Theoretical prediction for wavelet l_1-norm

Cosmology and Statistics Days (01 Feb 2024)

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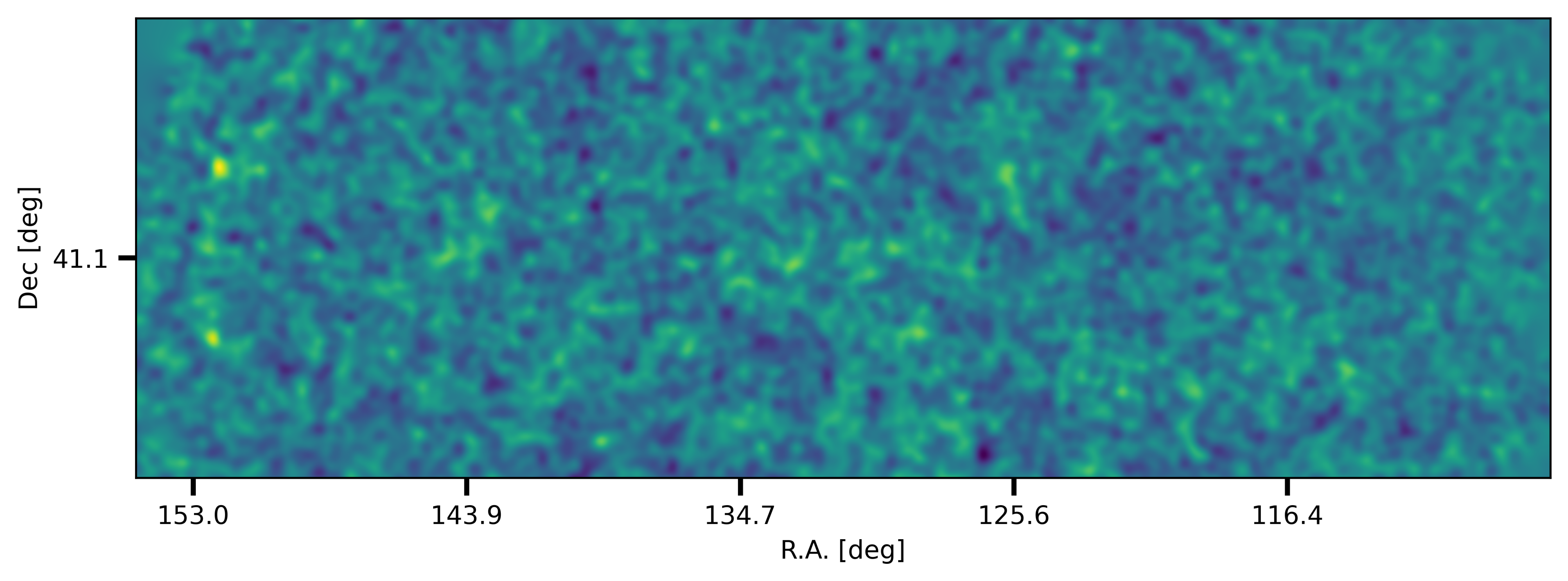
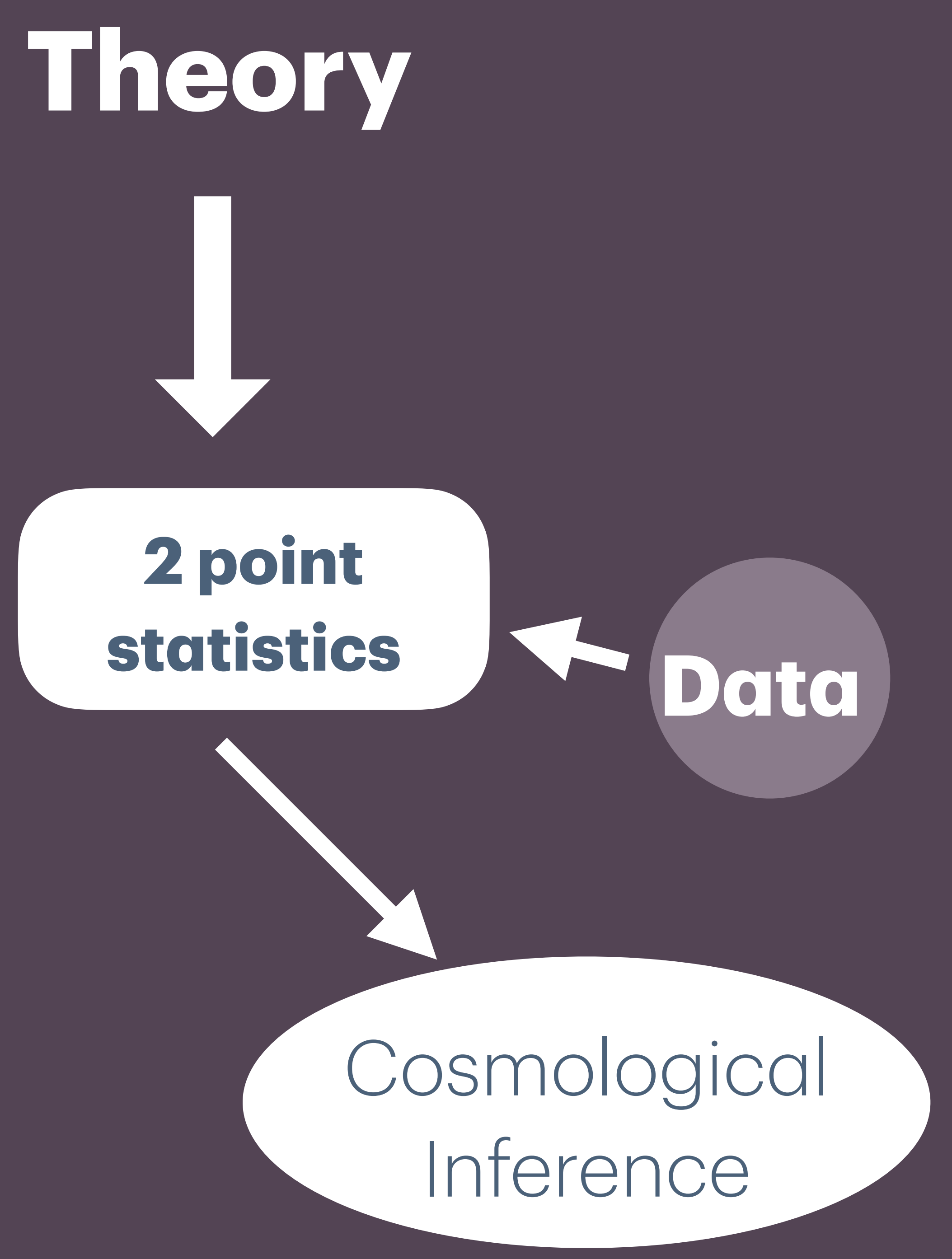
Weak lensing

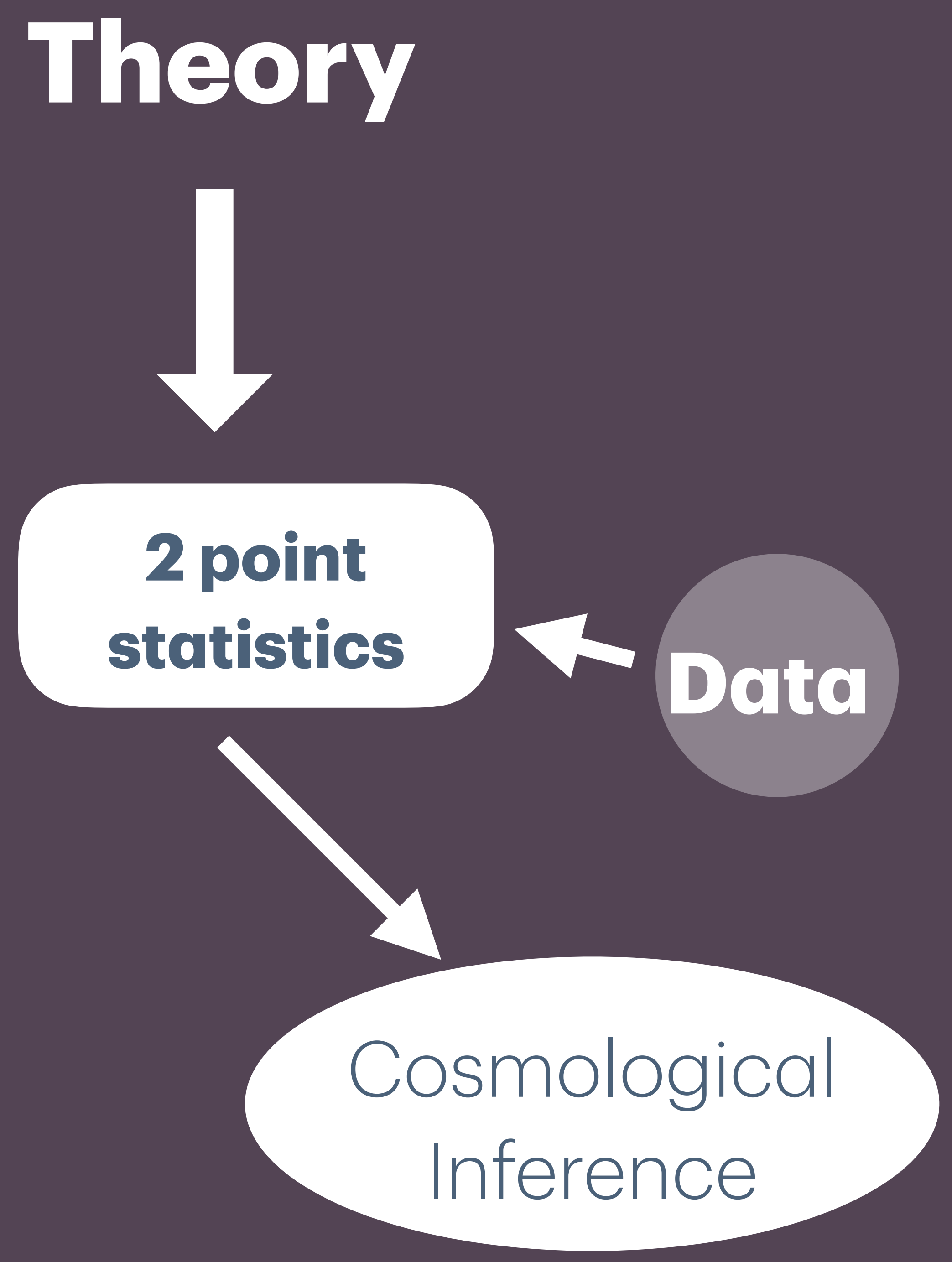
- Tracer of dark matter and dark energy
- Can help us in better understand the universe

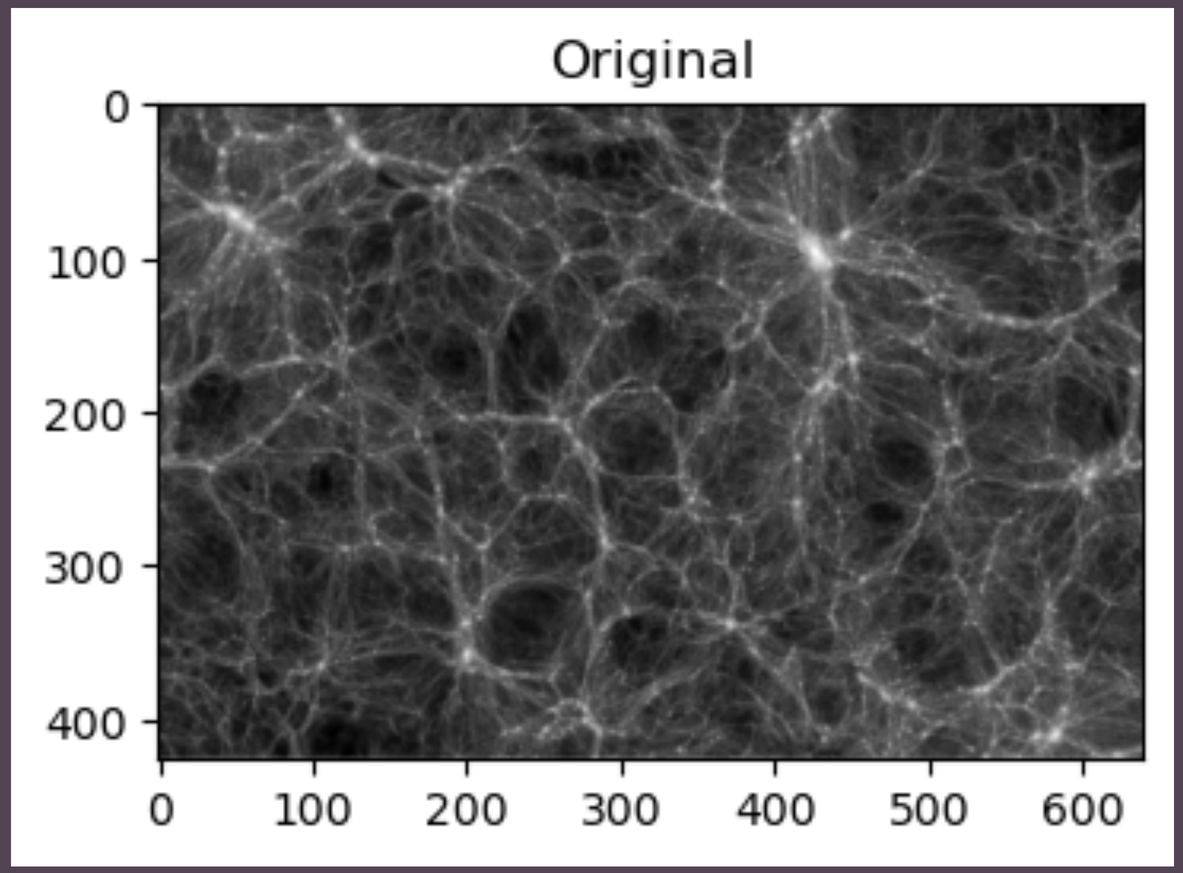


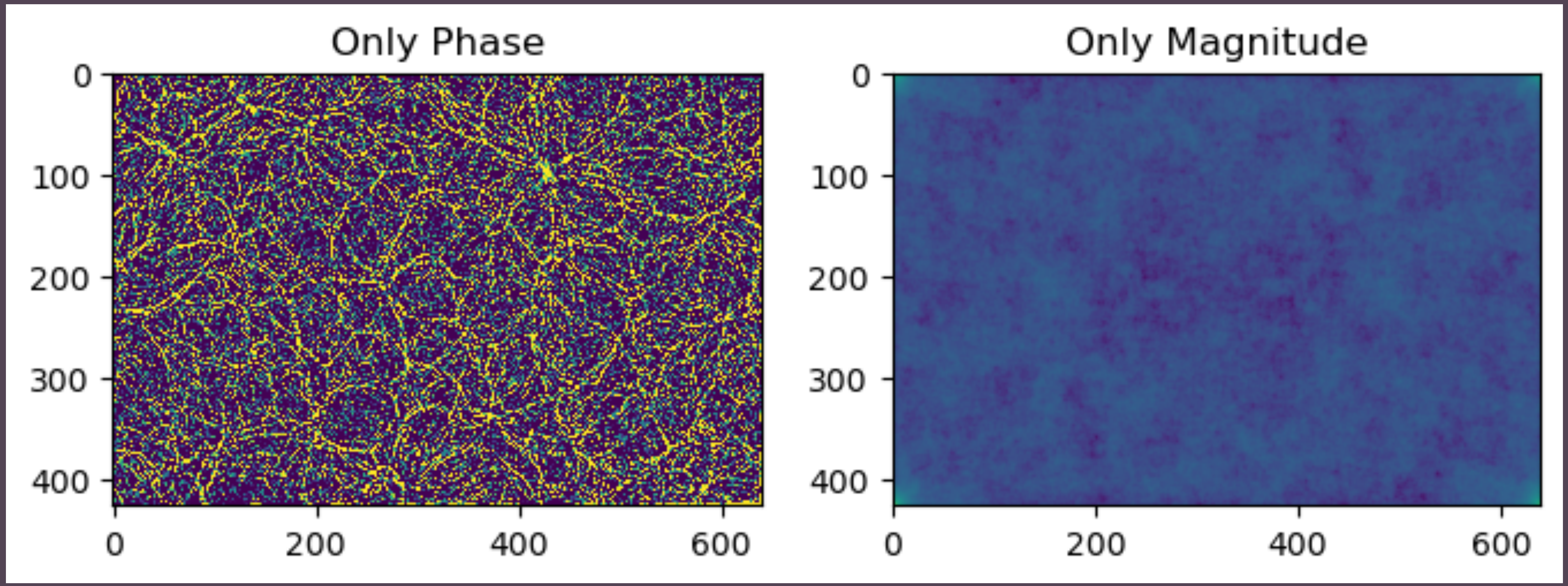
Source



But…is this sufficient for us?


Theory
2 point statistics
Data
Cosmological Inference

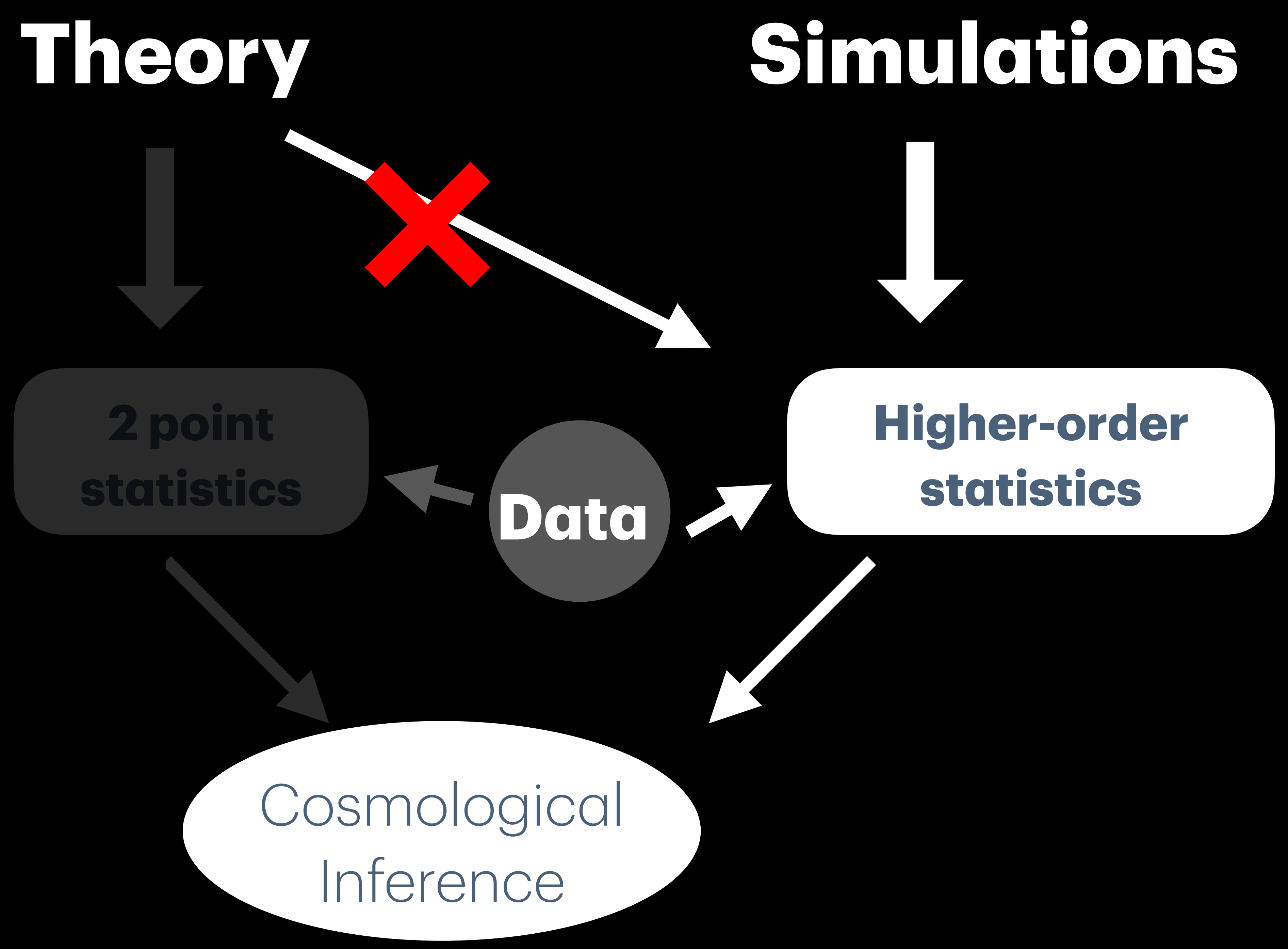

Original
0
100
200
300
400
0
100
200
300
400
500
600

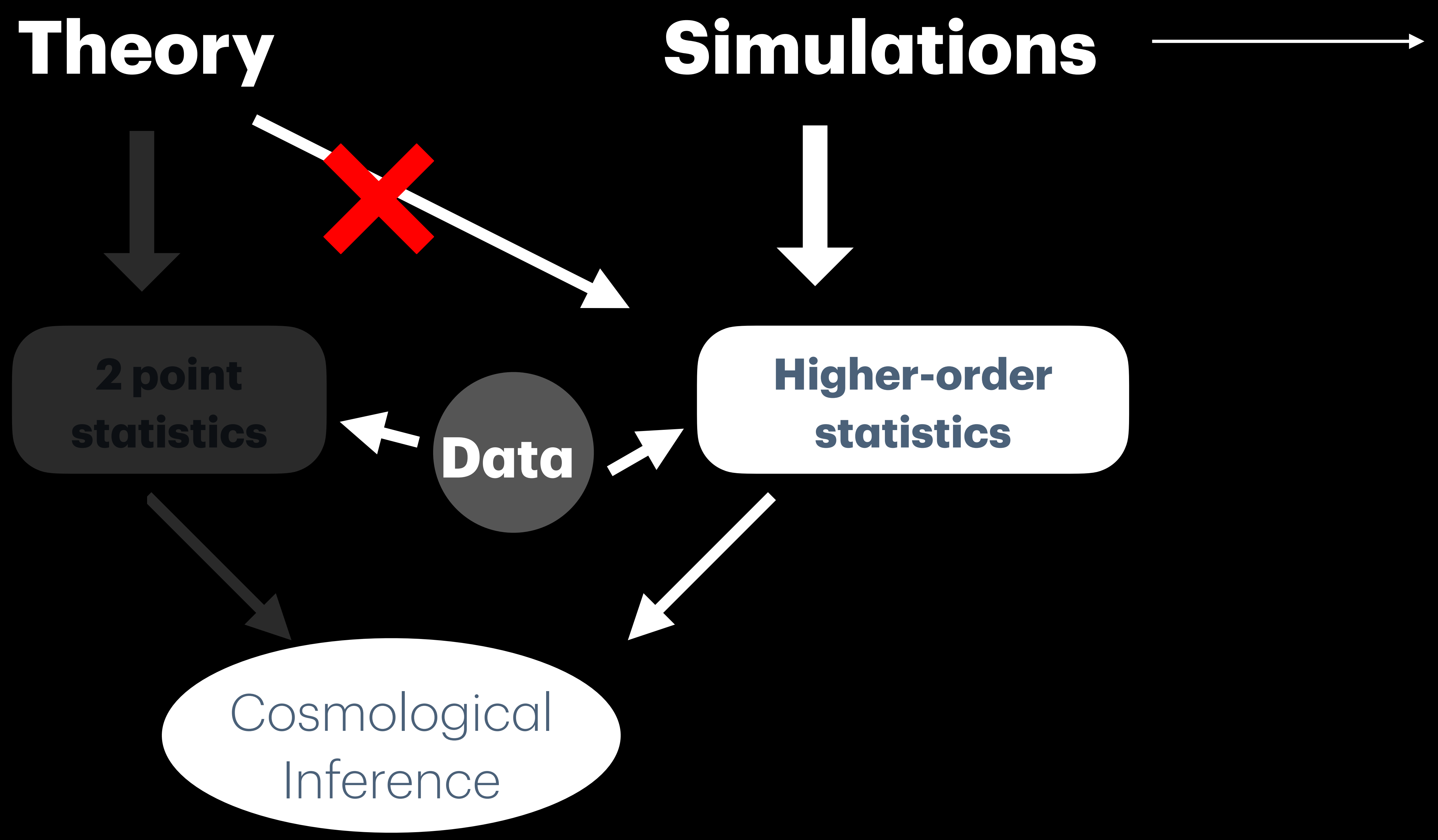

Only Phase
Only Magnitude
0
100
200
300
400
0
200
400
600

The alternative: Higher-order statistics

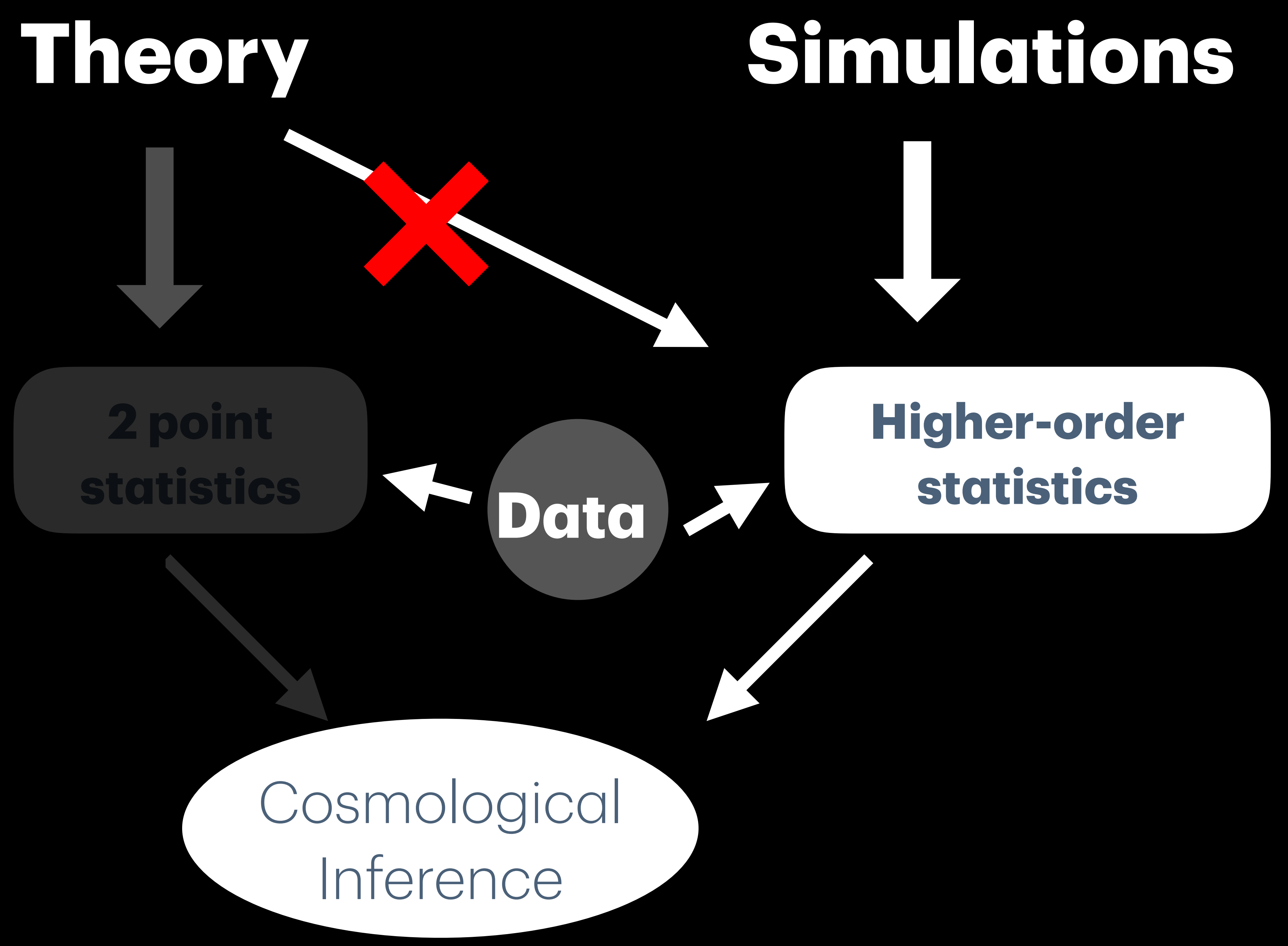
Statistics	Tomo	Systematics	Params	Forecasts (with II order)	Real data	Survey	References
Summary statistics employed in the analysis	If a tomographic analysis was performed	m = multiplicative bias c = additive bias photo-z = photometric redshifts bar = baryonic effects IA = intrinsic alignment	The cosmological parameters that are constrained	Improvement w.r.t 2PCF %=single parameter Number = 2D FoM	Constraining power > = better ~ = similar < = worst	Survey specs, name or sky coverage + galaxy number density	First author + year.
PDF	no	m, c	Ω_m, σ_8	2		DES-Y1	Patton + 2017
	yes	no	M_ν, A_s	35%, 61%		LSST	Liu, J.+ 2018
	no	no	M_ν, w_0	27%,40%+Planck		Euclid	Boyle+ 2020
Bispectrum	yes	no	$\sigma_8, w_a, w_0, \Omega_\Lambda$	3		4000 deg^2, 100 arcmin^{-2}	Takada+ 2005
	yes	no	Ω_m, σ_8	2		Euclid	Bergé+ 2010
	yes	no	M_ν, Ω_m, A_s	32%, 13%, 57%		LSST	Coulton+ 2019
MF	yes	no	Ω_m, σ_8, w_0	11%, 14%, 14%		LSST	Kratochvil+ 2012
	no	photo-z, m, c	Ω_m, σ_8		biased (syst.)	CFHTLenS	Petri+2015
	yes	no	M_ν, Ω_m, A_s	4		LSST	Marques+2018
	yes	IA, photo-z, m	Ω_m, σ_8	4.2		DES	Zürcher+ 2021
Moments	no	photo-z, m, c	Ω_m, σ_8		> 2PCF	CFHTLenS	Petri+ 2015
	yes	m, c	Ω_m, σ_8	2		3500 deg^2, 27 arcmin^{-2}	Vicinanza+ 2018
	yes	bar, IA, photo-z, m	S_8	20%		DES-Y3	Gatti+ 2019
Peaks	yes	photo-z, m, c	Ω_m, σ_8		~ 2PCF	CS82	Liu X.+ 2015
	yes	photo-z, m, c	Ω_m, σ_8		> 2PCF (2)	CFHTLenS	Liu J.+ 2015
	no	m,c, IA, boost, photo-z	Ω_m, σ_8		~ 2PCF	DES-Y1	Kacprzak+ 2016
	yes	m,c, IA, photo-z, bar	S_8		> 2PCF (20%)	KiDS-450	Martinet+ 2017
	yes	no	M_ν, Ω_m, A_s	39%, 32%, 60%		LSST	Li Z.+ 2018
	yes	no	M_ν, Ω_m, A_s	63%, 40%, 72%		Euclid	Ajani+ 2020
Minima	yes	IA, photo-z, m	Ω_m, σ_8	2.8		DE	Zürcher+ 2021
Minima+Peaks	yes	bar	M_ν, Ω_m, A_s	44%, 11%, 63%		LSST	Coulton+ 2020
Voids	no	no	Ω_m, S_8, h, w_0	$\gtrsim$ 2PCF		LSST	Davies+ 2020
1D M_{ap}	yes	no	Ω_m, S_8, w_0	57%, 46%, 68%		Euclid	Martinet+2020
M. Learning	no	no	Ω_m, σ_8	5		3500 deg^2, no noise	Gupta+ 2018
	no	no	Ω_m, σ_8	~45% (dep. noise)		KiDS-450	Fluri 2018
	yes	photo-z, m, c, IA	S_8		> 2PCF (30%)	KiDS-450	Fluri 2019
Scattering T.	yes	no	M_ν, Ω_m, w_0	40%, > 2PCF		LSST	Cheng S.+ 2021
Starlet ℓ_1- norm	yes	no	M_ν, Ω_m, A_s	72%, 60%, 75%		Euclid	Ajani+ 2021

Source: Ajani 2021


Theory
Simulations
2 point statistics
Data
Higher-order statistics
Cosmological Inference



Multiple realisations,
Highly resource
intensive!



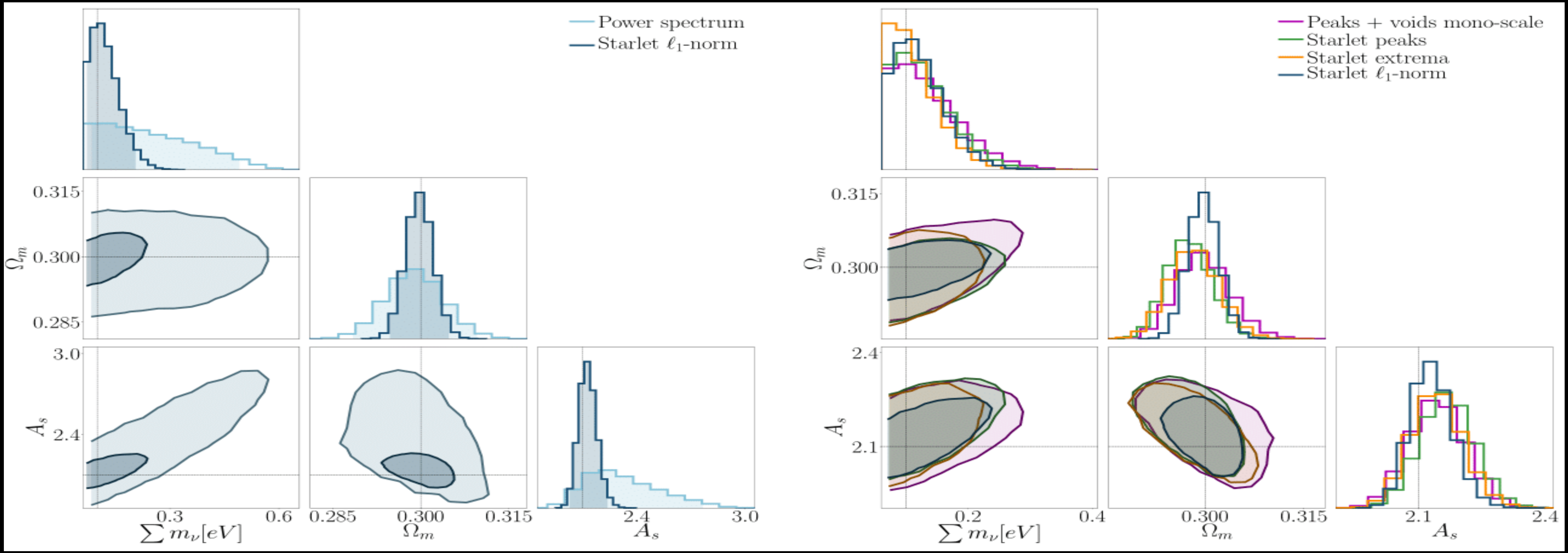
Multiple realisations, Highly resource intensive!

My work:

Have a theory for the wavelet l_1-norm

Wavelet ℓ_1-norm:

- Shown in Ajani et al. (2021) that it remarkably outperforms commonly used summary statistics,



Source: Ajani et al. (2021)

Wavelet ℓ_1-norm: mathematical formulation

$$w_j = <\kappa, \psi_j>$$
$$\psi_j = \varphi_j - \varphi_{j+1}$$
$$w_j = <\kappa, \varphi_j> - <\kappa, \varphi_{j+1}>$$

Scaling function

Where w_j is the wavelet coefficient for a scale j.

- information encoded in all pixels
- automatically includes peaks and voids
- multi-scale approach

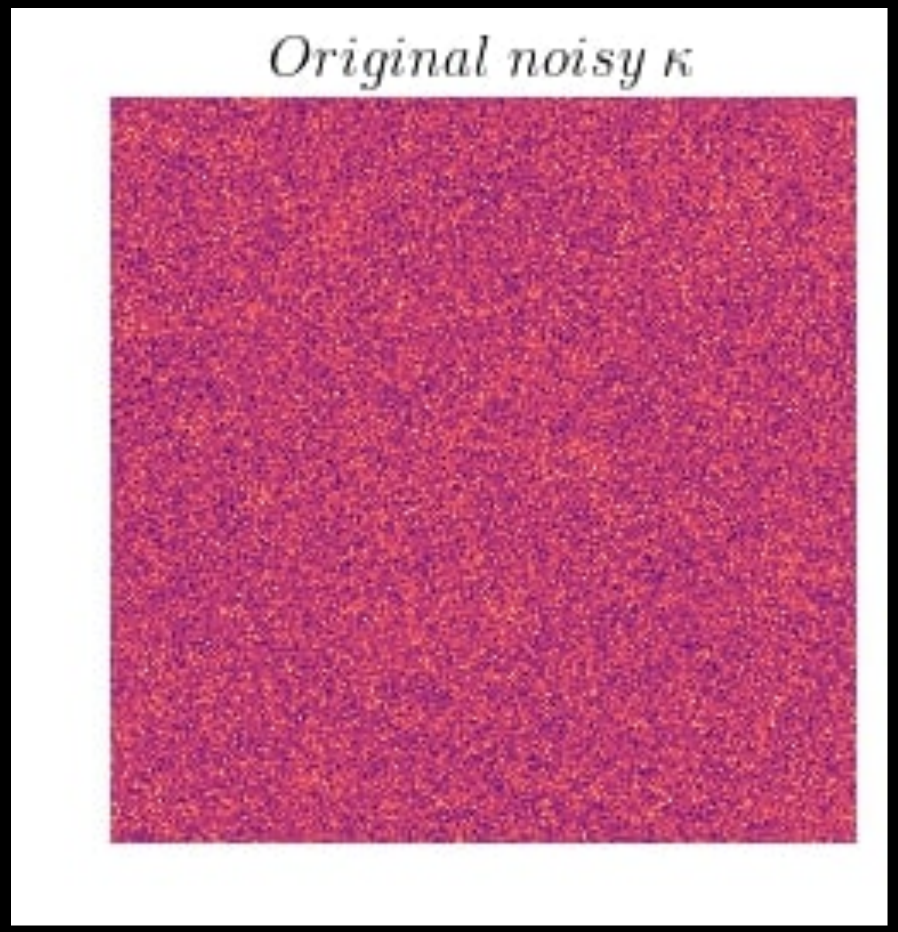
Wavelet ℓ_1-norm at scale j and bin B_i is

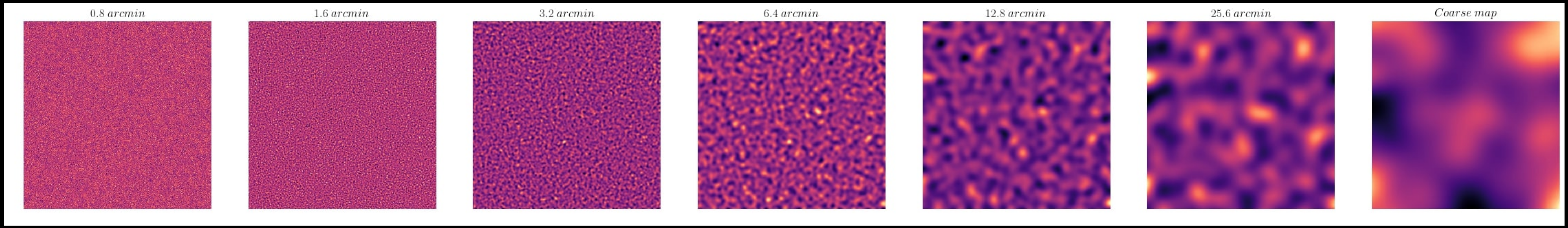
$$\ell_1^{j,i} = \sum_{u=1}^{\#coef(Sj,i)} |S_{j,i}[u]|$$

$$S_{j,i} = w_{j,k} / B_i < w_{j,k} < B_{i+1}$$

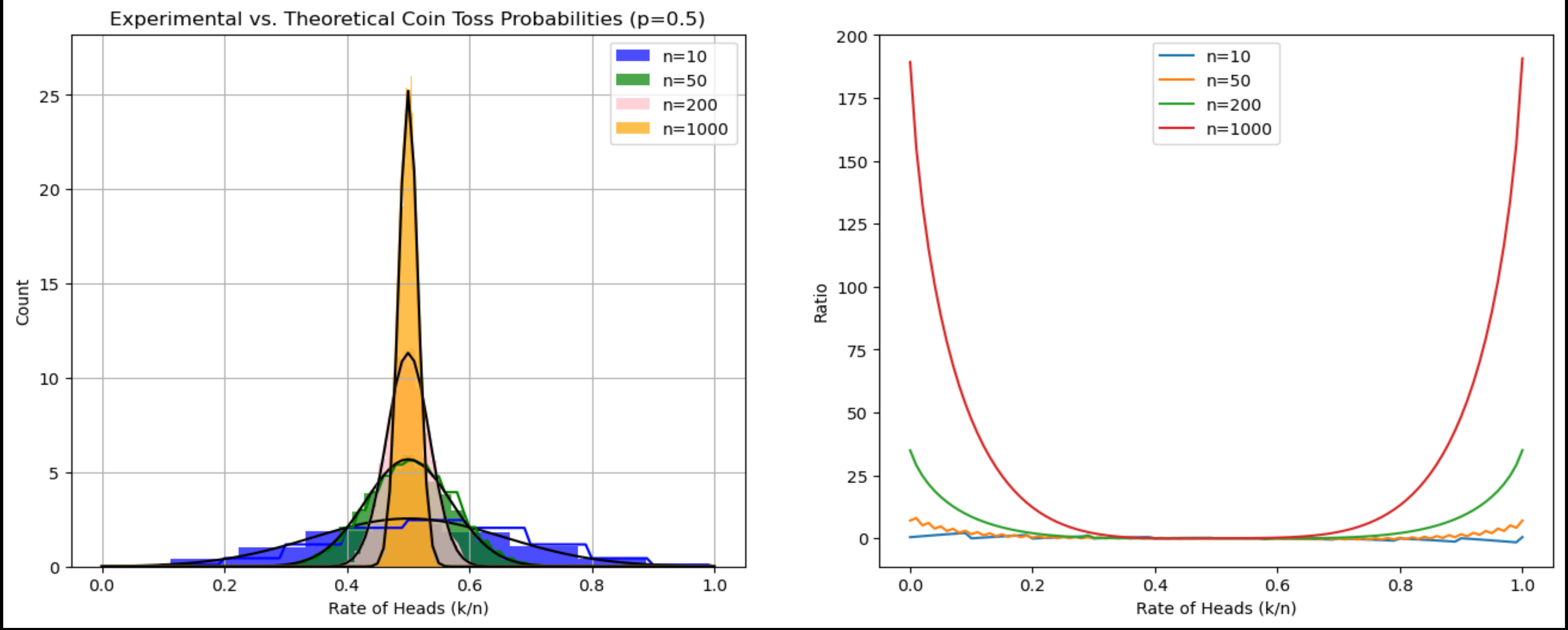
Where the wavelet coefficient, with k the pixel index at scale j is: $w_{j,k}$

An example: wavelet decomposition of a convergence map



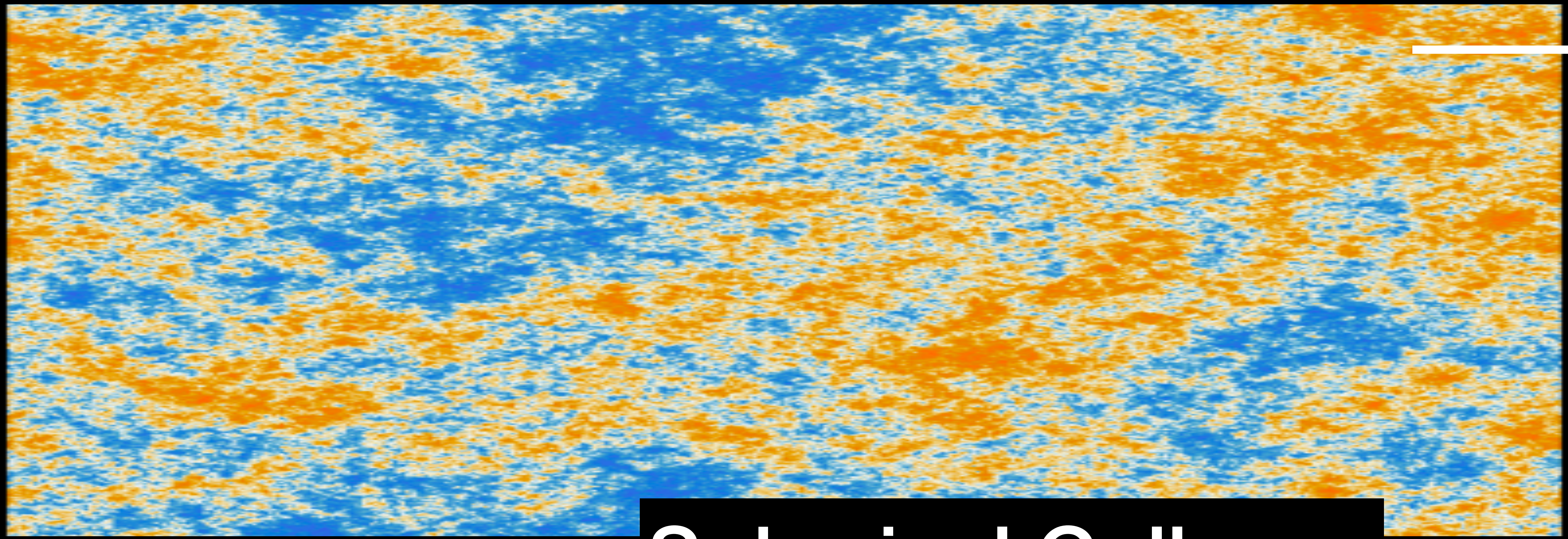


Large Deviation Theory: Intuition



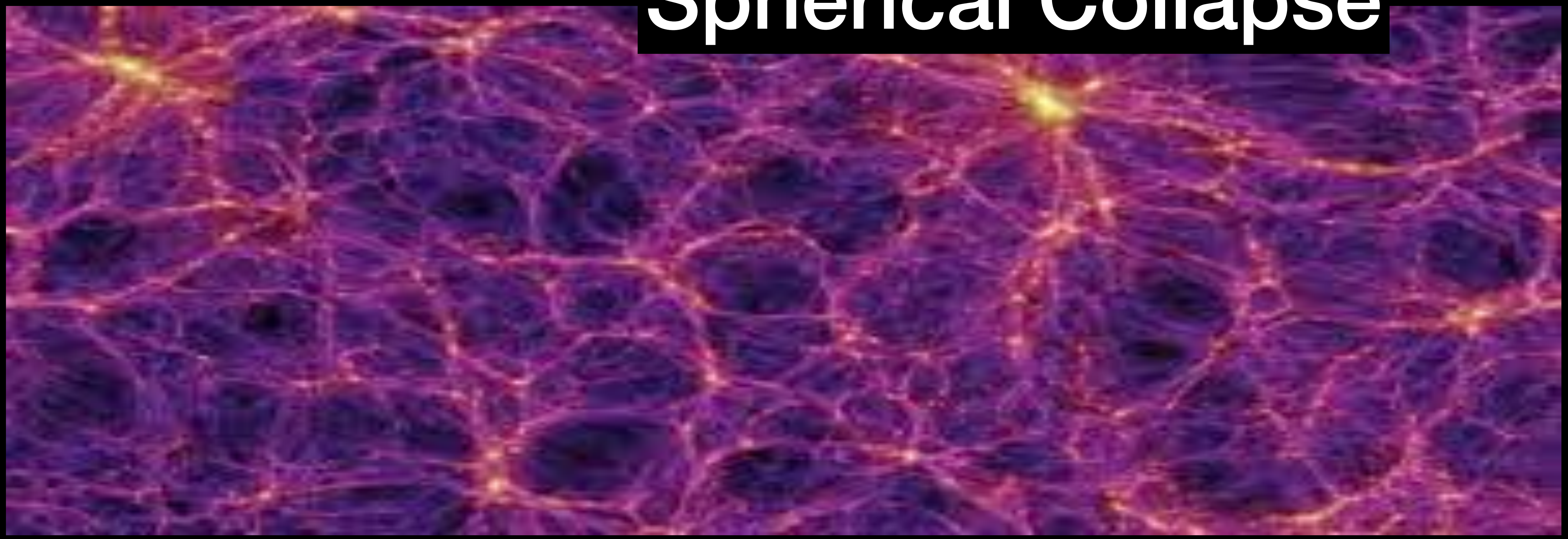
One-point PDF from Large Deviation Theory

Large Deviation Theory——>A framework to predict one-PDF in mildly non-linear regime from the 1st principles of Cosmology



→ We know the PDF

Spherical Collapse



LDT

Rate Function

↓

Probability Density Function

Deriving wavelet ℓ_1-norm from PDF

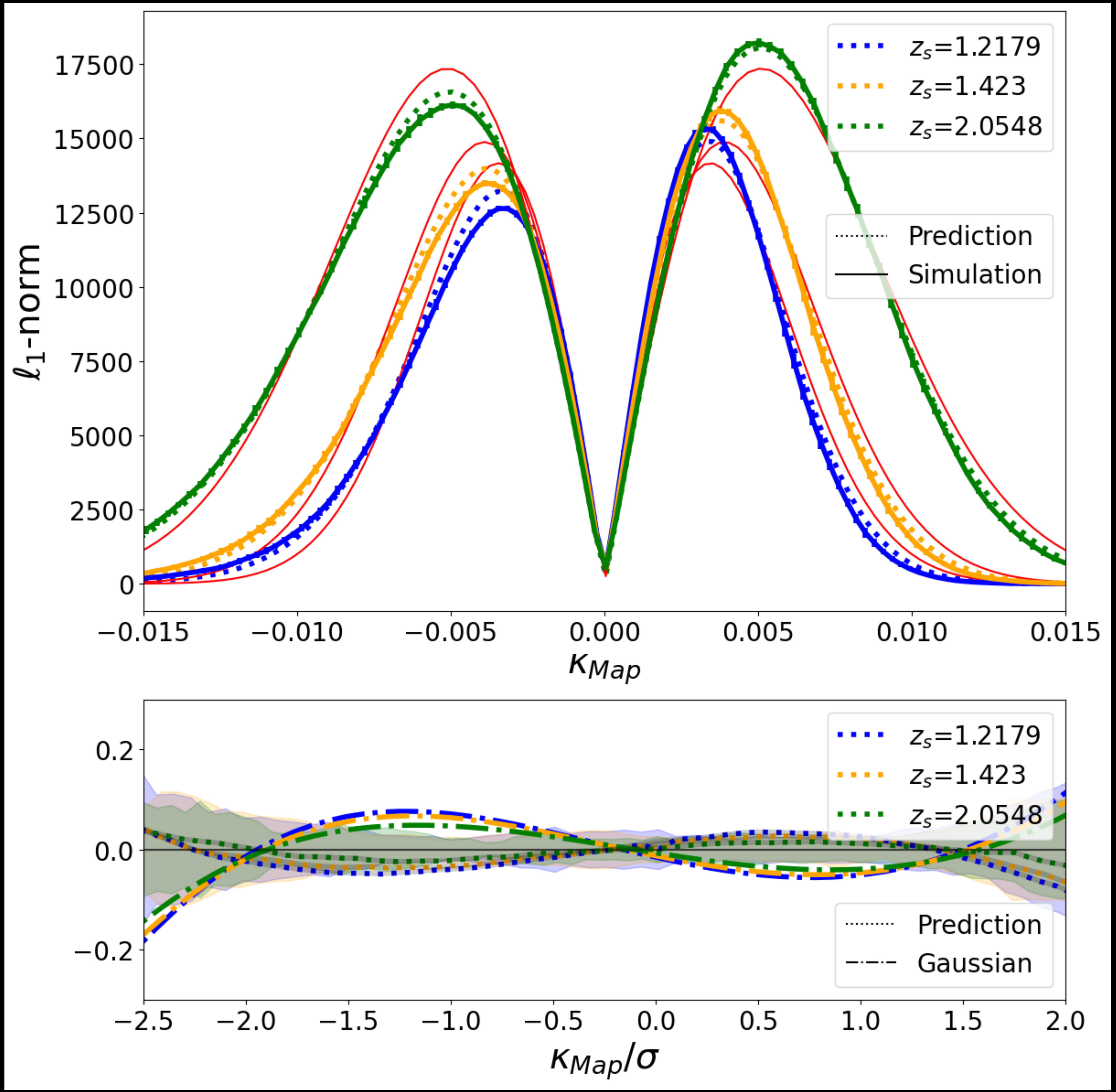
$$w_j = < \kappa, \varphi_{j+1} > - < \kappa, \varphi_j >$$

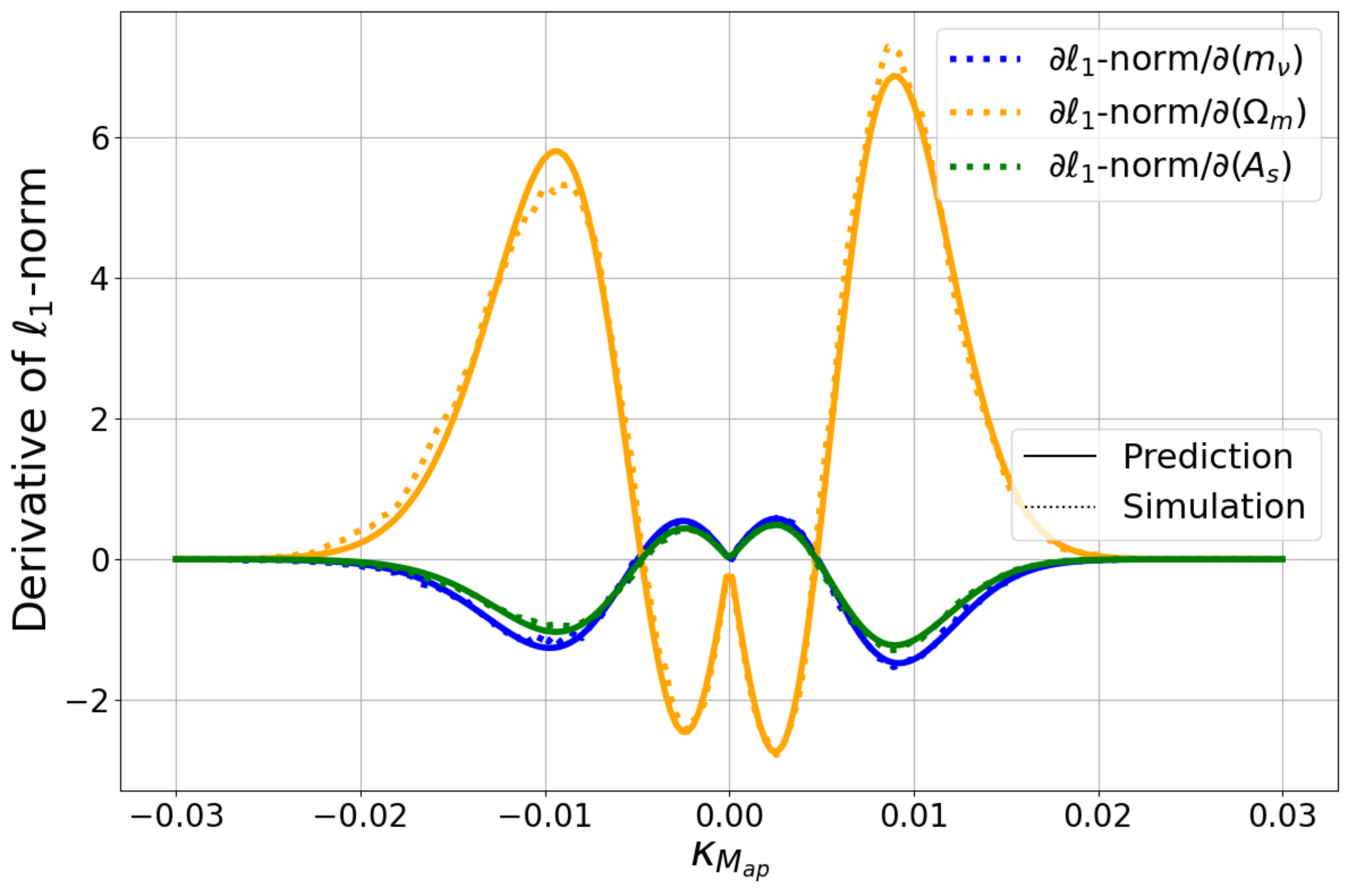
Apply this in the LDT framework to get the wavelet ℓ_1-norm of the wavelet coefficients w_j

Using LDT first get the $P(w_j)$

$$\ell_{1,pred}^{j,i} = P_i(w_j) \times |B_i| \text{ X scaling factor}$$

$$\ell_{1,theory}^{j,i} = \sum_{u=1}^{\#coef(Sj,i)} |S_{j,i}[u]|$$


z_s=1.2179
z_s=1.423
z_s=2.0548
Prediction
Simulation
ℓ_1-norm
17500
15000
12500
10000
7500
5000
2500
0
−0.015
−0.010
−0.005
0.000
0.005
0.010
0.015
κ_{Map}
z_s=1.2179
z_s=1.423
z_s=2.0548
Prediction
Gaussian
0.2
0.0
−0.2
−2.5
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
κ_{Map}/σ



$\partial\ell_1$-norm/$\partial(m_\nu)$
$\partial\ell_1$-norm/$\partial(\Omega_m)$
$\partial\ell_1$-norm/$\partial(A_s)$
Prediction
Simulation
Derivative of ℓ_1-norm
$\kappa_{M_{ap}}$
6
4
2
0
−2
−0.03
−0.02
−0.01
0.00
0.01
0.02
0.03

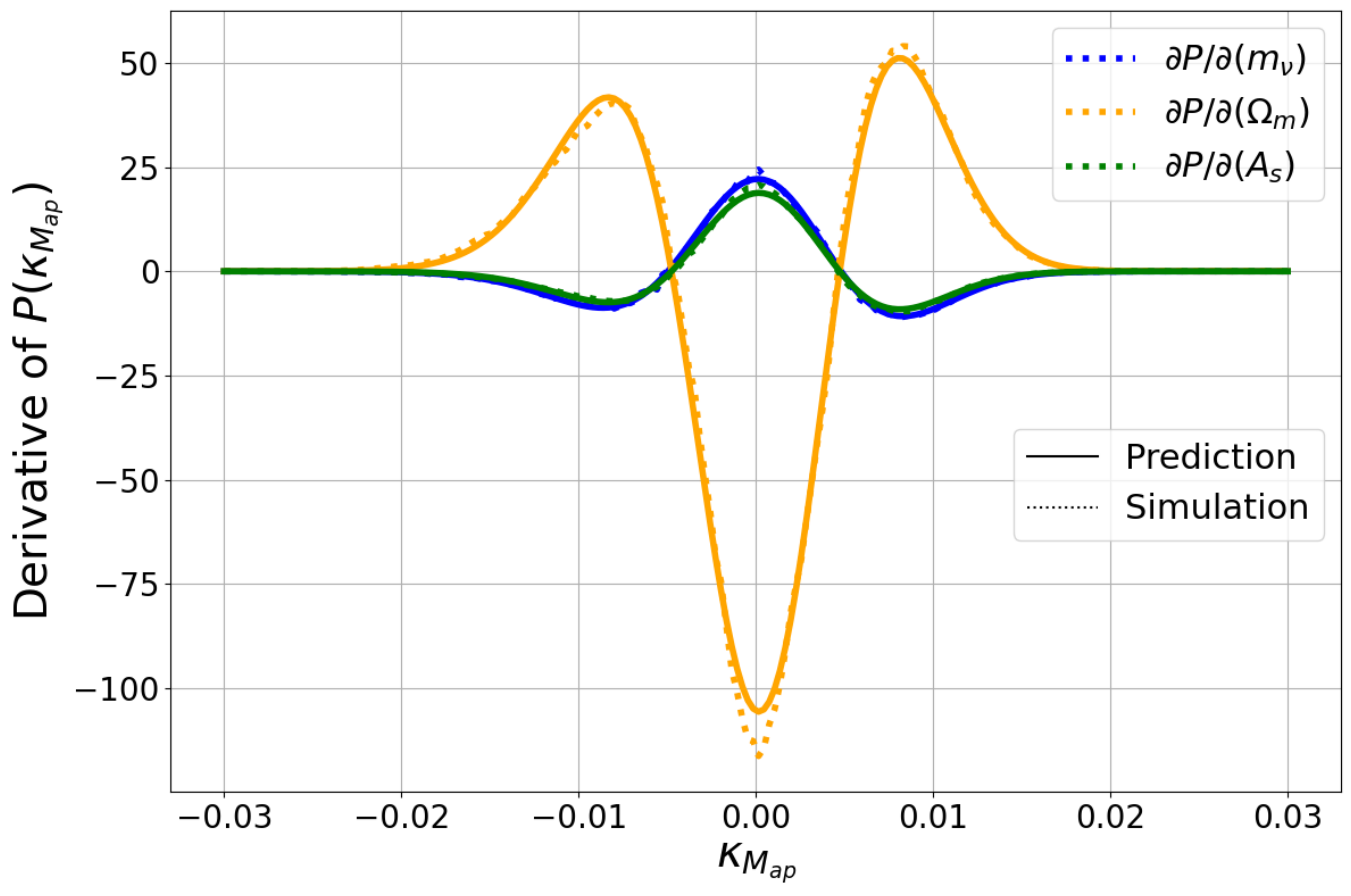
Conclusion

- We need different analytical methods to extract non-Gaussianities
  - Using Higher-Order statistics
  - Wavelet ℓ_1-norm is shown to be a better estimator in comparison to power spectrum, peaks and void statistics

- Current methods use simulations based approach —> **Highly resource intensive**
  - Need theoretical modelling

- Use LDT based approach to obtain the PDF for mass maps
  - Derived wavelet ℓ_1-norm from PDF

- Future work: Extending to any wavelet filter and building an emulator with these constraints

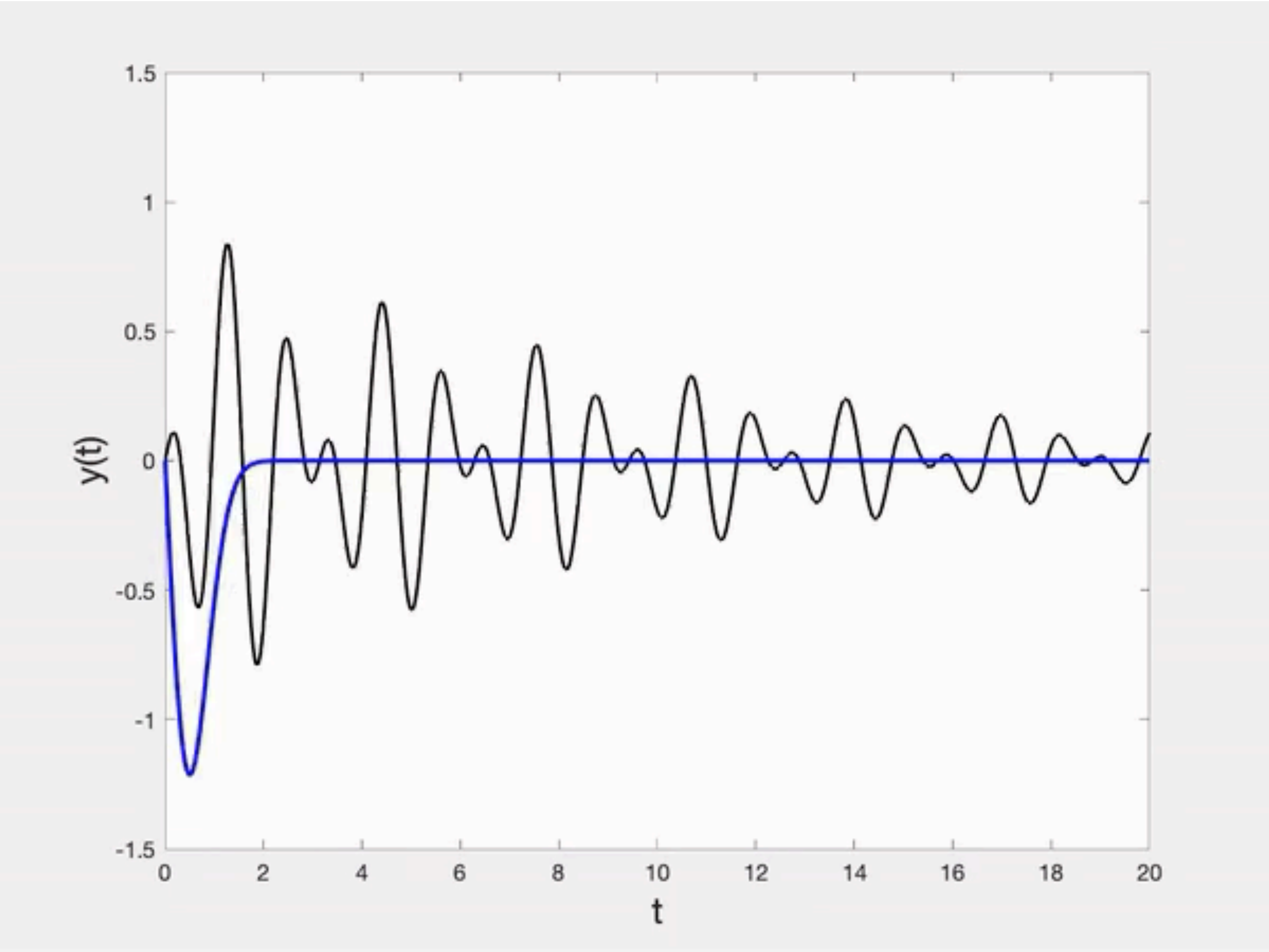


Derivative of $P(\kappa_{M_{ap}})$
50
25
0
−25
−50
−75
−100
−0.03
−0.02
−0.01
0.00
0.01
0.02
0.03
$\kappa_{M_{ap}}$
$\partial P/\partial(m_\nu)$
$\partial P/\partial(\Omega_m)$
$\partial P/\partial(A_s)$
Prediction
Simulation

Wavelets:

A set of mathematical function that is defined by the following properties:

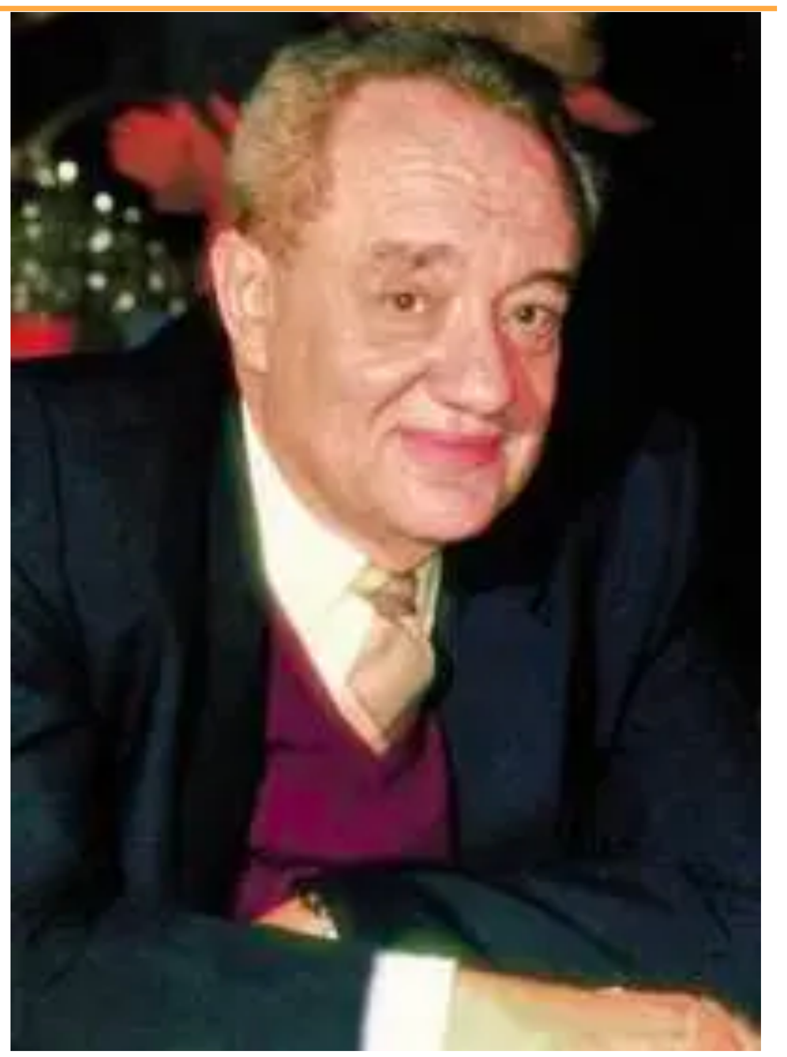
- Highly localized in space/time
- Has a vanishing mean



- A useful tool in analyzing signals where there are sharp spikes and discontinuities

The Continuous Wavelet Transform

$$W(a,b) = K \int_{-\infty}^{+\infty} \psi^*\left(\frac{x-b}{a}\right) f(x) dx$$



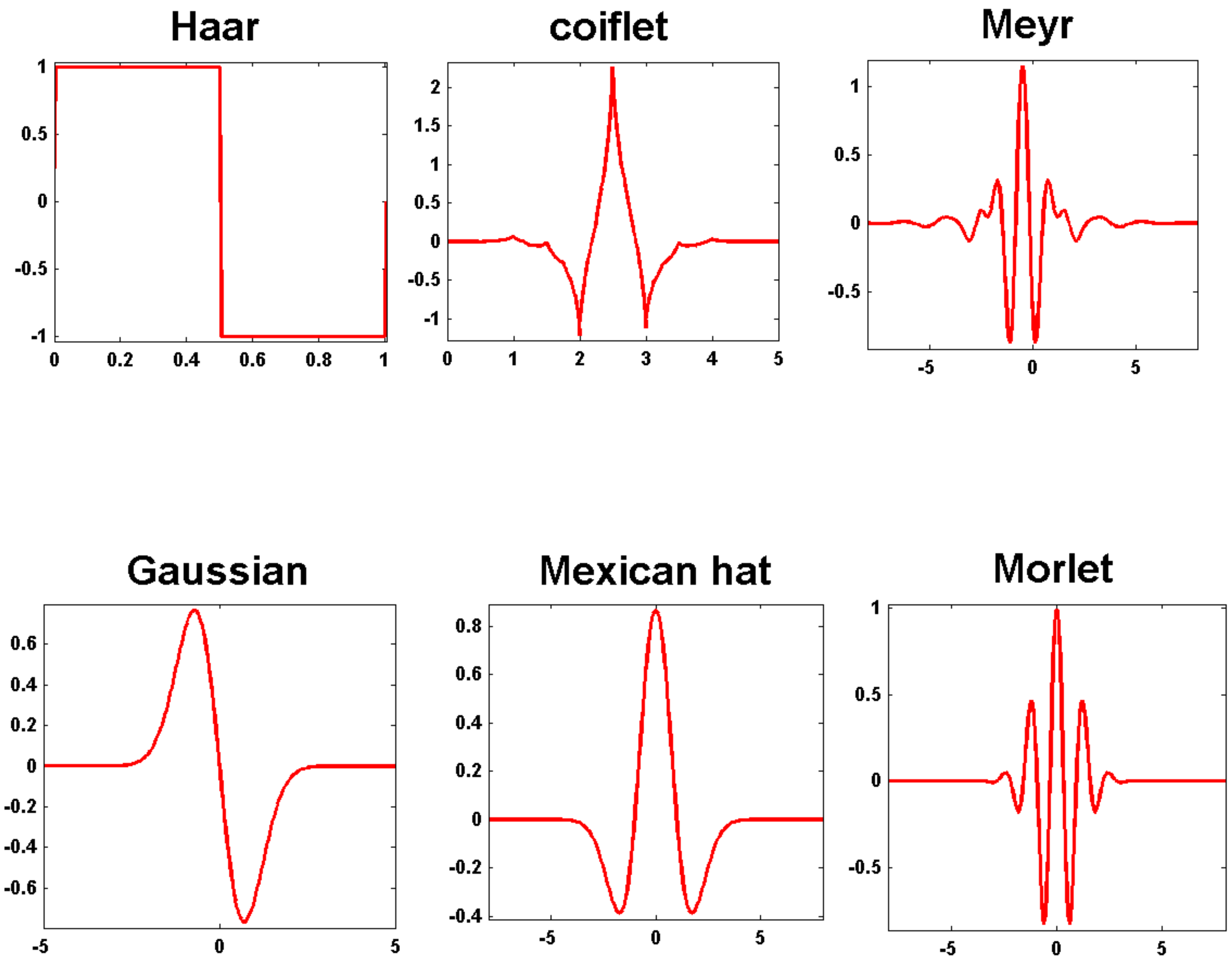
Jean Morlet

where:

- $W(a,b)$ is the wavelet coefficient of the function $f(x)$
- $\psi(x)$ is the analyzing wavelet
- a (> 0) is the scale parameter
- b is the position parameter

In Fourier space, we have: $\hat{W}(a,\nu) = \sqrt{a}\hat{f}(\nu)\hat{\psi}^*(a\nu)$
When the scale a varies, the filter $\hat{\psi}^*(a\nu)$ is only reduced or dilated while keeping the same pattern.

Some typical mother wavelets

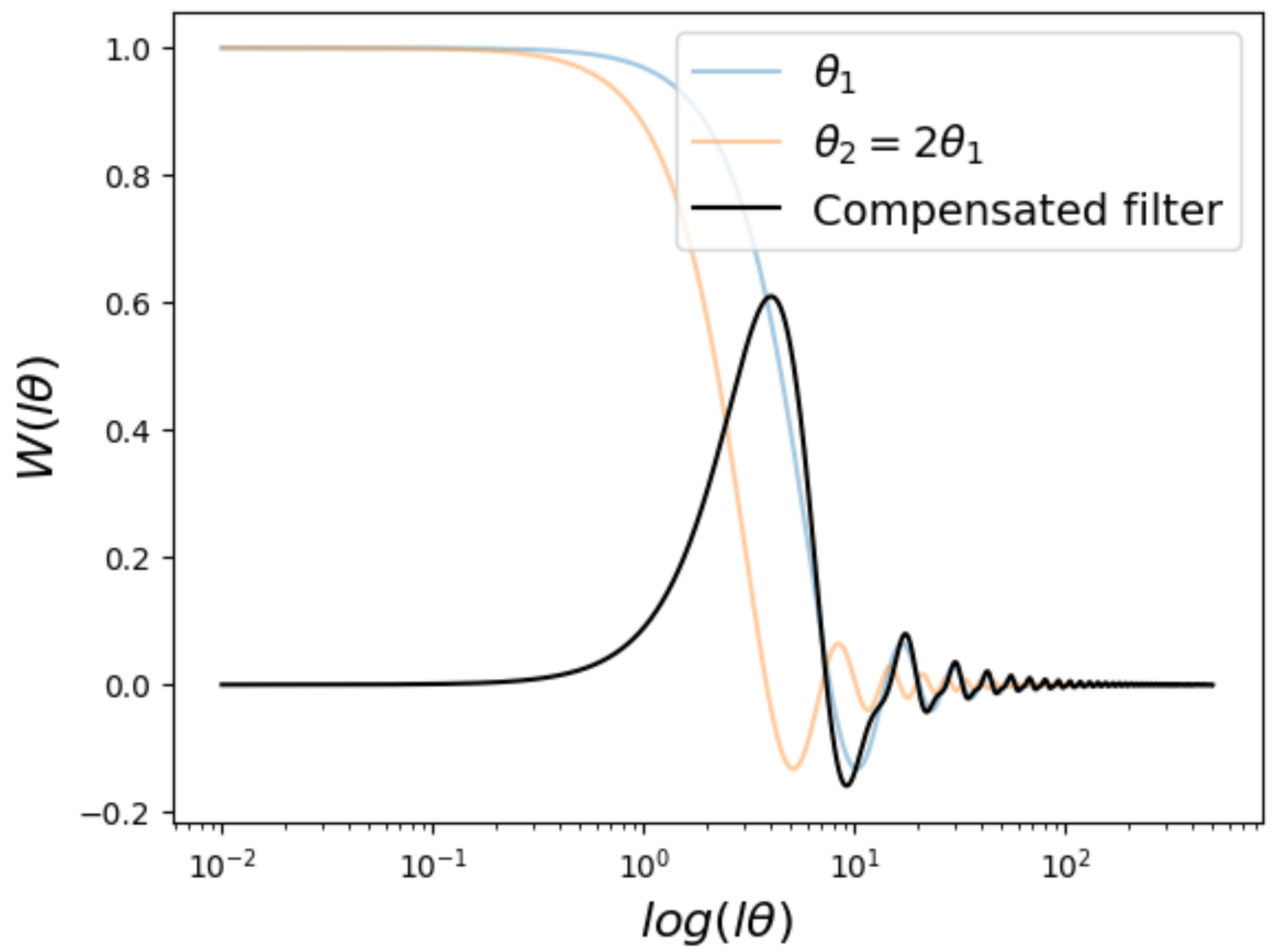


Filter Choice

- Mass sheet degeneracy → Use aperture mass statistic
  ↪Equivalent to using a compensated filter on the convergence map
- Wavelet filters are formally identical to aperture mass Leonard et al. (2012)

In this work: : use a function of concentric disks

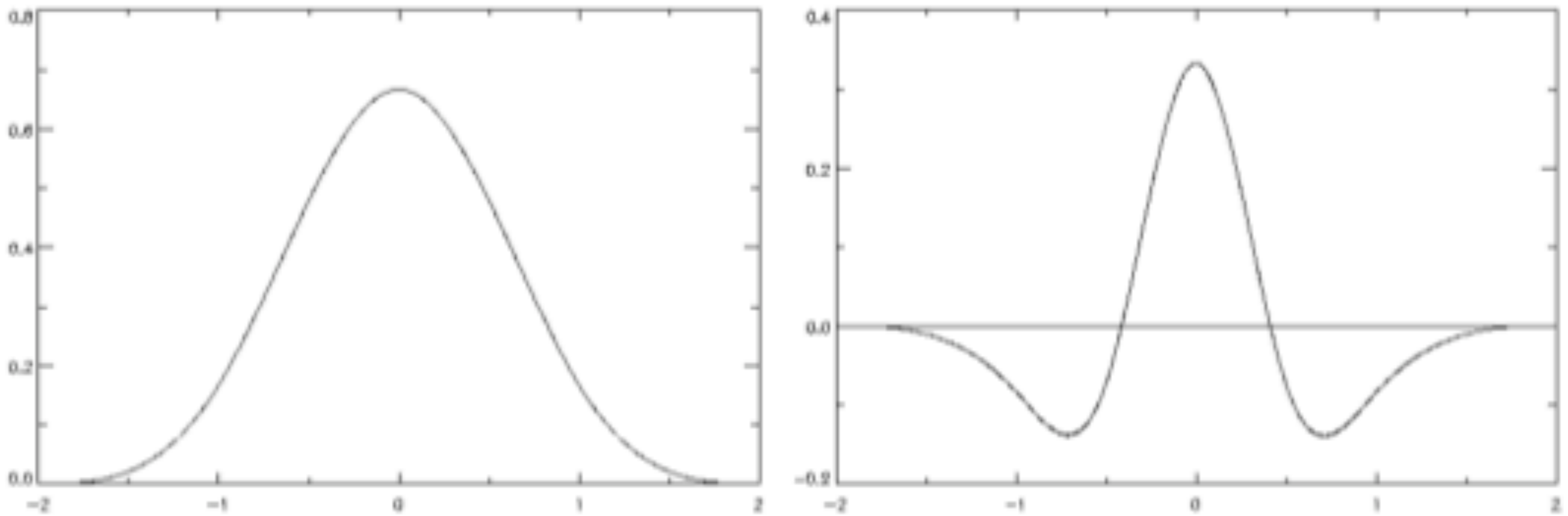
$$M_{ap}(\nu) = \kappa_{<\theta_2}(\nu) - \kappa_{<\theta_1}(\nu)$$

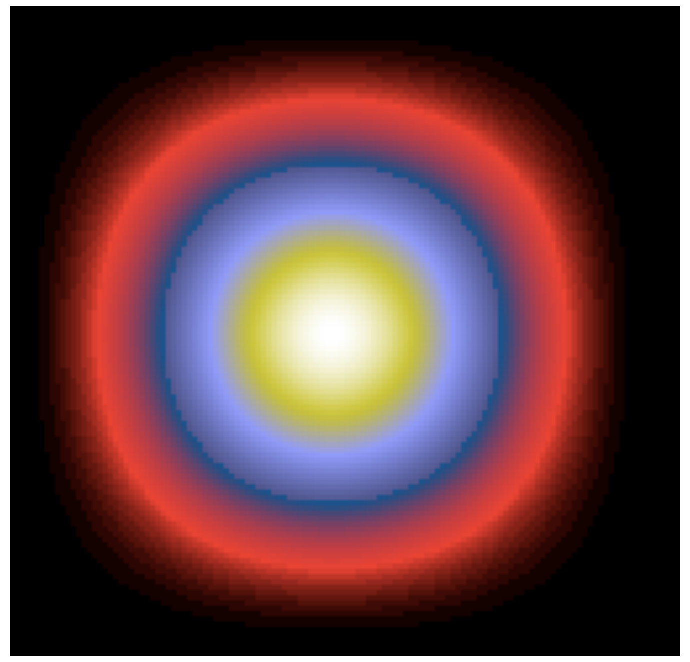


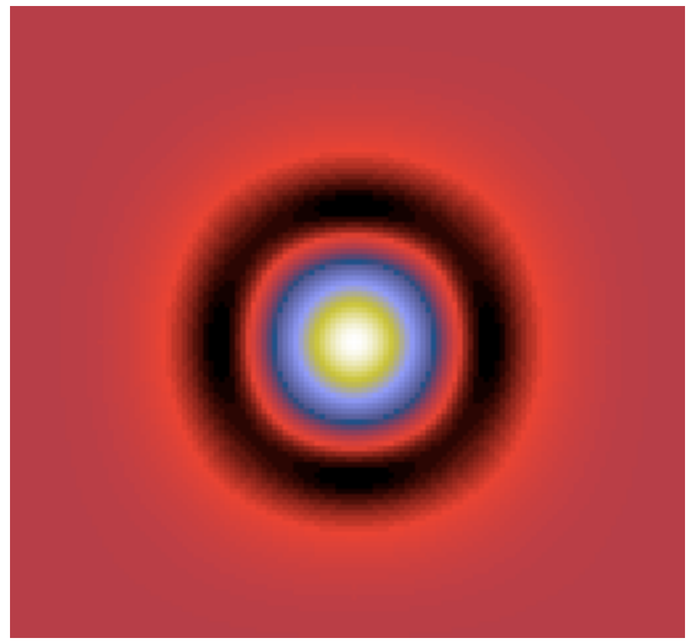
Wavelet Transform in Astronomy

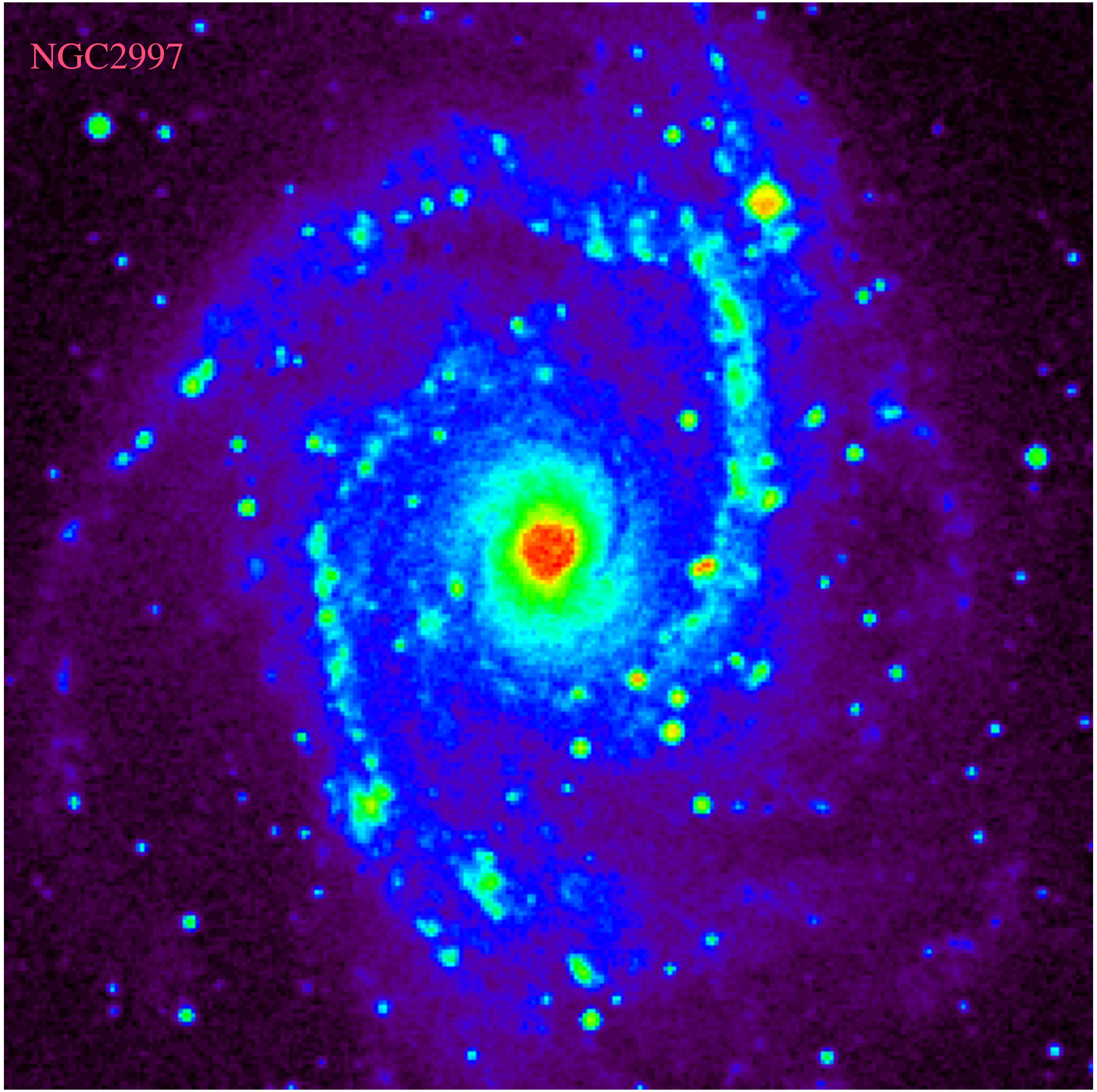
The Isotropic Wavelet and Scaling Functions

$$
\begin{aligned}
B_3(x) &= \frac{1}{12}(\mid x-2\mid^3 - 4\mid x-1\mid^3 + 6\mid x\mid^3 - 4\mid x+1\mid^3 + \mid x+2\mid^3) \\
\psi(x,y) &= B_3(x)B_3(y) \\
\frac{1}{4}\psi(\frac{x}{2},\frac{y}{2}) &= \phi(x,y) - \frac{1}{4}\phi(\frac{x}{2},\frac{y}{2})
\end{aligned}
$$

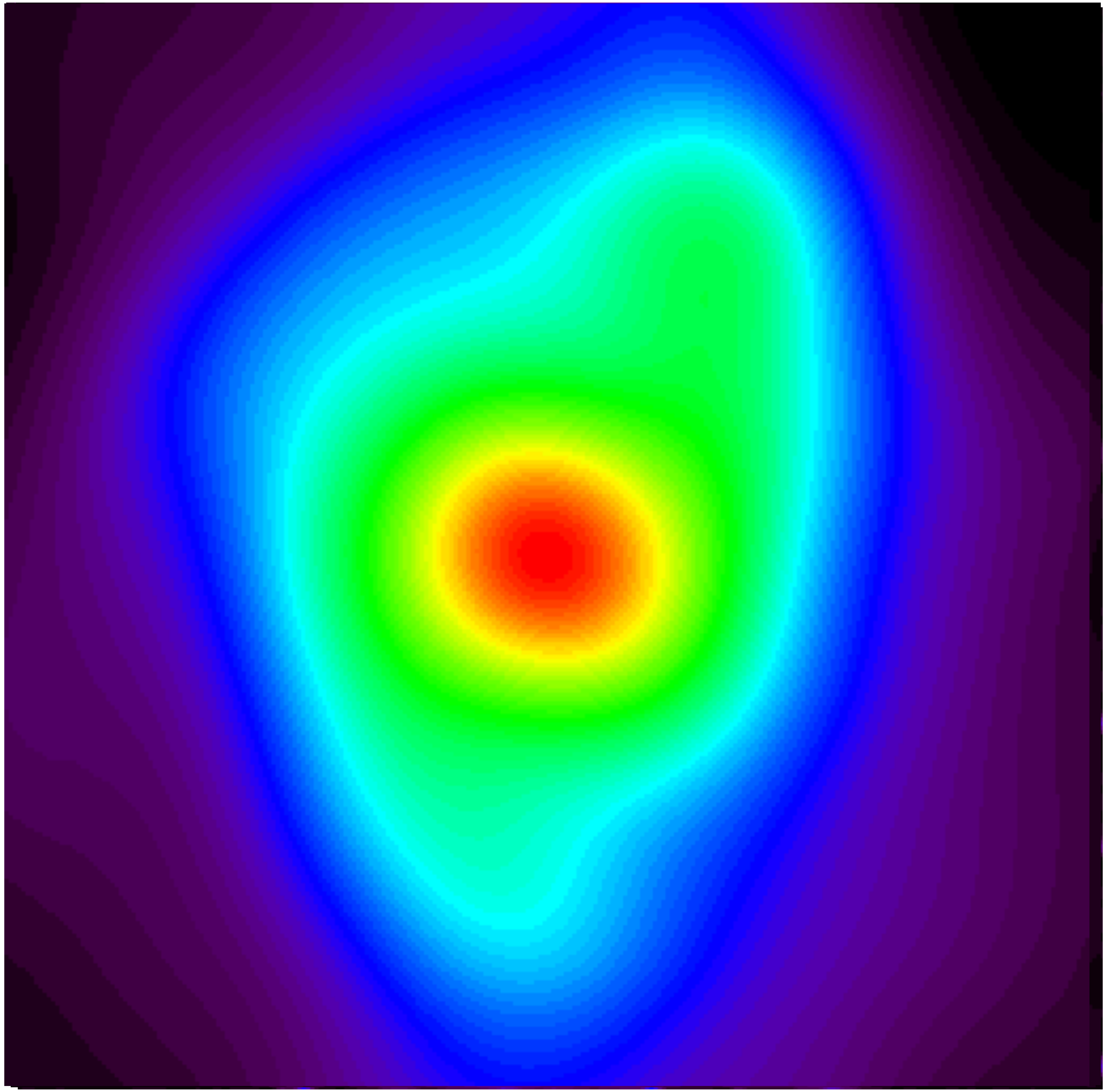







NGC2997

NGC2997



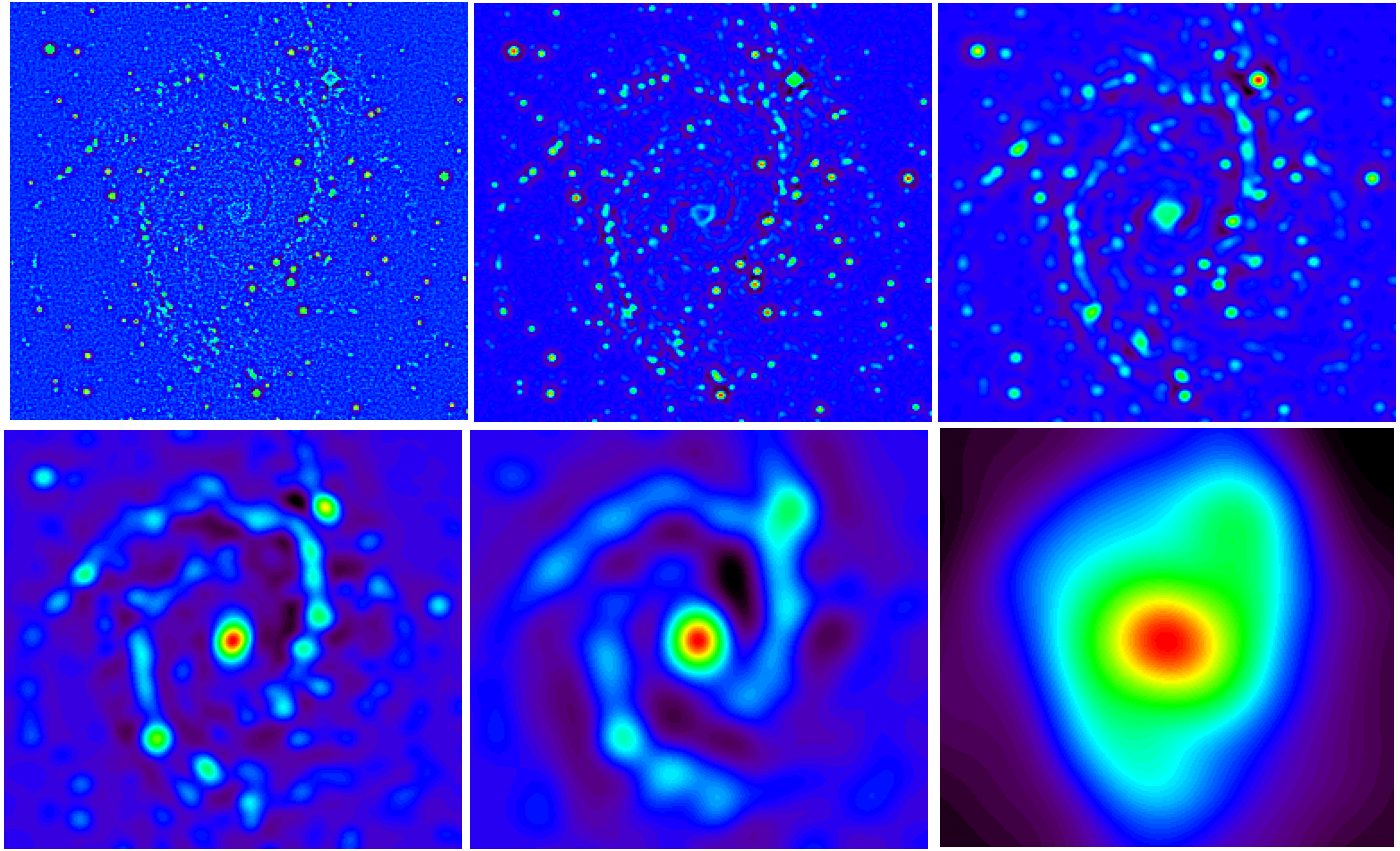
The STARLET Transform

Isotropic Undecimated Wavelet Transform (a trous algorithm)

$$\varphi = B_3 - \text{spline}, \quad \frac{1}{2}\psi\left(\frac{x}{2}\right) = \frac{1}{2}\varphi\left(\frac{x}{2}\right) - \varphi(x)$$

$$I(k,l) = c_{J,k,l} + \sum_{j=1}^{J} w_{j,k,l}$$

$$h = [1,4,6,4,1]/16, \quad g = \delta - h, \quad \tilde{h} = \tilde{g} = \delta$$



Aperture Mass and Wavelets

A. Leonard

$$\begin{cases} M_{ap}(\boldsymbol{\theta}) = \int d^2\boldsymbol{\vartheta}\ \gamma_t(\boldsymbol{\vartheta}) Q(|\boldsymbol{\vartheta}|) \\ Q(\vartheta) \equiv \frac{2}{\vartheta^2} \int_0^{\vartheta} \vartheta' U(\vartheta') d\vartheta' - U(\vartheta) \end{cases}$$

$$\begin{cases} \gamma(\boldsymbol{\theta}) = \frac{1}{\pi} \int d^2\boldsymbol{\theta}' \mathcal{D}(\boldsymbol{\theta} - \boldsymbol{\theta}') \kappa(\boldsymbol{\theta}') \\ \mathcal{W}_j(x,y) = \int_{-\infty}^{+\infty} \kappa(x,y) \psi_j(x,y) dx dy \end{cases}$$

$$M_{ap}(\theta) = (\boldsymbol{\Phi}^t \kappa)_\theta$$

$\Rightarrow$ Wavelets filters are formally **indentical** to Mass aperture

A. Leonard et al, "Fast Calculation of the Weak Lensing Aperture Mass Statistic", **MNRAS**, 423, pp 3405-3412, 2012.

but wavelets presents several advantages:
- compensated and **compact** support filters
- **all scales** processed in one step.
- **reconstruction** is possible
  ==> image restoration for peak counting

Fast calculation for both aperture and wavelet approaches if we grid the shear data and use the FFT.