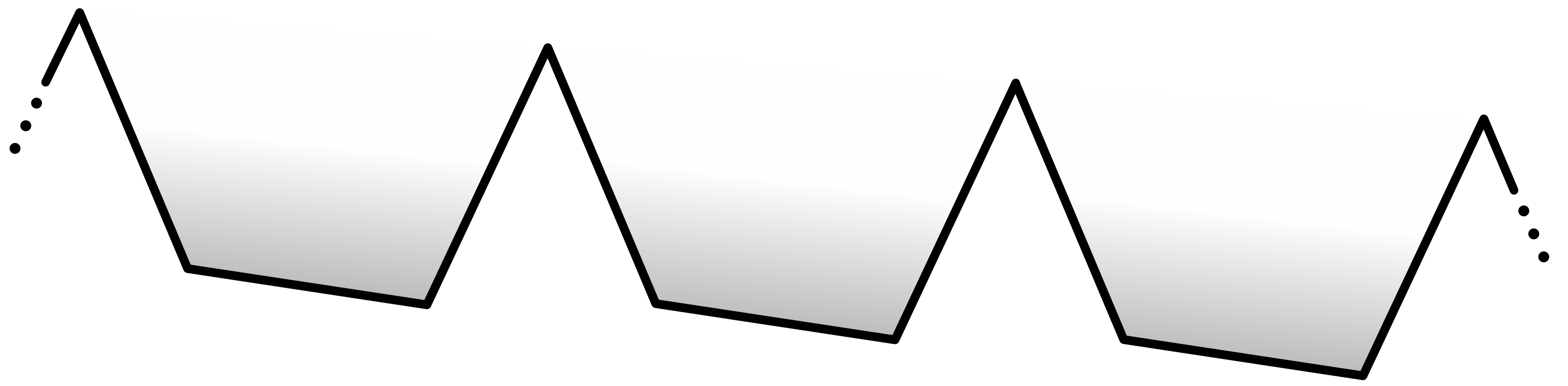
$$+\,\mathcal{O}\left(g^4\right)$$

Form Factor — Wilson loop duality for $\mathrm{Tr}\,\phi^3$

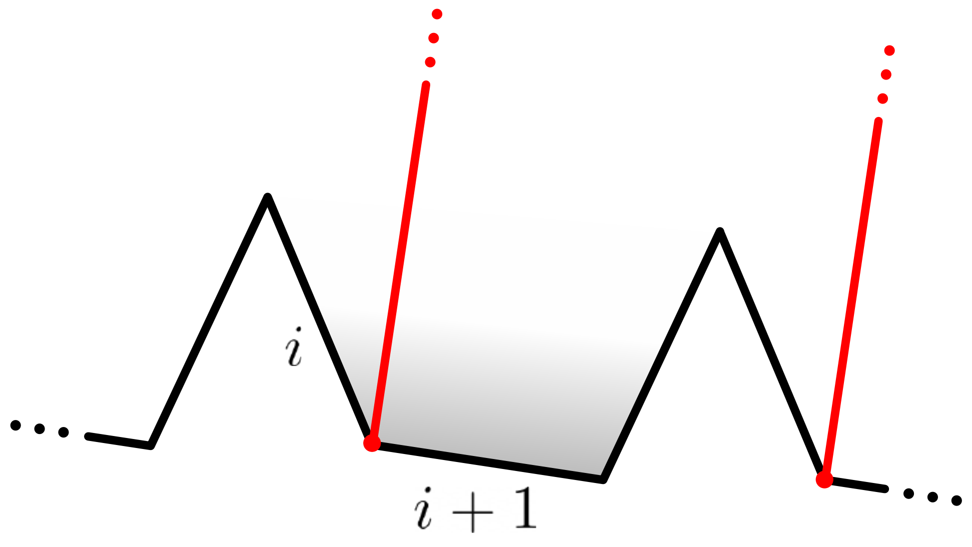


$\mathrm{Tr}\,\phi^3$

[Basso, AT ’23]

$\langle \phi(0) |$  $\rangle$

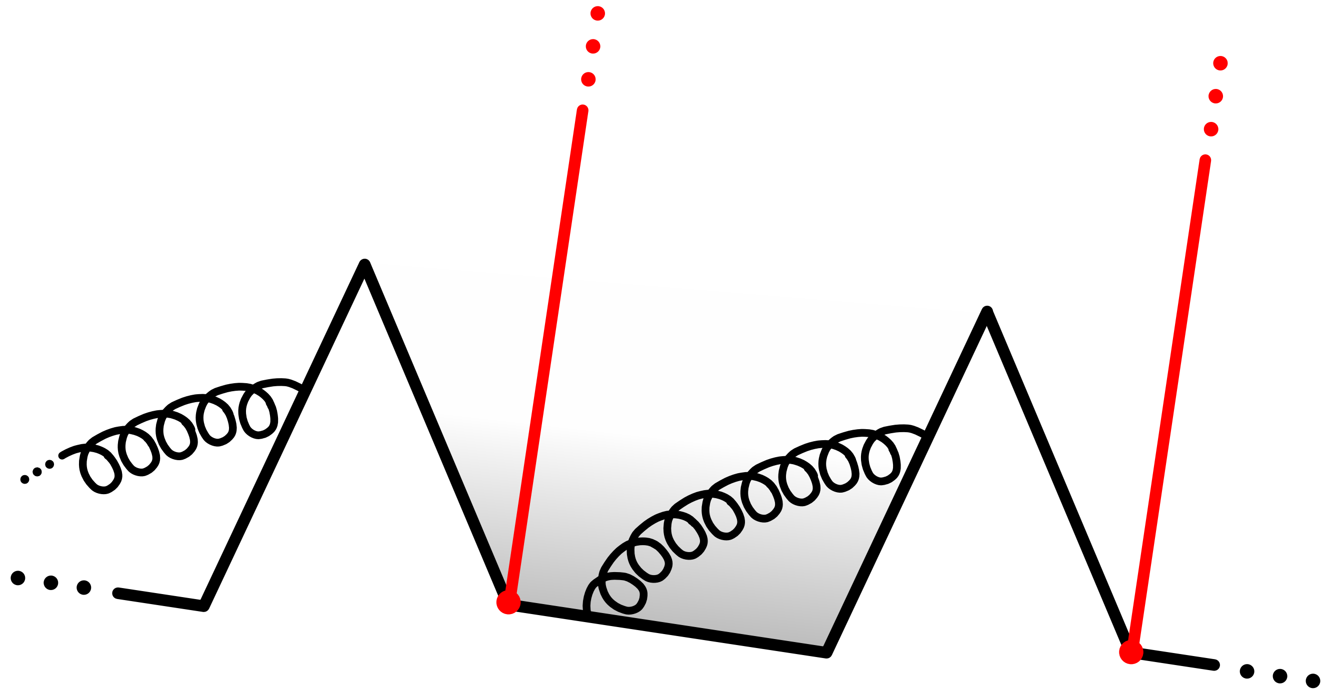
To compensate for the charge at the infinity, the Wilson loop needs to be “charged” accordingly. [Caron-Huot ’10]

$W_{3,n} = \sum_i$  $+ \mathcal{O}\left(g^2\right)$

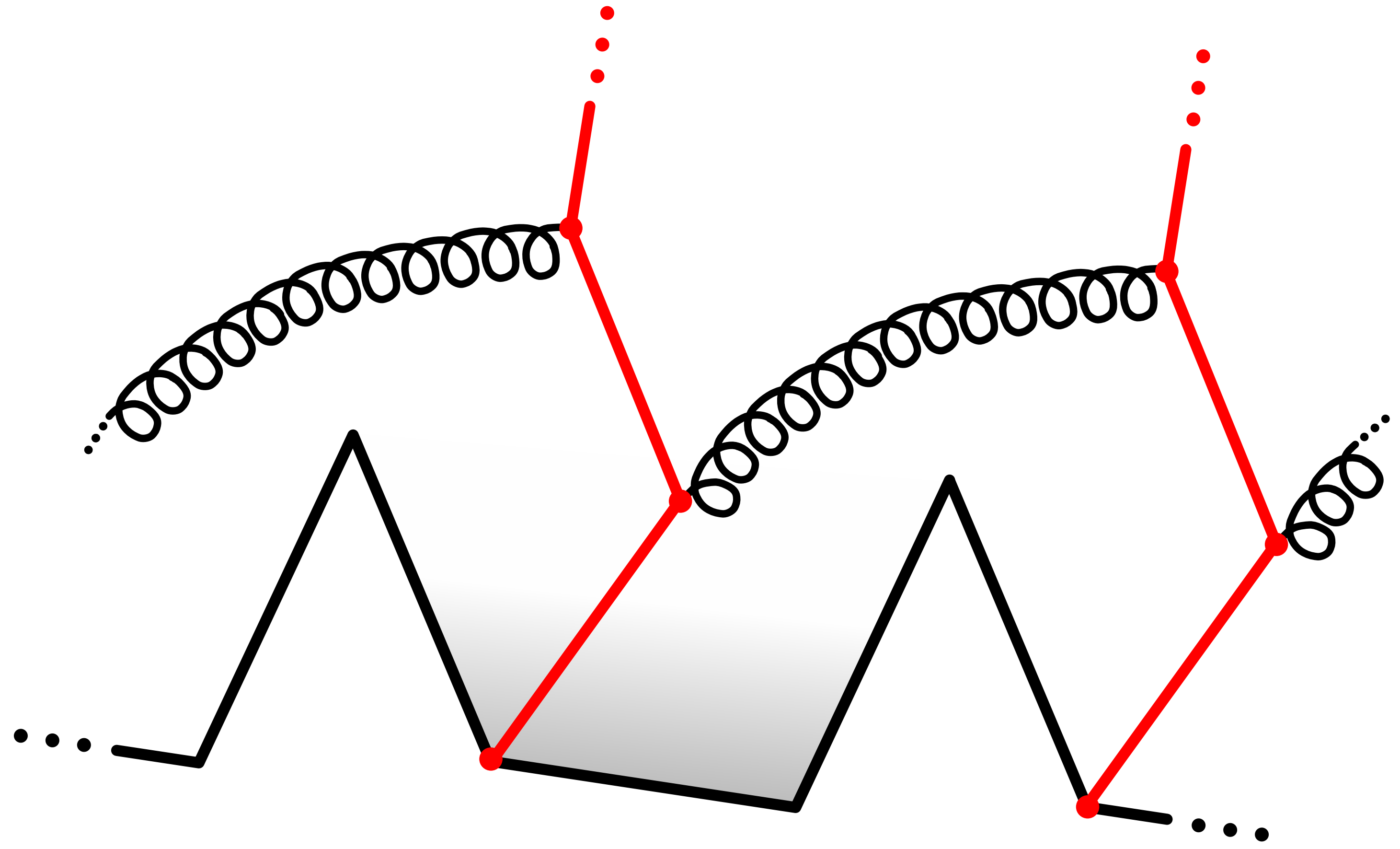
In the general case of $\mathrm{Tr}\,\phi^k$, the asymptotic state consists of $k-2$ zero-momentum scalars.

$\mathrm{Tr}\,\phi^3$ one-loop check

We also tested this duality at one loop. Two types of diagrams need to be considered:



Typical 1-loop form factor corrections

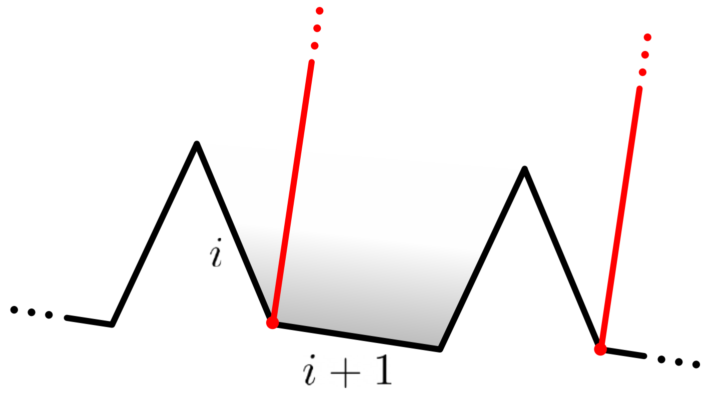


Operator renormalisation diagrams

Because the operator is protected, diagrams of the second kind add up to zero. The diagrams of the first kind add up perfectly to the expected result.

$m = 2$ amplituhedron

$$W_{3,n}^{\text{tree}} = \sum_{i=1}^{n}$$

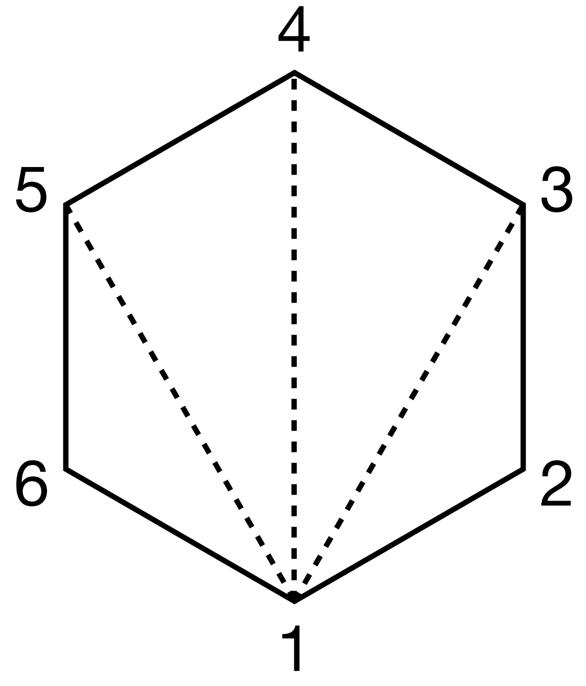


$$= -\sum_{i=2}^{n-1} (1ii+1)$$

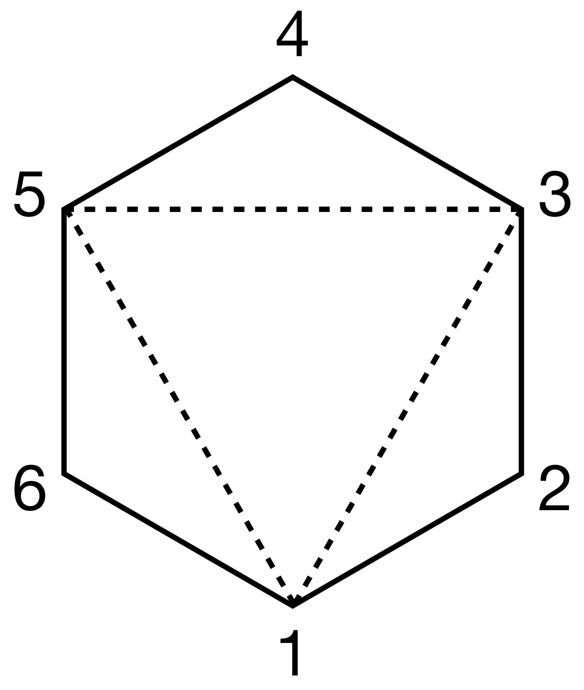
where $$(ijk) = \frac{\delta^{0|2}(\langle ij\rangle\eta_k^- + \langle jk\rangle\eta_i^- + \langle ki\rangle\eta_j^-)}{\langle ij\rangle\langle jk\rangle\langle ki\rangle}$$

These 3-brackets are $2D$ versions of the standard R-invariant $$[ijklm] = \frac{\delta^{0|4}\left(\langle[ijkl\rangle\eta_{m]}\right)}{\langle ijkl\rangle\langle jklm\rangle\langle klmi\rangle\langle lmij\rangle\langle mijk\rangle}$$

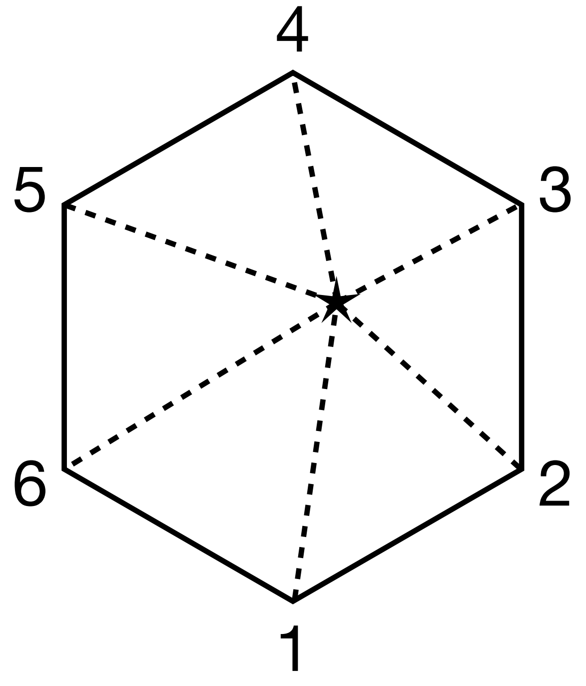
This result is nothing but a triangulation of a polygon:



$$W_{3,n}^{\text{tree}} = -\sum_{i=2}^{n-1}(1ii+1)$$



$$W_{3,n}^{\text{tree}} = -\sum_{T\in\mathcal{T}_n}(T_1T_2T_3)$$



$$W_{3,n}^{\text{tree}} = -\sum_{i=1}^{n}(*ii+1)$$

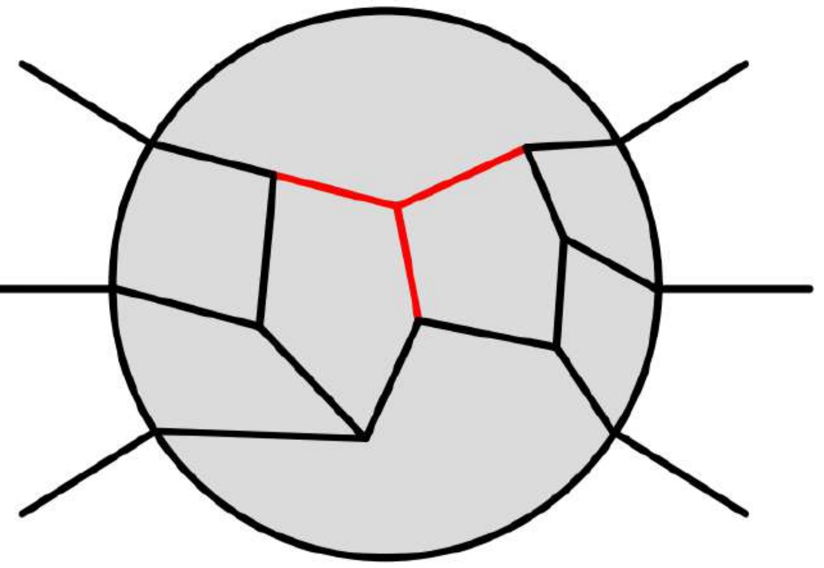
For general k we find $W_{k,n}^{\text{tree}} = \frac{1}{(k-2)!}\left(W_{3,n}^{\text{tree}}\right)^{k-2}$. This forms an amplituhedron $A_{m,n,k'}$ with $m = 2$ and $k' = k - 2$.

Wilson Loop OPE & Form Factor OPE

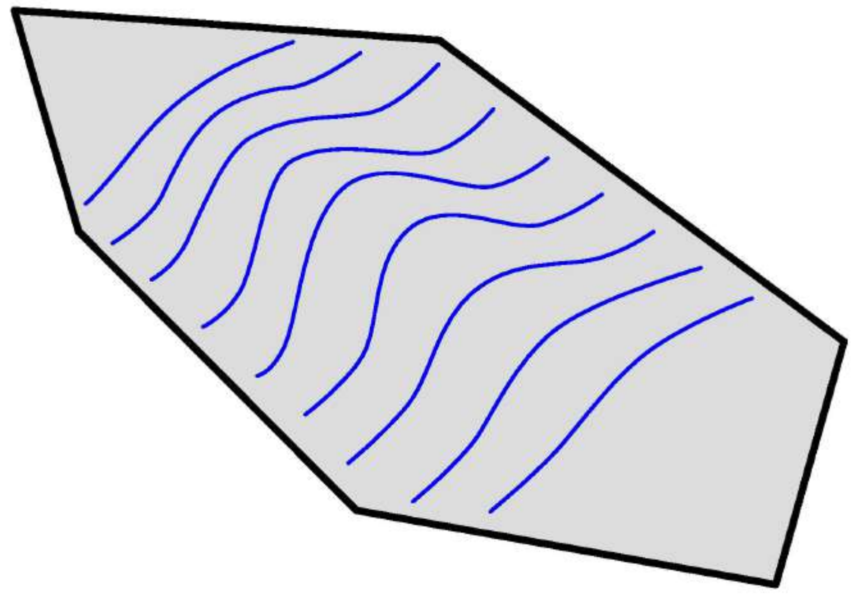
[Alday, Gaiotto, Maldacena, Sever, Vieira '11]

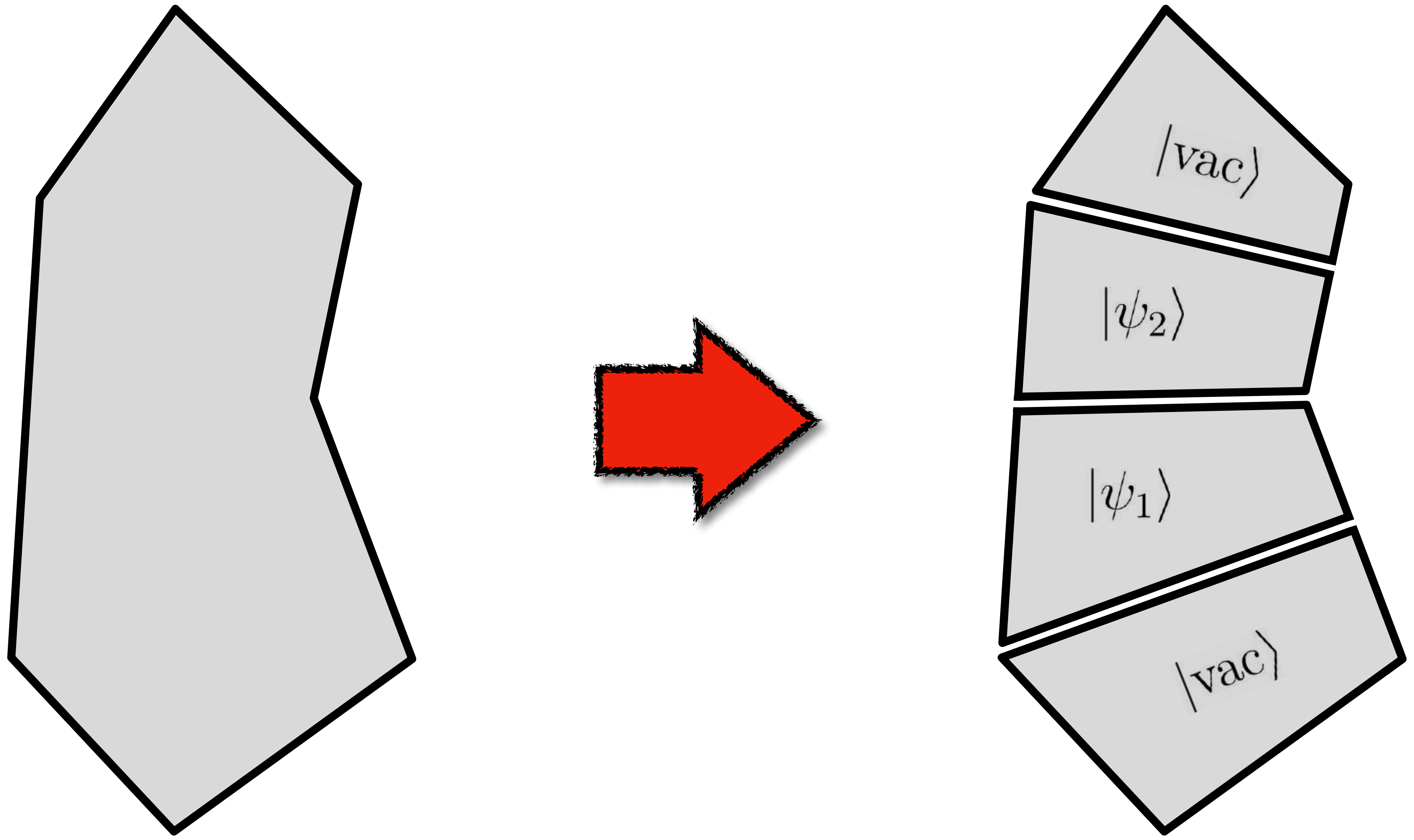
[Sever, AT, Wilhelm '20]

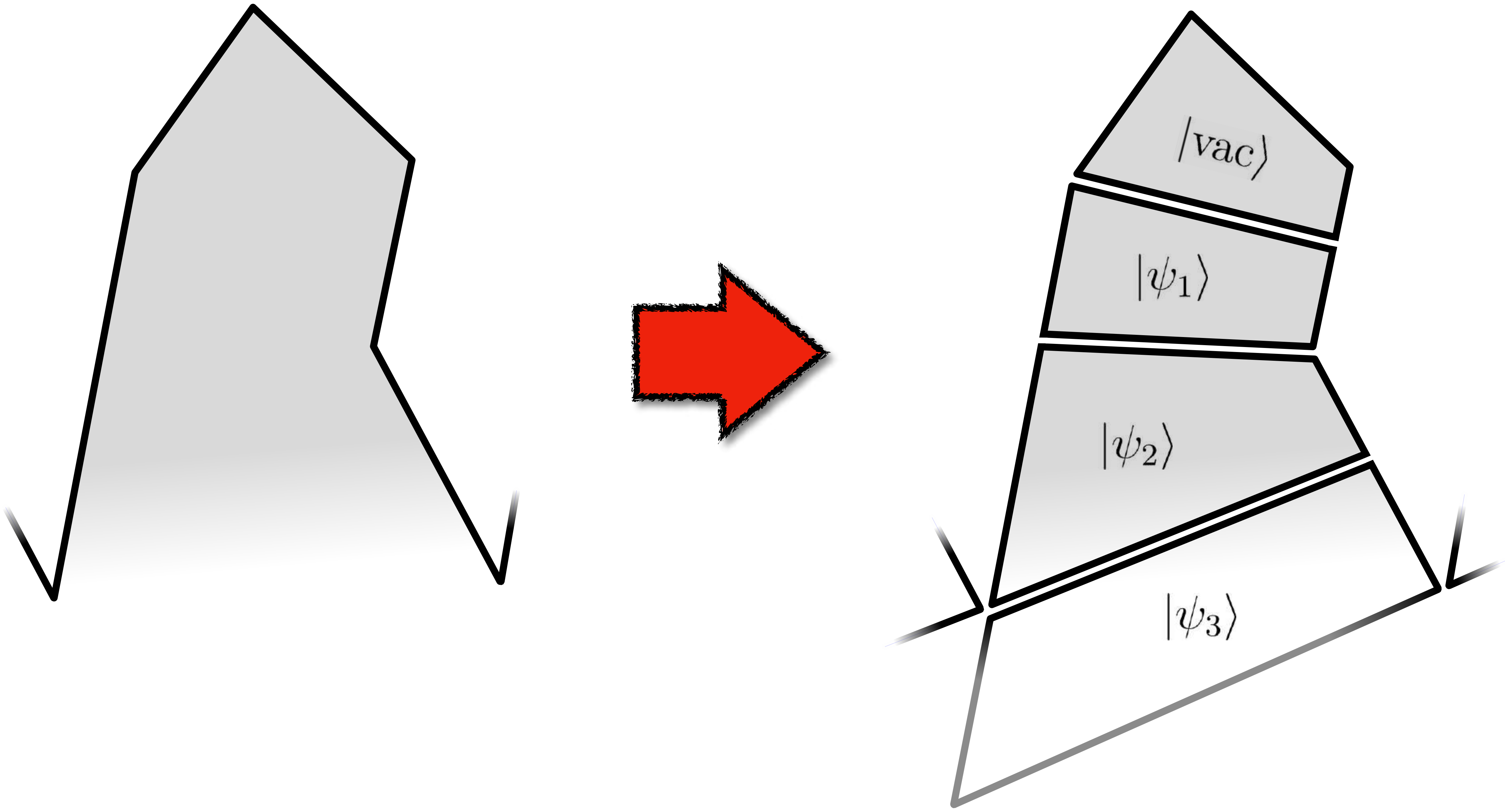
$$\sum_{L=0}^{\infty} g^{2L}$$



$$= \sum_{\psi} e^{-E(\psi)\tau}$$







$$\mathcal{W}_7 \sim \sum_{\psi_i} P(0|\psi_1)\, P(\psi_1|\psi_2)\, P(\psi_2|0)$$

$$\mathcal{W}_{5,\mathcal{O}} \sim \sum_{\psi_i} P(0|\psi_1)\, P(\psi_1|\psi_2)\, P(\psi_2|\psi_3)\, F_{\mathcal{O}}(\psi_3)$$

Back to bootstrapping: imposing the constraints

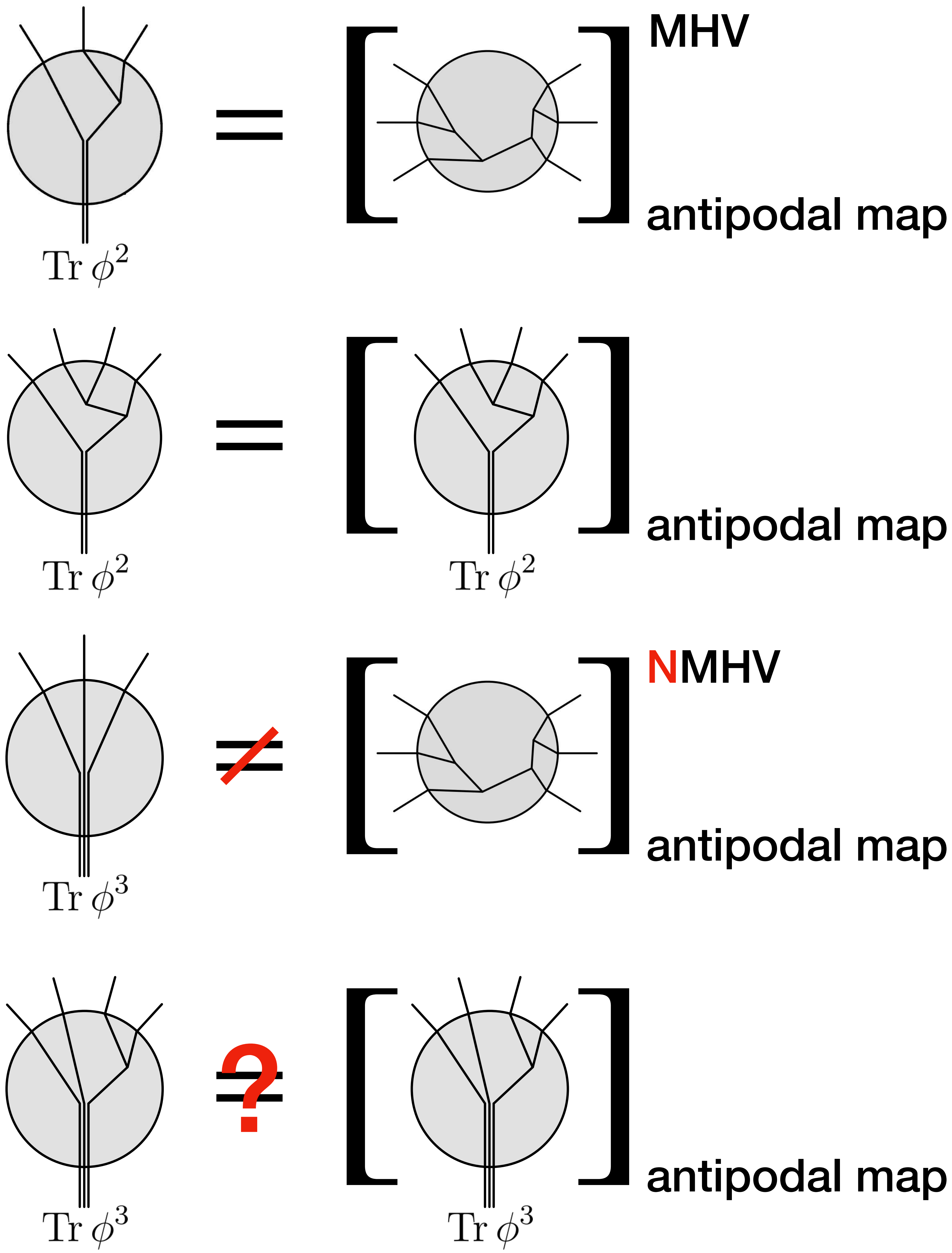
Tr ϕ^2

L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	191	191
L^{th} discontinuity	2	5	17	38	75	171	164
collinear limit	0	1	2	8	19	70	6
OPE $T^2 \ln^{L-1} T$	0	0	0	4	12	56	0
OPE $T^2 \ln^{L-2} T$	0	0	0	0	0	36	0
OPE $T^2 \ln^{L-3} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-4} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-5} T$	0	0	0	0	0	0	0

Tr ϕ^3

L	2	3	4	5	6
functions in $\mathcal{C}$	52	284	1495	$\sim$8000	?????
dihedral symmetry	13	63	302	$\sim$1400	????
$(L-1)$ final entries	4	15	47	190	407
$(L+1)^{\text{st}}$ discontinuity	3	13	43	182	394
OPE $T^1 \ln^{L} T$	2	10	38	171	???
OPE $T^1 \ln^{L-1} T$	1	6	31	158	???
OPE $T^1 \ln^{L-2} T$	0	2	20	137	322
OPE $T^1 \ln^{L-3} T$	0	0	4	103	272
OPE $T^1 \ln^{L-4} T$	0	0	0	50	190
OPE $T^1 \ln^{L-5} T$	0	0	0	0	64
OPE $T^1 \ln^{L-6} T$	0	0	0	0	0

Antipodal duality



[Dixon, Gurdogan, McLeod, Wilhelm ’21]

[Dixon, Gurdogan, Liu, McLeod, Wilhelm ’22]

Future goals

Short term

Three-point form factor of $\mathrm{Tr}\,\phi^3$ at two loops (90% done).

Three-point form factor of $\mathrm{Tr}\,\phi^4$ at three loops.

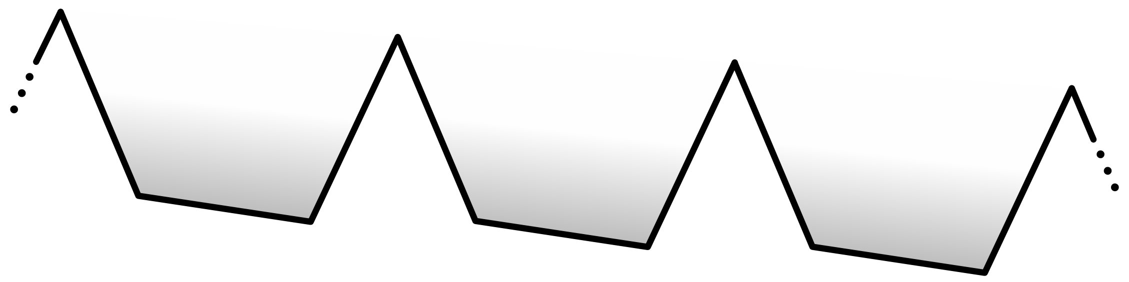
Unprotected operators (Konishi). Potential overlap with the Quantum Spectral Curve program.

Long term

Finding a critical amount of hidden constraints on the function spaces of amplitudes and form factors to make bootstrap work without integrability. If it's possible, we can turn loop calculations into a linear algebra problem!

Understanding the physical reasons behind the emergence of the antipodal duality, extended Steinmann-like relations and multiple-final-entry constraints.

Form Factor Transitions

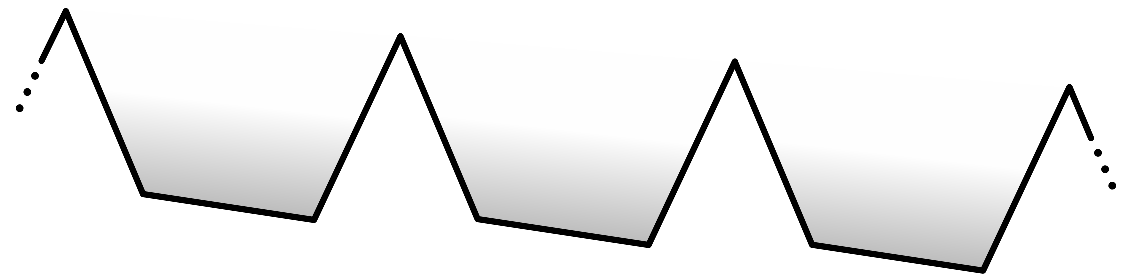
$\mathrm{Tr}\,\phi^2$ $\langle$  $\rangle$

$|\mathrm{vac}\rangle$ $|F\bar{F}\rangle$ $|\psi_A\bar{\psi}^A\rangle$ $|\phi_i\bar{\phi}^i\rangle$ $\ldots$

twist 0 T^0

twist 2 T^2

$$\mathcal{W}_{3,2}(S,T) = 1 + \mathcal{W}^{(2)}_{3,2}(S)\,T^2 + \mathcal{O}(T^4)$$

$\mathrm{Tr}\,\phi^3$ $\langle\phi(0)|$  $\rangle$

$|\phi_i\rangle$ $|F\phi_i\bar{F}\rangle$ $|\psi_A\phi_i\bar{\psi}^A\rangle$ $|\phi_j\phi_i\bar{\phi}^j\rangle$ $\ldots$

twist 1 T^1

twist 3 T^3

$$\mathcal{W}_{3,3}(S,T) = \mathcal{W}^{(1)}_{3,3}(S)\,T + \mathcal{W}^{(3)}_{3,3}(S)\,T^3 + \mathcal{O}(T^5)$$

Terms in this expansion can be computed exactly in the coupling. For example,

$$\mathcal{W}^{(1)}_{3,3}(S)\,T = \int \frac{du}{2\pi}\, T^{E_\phi(u)}\, S^{ip_\phi(u)}\, \sqrt{\mu_\phi(u)\nu_\phi(u)}$$

Where $E_\phi(u) = 1 + \mathcal{O}(g^2)$ and $p_\phi(u) = 2u + \mathcal{O}(g^2)$ are the flux tube energy and momentum of a scalar. [Basso '11]

$\mu_\phi(u)$ is the scalar flux tube measure, while $\nu_\phi(u)$ is the tilted scalar flux tube measure, which involves the octagon kernel.

Tilted Bessel Kernels

$$\mathbb{K}_{ij} = 2j(-1)^{ij+j}\int_0^\infty \frac{dt}{t}\frac{J_i(2gt)J_j(2gt)}{e^t-1}$$

[Beisert, Eden, Staudacher ’07]

$\alpha-$tilt

$$\mathbb{K}(\alpha) = 2\cos(\alpha)\begin{pmatrix}\cos(\alpha)\mathbb{K}_{\circ\circ} & \sin(\alpha)\mathbb{K}_{\circ\bullet}\\ \sin(\alpha)\mathbb{K}_{\bullet\circ} & \cos(\alpha)\mathbb{K}_{\bullet\bullet}\end{pmatrix}$$

[Basso, Dixon, Papathansiou ’20]

$\alpha = 0$ — octagon kernel

Pentagon transitions

$P(\psi|0)$

$\alpha = \frac{\pi}{4}$

$\alpha-$tilt

Form factor transitions

$F_{\mathcal{O}}(\psi)$

$\alpha = 0$