Towards a thermally complete study of inflationary predictions

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in collaboration with

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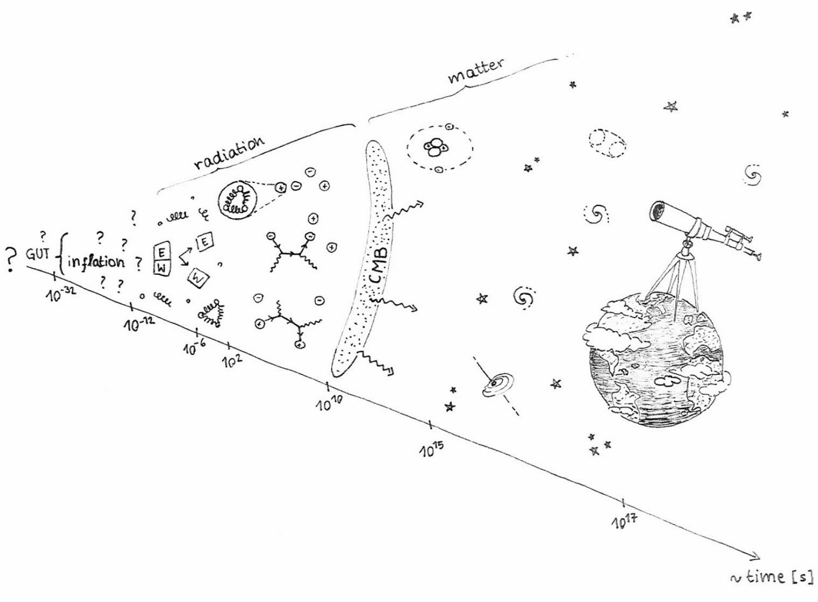
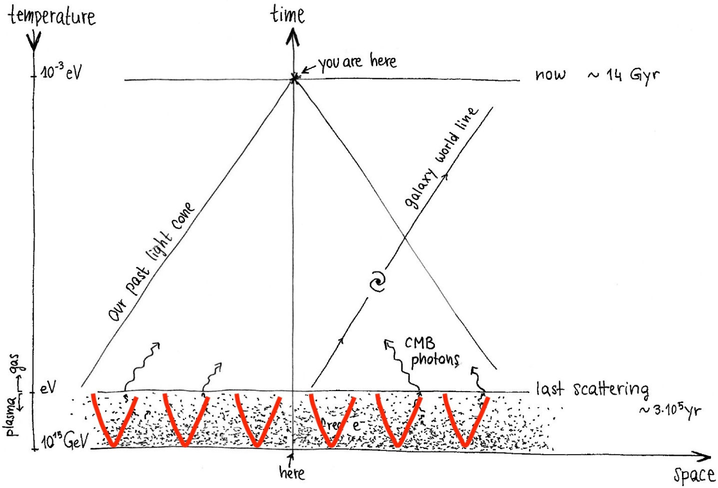
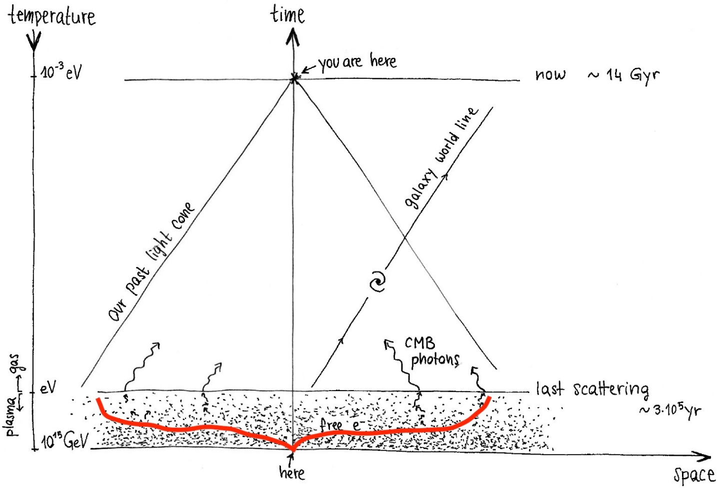
wonderful Cosmic Microwave Background!



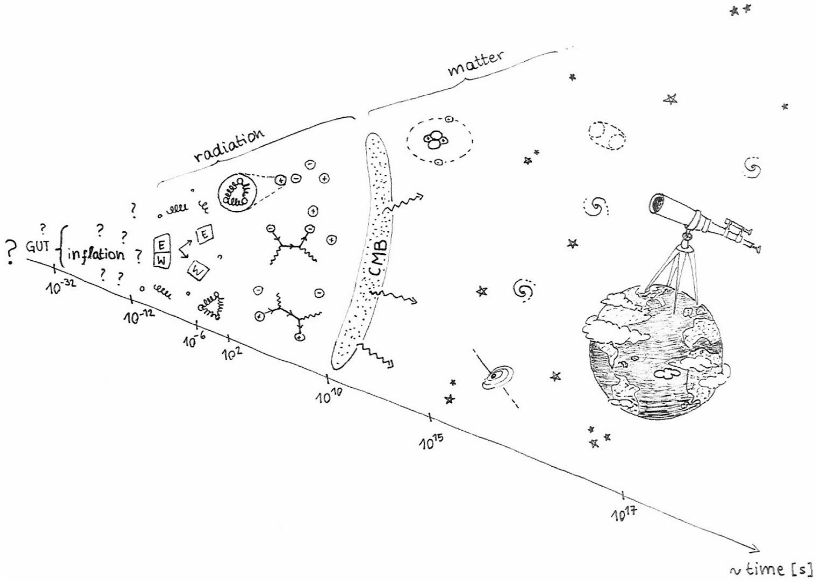
assume decelerated expansion history



inflation ≡ early period of accelerated expansion



looking for tests of inflation...



homogeneous and isotropic expanding universe

$$\mathrm{d}s^2 = \bar{g}_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = -\mathrm{d}t^2 + a(t)^2\,\mathrm{d}\mathbf{x}^2 \;, \quad H \equiv \frac{\dot{a}}{a} > 0\,, \quad \dot{a} \equiv \partial_t a$$

$$\bar{T}^\mu{}_\nu = \begin{pmatrix} \bar{e} & \\ & -\bar{p}\,\delta_{ij} \end{pmatrix} \;, \quad \bar{p} = \bar{p}(\bar{e})$$

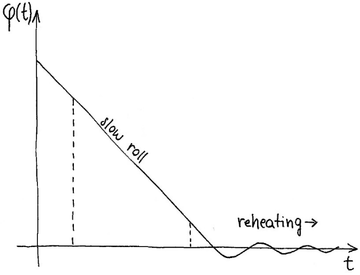
$$\curvearrowright \quad \boxed{G^\mu{}_\nu = \frac{8\pi}{m_{\mathrm{pl}}^2}\, T^\mu{}_\nu} \quad \Rightarrow \quad \begin{cases} H^2 = \frac{8\pi}{3m_{\mathrm{pl}}^2}\,\bar{e} \\ \frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\mathrm{pl}}^2}(\,\bar{e} + 3\bar{p}\,) > 0 \end{cases}$$

$\Rightarrow$ $p < 0$ during inflation

inflation $\equiv$ early period of accelerated expansion

$$\begin{cases} H^2 = \frac{8\pi}{3m_{\rm pl}^2}\,\bar{e} \\ \frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\rm pl}^2}\left(\bar{e}+3\bar{p}\right) > 0 \quad \Leftrightarrow \quad p<0 \end{cases}$$

parametrize with scalar field: $\mathcal{L} = \frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - V(\varphi)$



$$\bar{p} = \frac{\dot{\bar{\varphi}}^2}{2} - V(\bar{\varphi}) \approx -V(\bar{\varphi})$$

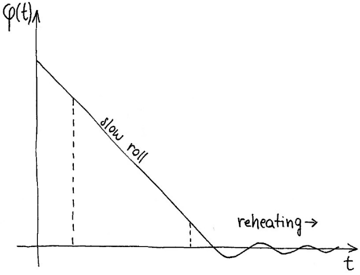
$$\bar{e} = \frac{\dot{\bar{\varphi}}^2}{2} + V(\bar{\varphi}) \approx V(\bar{\varphi})$$

$$\ddot{\bar{\varphi}} + 3H\dot{\bar{\varphi}} + \partial_\varphi V(\bar{\varphi}) = 0$$

inflation $\equiv$ early period of accelerated expansion

$$\begin{cases} H^2 = \frac{8\pi}{3m_{\rm pl}^2}\,\bar{e} \approx \text{const} & \Rightarrow \quad a \sim e^{Ht} \\ \frac{\ddot{a}}{a} = -\frac{4\pi}{3m_{\rm pl}^2}(\,\bar{e} + 3\bar{p}\,) > 0 & \Leftrightarrow \quad p < 0 \end{cases}$$

parametrize with scalar field: $\mathcal{L} = \frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - V(\varphi)$

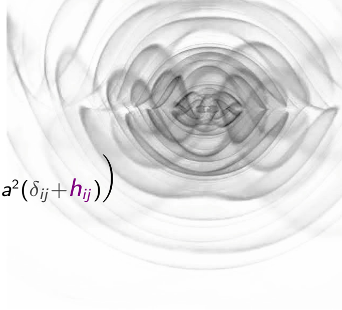


$$\bar{p} = \frac{\dot{\bar{\varphi}}^2}{2} - V(\bar{\varphi}) \approx -V(\bar{\varphi})$$

$$\bar{e} = \frac{\dot{\bar{\varphi}}^2}{2} + V(\bar{\varphi}) \approx \text{const}$$

$$\ddot{\bar{\varphi}} + 3H\dot{\bar{\varphi}} + \partial_\varphi V(\bar{\varphi}) = 0$$

switch on perturbations:



$$g_{\mu\nu} = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\mathbf{x}) + \mathcal{O}(\delta^2) = \begin{pmatrix} -1 & \\ & a^2(\delta_{ij}+h_{ij}) \end{pmatrix}$$

$$T^{\mu}{}_{\nu} = \bar{T}^{\mu}{}_{\nu}(t) + \delta T^{\mu}{}_{\nu}(t,\mathbf{x}) + \mathcal{O}(\delta^2)$$

$\curvearrowright$ $$\boxed{G^{\mu}{}_{\nu} = \frac{8\pi}{m_{\rm pl}^2} T^{\mu}{}_{\nu}}$$ $\Rightarrow$ tensor perturbations are gravitational waves

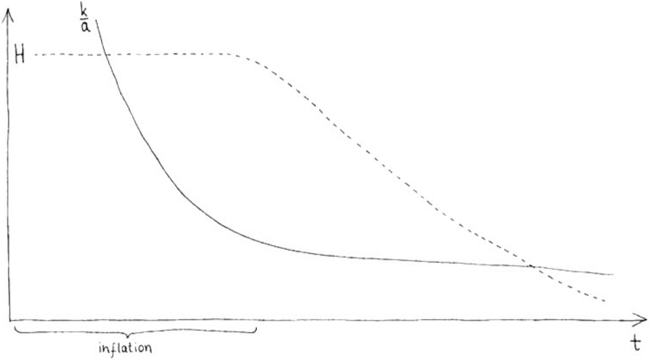
$\overset{\mathbf{x}\to\mathbf{k}}{\curvearrowright}$ $$\boxed{\left(\partial_t^2 + 3H\partial_t + \frac{k^2}{a^2}\right) h_{ij}(t,\mathbf{k}) = \frac{8\pi}{m_{\rm pl}^2} \mathbb{L}_{ijmn}(\hat{\mathbf{k}})\delta T_{mn}(t,\mathbf{k})}$$

figure from www.esa.int/ESA Multimedia/If our eyes could see gravitational waves

gravitational waves during inflation

originate from vacuum fluctuations

$$\mathbb{L}_{ijmn}(\hat{\mathbf{k}})\delta T_{mn}(t,\mathbf{k}) = 0 \quad \Rightarrow \quad \boxed{\left(\partial_t^2 + 3H\partial_t + \frac{k^2}{a^2}\right) h_{ij}(t,\mathbf{k}) = 0}$$

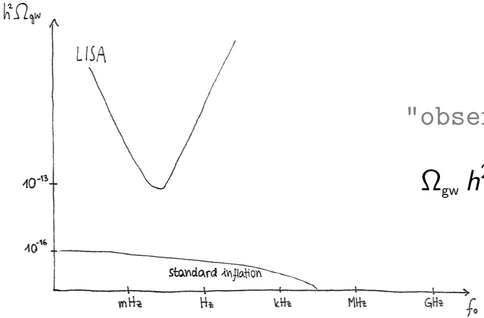


gravitational waves during inflation

$$\left(\partial_t^2 + 3H\partial_t + \frac{k^2}{a^2}\right) h_{ij}(t,\mathbf{k}) = 0$$

mean amplitude

$$\curvearrowright \quad \langle h_{ij}(t,\mathbf{x})\, h_{ij}(t,\mathbf{0})\rangle = \int_{-\infty}^{\infty} \mathrm{d}\ln k\, \mathcal{P}_{\mathrm{T}}(t,k)$$



"observable" spectrum $\underbrace{\text{today}}_{}^{f_0 \equiv \frac{2\pi k}{a_0}}$

$$\Omega_{\mathrm{gw}}\, h^2 = \mathcal{T}_{\mathrm{T}}(t_0, t_*, k)\; \underbrace{\mathcal{P}_{\mathrm{T}}(t_*, k)}_{= \frac{16}{\pi}\left(\frac{H}{m_{\mathrm{pl}}}\right)^2}$$

Towards a thermally complete study of inflationary predictions

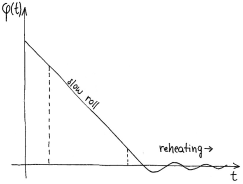
1. Interaction between inflaton and medium
   linear response
   evolution equations
2. Example: non-Abelian axion inflation
   Numerical benchmark results
3. Gravitational wave production
   inflation
   reheating
4. Summary and outlook

inflation + medium

$$\mathcal{L} = -\tfrac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - V(\varphi) - \varphi J + \mathcal{L}_{\texttt{bath}}$$

$$\downarrow$$

$$\ddot{\bar{\varphi}} + 3H\dot{\bar{\varphi}} + \partial_\varphi V(\bar{\varphi}) + \underbrace{\langle J(t)\rangle}_{\text{medium response}} = 0$$



expect: $$\ddot{\bar{\varphi}} + (3H + \Upsilon)\dot{\bar{\varphi}} + \partial_\varphi V(\bar{\varphi}, m_{\mathrm{T}}) \approx 0$$

as effective evolution equation at the end of inflation[1]

[1] M. Laine and S. Procacci, JCAP 06 (2021) 031.

$\langle J(t) \rangle$: response of medium to small perturbation

Hamiltonian: $\hat{H} = \hat{H}_{\mathtt{bath}} + \bar{\varphi} J$

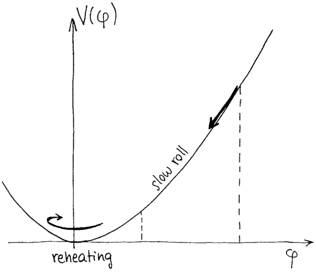
Heat bath density matrix: $\rho(t)\ ,\quad [\hat{H}_{\mathtt{bath}}, \rho(0)] = 0$

$$i\partial_t \dot{\rho}(t) = [\hat{H}(t), \rho(t)]$$

$$\overset{\text{linear}}{\underset{\text{response}}{\Longrightarrow}} \quad \langle J(t) \rangle = -\int_0^t dt_1\, \bar{\varphi}(t_1) \underbrace{G_{\mathrm{R}}(t - t_1)}_{\substack{\text{retarded} \\ \text{correlator}}} + \mathcal{O}(J^3)$$

$$\equiv \theta(t - t_1)\, \langle i[J(t), J(t_1)] \rangle_0$$

$\overset{\text{eom}}{\curvearrowright}$ $\overset{t\to\omega}{\curvearrowright}$ $\varphi(t)\theta(t) \approx \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}\,\mathcal{G}[\omega,\varphi^{(n)}(0)]}{\omega^2 + 3iH\omega - m^2 + G_{\mathrm{R}}(\omega)}$



$$V(\bar{\varphi}) \approx \frac{1}{2} m^2 \bar{\varphi}^2 + \mathcal{O}(\bar{\varphi}^4)$$

thermal corrections relevant at $\omega \sim m \Rightarrow G_{\mathrm{R}} \to G_{\mathrm{R}}(m)$

evolution equations

inflaton:

$$\ddot{\bar{\varphi}} + (3H + \Upsilon)\dot{\bar{\varphi}} + \partial_\varphi V(\bar{\varphi}, m_{\mathrm{T}}) \approx 0$$

$$\Upsilon \approx \frac{\mathrm{Im}\, G_{\mathrm{R}}(m)}{m} \;, \qquad m_{\mathrm{T}}^2 \approx m^2 - \mathrm{Re}\, G_{\mathrm{R}}(m)$$

medium ("radiation"):

$$\dot{e}_r + 3H(e_r + p_r - T\partial_T V) - T\partial_T \dot{V} = \Upsilon \dot{\bar{\varphi}}^2$$

parametrize $e_r = e_r(T)$, $p_r = p_r(T)$

example: non-Abelian axion-like inflation[2]

$$\mathcal{L} = -\frac{1}{2}\,\partial^\mu\varphi\,\partial_\mu\varphi - V(\varphi) - \varphi J + \mathcal{L}_{\texttt{bath}}$$

* topological interaction term: $J = \frac{\alpha}{16\pi f_a}\epsilon^{\mu\nu\rho\sigma}F^c_{\mu\nu}F^c_{\rho\sigma}$

  f_a decay const., $c \in \{1, \ldots, N_c^2 - 1\}$

* periodic potential: $V(\varphi) = m^2 f_a^2\left[1 - \cos\left(\frac{\varphi}{f_a}\right)\right]$

$\Rightarrow$ $\varphi \to \varphi + 2\pi f_a$ symmetry, corrections non-perturbative

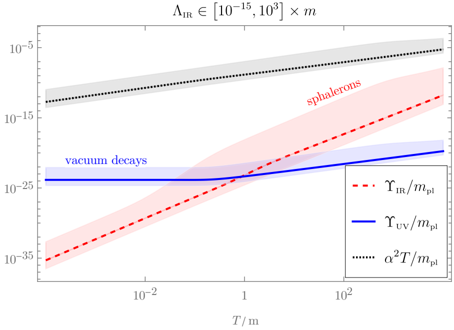
[2] e.g. W. De Rocco, P.W. Graham and S. Kalia, JCAP 11 (2021) 011

friction coefficient $\Upsilon \approx \Upsilon_{\text{IR}} + \Upsilon_{\text{UV}}$

* $m \lesssim \alpha N_c T$: non-perturbative sphaleron dynamics[3]

$$\Upsilon_{\text{IR}} \sim \frac{\alpha^5 (N_c T)^3}{f_a^2} \left[1 + \left(\frac{m}{c_{\text{IR}} \alpha^2 N_c^2 T}\right)^2\right] \left[1 + \left(\frac{m}{c_{\text{M}} \alpha^2 N_c^2 T}\right)^2\right]^{-1}$$

* $m \gg \pi T$: perturbative decays $\varphi \to gg$,[4,5] $\Upsilon_{\text{UV}} \sim \frac{\alpha^2 m^3}{f_a^2}$



[3] M. Laine, L. Niemi, S. Procacci and K. Rummukainen, JHEP 11 (2022) 126

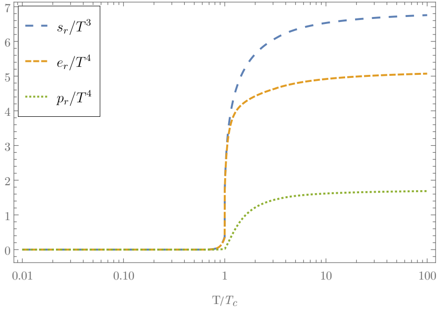
[4] S. Caron-Huot, Phys. Rev. D 79 (2009) 125009

[5] A. L. Kataev, N. V. Krasnikov and A. A. Pivovarov, Nucl. Phys. B 198 (1982) 508

non-Abelian gauge plasma with coupling α

* medium thermalizes quickly[6] $\sim \alpha^2 T$
* confinement scale Λ_{IR} $\Rightarrow$ phase transition at $T_c \sim \Lambda_{\text{IR}}$
* non-perturbative thermodynamics:

fit to lattice results[7] for $N_c = 3$



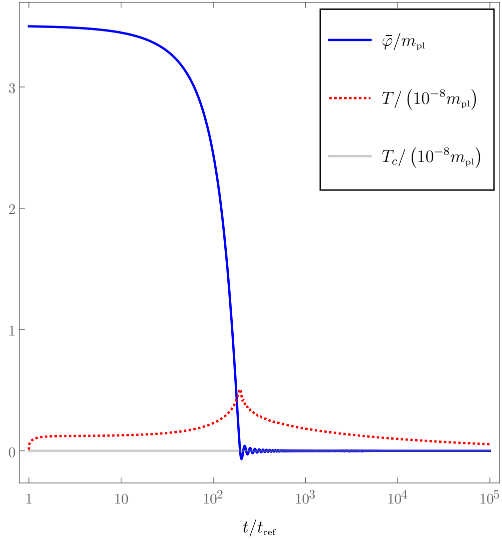
6 e.g. Y. Fu, J. Ghiglieri, S. Iqbal and A. Kurkela, Phys. Rev. D 105 (2022) 054031

7 L. Giusti and M. Pepe, Phys. Lett. B 769 (2017) 385

benchmark solutions: $f_a \sim m_{\rm pl}\,,\ \ m \sim 10^{-6} m_{\rm pl}\,,\ \ t_{\rm ref} = H_{\rm ref}^{-1}$

weakly-coupled plasma:

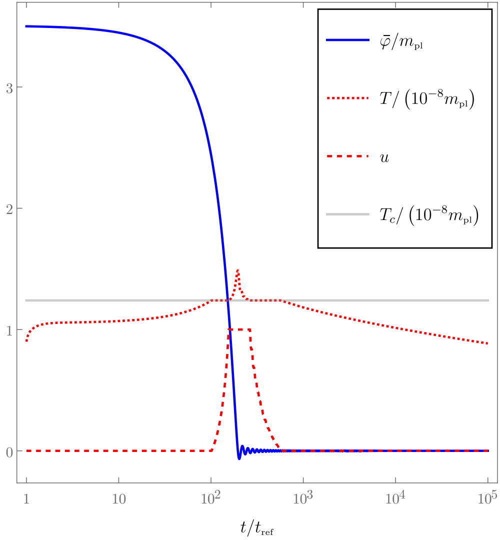
$\Lambda_{\rm IR} = 2 \times 10^{-20} m_{\rm pl}$



benchmark solutions: $f_a \sim m_{\text{pl}}\,,\ \ m \sim 10^{-6} m_{\text{pl}}\,,\ \ t_{\text{ref}} = H_{\text{ref}}^{-1}$

fine-tuned coupling:

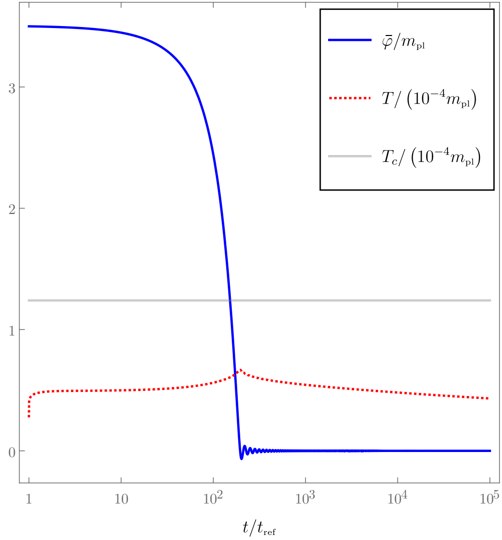
$\Lambda_{\text{IR}} = 1 \times 10^{-8} m_{\text{pl}}$



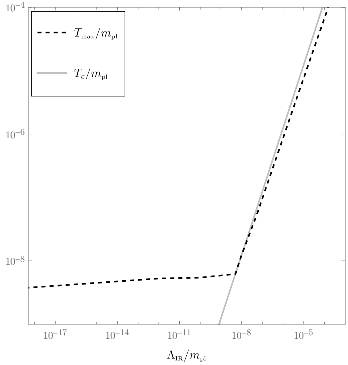
benchmark solutions: $f_a \sim m_{\rm pl}\,,\ \ m \sim 10^{-6} m_{\rm pl}\,,\ \ t_{\rm ref} = H_{\rm ref}^{-1}$

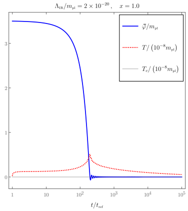
strongly-coupled plasma:

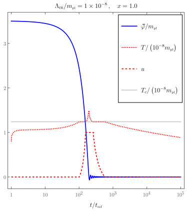
$\Lambda_{\rm IR} = 1 \times 10^{-4} m_{\rm pl}$

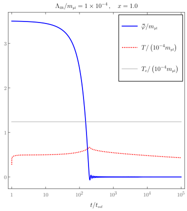


T_{max} depends on the self-coupling of the plasma[8]









8 H. Kolesova, M. Laine and S. Procacci, JHEP 05 (2023) 239

consequences for gravitational waves:

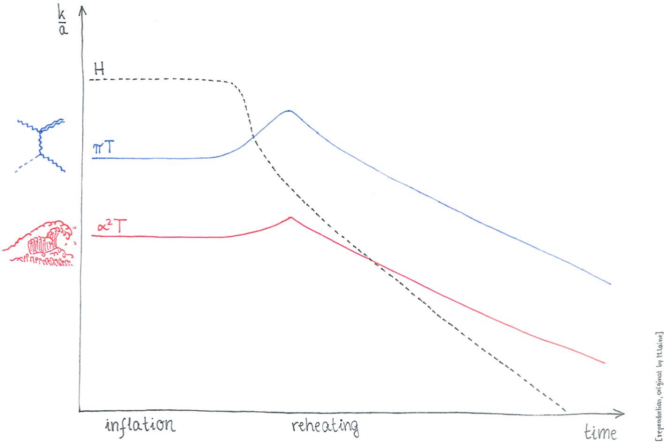
$$\left(\partial_t^2 + 3H\partial_t + \frac{k^2}{a^2}\right) h_{ij}(t, \mathbf{k}) = \frac{8\pi}{m_{\mathtt{pl}}^2} \mathbb{L}_{ijmn}(\hat{\mathbf{k}}) \delta T_{mn}(t, \mathbf{k})$$

sourced by anisotropies in the heat bath

$$\curvearrowright \quad \langle h_{ij}(t, \mathbf{x})\, h_{ij}(t, \mathbf{0}) \rangle_{\mathtt{vacuum\ +\ thermal}} = \ldots$$

$$\Rightarrow \Delta\mathcal{P}_{\mathtt{T}}^{\mathtt{thermal}} = \frac{(32)^2 k^3}{m_{\mathtt{pl}}^4} \int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \overbrace{G_{\mathtt{R}}(t, t_1, k)}^{\text{Green's function}} G_{\mathtt{R}}(t, t_2, k) \times \mathbb{L}_{ijmn} \langle \delta T_{ij}(t_1, \mathbf{k})\, \delta T_{mn}^*(t_2, \mathbf{k}) \rangle_{\mathtt{T}}$$

evolution of the different scales



during inflation the dilution is efficient:

$\frac{k}{a} \ll \pi T < H \quad \Rightarrow$ hydrodynamic regime 

$$T^{\text{hydro}}_{\mu\nu} = \underbrace{\bar{T}_{\mu\nu}}_{(\bar{T},0)} + \underbrace{\delta T^{\text{hydro}}_{\mu\nu}}_{(\delta T, v^i)} + \underbrace{S_{\mu\nu}}_{\text{thermal noise}} + \mathcal{O}(\delta^2)$$

∗ macroscopic scales ↔ collective phenomena

∗ $S_{\mu\nu}$: fluctuation-dissipation relation

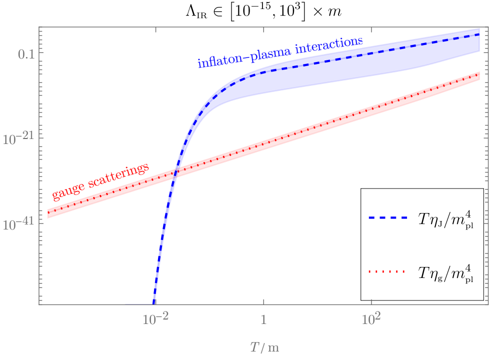
∗ $\langle S_{ij}\, S_{mn} \rangle \sim T \left[\underset{\text{shear}}{\eta} \left(\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\right) + \left(\underset{\text{bulk}}{\zeta} - \frac{2\eta}{3}\right) \delta_{ij}\delta_{mn} \right]$

$$\Rightarrow \quad \boxed{\mathbb{L}_{ijmn} \langle T^{\text{hydro}}_{ij}\; T^{\text{hydro}*}_{mn} \rangle \sim T\eta}$$

during inflation[9] the dilution is efficient: $k/a \ll H$

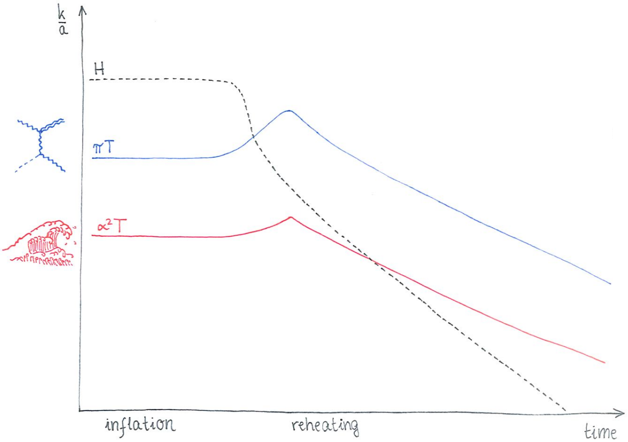
$$\mathcal{P}_{\mathrm{T}}(k) = \frac{16}{\pi}\left(\frac{H_*}{m_{\mathrm{pl}}}\right)^2 + \overbrace{\frac{(32)^2 k^3}{m_{\mathrm{pl}}^4}}^{\text{model independent}} \int_0^{t_e} \mathrm{d}t_{\mathrm{i}} \overbrace{G_{\mathrm{R}}^2(t_e, t_{\mathrm{i}}, k)}^{\text{independent of k}} \underbrace{T\eta(t_{\mathrm{i}})}_{\text{model dependent}}$$

$(\partial_t^2 + 3H\partial_t)G_{\mathrm{R}} \approx 0$



[9] P. Klose, M. Laine and S. Procacci JCAP 12 (2022) 020

evolution of the different scales



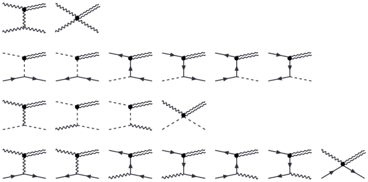
[reproduction, original by M.Laine]

during reheating new fluctuations sourced at $k/a \gg H$

* thermal equilibrium justified: $\alpha^2 T > H$
* $k/a \ll \pi T$: more hydrodynamic modes
* $k/a \sim \pi T$: microscopic scales $\leftrightarrow$ scatterings

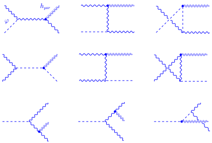
$$\partial_t \, \Delta\mathcal{P}_{\mathrm{T}}^{\text{thermal}} \sim \overbrace{k^3 \, n_{\mathrm{B}}(k)}^{\substack{\text{model}\\ \text{independent}}} \oplus \text{scat}_{n\to m} \left| \begin{matrix}\text{matrix}\\ \text{elements}\end{matrix} \right|^2$$

SM contribution known up to LO:[10]

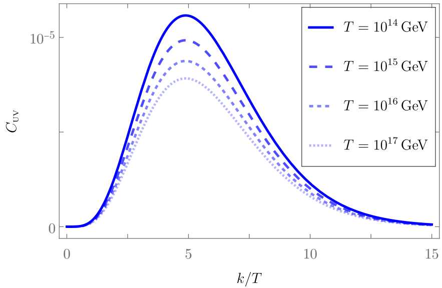


10 J. Ghiglieri, G. Jackson, M. Laine and Y. Zhu, JHEP 07 (2020) 092

add contribution from φ-plasma interactions[11]



$$\partial_t \, \Delta\mathcal{P}_{\mathtt{T}}^{\mathtt{thermal}} \approx \overbrace{C_{\mathtt{UV}}(k,T)\, \frac{T^9}{f_a^2 m_{\mathtt{pl}}^2}} + C_{\mathtt{SM}}(k,T)\, \frac{T^7}{m_{\mathtt{pl}}^2}$$



$C_{\mathtt{SM}}^{\mathtt{max}} \sim 10^4 \, C_{\mathtt{UV}}^{\mathtt{max}}$

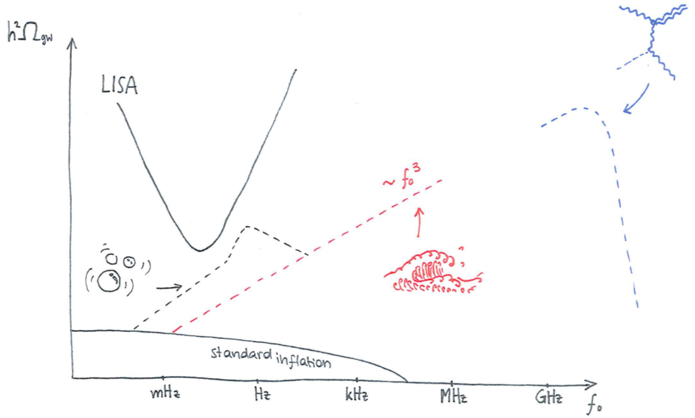
$\Downarrow$

SM contribution dominates

[11] P. Klose, M. Laine and S. Procacci, JCAP 05 (2022) 021

gravitational waves from

vacuum + thermal processes at micro/macro scopic scales

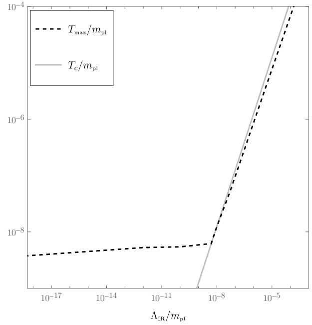


* shapes are model-independent
* amplitudes depend on $T_{\max}$ reached during reheating

summary:

* dynamical reheating after inflation requires careful implementation

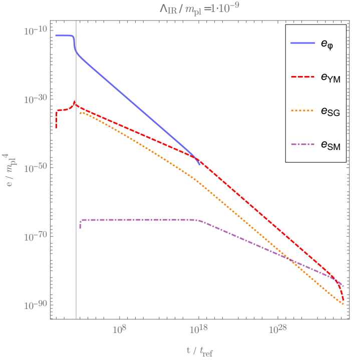
* example rich of phenomenological implications: $\mathcal{L} \supset -\varphi \frac{\alpha(\Lambda_{\rm IR})}{16\pi f_a} F\tilde{F}$



dark matter vs. gravitational waves

outlook: reheating via strongly-coupled dark sector[12]

- at $T < T_c$ dark glueballs connected to SM sector
- different lifetimes $\Rightarrow$ reheating + relic DM

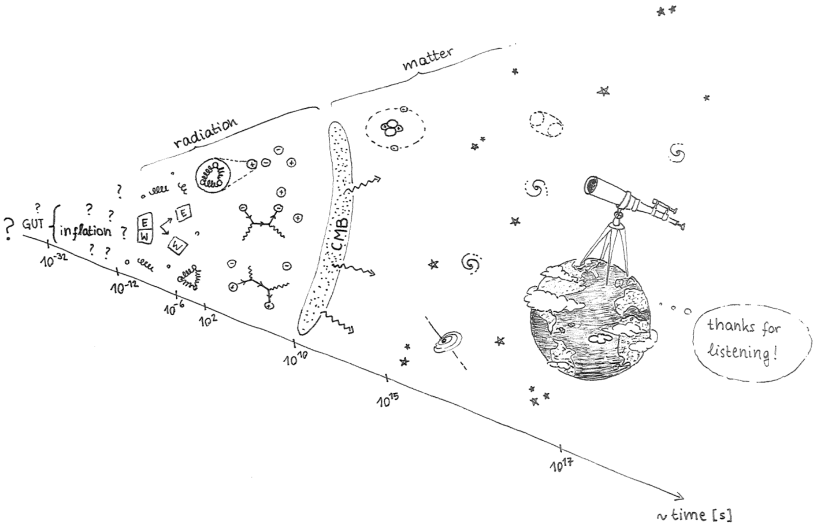


$$\dot{e}_\varphi + 3He_\varphi = -\Upsilon e_\varphi$$

$$\dot{e}_{\rm YM} + 3H(e_{\rm YM} + p_{\rm YM}) = \Upsilon e_\varphi - B_{\rm U}\Gamma e_{\rm YM}$$

$$\dot{e}_{\rm SM} + 4He_{\rm SM} = B_{\rm U}\Gamma e_{\rm YM}$$

[12] S. Biondini, H. Kolesova, S. Procacci, in preparation.

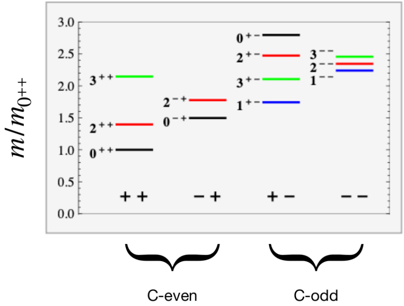

radiation
matter
?
GUT
inflation
E
W
CMB
10^{-32}
10^{-12}
10^{-6}
10^{2}
10^{10}
10^{15}
10^{17}
~ time [s]
thanks for listening!

backup

glueballs dark matter??[13]

if dark sector is also charged under SM gauge group

J. Juknevich, D. Melnikov, M. Strassler, JHEP 07 (2009) 055



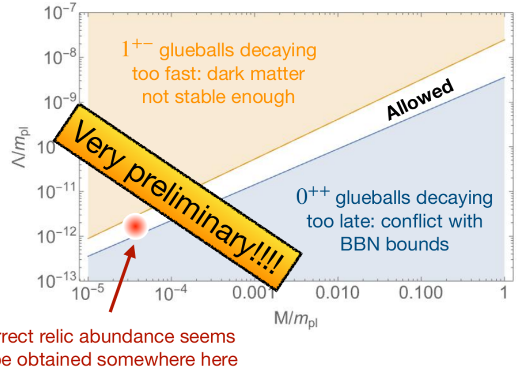
at scales $\Lambda_{\rm IR} \ll M$ glueballs decay to the SM sector:

$$\Gamma_{0^{++}} \sim \frac{\Lambda_{\rm IR}^5}{M^4}\,, \qquad \Gamma_{1^{+-}} \sim \frac{\Lambda_{\rm IR}^9}{M^8}$$

13 see e.g. L. Forestell, D. Morrissey, K. Sigurdson Phys. Rev. D 95, 015032

SU(3) sector coupled to inflaton:[14]

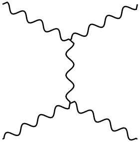
* C-odd ↔ stable glueballs form DM
* C-even ↔ unstable glueballs reheat the SM



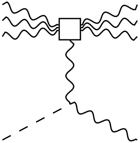
[14] S. Biondini, H. Kolesova, S. Procacci, in preparation.

how about the thermalization assumption?

lesson from heavy-ion studies: effective kinetic theory



$\sim \alpha^2 T$



$\sim \Upsilon$

a posteriori: thermalization rate of φ suppressed, $\Gamma_\varphi \sim \Upsilon \ll \alpha^2 T \sim \Gamma_g$.

what are Γ_φ and Γ_g if the plasma is strongly coupled?

perturbative thermodynamics for thermalized inflaton

$$V \approx \underbrace{V_0\,\big|_{m^2}}_{\substack{\text{UV physics}\\ \text{from } J}} + \underbrace{V_0\,\big|_{m^2 \to m_{\mathrm{T}}^2 - m^2}}_{\substack{\text{thermal mass}\\ \text{from } J}} + \underbrace{V_{\mathrm{eff}}^{(1)}}_{\substack{\text{thermalized}\\ \varphi}}$$

1-loop free energy of weakly coupled massive scalar field

$$V_{\mathrm{eff}}^{(1)} = -\lim_{L \to \infty} \frac{T}{L^3} \ln \mathcal{Z}^{(1)}$$

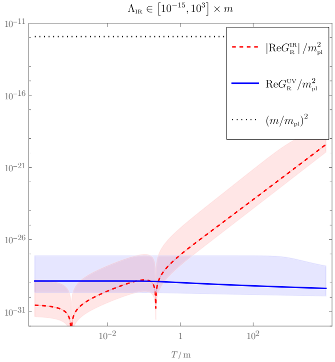
$$\Rightarrow \; p_\varphi + V_0 \supset \frac{m^2 T^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2\left(\frac{nm}{T}\right),$$

$$e_\varphi - V_0 \supset \dots,$$

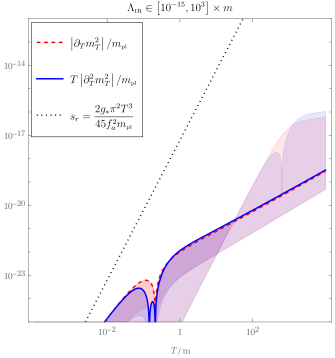
$$-T\partial_T \dot{V} \supset \dots$$

exponentially small contributions at $m \ll \pi T$

m_{T} is negligible



(a) UV and IR contributions to the thermal mass m_{T}^2



(b) thermal corrections to total entropy density and heat capacity.

cusps at $m \approx 5.2\,T$ and $m \approx 1023.0\,T$ mark the change of sign in the IR contributions.

induced potential

$$\mathcal{L} \supset -\frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - \varphi J + \mathcal{L}_{\text{bath}}\ , \quad \int \mathrm{d}^4x\,\sqrt{-g}\,J = -\frac{Q}{f_a}\ ,\ Q \in \mathbb{Z}$$

treat $\theta \equiv \varphi/f_a$ as free parameter, Wick-rotate and integrate over fields in $\mathcal{L}_{\text{bath}}$:

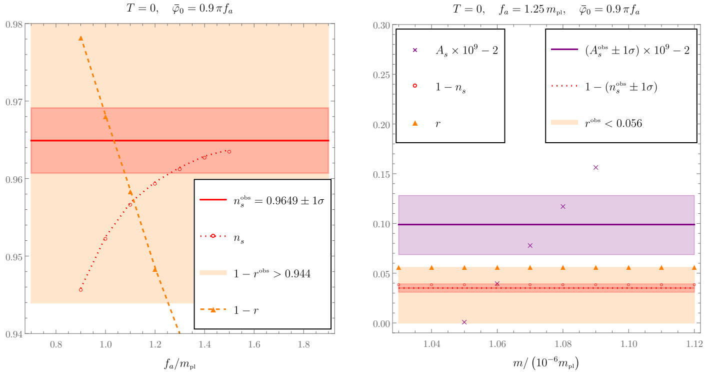
$$V(\theta) = -\lim_{L\to\infty}\frac{T}{L^3}\,\ln\left(\frac{\mathcal{Z}(\theta)}{\mathcal{Z}(0)}\right)$$

dilute instanton-gas approximation

$$\mathcal{Z}(\theta) \simeq \sum_{n,\bar{n}=0}^{\infty}\frac{[\mathcal{Z}_1 e^{i\theta}]^n}{n!}\frac{[\mathcal{Z}_1 e^{-i\theta}]^{\bar{n}}}{\bar{n}!} = \exp[2\mathcal{Z}_1\cos\theta]$$

$$\Rightarrow\ V(\varphi) \simeq \Lambda_{\text{UV}}^4\left[1-\cos\left(\frac{\varphi}{f_a}\right)\right]$$

inflaton mass m and decay rate f_a from CMB constraints



A_s, n_s and r parametrize the shape and amplitude of the perturbation spectrum observed at the CMB,

$$\mathcal{P}_{\rm T} = A_{\rm T}\left(\frac{k}{k_*}\right)^{n_{\rm T}-1}, \quad \mathcal{P}_{\mathcal{R}} = A_s\left(\frac{k}{k_*}\right)^{n_s-1}, \quad r \equiv \frac{A_{\rm T}}{A_s}$$