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PREX and CREX： Evidence for Strong Isovector Spin-orbit Interaction

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Lead (^{208}Pb) Radius Experiment: PREX

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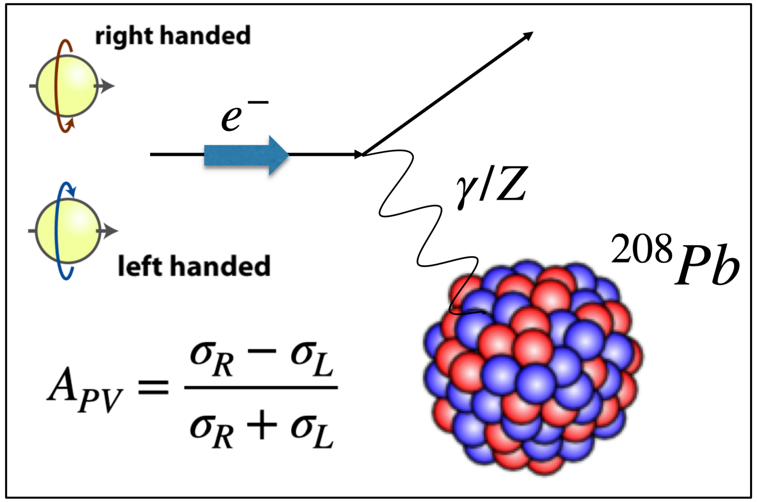
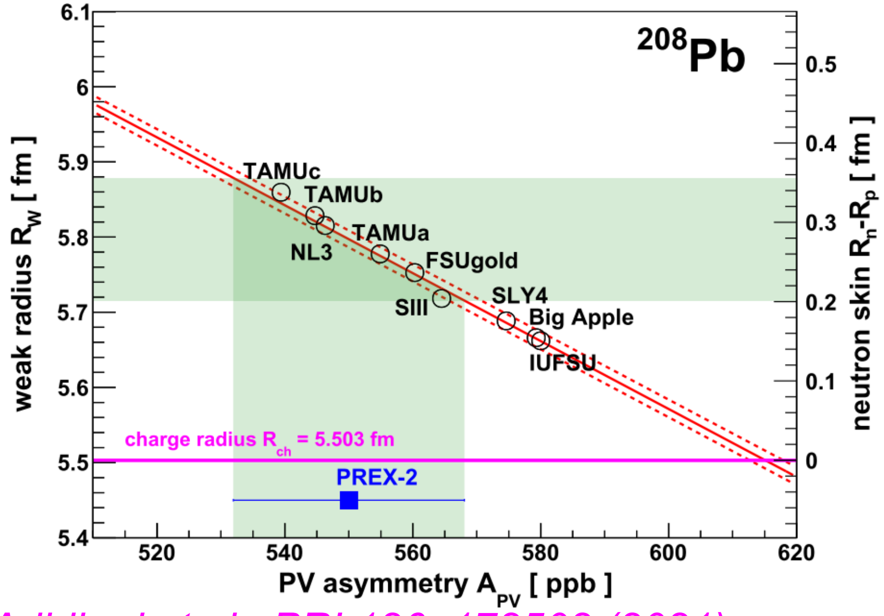
$Q_p^{(W)} = 0.0713$

$Q_n^{(W)} = -0.9888$





Adhikari et al., PRL126, 172502 (2021)

- Parity-violating asymmetry in longitudinally polarized elastic electron scattering :

$$A_{PV} = \frac{\sigma_R - \sigma_L}{\sigma_R + \sigma_L} \approx \frac{G_F Q^2 |Q_W|}{4\sqrt{2}\pi\alpha Z} \frac{F_W(Q^2)}{F_{ch}(Q^2)}$$

Donnelly et al., NPA503, 589 (1989);
Horowitz et al., PRC63, 025501 (2001).

- Free from most strong interaction uncertainties.

- PREX-2 results ($\langle Q^2 \rangle = 0.00616\ \mathrm{GeV}^2$) :

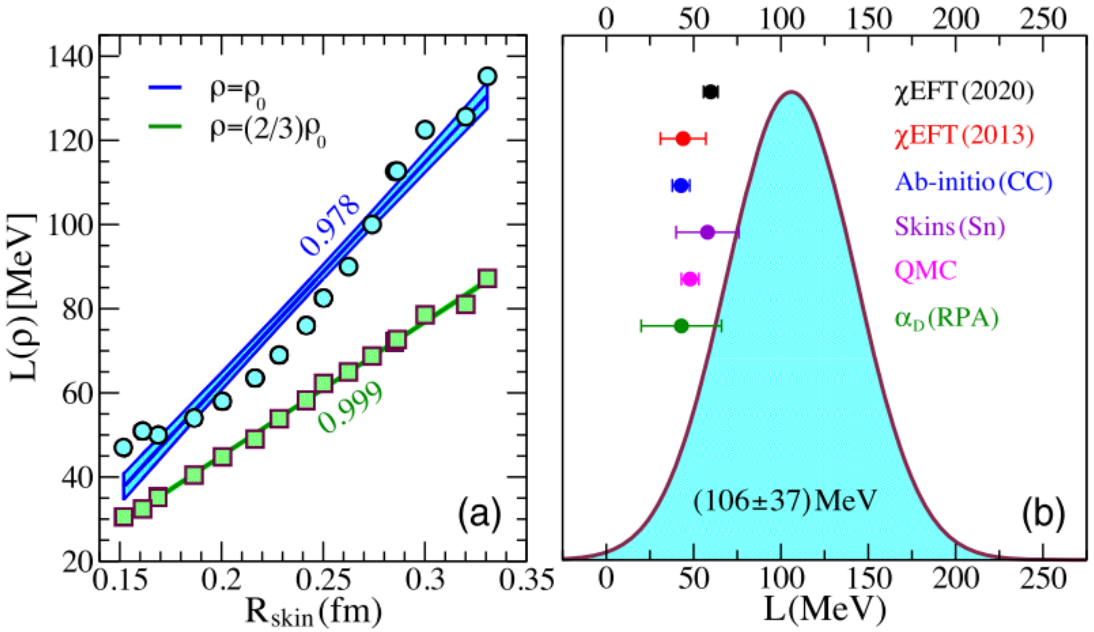
$$A_{\mathrm{PV}}^{\mathrm{meas}} = 550 \pm 16(\text{ stat }) \pm 8(\text{ syst })\mathrm{ppb}$$

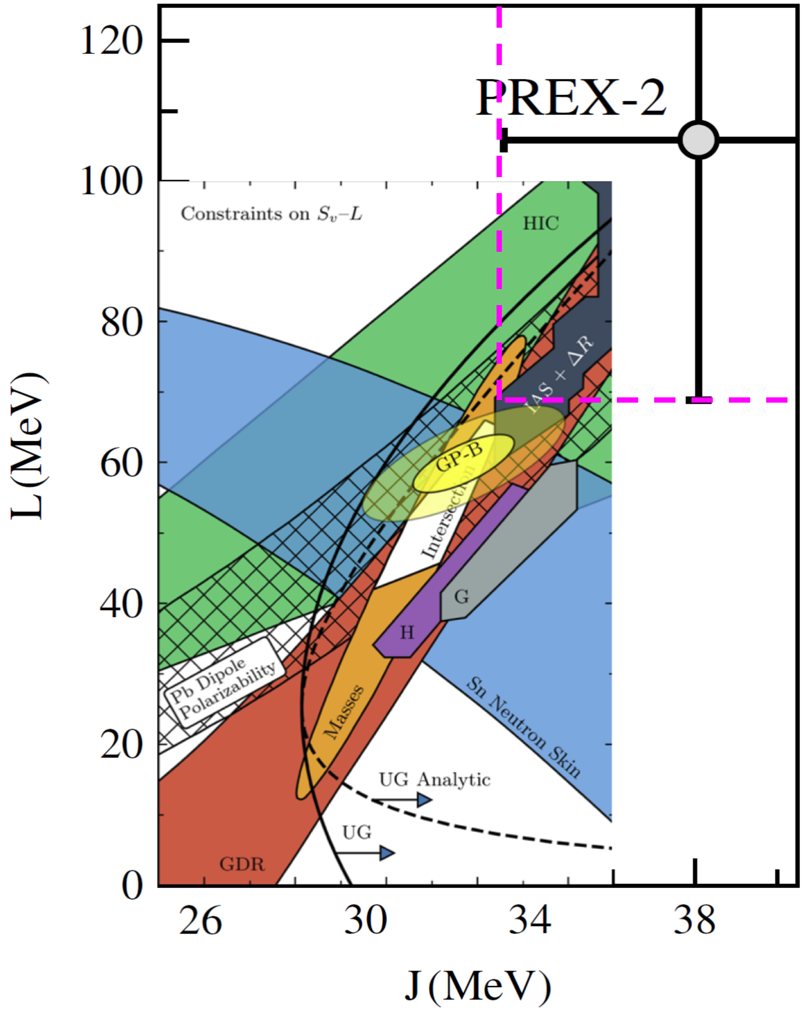
$$F_W(\langle Q^2 \rangle) = 0.368 \pm 0.013(\exp) \pm 0.001(\text{ theo })$$

^{208}Pb Parameter	Value
Weak radius (R_W)	5.800 ± 0.075 fm
Interior weak density (ρ_W^0)	-0.0796 ± 0.0038 fm^{-3}
Interior baryon density (ρ_b^0)	0.1480 ± 0.0038 fm^{-3}
Neutron skin ($R_n - R_p$)	0.283 ± 0.071 fm

PREX-2

Reed et al., PRL126, 172503 (2021)





- **Superstiff symmetry energy from relativistic EDF analysis:**

$$J = (38.1 \pm 4.7)\ \mathrm{MeV},$$

$$L = (106 \pm 37)\ \mathrm{MeV},$$

- **Challenge our understanding of the symmetry energy.**

Calcium (^{48}Ca) Radius Experiment: CREX

CREX, PRL129, 042501 (2022)

PHYSICAL REVIEW LETTERS **129,** 042501 (2022)

Editors' Suggestion

Precision Determination of the Neutral Weak Form Factor of ^{48}Ca

- Model-independent determination of charge-weak form factor difference:

$$\Delta F^{48}_{\mathrm{CW}}(q) = 0.0277 \pm 0.0055, \quad q = 0.8733\mathrm{fm}^{-1}, CREX$$

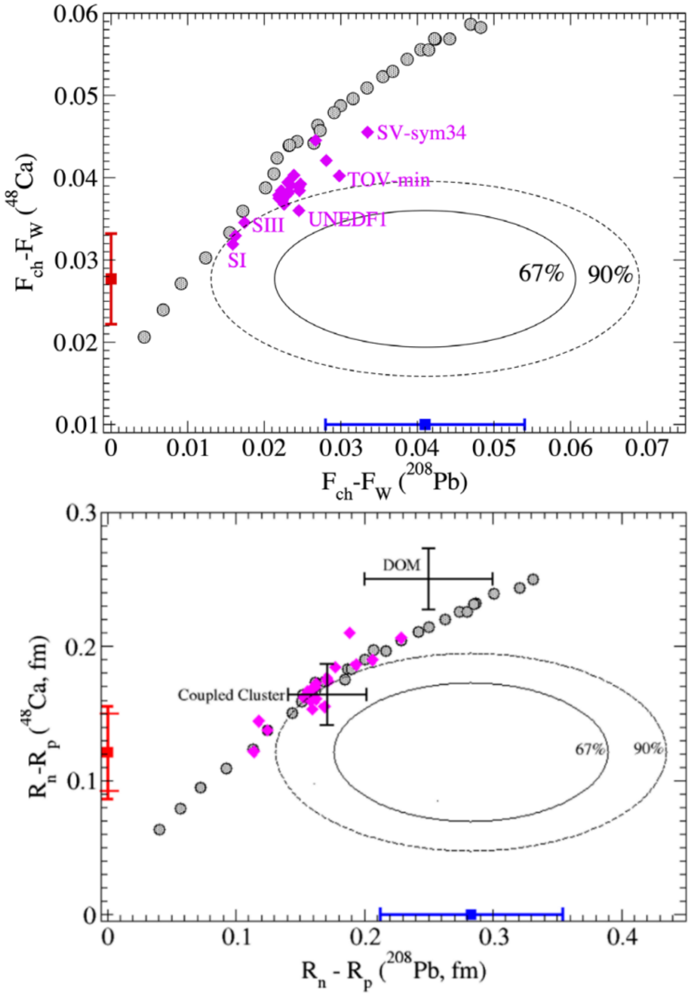
$$\Delta F^{208}_{\mathrm{CW}}(q) = 0.041 \pm 0.013, \quad q = 0.3977\mathrm{fm}^{-1}, PREX$$

- Extracted neutron skin of Ca48

Quantity	Value ± (exp) ± (model) (fm)
$R_W - R_{\mathrm{ch}}$	0.159 ± 0.026 ± 0.023
$R_n - R_p$	0.121 ± 0.026 ± 0.024

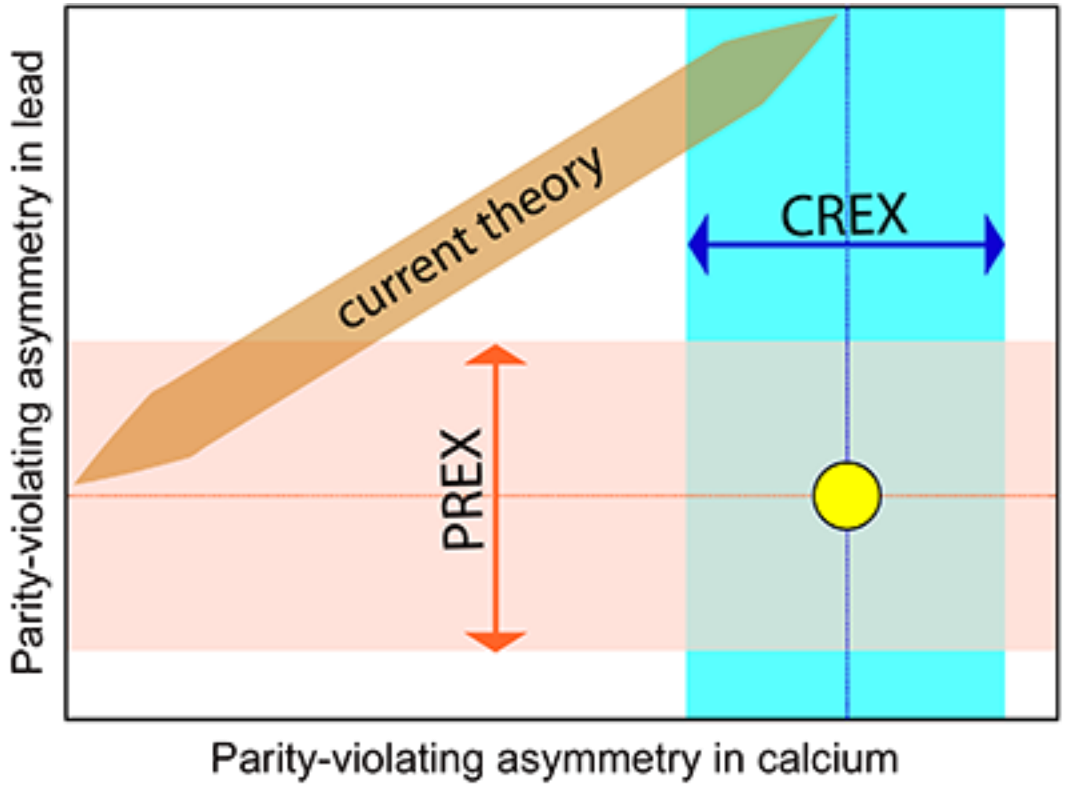
- **Strong tension between CREX and PREX-2 results?**

Too small Nskin of ^{48}Ca or too large Nskin of ^{208}Pb



Challenging modern nuclear EDF theory!

PREX-CREX Puzzle



https://frib.msu.edu/news/2022/prl-paper.html

- Tension between the results of CREX and PREX measurements and the predictions of current global models.

Spin-orbit interaction and neutron skin

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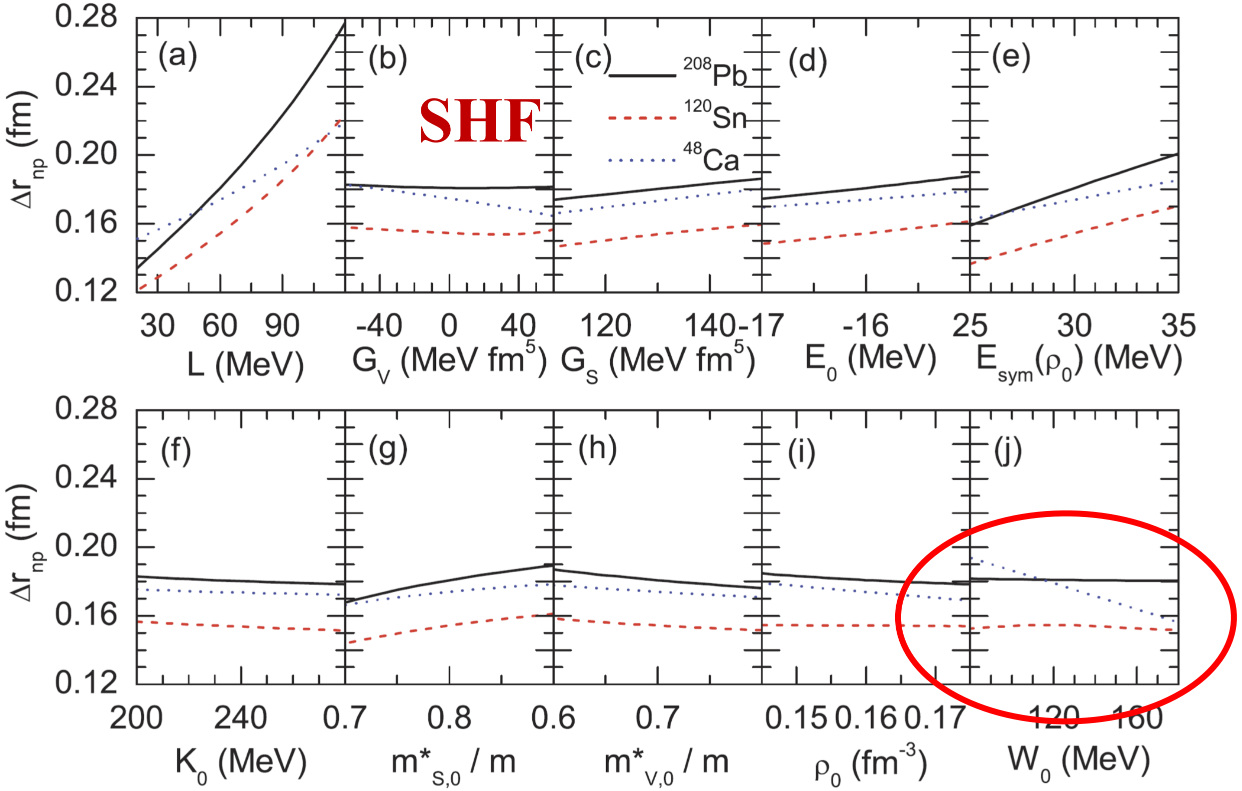
Chen, Ko, Li, &Xu, PRC82, 024321 (2010)



Horowitz & Piekarewitz, PRC86, 045503 (2012)

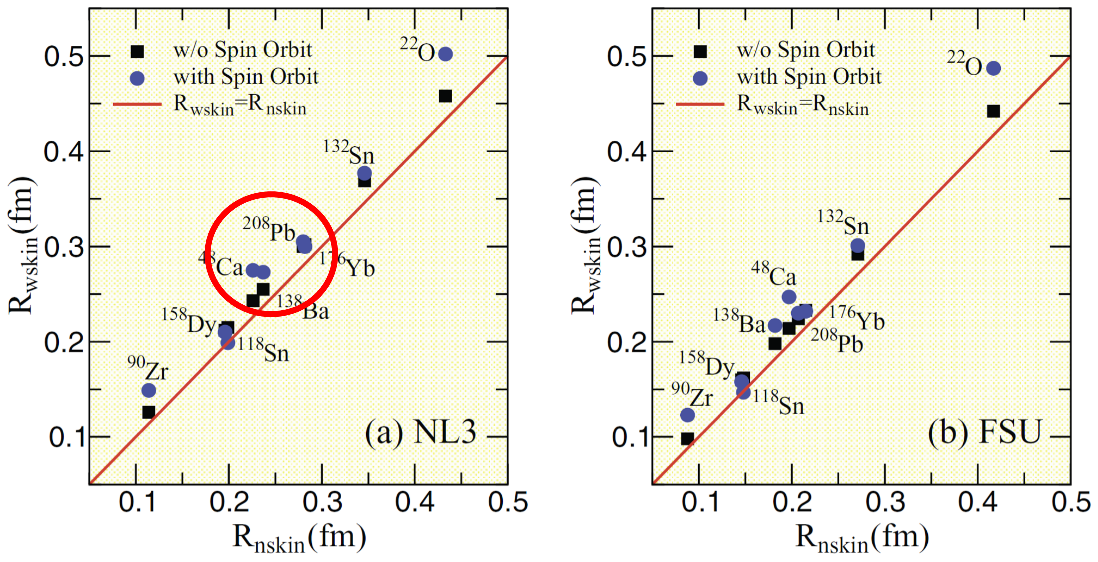
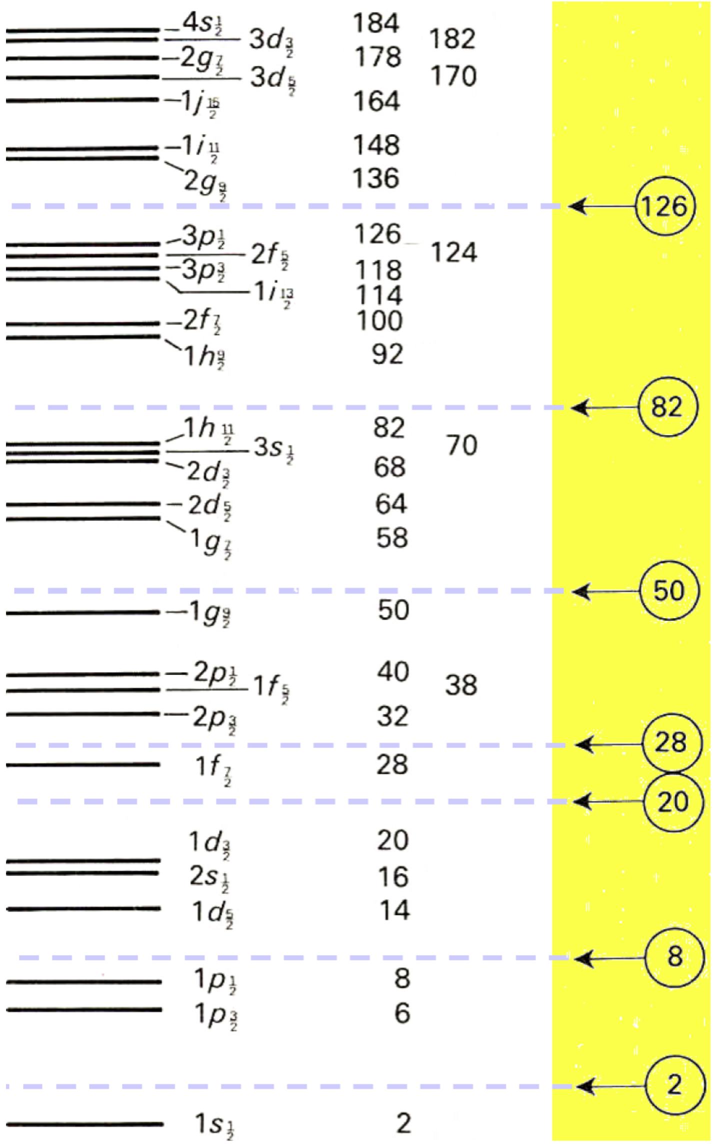


FIG. 2. (Color online) Electroweak skin (R_{wk}-R_{ch}) with and without spin-orbit corrections as a function of neutron skin (R_n-R_p) for the various neutron-rich nuclei considered in this work. Predictions are made using both the (a) NL3 and (b) FSU interactions.

- ❖The Nskin of Ca48 is sensitive to spin-orbit coupling W0 in the standard SHF!
- ❖ Spin-orbit coupling makes significant contribution to electroweak skin Rwk-Rch .
- ❖Ca48 and Pb208 have different shell and surface structures – Both are related to Spin-Orbit interaction

Spin-orbit interaction



❖Strong spin-orbit interaction-> magic numbers

$$V(r) \rightarrow V(r) + W(r) L \cdot S$$

$$W(r) = -|V_{LS}| \left(\frac{\hbar}{m_\pi c} \right)^2 \frac{1}{r} \frac{dV(r)}{dr}$$

Relativistic effects

(Duerr, PR103), 469(1956)

Mayer and Jensen (1949)

Nobel Prize, 1963 (Also Wigner)

❖Naturally introduced in Relativistic mean-field models.

❖Nonrelativistic energy density functionals (Skyrme):

Spin-orbit interaction: $iW_0 \boldsymbol{\sigma} \cdot \left[\boldsymbol{P}' \times \delta(\boldsymbol{r}) \boldsymbol{P} \right]$

Spin-orbit energy density: *Chabanat, et al., NPA 627, 710 (1997)*

$$\mathscr{H}_{\rm so} = \frac{1}{2} W_0 \left[J \cdot \nabla \rho + J_p \nabla \rho_p + J_n \nabla_n \right]$$

❖Is the neutron spin-orbit interaction the same with that of proton?

Isovector spin-orbit interaction

Hamiltonian Density from Spin-Orbit Interaction:

$$E_{\mathrm{so}} = \int d^3r \left[\frac{b_{\mathrm{IS}}}{2} \boldsymbol{J} \cdot \nabla \rho + \frac{b_{\mathrm{IV}}}{2} (\boldsymbol{J}_n - \boldsymbol{J}_p) \cdot \nabla (\rho_n - \rho_p) \right]$$

Isovector

Standard Skyrme EDF:

$b_{\mathrm{IV}} = b_{\mathrm{IS}}/3 = W_0/2 \approx 60\ \mathrm{MeV} \cdot \mathrm{fm}^5$

Reinhard and Flocard, NPA 584, 467488 (1995)
Bender, Heenen, and Reinhard, Rev. Mod. Phys. 75, 121 (2003).
Ebran, Mutschler, Khan, and Vretenar, PRC 94, 024304 (2016).

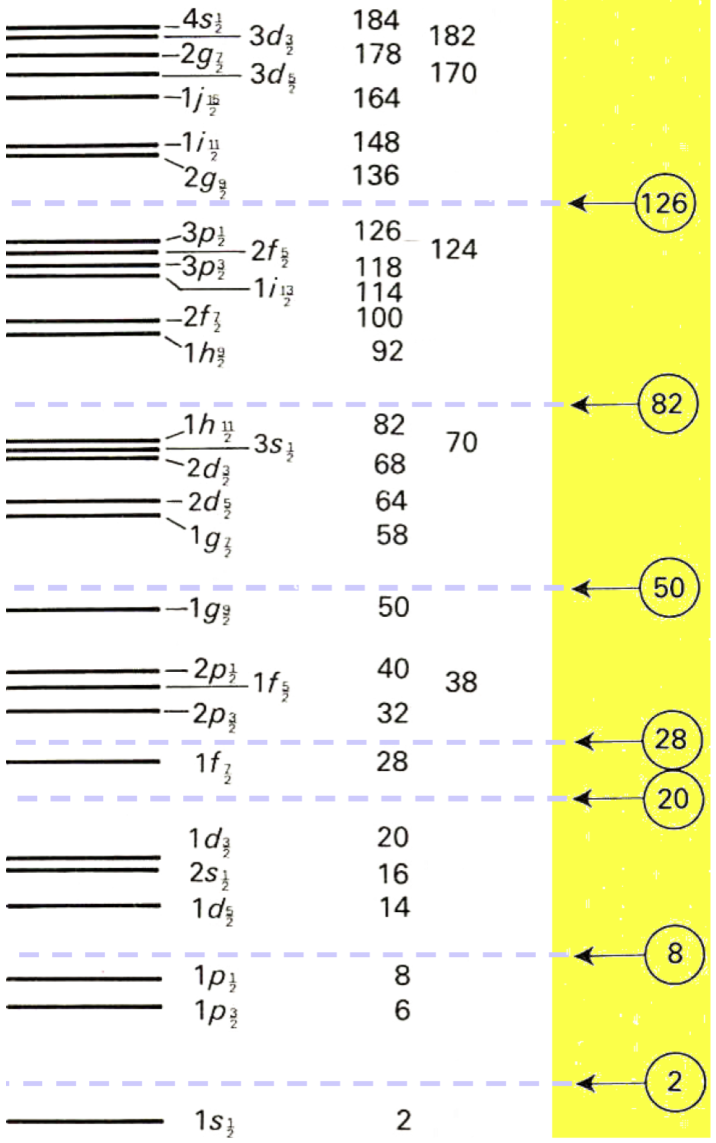
Relativistic mean field model (nonrelativistic reduction):

$b_{\mathrm{IV}} \approx 0$

Lack experimental probes to constraint b_{IV}

- The isovector spin-orbit coupling b_{IV} is expected to have significant effects on lighter nuclear with larger $\boldsymbol{J}_n - \boldsymbol{J}_p$.

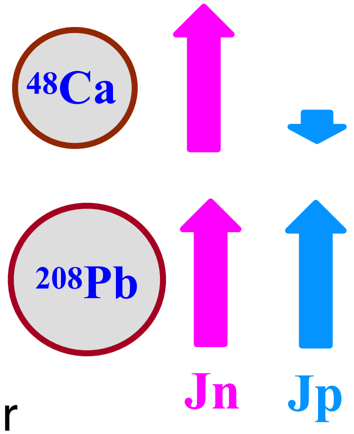
Spin-orbit density in ^{48}Ca and ^{208}Pb



Spin-Orbit density (spherical nuclei):

$$J_q(r) = \frac{1}{4\pi r^3} \sum_i v_i^2 \, (2j_i + 1) \times \left[j_i\,(j_i + 1) - l_i\,(l_i + 1) - \frac{3}{4} \right] R_i^2(r)$$

Colo et al., Phys. Lett. B 646, 227 (2007)



❖Contributions from $j_>$ and $j_<$ largely cancel with each other

- $j_> = l + 1/2$: positive contribution
- $j_< = l - 1/2$: negative contribution

❖ $|J_n - J_p|$ is large in ^{48}Ca, but relatively small in ^{208}Pb

- Ca40: $J_p \approx 0,\ J_n \approx 0$
- Ca48: $J_p \approx 0,\ J_n >> 0$ due to the 8 $1f_{\frac{7}{2}}$ neutrons of unpaired l·s partner
- Pb208: $J_p \approx J_n >> 0$ due to 14 $1i_{\frac{13}{2}}$ neutrons and 12 $1h_{\frac{11}{2}}$ protons

✦ The isovector spin-orbit coupling $b_{\rm IV}$ is expected to have significant effect on Ca48 while essentially no influence on Pb208!

(Extended) Skyrme EDF with tensor force

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Standard Skyme interaction:

$$v(r_1, r_2) = t_0(1 + x_0 P_\sigma)\delta(\mathrm{r}) + \frac{1}{2}t_1(1 + x_1 P_\sigma)\left[\mathrm{k}'^2\delta(\mathrm{r}) + \delta(\mathrm{r})\mathrm{k}^2\right] + t_2(1 + x_2 P_\sigma)\mathrm{k}' \cdot \delta(\mathrm{r})\mathrm{k}$$
$$+ \frac{1}{6}t_3(1 + x_3 P_\sigma)[\rho(\mathbf{R})]^\alpha \delta(\mathrm{r}) + iW_0(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \cdot [\mathrm{k}' \times \delta(\mathrm{r})\mathrm{k}],$$

Chabanat, et al., NPA 627, 710 (1997)

Momentum-dependent three-body interaction:

$$v' = v + \frac{1}{2}t_4(1 + x_4 P_\sigma)\left[\mathrm{k}'^2 \rho(R)^\beta \delta(\mathrm{r}) + \delta(\mathrm{r})\rho(R)^\beta \mathrm{k}^2\right] + t_5(1 + x_5 P_\sigma)\mathrm{k}' \cdot \rho(R)^\gamma \delta(\mathrm{r})\mathrm{k}.$$

Zhang & Chen, PRC 94, 064326 (2016)

Zero-range tensor force:

$$V_T = \frac{1}{2}T\left\{\left[(\sigma_1 \cdot \mathrm{k}')(\sigma_2 \cdot \mathrm{k}') - \frac{1}{3}\mathrm{k}'^2(\sigma_1 \cdot \sigma_2)\right]\delta(\mathrm{r}) + \delta(\mathrm{r})\left[(\sigma_1 \cdot \mathrm{k})(\sigma_2 \cdot \mathrm{k}) - \frac{1}{3}\mathrm{k}^2(\sigma_1 \cdot \sigma_2)\right]\right\}$$
$$+ U\left\{(\sigma_1 \cdot \mathrm{k}')\delta(\mathrm{r})(\sigma_2 \cdot \mathrm{k}) - \frac{1}{3}(\sigma_1 \cdot \sigma_2)[\mathrm{k}' \cdot \delta(\mathrm{r})\mathrm{k}]\right\},$$

Stancu, Brink, and Flocard, PLB 68, 108 (1977)

(Extended) Skyrme EDF with tensor force

Energy density functional:

$$\begin{aligned}\mathcal{E}_{\text{Skyrme}} &= \frac{B_0 + B_3\rho^\alpha}{2}\rho^2 - \frac{B_0' + B_3'\rho^\alpha}{2}\tilde{\rho}^2 + (B_1 + B_4\rho^\beta + B_5\rho^\gamma)\rho\tau - (B_1' + B_4'\rho^\beta + B_5'\rho^\gamma)\tilde{\rho}\tilde{\tau} \\ &+ \frac{2B_2 + (2\beta+3)B_4\rho^\beta - B_5\rho^\gamma}{4}(\nabla\rho)^2 - \frac{2B_2' + 3B_4'\rho^\beta - B_5'\rho^\gamma}{4}(\nabla\tilde{\rho})^2 - \frac{\beta B_4'}{2}\rho^{\beta-1}\tilde{\rho}\nabla\rho\cdot\nabla\tilde{\rho} \\ &+ \frac{C_1 + C_2\rho^\beta + C_3\rho^\gamma}{2}\boldsymbol{J}^2 + \frac{C_1' + C_2'\rho^\beta + C_3'\rho^\gamma}{2}\tilde{\boldsymbol{J}}^2 \\ &+ \frac{b_{\text{IS}}}{2}\nabla\rho\cdot\mathbf{J} + \frac{b_{\text{IV}}}{2}\nabla\tilde{\rho}\cdot\tilde{\mathbf{J}} + \frac{\alpha_T + \beta_T}{4}\boldsymbol{J}^2 + \frac{\alpha_T - \beta_T}{4}\tilde{\boldsymbol{J}}^2.\end{aligned}$$

$$\begin{aligned}\rho_q(\mathbf{r}) &= \sum_i v_i^2 \left|\varphi_i(\mathbf{r})\right|^2, \\ \tau_q(\mathbf{r}) &= \sum_i v_i^2 \left|\nabla\varphi_i\mathbf{r})\right|^2, \\ \mathbf{J}_q(\mathbf{r}) &= -\mathrm{i}\sum v_i^2\varphi_i^+(\mathbf{r})\nabla\times\hat{\sigma}\varphi_i(\mathbf{r}).\end{aligned}$$

$$\rho = \rho_n + \rho_p, \quad \tau = \tau_n + \tau_p, \quad \boldsymbol{J} = \boldsymbol{J}_n + \boldsymbol{J}_p,$$

$$\tilde{\rho} = \rho_n - \rho_p, \quad \tilde{\tau} = \tau_n - \tau_p, \quad \tilde{\boldsymbol{J}} = \boldsymbol{J}_n - \boldsymbol{J}_p,$$

❖ Construct 6 new EDFs to simultaneously fit CREX and PREX results, ground- and excited-state of a number of typical (semi-)closed-shell nuclei, and constraints on EOS of nuclear matter.
(e)S500T, (e)S240T, and (e)S240. [e: extended, T: tensor force, number: the value of b_{IV}

Six New EDFs with strong isovector spin-orbit interaction

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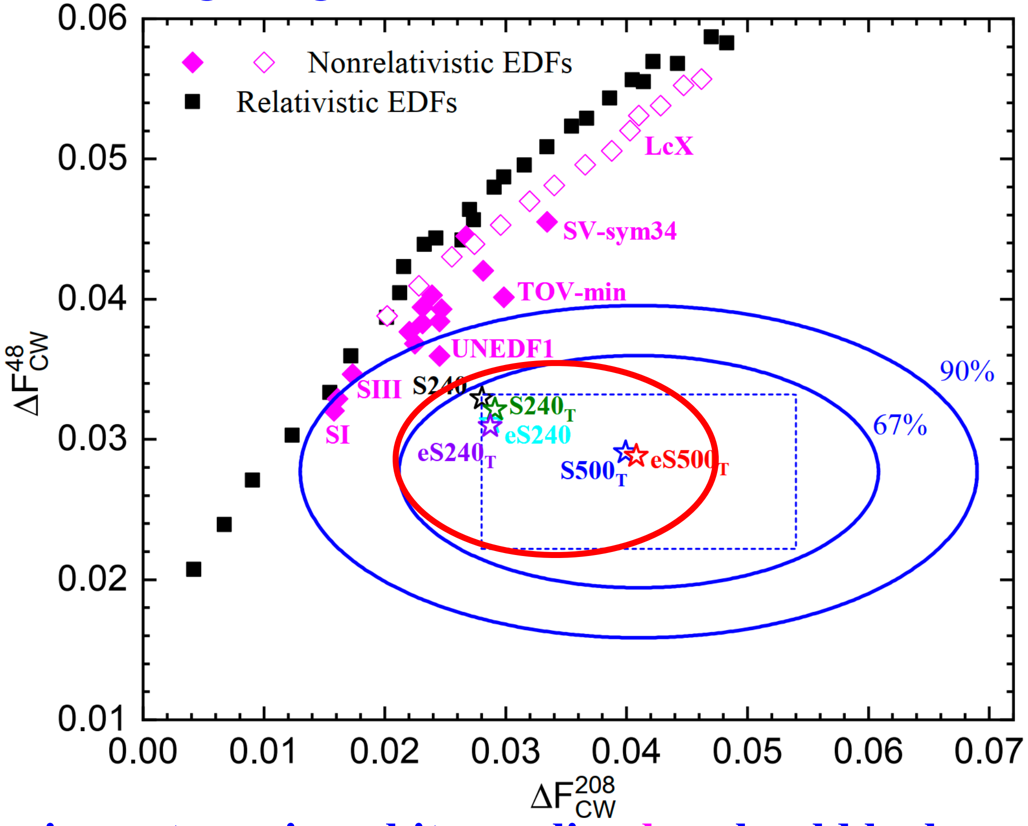
Tong-Gang Yue, ZZ, Lie-Wen Chen, arXiv:2406.03844



	S240	eS240	$S240_T$	$eS240_T$	$S500_T$	$eS500_T$
ρ_0	0.16359	0.15580	0.16498	0.15442	0.16342	0.15089
E_0	-16.147	-16.170	-16.220	-16.190	-16.288	-15.957
$\bar{m}_{s,0}$	0.982	0.939	0.993	0.865	1.022	0.921
$\bar{m}_{v,0}$	0.816	0.898	0.883	0.765	0.602	0.662
S	34.08	34.45	35.19	34.06	39.03	36.96
L	46.6	60.5	52.7	57.4	99.7	80.6
K_{sym}	-207.4	-87.3	-190.4	-133.1	-101.1	-189.5
ΔF^{208}_{CW}	0.0280	0.0288	0.0291	0.0287	0.0400	0.0408
ΔF^{48}_{CW}	0.0329	0.0312	0.0321	0.0310	0.0291	0.0288
Δr^{208}_{np}	0.189	0.195	0.194	0.195	0.263	0.273
Δr^{48}_{np}	0.139	0.090	0.128	0.099	0.100	0.105
α^{208}_{D}	19.35	20.15	19.51	20.20	22.77	22.98
α^{48}_{D}	2.29	2.29	2.29	2.23	2.68	2.85

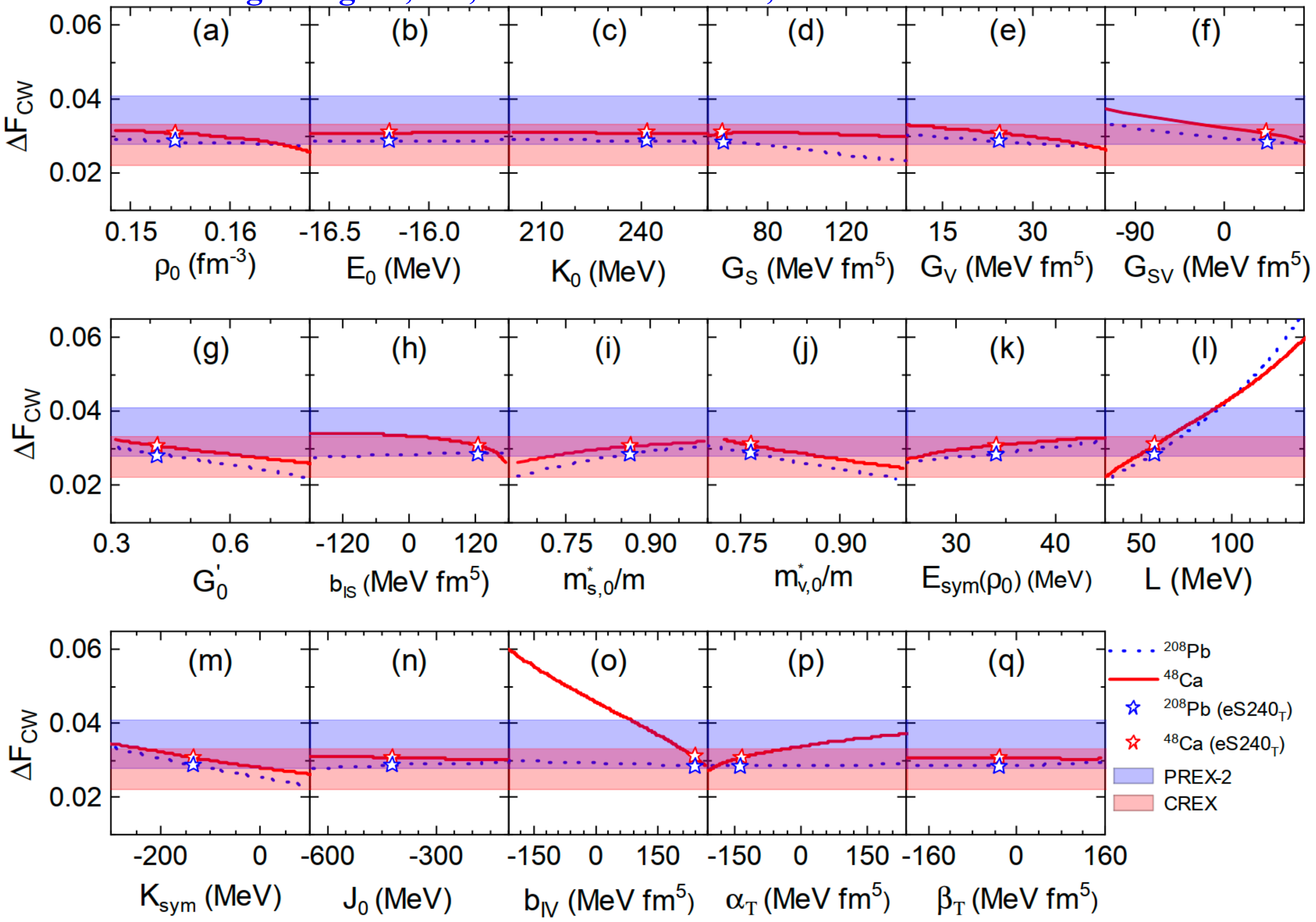
The isovector spin-orbit coupling b_{IV} should be larger than ~ 240 MeV fm⁵ to fit CREX/PREX data (b_{IV} ~60 MeV fm⁵ in conventional non-relativistic EDFs. Note: b_{IS} ~120 MeV fm⁵)

✦Strong isovector dependence of spin-orbit interaction (b_{IV}~ 240 MeV fm⁵ versus b_{IS} ~120 MeV fm⁵)

- **S500T and eS500T overpredict the measured electric dipole polarizability alphaD at RCNP**
- **S240/eS240/S240T/eS240T:**
  Nskin(Pb208) ~ 0.19 fm, Nskin(Ca48) ~ 0.12 fm
  $E_{sym}(\rho_0)$ ~ 34 MeV, L ~ 55 MeV
  (Nicely agree with World Average Values!)

Correlation analysis for ΔF_{CW} in Ca48 and Pb208

Tong-Gang Yu, ZZ, and Lie-Wen Chen, arXiv:2406.03844



- ΔF_{CW} of ^{208}Pb is only sensitive to L.
- ΔF_{CW} of ^{48}Ca is positively(negatively) correlated to $L(b_{\mathrm{IV}})$
- A large L can still reproduce the CREX result with a large b_{IV}

Ground-state properties: mass, radius, spin-orbit splitting

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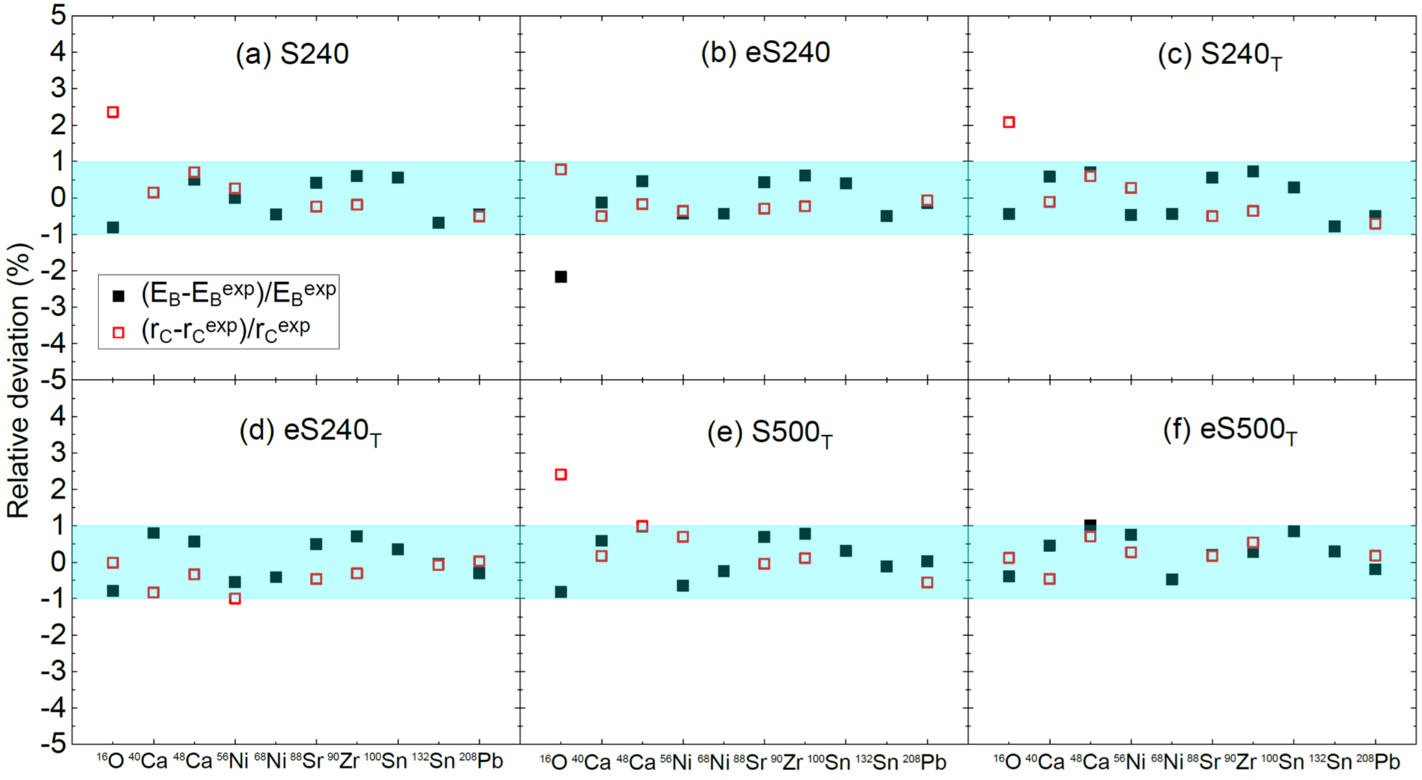
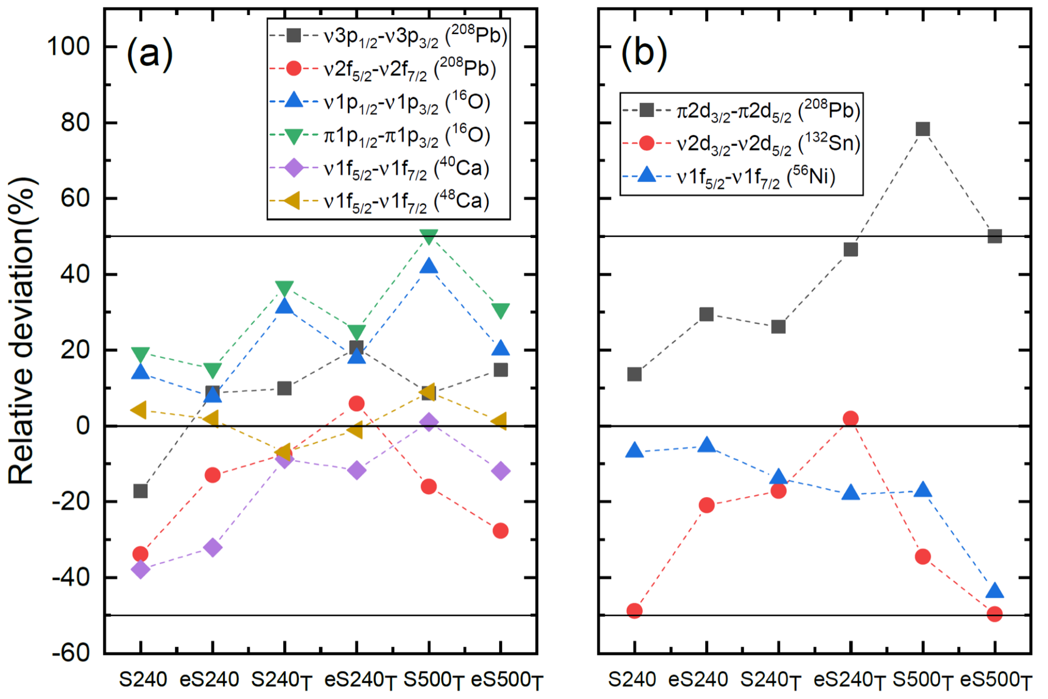
Tong-Gang Yu, ZZ, and Lie-Wen Chen, arXiv:2406.03844





❖The new EDFs with strong isovector spin-orbit interaction can well describe the nuclear global properties!

EOS of symmetric nuclear matter and neutron matter

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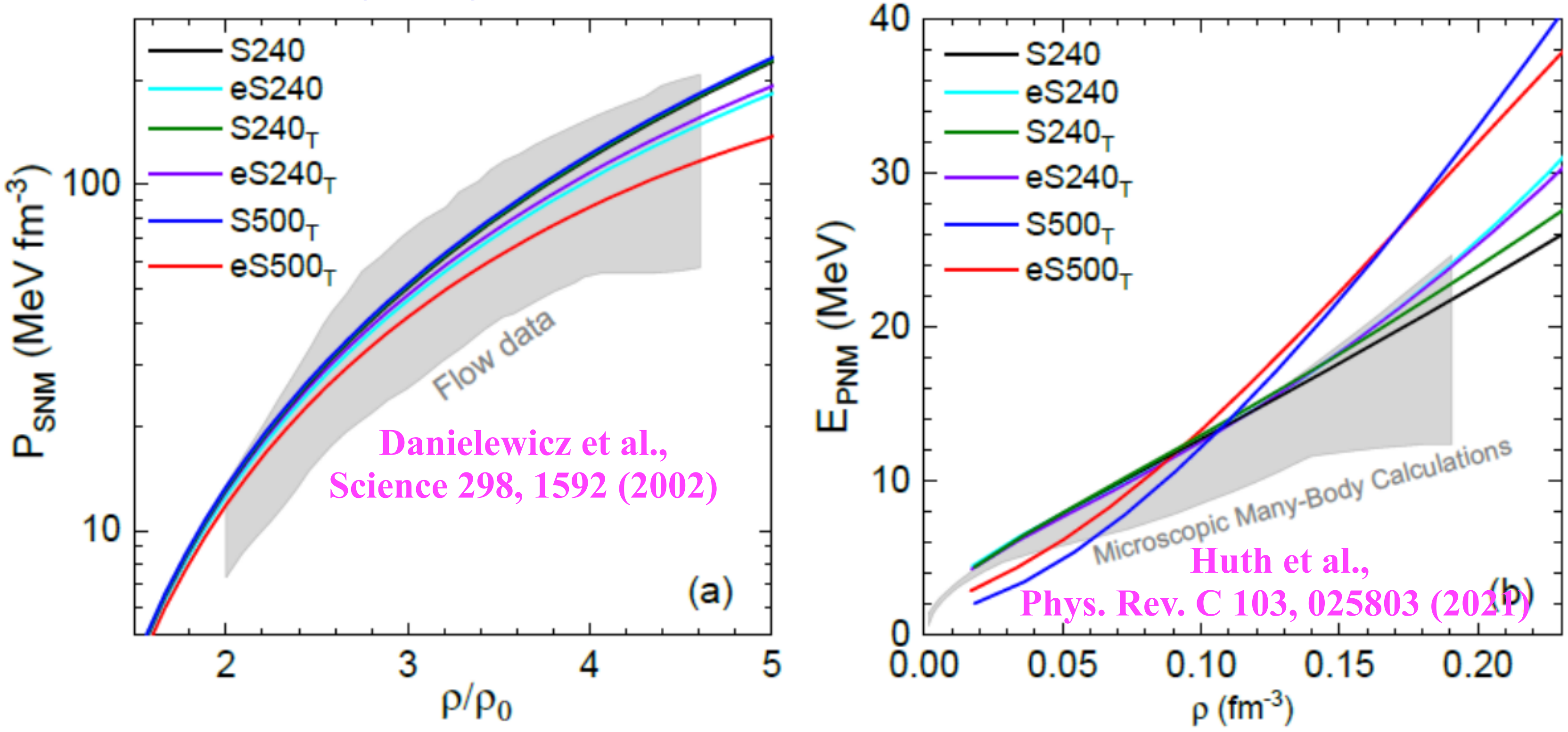
Tong-Gang Yu, ZZ, Lie-Wen Chen, arXiv:2406.03844



❖The new EDFs with strong isovector spin-orbit interaction can well describe the empirical EOS of SNM and PNM! (but S500T and eS500T predict too stiff PNM EOS)

Summary

- PREX-CREX puzzle can be resolved by introducing a strong isovector spin-orbit interaction.

- Such a strong isovector spin-orbit interaction is expected to have significant impacts on essentially all properties of neutron-rich nuclei: The location of neutron-drip line, shell evolution in exotic nuclei, the new magic number, the properties of superheavy nuclei, …

- Future PVES for some stable nuclei (MREX/MESA):
  Pb208, Ni60,…: Not sensitive to the isovector Spin-Orbit interactions (Esym);
  Ca48, Zr90,…: Sensitive to the isovector Spin-Orbit interactions (b_{IV})

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Thanks for your attention

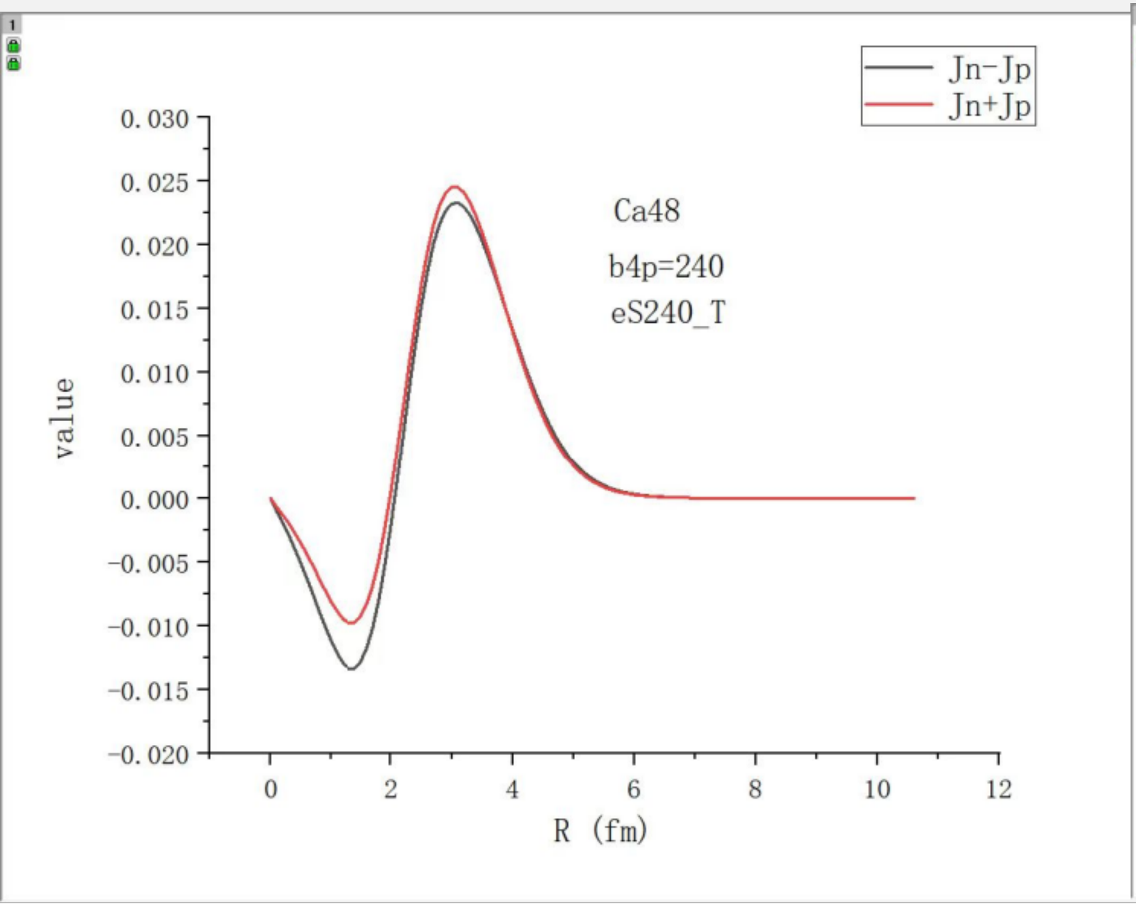
TABLE III. Experimental data and adopted errors for nuclear structure observables used in the sampling (see Sec. III for details). The second line shows the globally adopted error for each observable. That error is multiplied for each observable by a further integer weight factor given in the parenthesis next to the data value.

Nuclei	E_{B} (1 MeV)	r_c (0.02 fm)	R_d (0.04 fm)	σ (0.04 fm)	$\Delta\epsilon_{ls}$ (10%)	$\Delta\epsilon_F$ (1.2 MeV)	E_{GMR} (0.074 MeV)	ΔF_{CW} (0.002)
^{16}O	−127.620(1)	2.701(1)	2.777(1)	0.839(1)	6.32(3)	−3.53(1)		
					6.17(3)			
^{40}Ca	−342.051(1)	3.478(1)	3.845(1)	0.978(1)	6.80(1)	−7.31(1)		
^{48}Ca	−415.990(1)	3.479(1)	3.964(1)	0.881(1)	8.80(1)	6.10(1)		0.0277(1)
^{56}Ni	−483.990(1)	3.750(1)			7.16(1)	−9.47(1)		
^{68}Ni	−590.430(1)							
^{100}Sn	−825.800(1)							
^{132}Sn	−1102.900(1)				1.66(3)	8.40(1)		
^{208}Pb	−1636.446(1)	5.504(1)	6.776(1)	0.913(1)	1.46(3)	0.64(1)	13.614(1)	
					0.89(3)			0.0410(1)
					2.13(3)			

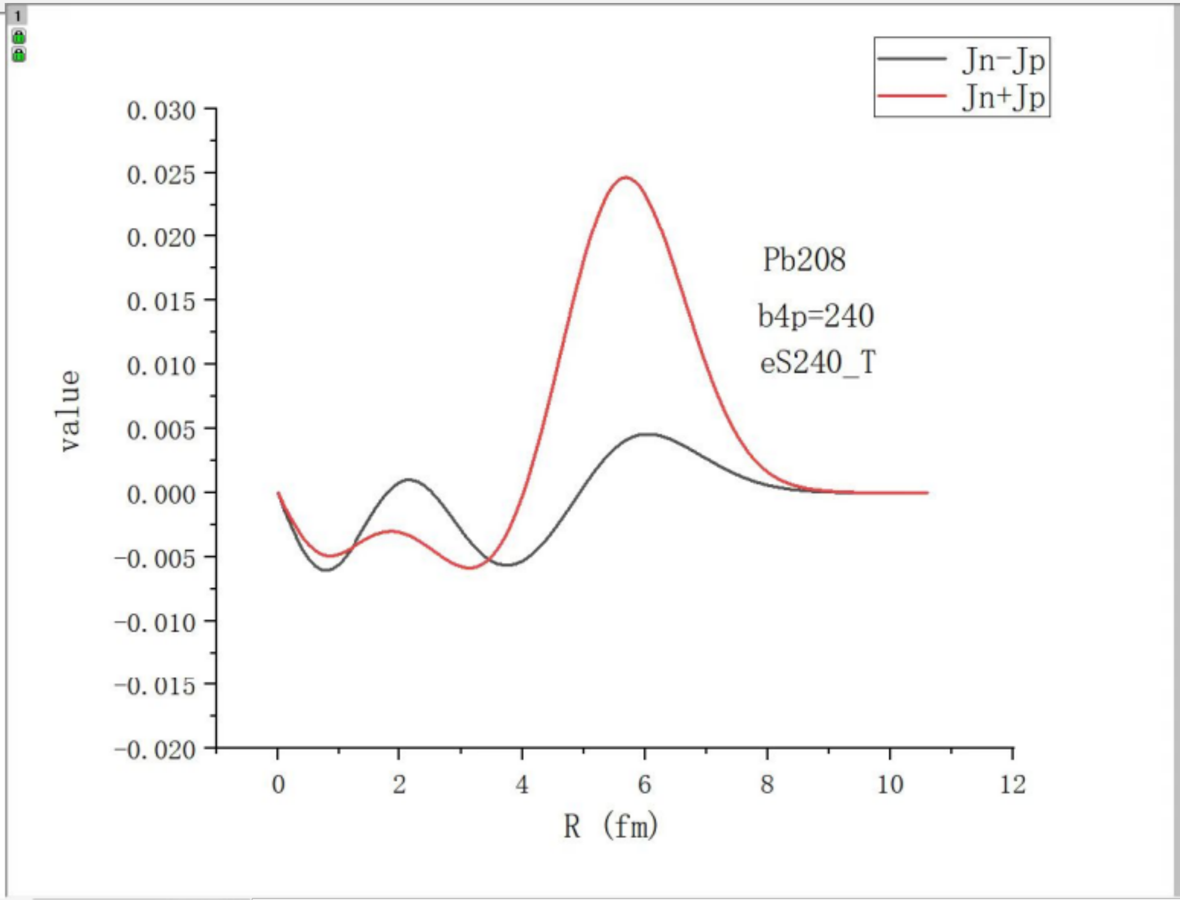
Note. $\Delta\epsilon_{ls}$ data are for ^{16}O($1p_p$, $1p_n$), ^{40}Ca($1f_n$), ^{48}Ca($1f_n$), ^{56}Ni($1f_n$), ^{132}Sn($2d_n$), and ^{208}Pb($2d_p$, $3p_n$, $2f_n$), respectively.

	S240	eS240	$S240_T$	$eS240_T$	$S500_T$	$eS500_T$
t_0	-2029.4	-1777.4	-2013.2	-1707.4	-1942.9	-1632.7
t_1	319.969	534.945	312.255	576.127	310.4	545.566
t_2	328.400	-44.585	-11.507	128.93	604.89	-946.38
t_3	13713.8	12882.5	13652.8	12107.0	13579.5	12807.8
t_4	–	-1608.5	–	-1718.0	–	-1815.3
t_5	–	-1983.3	–	-1921.9	–	7734.2
x_0	0.2655	0.3706	0.2808	0.3077	-0.4972	-0.1312
x_1	-1.3375	-0.7830	-0.8068	-0.8494	-0.5167	0.8692
x_2	-1.9532	6.6536	18.785	-3.8262	-1.6533	-0.9003
x_3	0.1038	0.3573	0.1579	0.2180	-1.4670	-0.9644
x_4	–	-1.6562	–	-1.7790	–	1.6947
x_5	–	-1.8112	–	-1.8660	–	-1.1009
α	0.27317	0.35737	0.27604	0.37750	0.29794	0.43201
β	–	1	–	1	–	1
γ	–	1	–	1	–	1
b_{IS}	160.81	149.72	123.18	124.78	100.56	118.87
b_{IV}	240	240	240	240	500	500
α_T	0	0	-200.163	-132.004	-121.47	-1.6178
β_T	0	0	-51.2445	-32.56077	-295.74	-244.841

ρ_0	0.16359	0.15580	0.16498	0.15442	0.16342	0.15089
E_0	-16.147	-16.170	-16.220	-16.190	-16.288	-15.957
K_0	226.78	234.60	227.90	241.17	231.6	236.22
J_0	-408.13	-436.91	-410.56	-423.94	-408.39	-463.64
S_0	34.08	34.45	35.19	34.06	39.03	36.96
L	46.6	60.5	52.7	57.4	99.7	80.6
K_{sym}	-207.4	-87.3	-190.4	-133.1	-101.1	-189.5
$\bar{m}_{s,0}$	0.982	0.939	0.993	0.865	1.022	0.921
$\bar{m}_{v,0}$	0.816	0.898	0.883	0.765	0.602	0.662
G'_0	1.037	0.175	0.549	0.411	1.491	0.951
G_S	118.86	55.35	116.64	56.35	117.79	44.66
G_V	-80.07	31.33	-31.83	24.42	-44.57	7.19
G_{SV}	0	36.22	0	42.41	0	-75.15
E^{208}_{GMR}	13.592	13.449	13.628	13.565	13.561	13.546
r^{48}_c	3.501	3.471	3.498	3.465	3.511	3.501
r^{208}_c	5.473	5.497	5.462	5.497	5.471	5.511
r^{48}_m	3.514	3.461	3.503	3.459	3.501	3.482
r^{208}_m	5.537	5.561	5.529	5.562	5.580	5.621
ΔF^{208}_{CW}	0.0280	0.0288	0.0291	0.0287	0.0400	0.0408
ΔF^{48}_{CW}	0.0329	0.0312	0.0321	0.0310	0.0291	0.0288
Δr^{208}_{np}	0.189	0.195	0.194	0.195	0.263	0.273
Δr^{48}_{np}	0.139	0.090	0.128	0.099	0.100	0.105
α^{208}_D	19.35	20.15	19.51	20.20	22.77	22.98
α^{48}_D	2.29	2.29	2.29	2.23	2.68	2.85



Jn-Jp
Jn+Jp
Ca48
b4p=240
eS240_T
value
R (fm)



Jn-Jp
Jn+Jp
Pb208
b4p=240
eS240_T
value
R (fm)

$$
\begin{aligned}
\rho_q(\mathbf{r}) &= \sum_i v_i^2 \left|\varphi_i(\mathbf{r})\right|^2 , \\
\tau_q(\mathbf{r}) &= \sum_i v_i^2 \left|\nabla\varphi_i \mathbf{r})\right|^2 , \\
\mathbf{J}_q(\mathbf{r}) &= -\mathrm{i} \sum v_i^2 \varphi_i^{+}(\mathbf{r}) \nabla \times \hat{\sigma} \varphi_i(\mathbf{r}).
\end{aligned}
$$

$$\rho = \rho_n + \rho_p, \quad \tau = \tau_n + \tau_p, \quad \boldsymbol{J} = \boldsymbol{J}_n + \boldsymbol{J}_p,$$

$$\tilde{\rho} = \rho_n - \rho_p, \quad \tilde{\tau} = \tau_n - \tau_p, \quad \tilde{\boldsymbol{J}} = \boldsymbol{J}_n - \boldsymbol{J}_p,$$

$$\frac{\hbar^2}{2m_q^*} = \frac{\hbar^2}{2m} + (B_1 + B_4\rho^\beta + B_5\rho^\gamma)\rho - t_q(B_1' + B_4'\rho^\beta + B_5'\rho^\gamma)\tilde{\rho},$$

$$\begin{aligned} U_q = {} & B_0\rho - t_q B_0'\tilde{\rho} + B_1\tau - t_q B_1'\tilde{\tau} + \frac{\alpha+2}{2}B_3\rho^{\alpha+1} - \frac{B_3'}{2}(\alpha\tilde{\rho} + 2t_q\rho)\rho^{\alpha-1}\tilde{\rho} \\ & +(\beta+1)B_4\rho^\beta\tau + (\gamma+1)B_5\rho^\gamma\tau - (B_4'\beta\rho^\beta + B_5'\gamma\rho^\gamma)\tilde{\rho}\tilde{\tau} - t_q(B_1' + B_4'\rho^\beta + B_5'\rho^\gamma)\tilde{\tau} \\ & -\frac{\beta(2\beta+3)B_4\rho^{\beta-1} - B_5\gamma\rho^{\gamma-1}}{4}(\nabla\rho)^2 - \frac{2B_2 + (2\beta+3)B_4\rho^\beta - B_5\rho^\gamma}{2}\nabla^2\rho \\ & -\frac{3\beta B_4'\rho^{\beta-1} - \gamma B_5'\rho^{\gamma-1}}{4}\nabla\tilde{\rho}\cdot(2t_q\nabla\rho + \nabla\tilde{\rho}) - \frac{2B_2' + 3B_4'\rho^\beta - B_5'\rho^\gamma}{2}t_q\nabla^2\tilde{\rho} \\ & +\frac{\beta B_4'}{2}\rho^{\beta-1}(\nabla\tilde{\rho})^2 + \frac{\beta B_4'}{2}\rho^{\beta-1}\tilde{\rho}\nabla^2\tilde{\rho} + \frac{\beta(\beta-1)B_4'}{2}\rho^{\beta-2}\tilde{\rho}(\nabla\rho)^2 t_q + \frac{\beta B_4'}{2}\rho^{\beta-1}\tilde{\rho}(\nabla^2\rho)t_q \\ & +\frac{\beta C_2\rho^{\beta-1} + \gamma C_3\rho^{\gamma-1}}{2}\boldsymbol{J}^2 + \frac{\beta C_2'\rho^{\beta-1} + \gamma C_3'\rho^{\gamma-1}}{2}\tilde{\boldsymbol{J}}^2 - \frac{b_{\rm IS}}{2}\nabla\cdot\boldsymbol{J} - t_q\frac{b_{\rm IV}}{2}\nabla\cdot\tilde{\boldsymbol{J}}, \end{aligned}$$

$$\boldsymbol{W}_q = \frac{b_{\rm IS}}{2}\nabla\rho + t_q\frac{b_{\rm IV}}{2}\nabla(\rho_n - \rho_p) + \frac{\alpha_{\rm J}+\beta_{\rm J}}{2}\boldsymbol{J} + t_q\frac{\alpha_{\rm J}-\beta_{\rm J}}{2}(\boldsymbol{J}_n - \boldsymbol{J}_p),$$

$$\alpha_{\rm T} = \frac{5}{12}U, \quad \beta_{\rm T} = \frac{5}{24}(T+U),$$

$$\alpha_{\rm C} = 2C_1 + 2C_2\rho^\beta + 2C_3\rho^\gamma, \quad \beta_{\rm C} = 2C_1' + 2C_2'\rho^\beta + 2C_3'\rho^\gamma.$$

$$B_0 = \frac{3}{4}t_0, \qquad B_0' = \frac{1}{2}t_0(\frac{1}{2} + x_0),$$

$$B_1 = \frac{3}{16}t_1 + \frac{5}{16}t_2 + \frac{1}{4}t_2x_2, \qquad B_1' = \frac{1}{8}\left[t_1\left(\frac{1}{2} + x_1\right) - t_2\left(\frac{1}{2} + x_2\right)\right],$$

$$B_2 = \frac{9}{32}t_1 - \frac{5}{32}t_2 - \frac{1}{8}t_2x_2, \qquad B_2' = \frac{1}{16}\left[3t_1\left(\frac{1}{2} + x_1\right) + t_2\left(\frac{1}{2} + x_2\right)\right],$$

$$B_3 = \frac{1}{8}t_3, \qquad B_3' = \frac{1}{12}t_3(\frac{1}{2} + x_3),$$

$$B_4 = \frac{3}{16}t_4, \qquad B_4' = \frac{1}{8}t_4\left(\frac{1}{2} + x_4\right),$$

$$B_5 = \frac{5}{16}t_5 + \frac{1}{4}t_5x_5, \qquad B_5' = -\frac{1}{8}t_5\left(\frac{1}{2} + x_5\right),$$

$$C_1 = \eta_{\rm tls}\frac{1}{8}\left[t_1\left(\frac{1}{2} - x_1\right) - t_2\left(\frac{1}{2} + x_2\right)\right], \qquad C_1' = \eta_{tls}\frac{1}{16}(t_1 - t_2),$$

$$C_2 = \eta_{\rm tls}\frac{1}{8}t_4\left(\frac{1}{2} - x_4\right), \qquad C_2' = \eta_{tls}\frac{1}{16}t_4,$$

$$C_3 = -\eta_{\rm tls}\frac{1}{8}t_5\left(\frac{1}{2} + x_5\right), \qquad C_3' = -\eta_{tls}\frac{1}{16}t_5.$$

Fw