How to explore the hadronic equation of state?

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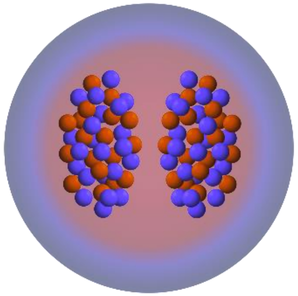


Motivation

The hadronic equation of state (EOS) is not an observable
➔ it can only be explored by a collaboration of theory and experiment

- **Which sources of information we have?**

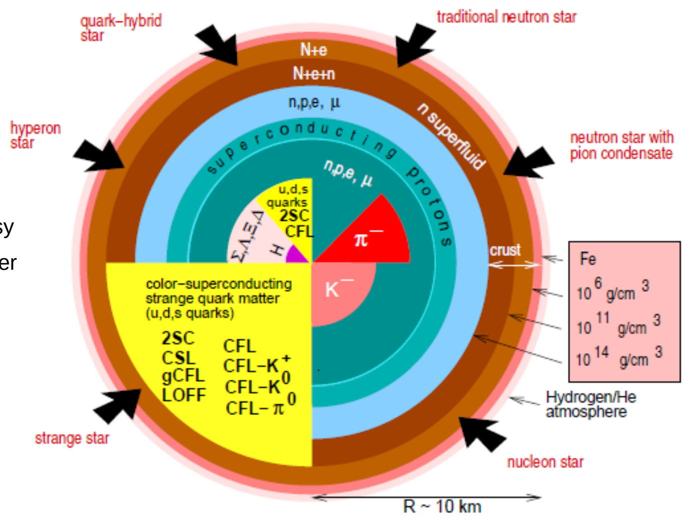
- **Which tools do we have on the theoretical side?**

  Transport approaches (BUU and QMD type)

- **What is our present knowledge of the EOS?**

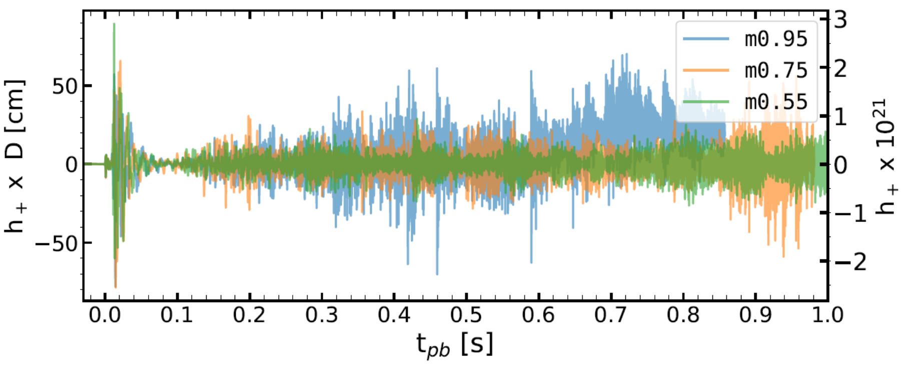
Sources of knowledge of EoS

- **Neutron stars** (very complex)

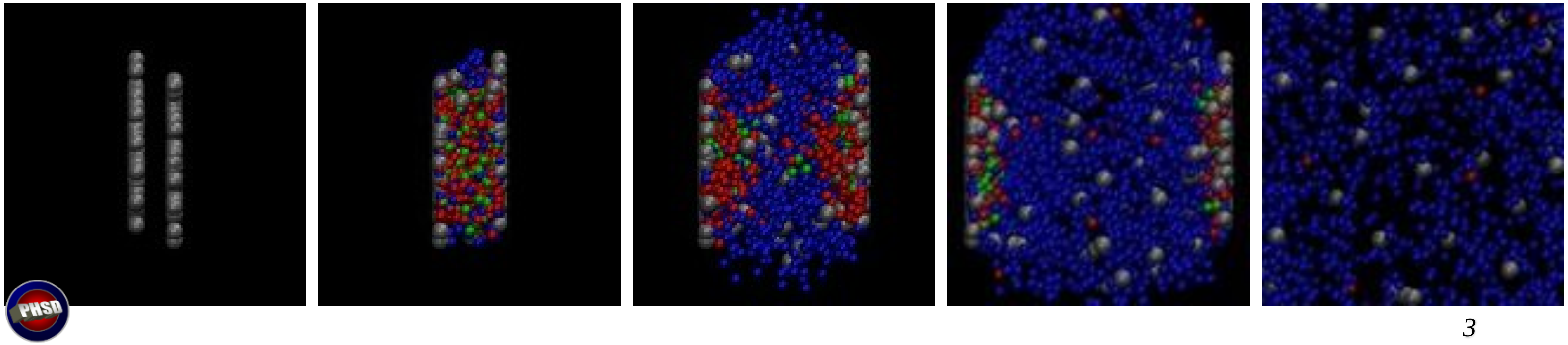


by courtesy of F.Weber

- **Gravitational waves** from neutron star mergers (2106.09734)



- **Heavy-ion collisions** - the only source for which conditions can be tuned (different energy, system size, impact parameter)



however: surface important, signals small

Why Transport Approaches ?

In heavy-ion collisions high densities phase (> ρ_0) existes only for a short time
→ We have to study how signals from the high density reach detector
→ We need a full microscopic description of the expanding system which is only possible by transport approaches!

Hydrodynamics: not applicable at beam energies sensitive to the EOS

- ❑ **How we can formulate the transport approaches*?**

I. Mean-field EoM:
BUU/VUU type approaches
- models: BUU, HSD,GiBUU, AMPT, SMASH, …
Kadanoff-Baym - PHSD

II. QMD type approaches
- models: IQMD, UrQMD, AMD, PHQMD, …

*** In this talk I limit myself to the most simple nonrelativisitc versions of them**

Basis of the BUU/VUU

Starting point for a quantal particle is the Schrödinger equation for a particle in a time dependent potential

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H} |\psi(t)\rangle$$

We can construct a density matrix

$$\rho(\mathbf{r}, \mathbf{s}, t) = \langle \mathbf{r} + \mathbf{s}/2 | \psi(t) \rangle \langle \psi(t) | \mathbf{r} - \mathbf{s}/2 \rangle$$

and the density matrix can be transformed in a Wigner density:

$$W(\mathbf{r}, \mathbf{p}, t) = \int_{-\infty}^{\infty} d\mathbf{s}\ e^{-i\mathbf{s}\mathbf{p}/\hbar} \langle \mathbf{r} + \mathbf{s}/2 | \psi(t) \rangle \langle \psi(t) | \mathbf{r} - \mathbf{s}/2 \rangle$$

It contains the same information but is a function of p and r, hence of the classical phase space variables.

Basis of the BUU/VUU: EoM

Time evolution equation of W:

$$\frac{\partial}{\partial t}\rho = \frac{-i}{\hbar}[H,\rho] \Longrightarrow \frac{\partial}{\partial t}W(\mathbf{r},\mathbf{p},t) + \frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{r}}W(\mathbf{r},\mathbf{p},t) + \underbrace{\sum_{m=0}(-\hbar^2)^m \frac{1}{(2m+1)!}\left(\frac{1}{2}\right)^{2m}\left[\frac{\partial^{2m+1}}{\partial \mathbf{r}^{2m+1}}V(\mathbf{r})\right]\left(\frac{\partial}{\partial \mathbf{p}}\right)^{2m+1}W(\mathbf{r},\mathbf{p},t)}_{\hbar\to 0 \;\Longrightarrow\; \frac{\partial V(\mathbf{r})}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}}W(\mathbf{r},\mathbf{p},t)}$$

turns out to be in the semiclassical limit ($\hbar \to 0$) a Vlasov eq.

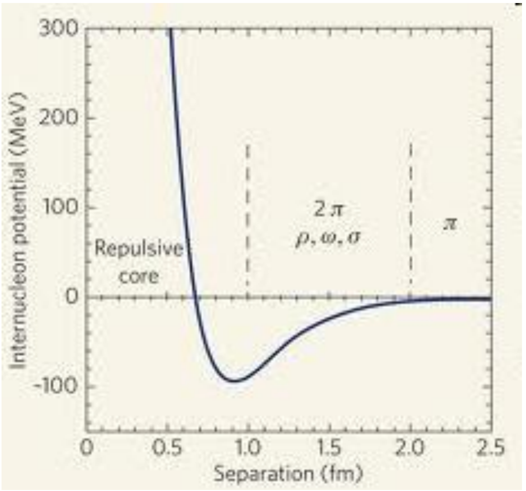
$$\frac{\partial}{\partial t}W(\mathbf{r},\mathbf{p},t) + \frac{\mathbf{p}}{m}\frac{\partial}{\partial \mathbf{r}}W(\mathbf{r},\mathbf{p},t) + \frac{\partial V(\mathbf{r})}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}}W(\mathbf{r},\mathbf{p},t) = 0$$

So why do we not solve a Vlasov equation which is at least correct if the gradient of the potential is small and the momentum distribution of the nucleons not too sharp,
means $\hbar \cdot \frac{\partial V(\mathbf{r})}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}}W(\mathbf{r},\mathbf{p},t) << 1$ **?**

Basis of the BUU/VUU: hard core

Because this does not work as proven in the 70' (in QMD as in BUU)

The Hamiltonian (in the Schrödinger equation)
contains V = NN potential



The NN potential has a hard core :

- makes TDHF calculations impossible
- makes also Vlasov transport calculations impossible (Bodmer 75)

Note: hard core → hard scattering
has in reality not been observed in low energy collisions

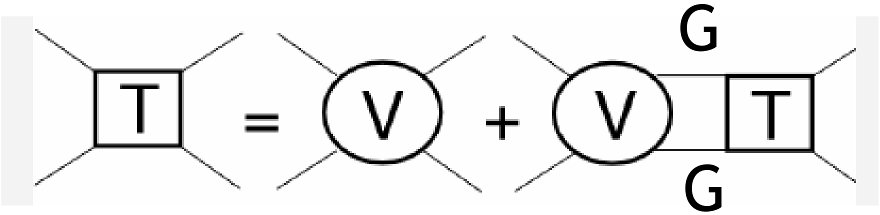
In a nuclear environment we have an effective potential which can be determined by many body techniques.
This potential enters the time dependent Schrödinger equation.

Colloquially we say: « Collisions are Pauli blocked », in reality in a nucleus complicated many body correlations are created.
Difficult to handle in transport approaches, several approximations possible

V_{NN}: Brueckner theory

Solution (taken over from TDHF):

In a many body system the NN potential V_{NN} has to be replaced by solution of the T-matrix approach (Brueckner)



$$T_\alpha(E;q,q') = V_\alpha(q,q') + \int k^2 dk\; V_\alpha(q,k)\; G^0_{Q\overline{Q}}(E,k)\; T_\alpha(E;k,q')$$

Consequences:

V_{NN} is real → T is complex = ReT + i Im T

ReT: Replaces V_{NN} in Hamiltonian Is smooth (Skyrme)

Im T: σ_{elast} collisions done identically

BUU (testp.) and QMD (part)

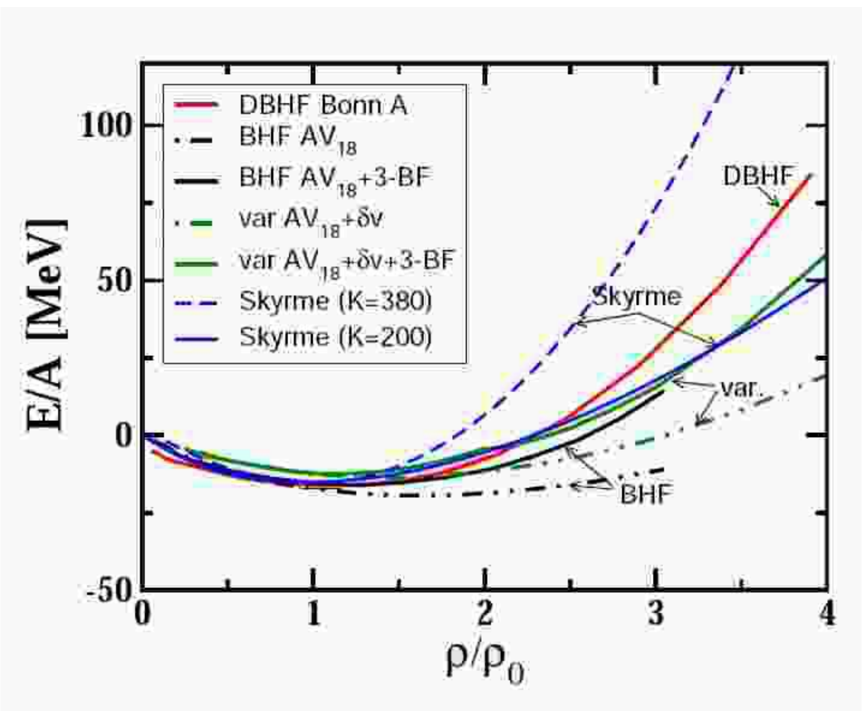
To this one adds inelastic collisions (BUU and QMD same way) !

Basis of the BUU/VUU

Problem 1:

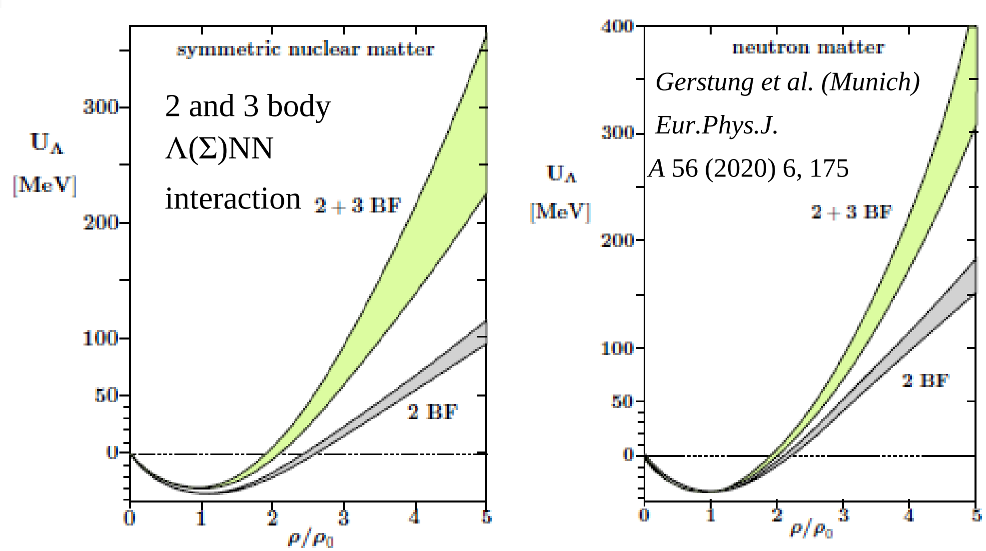
Brueckner – Matrix is an expansion of the of the many body amplitudes in $\frac{a p_F}{\hbar}$; a=range of NN pot, p_F =Fermi momentum

Above $\frac{a p_F}{\hbar}$ =1 other diagrams (like 3-body and NNbar interactions) important



Different flavors of Brückner and Skyrme parametrization

3-body ΛNN potential in symmetric and asymmetric matter



Basis of the QMD

Roots in Quantum Mechanics:

Remember QM cours when you faced the problem

- **we have a Hamiltonian**

$$\hat{H} = -\frac{\hbar^2\nabla^2}{2m} + V$$

- **the Schrödinger eq.**

$$\hat{H}|\psi_j >= E_j|\psi_j >$$

has no analytical solution

- **we look for the ground state energy**

Walther Ritz

Ritz variational principle:

Assume a trial function $\psi(q,\alpha)$ **which contains one adjustable parameter α, which is varied to find the lowest energy expectation value:**

$$\frac{d}{d\alpha} < \psi|\hat{H}|\psi >= 0 \to \alpha_{min}$$

determines α for which $\psi(q,\alpha)$ **is closest to the true ground state wf and** $< \psi(\alpha_{min})|\hat{H}|\psi(\alpha_{min}) >= E_0(\alpha_{min})$ **closest to true ground state E**

QMD time evolution

Dirac-Frenkel-McLachlan approach
A. Raab, Chem. Phys. Lett. 319, 674
J. Broeckhove et al., Chem. Phys. Lett. 149, 547

- **Generalized Ritz variational principle**: $$\delta \int_{t_1}^{t_2} dt < \psi(t)| i\frac{d}{dt} - H |\psi(t) >= 0.$$

Many-body wave function:

Assume that $\psi(t) = \prod_{i=1}^{N} \psi(\mathbf{r}_i, \mathbf{r}_{i0}, \mathbf{p}_{i0}, t)$ **for N particles (neglecting antisymmetrization !)**

Ansatz: trial wave function for one particle "*i*" :
Gaussian with width *L* centered at r_{i0}, p_{i0}

[Aichelin, Phys. Rept. 202 (1991)]

$$\psi(\mathbf{r}_i, \mathbf{r}_{i0}, \mathbf{p}_{i0}, t) = C\ \mathrm{e}^{-\frac{1}{4L}\left(\mathbf{r}_i - \mathbf{r}_{i0}(t) - \frac{\mathbf{p}_{i0}(t)}{m}t\right)^2} \cdot \mathrm{e}^{i\mathbf{p}_{i0}(t)(\mathbf{r}_i - \mathbf{r}_{i0}(t))} \cdot \mathrm{e}^{-i\frac{\mathbf{p}_{i0}^2(t)}{2m}t}$$

L=4.33 fm^2

- **Equations-of-motion (EoM) in coordinate and momentum space:**

$$\dot{r}_{i0} = \frac{\partial \langle H \rangle}{\partial p_{i0}} \qquad \dot{p}_{i0} = -\frac{\partial \langle H \rangle}{\partial r_{i0}}$$

Many-body Hamiltonian: $$H = \sum_i H_i = \sum_i (T_i + V_i) = \sum_i (T_i + \sum_{j \neq i} V_{i,j})$$

2-body potential: $Re\ T_{i,j} = V_{i,j} = V(\mathbf{r}_i, \mathbf{r}_j, (\mathbf{p}_i, \mathbf{p}_j), t)$

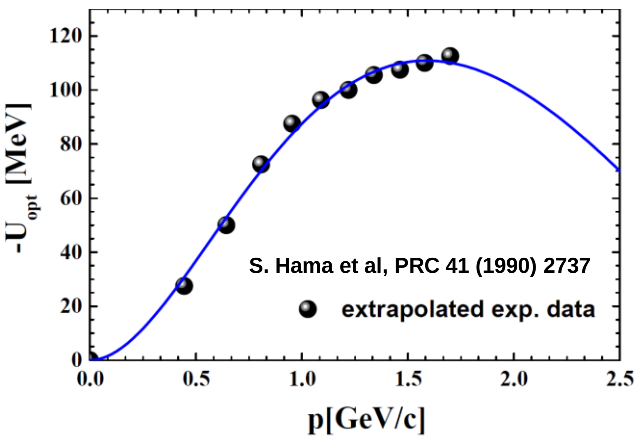
Antisymmetrization is neglected since impossible to formulate collision term consistently

Momentum dependent potential → EoS in PHQMD

2) Momentum dependent potential :

$$V(\mathbf{r}_1,\mathbf{r}_2,\mathbf{p}_{01},\mathbf{p}_{02}) = (a\Delta p + b\Delta p^2)\ exp[-c\sqrt{\Delta p}]\ \delta(\mathbf{r}_1-\mathbf{r}_2)$$

$$\Delta p = \sqrt{(\mathbf{p}_{01}-\mathbf{p}_{02})^2}$$



Parameters a, b, c are fitted to the "optical" potential (Schrödinger equivalent potential U_{SEP}) extracted from elastic scattering data in pA:

$$U_{SEQ}(p) = \frac{\int^{p_F} V(\mathbf{p}-\mathbf{p}_1)dp_1^3}{\frac{4}{3}\pi p_F^3}$$

❖ **In infinite matter a potential corresponds to the EoS:**

$$E/A(\rho) = \frac{3}{5}E_F + V_{Skyrme\ stat}(\rho) + V_{mom}(\rho)$$

$$V_{mom} = (a\Delta p + b\Delta p^2)\ exp(-c\sqrt{\Delta p})\ \frac{\rho}{\rho_0}$$

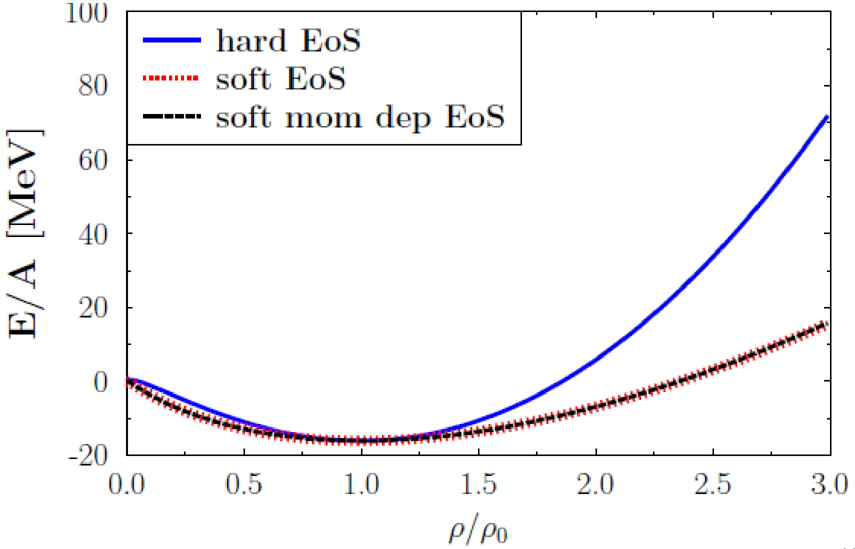
$$V_{Skyrme} = \alpha\frac{\rho}{\rho_0} + \beta\frac{\rho}{\rho_0}^{\gamma}$$

compression modulus K of nuclear matter:

$$K = -V\frac{dP}{dV} = 9\rho^2\frac{\partial^2(E/A(\rho))}{(\partial\rho)^2}|_{\rho=\rho_0}.$$

E.o.S.	$\alpha[MeV]$	$\beta[MeV]$	γ	K [MeV]
S	-383.5	329.5	1.15	200
H	-125.3	71.0	2.0	380
SM	-478.87	413.76	1.10	200
	a $[MeV^{-1}]$	b$[MeV^{-2}]$	c$[MeV^{-1}]$	
	236.326	-20.73	0.901	

EoS for infinite cold nuclear matter at rest



Momentum dependent interactions act differently in BUU and QMD

Green test particle (p_z) enters a cell with the same number of projectile (red, p_z) and target (blue,-p_z) nucleons

BUU: calculate average momentum of test particles in the cell:

$< p_z> =4^* p_z +4^*-p_z =0$

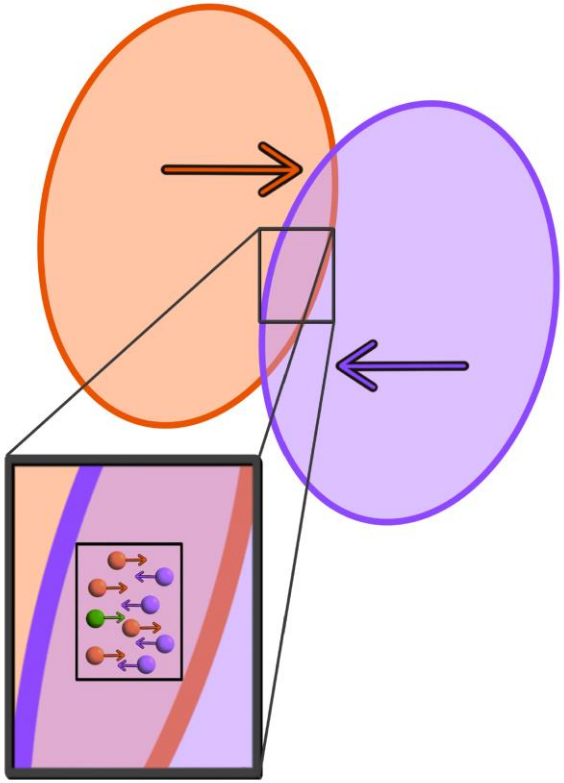
$V_{mom} = V(p_z -< p_z >) = V(p_z)$

QMD: $V_{mom} = 1/8[4V(2p_z) +4V(0)]$

$=[V(2p_z) +V(0)]/2$

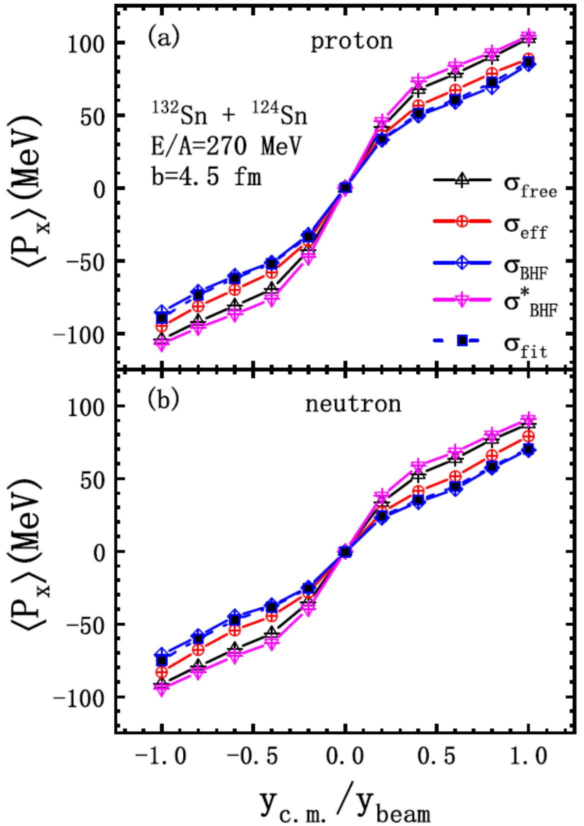
➔ Only if $[V(2p_z) +V(0)]/2 = V(p_z)$ we expect the same results for BUU and QMD

(not systematically studied yet, but has to be done if one wants to compare BUU and QMD type results)

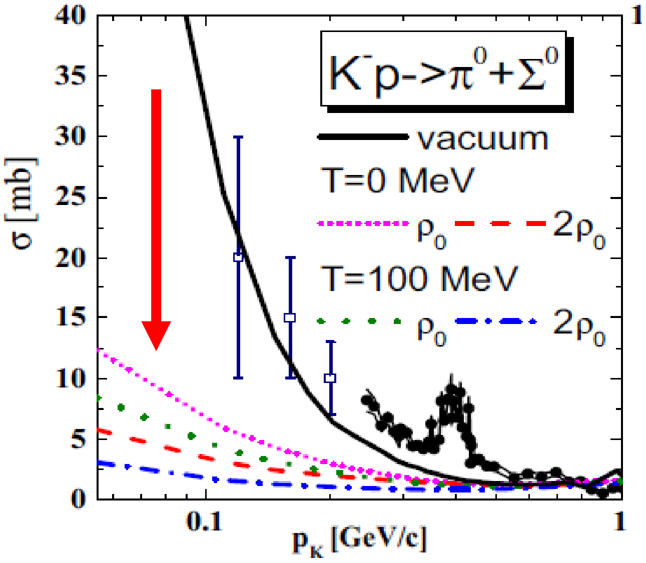


Collision integral (same in QMD and BUU)

- ❑ Usually free cross sections or theoretical results are used in the codes
- ❑ In medium cross sections have been calculated: Bruckner, G-matrix
- ➢ For elastic collisions: studied in NPA536(1992)201 → change the QMD results only little; more modern cross sections calc. not used yet at SIS energies



- ➢ In the strangeness sector the in medium modifications (G-matrix) are dramatic



PHSD: Song, PRC103,044901

BUU/LV/VUU — Summary — QMD/IQMD/AMD

BUU/LV/VUU	QMD/IQMD/AMD
$W_1(\mathrm{r},\mathrm{p},t)$	$W_N(\mathrm{r}_1,\mathrm{p}_1,...,\mathrm{r}_N,\mathrm{p}_N,t)$
Can predict correlations only if $W_2 = W_1 \cdot W_1$	**Can predict all correlations** $W_2 = \int \prod_{i\geq 3} d^3r_i d^3p_i W_N$
• **correct if the system is in global equilibrium**	**all HBT correlations**
deuteron density = neutron dens * proton dens (what is rarely the case) avoided in the coalescence model	**deuterons (2 body correl.) can be identified**

Parameters: grid size — **width L of wf**

determine the range of the Re T

We expect that

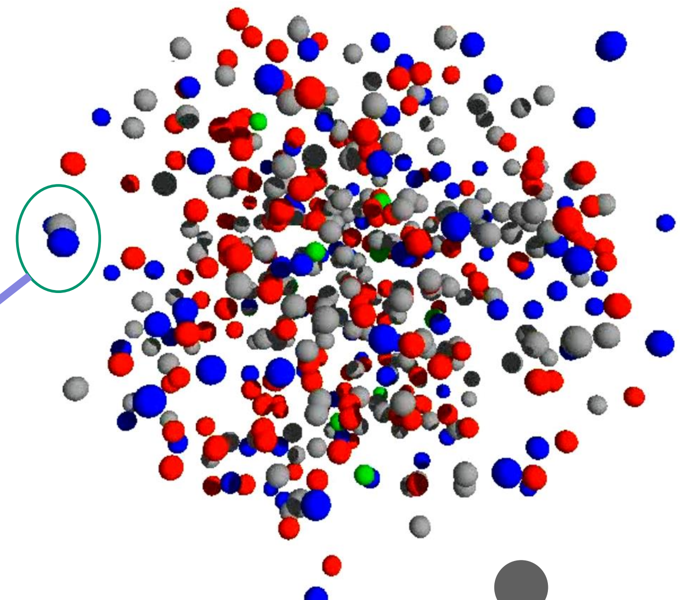
- **1 body observables like (p,n),Λ,K,π spectra are very similar but not identical due to the different realization of the forces**
  They have to be sufficiently similar to extract the EoS (we will see)
- **N body observables differ: d, t, He, HBT**

Cluster formation: QMD vs MF

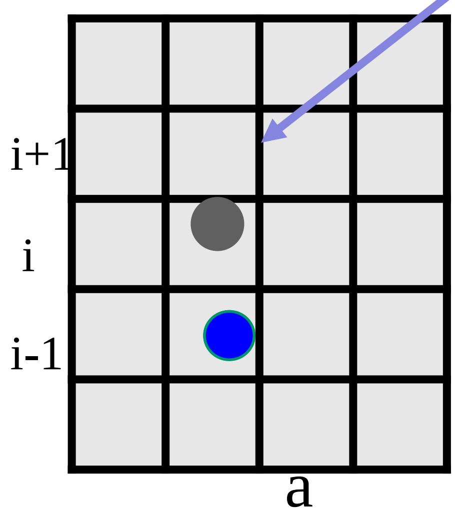
BUU:
Solves the one particle f(r,p,t) by test particle method.

→ 1 nucleon represented by N_T → ∞ point like test particles

Potential is a fct of the density; $U(\rho)$



BUU

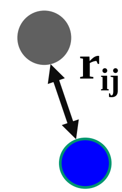


Circles are test particles

Calculate density ρ in each cell
Calculate force $F_i = (U(\rho_{i+1}) - U(\rho_{i-1}))/2a$

Problem: circles are test particles and present a $1/N_T$ fraction of a nucleon therefore density is $1/N_T$ lower therefore force is 1/N weaker and cannot bind nucleons (form clusters)

QMD



circles are nucleons

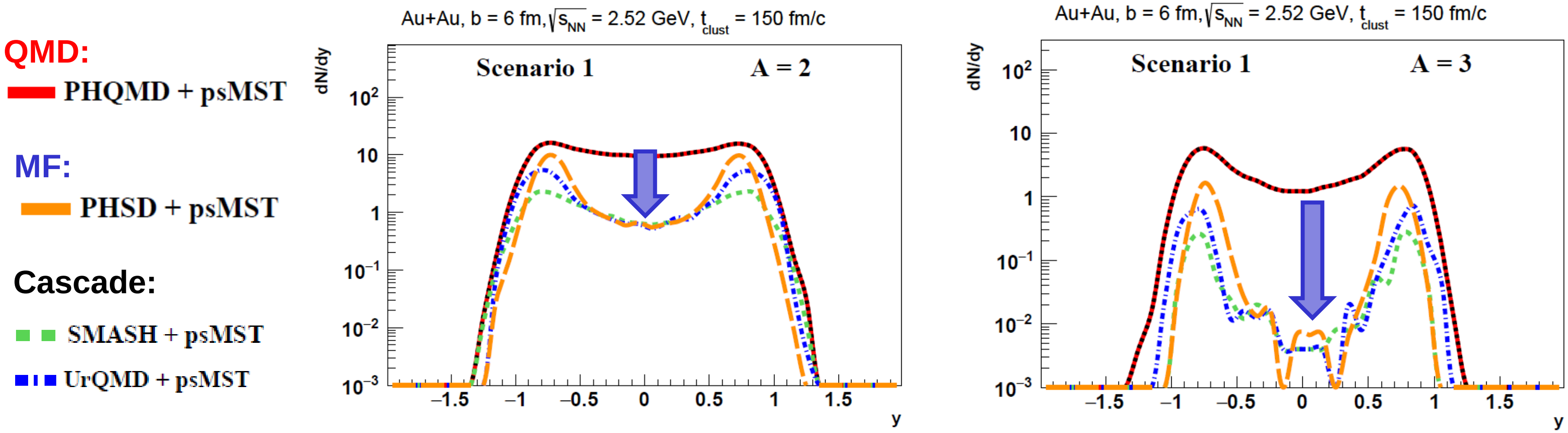
$F_i = \nabla V(r_{ij})$ is the full force between nucleons

→ Nucleons can form clusters

➔ **Mean field calculations not suited for cluster production if the NN potential plays a role in the cluster formation**

Cluster formation by potential: QMD vs MF

Example: **Cluster stability over time:**



V. Kireyeu, Phys.Rev.C 103 (2021) 5

- Cluster formation (by potential) is sensitive to nucleon dynamics

➔ One needs to keep the nucleon correlations (initial and final) by realistic nucleon-nucleon interactions in transport models

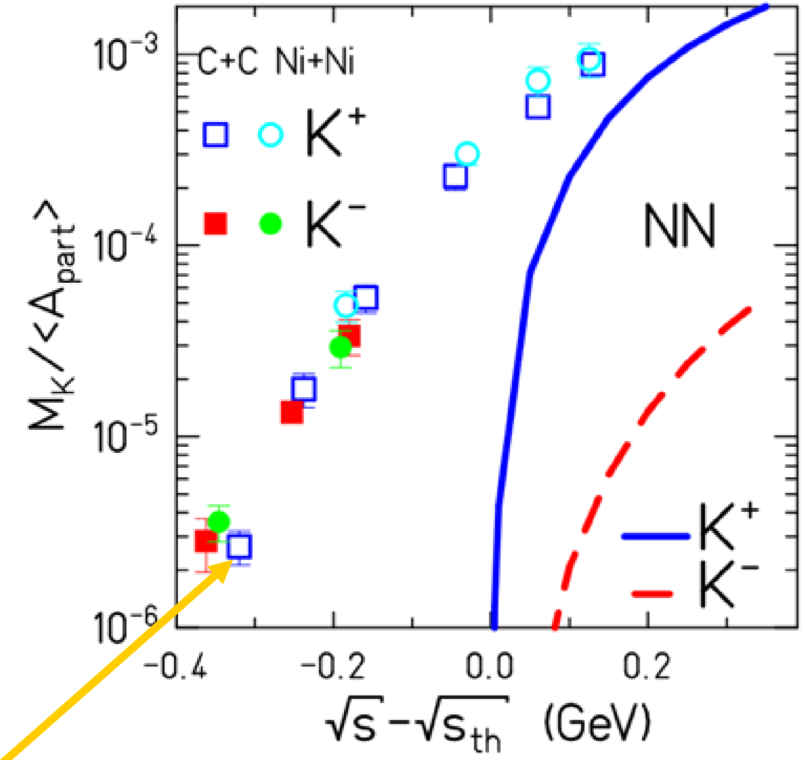
- QMD (quantum-molecular dynamics) – allows to keep correlations
- MF (mean-field based models) – no nontrivial correlations
- Cascade – no correlations by potential interactions (n-body)

➔ n-body QMD dynamics needed for the description of cluster production!

The two (presently) most important sources of our knowledge about the nuclear EoS:

- **K^+ yield in central HI reaction**
- **In plane and elliptic flow of protons in semi-peripheral HI collisions**

Strangeness and the EOS



- **AA collisions:** **experimental observation of K^+,K^- production below the NN-threshold**

- **NN: Excitation function of K^+ and K^- quite different**
- **AA: Excitation function of K^+ and K^- quite similar**
- **Fermi motion cannot explain very subthreshold production**

- **Conclusion:**
  AA: new mechanisms for strangeness production

(C. Hartnack et al., Phys. Rept. 510,119)

Near threshold strangeness production in AA

I. Strangeness production channels at low energies

- **baryon-baryon collisions:**

$$B + B \rightarrow B + Y + K$$

$$B + B \rightarrow B + B + K + \overline{K}$$

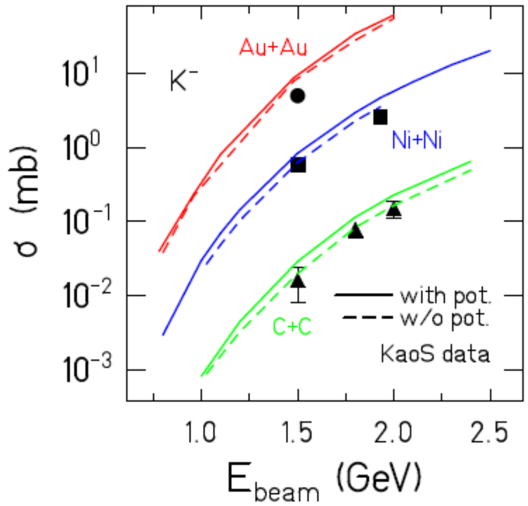
$$B + Y \rightarrow B + B + \overline{K}$$

$K = (K, K^0)$

$\overline{K} = (K^-, \overline{K}^0)$

$B = (N, \Delta, ...)$

$Y = (\Lambda, \Sigma)$



meson-baryon collisions:

$$\pi + B \rightarrow Y + K$$

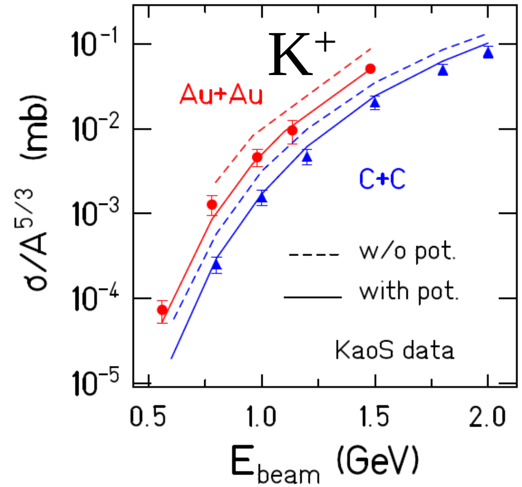
$$\pi + B \rightarrow B + K + \overline{K}$$

$$\pi + Y \rightarrow B + \overline{K}$$

→ dominant channel for low energy K^- production!

- **meson-meson collisions:** $\pi + \pi \rightarrow K + \overline{K}$

- **resonance decays:** $K^* \rightarrow \pi + K, ...,\quad \phi \rightarrow K + \overline{K}$



II. Strangeness rescattering

= (quasi-)elastic scattering with baryons and mesons

III. K^+ (K^-)-Nucleus potential V(ρ)

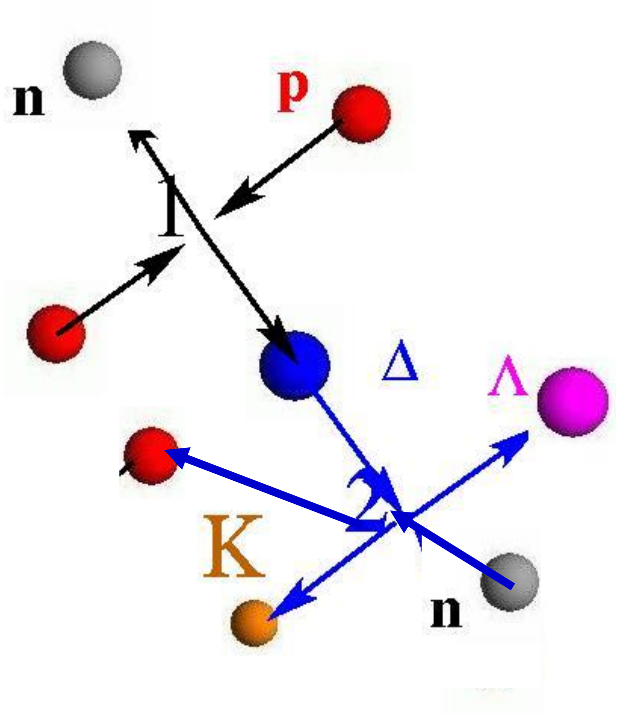
Origin of difference of pp and AA excitation functions

Dominant for K^+ in AA: Two step process $NN \rightarrow N\Delta$ $N\Delta \rightarrow K^+\Lambda N$

lowers the effective threshold

enhances K^+ below NN threshold

two step process more probable in central collision



Theory and simulations:

soft EoS: system gets to higher densities

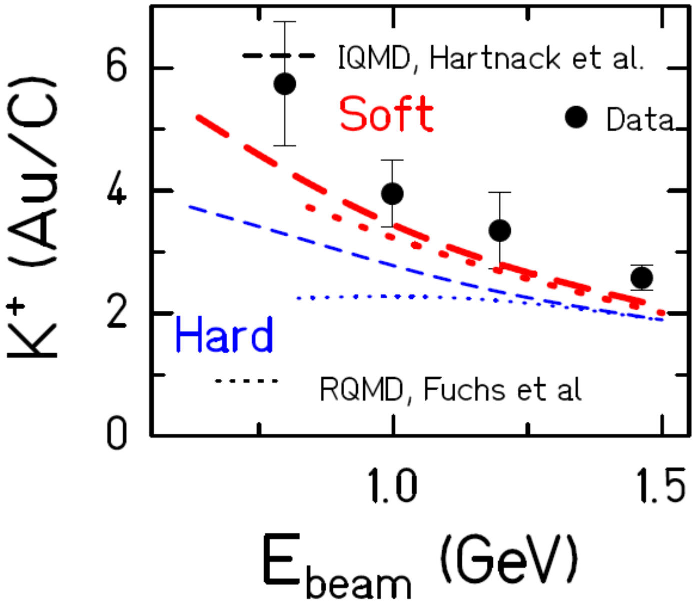
→ shorter mean free path for $N\Delta \rightarrow K^+\Lambda N$

$N\Delta \rightarrow K^+\Lambda N$ competes with Δ decay

→ for a soft EOS we expect more $N\Delta \rightarrow K^+\Lambda N$ collisions and hence more K^+

Strangeness production and the nuclear EoS

(Phys. Rept. 510,119)



Comparison with experiment

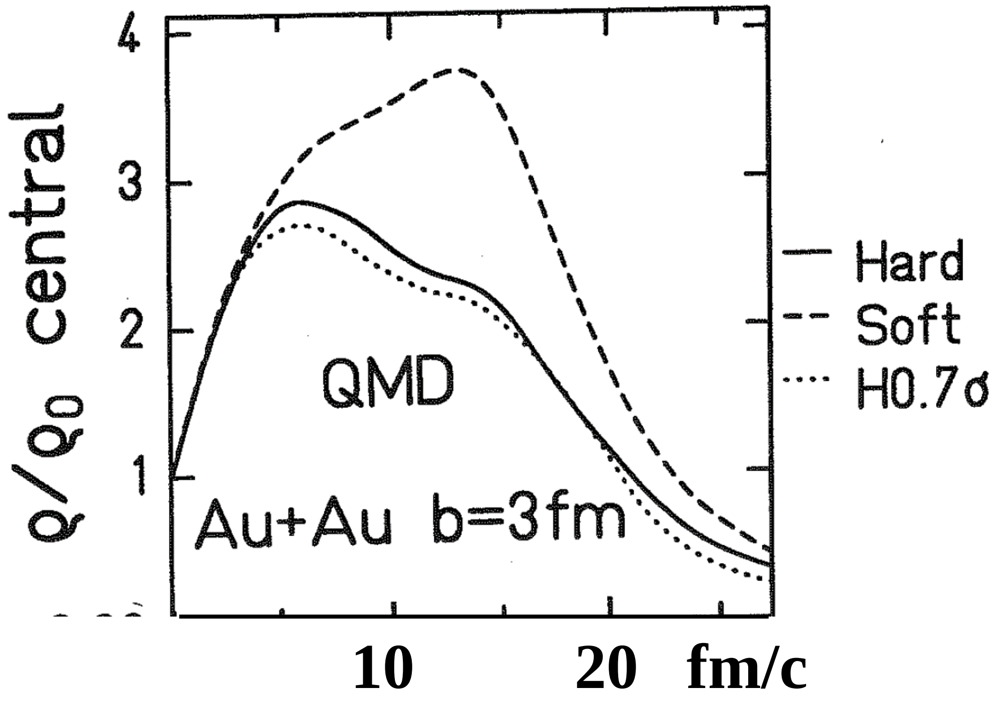
- **confirmes the EoS dependence of K^+ yield**
- **soft EoS: best agreement with data**

Up to today the ratio K^+(Au/C) is an observable which shows the strongest EoS dependence

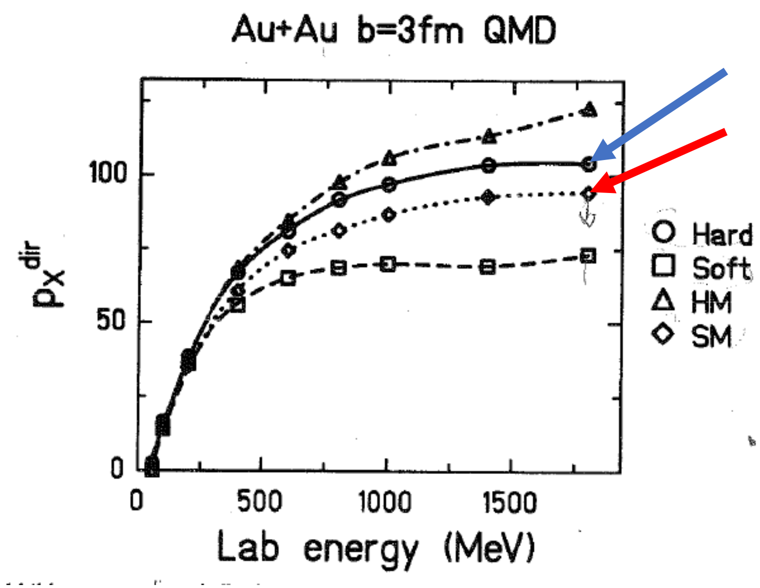
- **Perspectives: FAIR and NICA (Russia) have higher beam energies**
  excitation functions of Ξ and Ω become available
  sensitive probes for studying the reaction mechanism and EoS

Flow and the EOS

EOS dependence of flow known since 30 years:
the problem is to quantify this due to the complicated reaction scenario



Hartnack PhD thesis (1987)
PRL 58 (1987) 1926



Larger density→ larger density gradient → larger force→ larger flow

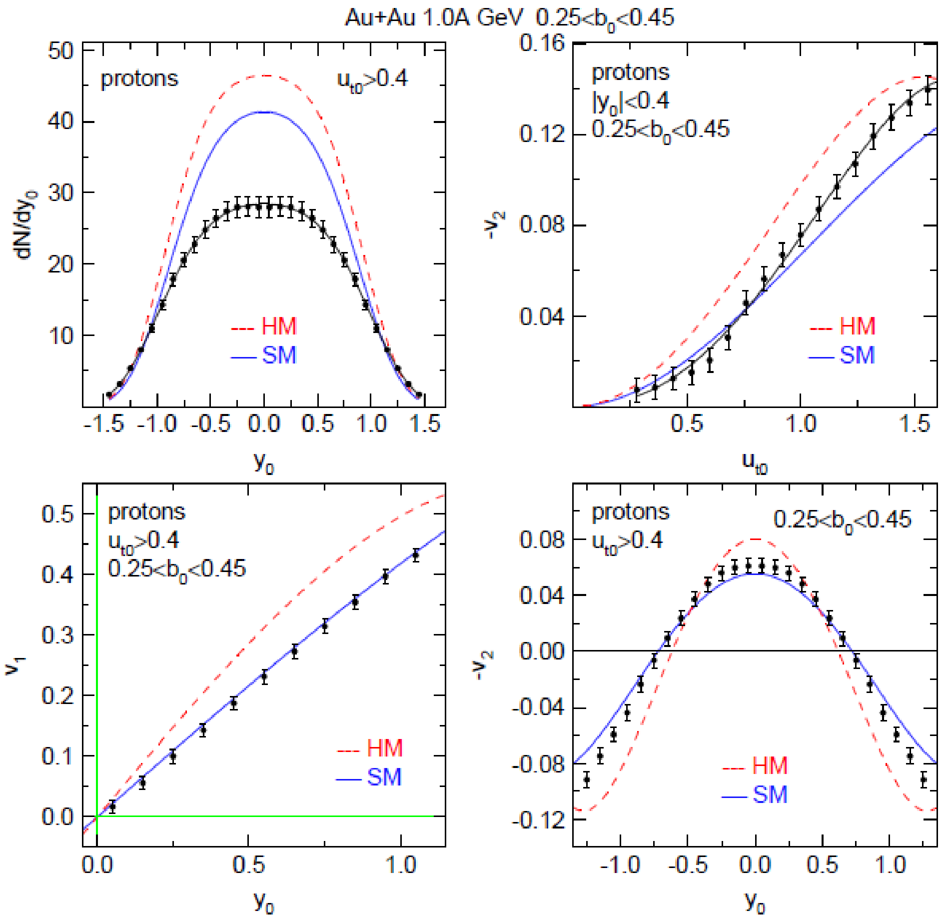
$$\frac{dN}{d\phi} \propto 1 + 2v_1 cos\phi + 2v_2 cos(2\phi).....$$

$$p_x^{dir} = \Sigma \, sign(y^i) \, p_x^i$$

Flow and the EOS

Comparison of IQMD with FOPI experiment

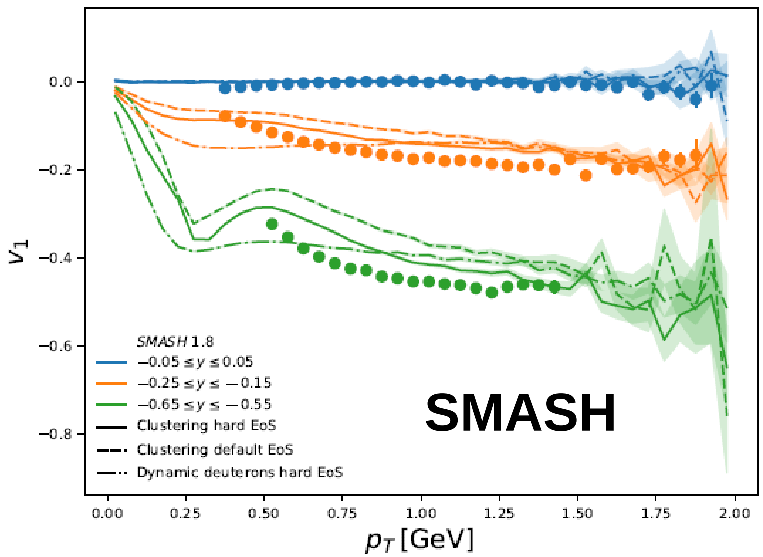
Reisdorf et al. NPA 876 (2012) 1



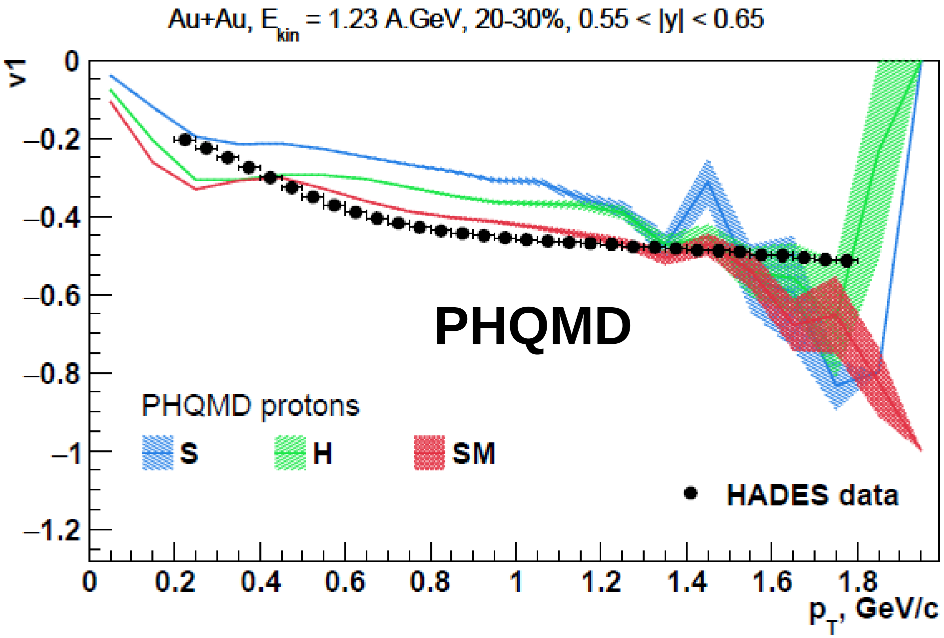
- **Comparison with data at one single beam energy and at one centrality bin gives reasonable agreement**
- **What lacks is a systematics with many reaction systems at different energies all impact parameters for different cluster d,t…**
- **New data HADES and STAR will allow for progress**

Flow and the EOS

Comparison of the state of the art transport models with the most precise data



Mohs, PRC 105 (2022) 3



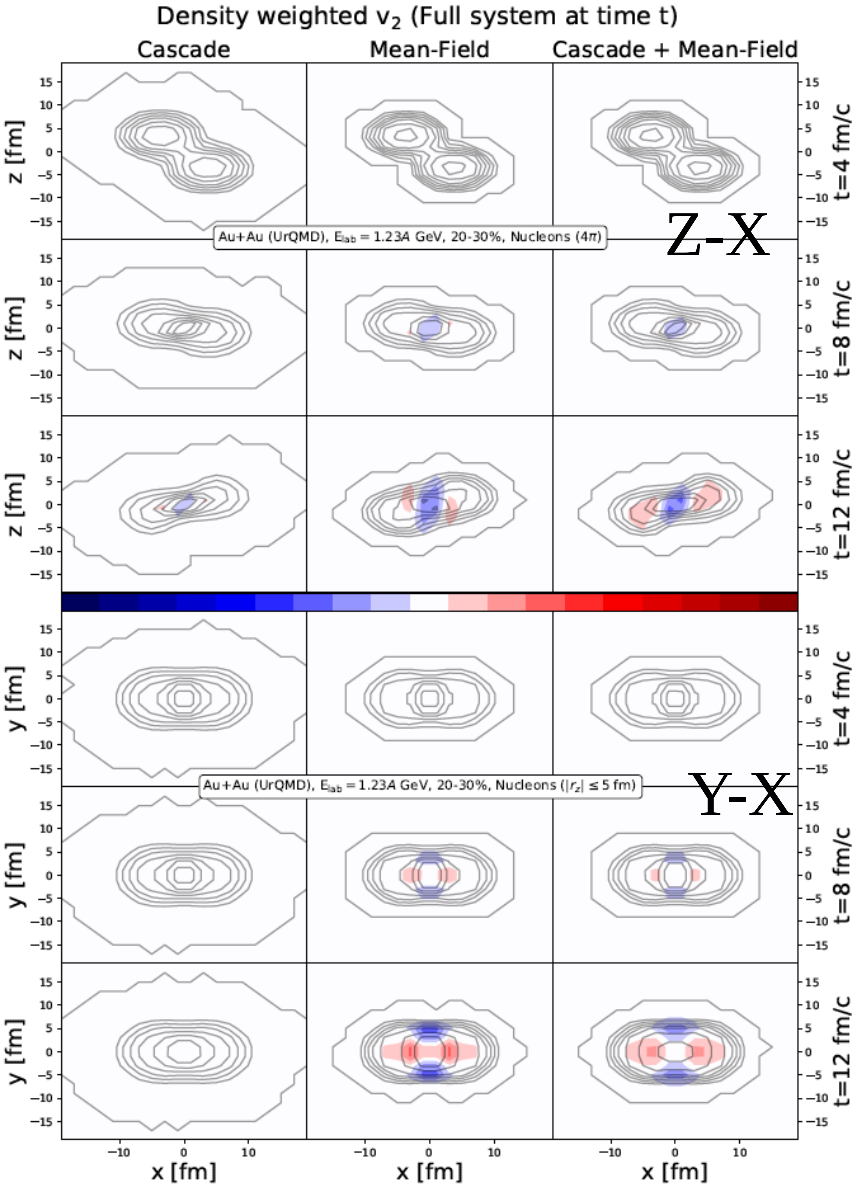
Kireyeu (to be published)

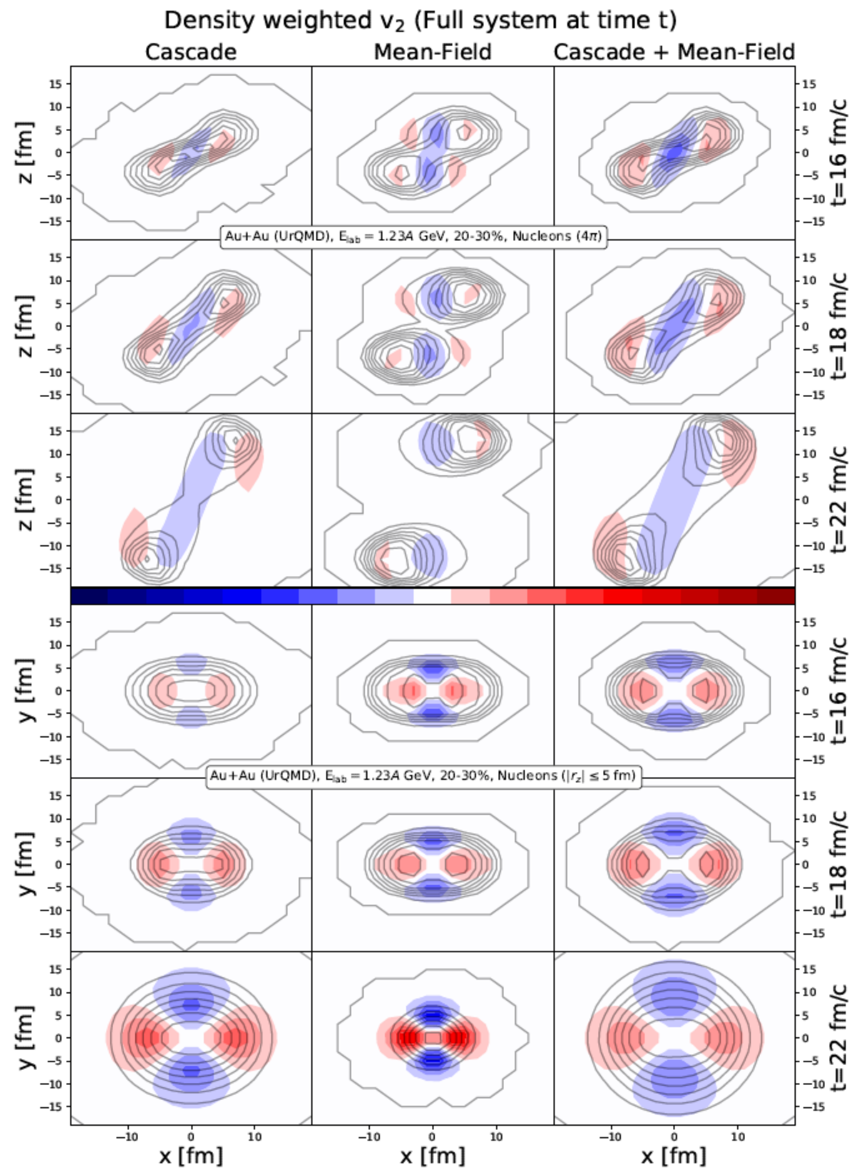
- **Differences between the models are tiny but as large as the difference between different EOS**
- **One has first to understand where the differences come from before flow can be used to nail down the EOS**
- **In addition systemtic experimental studies (A,E,y,p_T) needed**

First steps i this direction have been taken (Trento workshop, Wolter initiative)

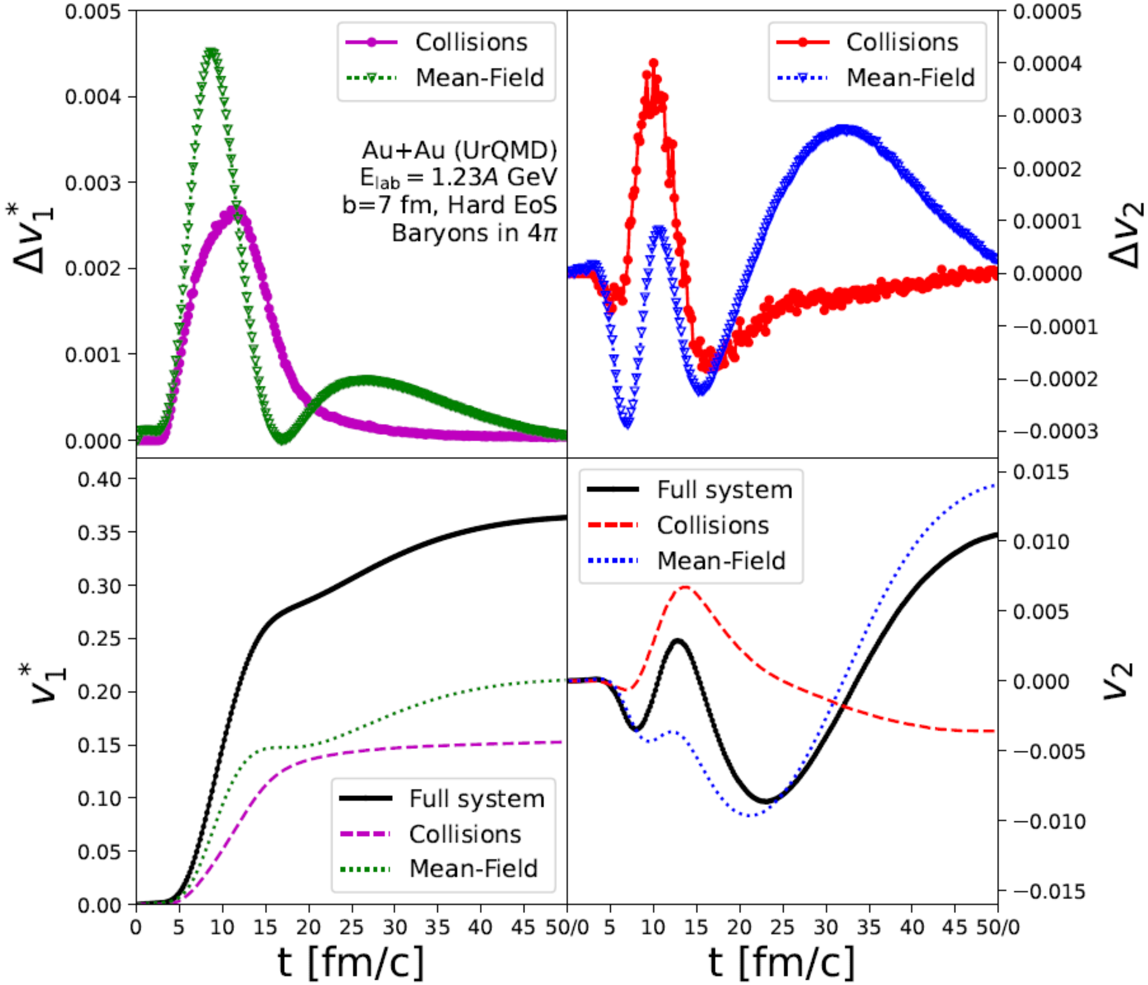
Time evolution of v_2

T. Reichert to be published





v_1 , v_2 : complex interplay between collisions and potential



A lot remains still to be done before this complexity is understood

Summary

- Transport approaches have done a great job to understand many experimental results and are the basis of the understanding of HI collisions:

- Now we attack the most complex problem, the determination of the EoS where the signal is of the same order as the noise
  (difference between different approaches)
  For this we
  - need to understand the differences in detail
  - need to update (in medium σ, potentials, initial cond., mom. dependence)
  (→Wolter, Colonna)

- If one wants to use clusters we have to study theoretically how clusters can be produced in BUU

- Despite of all these shortcomings there is a tentative agreement that a soft momentum dependent interaction describes data best

- Last but not least: It is certainly worthwhile to think how (the conceptually 40 year old) transport approaches can be theoretically and technically further developed