Vertex Operator Super Algebras on a Riemann Surface

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Introduction

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1. Sewing Tori to Form a Genus Two Riemann Surface

Consider two oriented tori $\Sigma_a = \mathbb{C}/\Lambda_{\tau_a}$ with $a = 1, 2$ for $\Lambda_{\tau_a} = 2\pi i(\mathbb{Z} \oplus \tau_a\mathbb{Z})$ for $\tau_a \in \mathbb{H}_1$, the complex upper half plane.

For $z_a \in \Sigma_a$ the closed disk $|z_a| \leq r_a$ is contained in Σ_a provided $r_a < \frac{1}{2}D(\tau_a)$ where

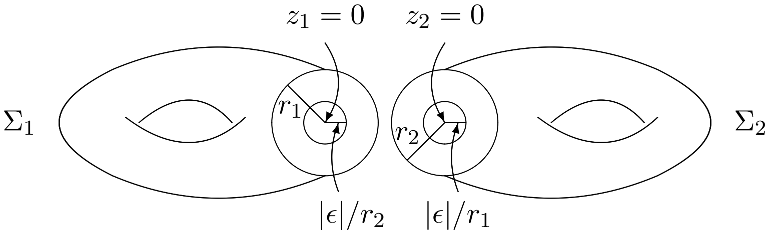
$$D(\tau_a) = \min_{\lambda \in \Lambda_{\tau_a}, \lambda \neq 0} |\lambda| = \text{minimal lattice distance.}$$

Introduce a sewing parameter $\epsilon \in \mathbb{C}$ and excise the disks $|z_1| \leq |\epsilon|/r_2$ and $|z_2| \leq |\epsilon|/r_1$ where

$$|\epsilon| \leq r_1 r_2 < \frac{1}{4}D(\tau_1)D(\tau_2).$$

Identify annular regions $|\epsilon|/r_2 \leq |z_1| \leq r_1$ and $|\epsilon|/r_1 \leq |z_2| \leq r_2$ via the sewing relation

$$z_1 z_2 = \epsilon.$$



Gives a genus two Riemann surface $\Sigma^{(2)}$ parameterized by the domain

$$\mathcal{D}^{\epsilon} = \{(\tau_1, \tau_2, \epsilon) \in \mathbb{H}_1 \times \mathbb{H}_1 \times \mathbb{C} \mid |\epsilon| < \frac{1}{4} D(\tau_1) D(\tau_2)\}.$$

Structures on $\Sigma^{(2)}$ Constructed from Genus One Data

Yamada (1980) describes how to compute the period matrix and other structures on a genus g Riemann surface in terms of lower genus data.

For standard homology basis a_i,b_j with $i = 1, \ldots, g$ on a genus g Riemann surface consider the *normalized differential of the second kind* which is a symmetric meromorphic form with

$$\omega(x,y) \sim \frac{dxdy}{(x-y)^2} \quad \text{for local coordinates } x \sim y,$$

where $\int_{a_i} \omega(x,\cdot) = 0$.

A normalized basis of holomorphic 1-forms ν_i and the period matrix Ω_{ij} are given by

$$\begin{aligned} \nu_i(x) &= \oint_{b_i} \omega(x,\cdot), \\ \Omega_{ij} &= \frac{1}{2\pi i} \oint_{b_i} \nu_i. \end{aligned}$$

$\omega^{(2)}$ on the Sewn Surface $\Sigma^{(2)}$

$\omega^{(2)}$ can be determined from $\omega^{(1)}$ on each torus in Yamada's sewing scheme [Yamada, Mason-Tuite].
For a torus $\Sigma^{(1)} = \mathbb{C}/\Lambda_\tau$ the differential is

$$\begin{aligned} \omega^{(1)}(x,y) &= P_2(x-y,\tau)\, dx\, dy, \\ P_2(z,\tau) &= \wp(z,\tau) + E_2(\tau), \end{aligned}$$

for Weierstrass function

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{k\geq 4} (k-1) E_k(\tau) z^{k-2},$$

and Eisenstein series for $k \geq 2$

$$E_k(\tau) = \frac{1}{(2\pi i)^k} \sum_m \left[{\sum_n}' \frac{1}{(m\tau+n)^k} \right].$$

E_k vanishes for odd k and is a weight k modular form for $k \geq 4$.
E_2 is a quasi-modular form.

Expanding

$$P_2(x-y,\tau) = \frac{1}{(x-y)^2} + \sum_{k,l\geq 1} C(k,l)x^{k-1}y^{l-1},$$

where

$$C(k,l) = C(k,l,\tau) = (-1)^{k+1}\frac{(k+l-1)!}{(k-1)!(l-1)!}E_{k+l}(\tau),$$

we compute $\omega^{(2)}(x,y)$ in the sewing scheme in terms of the following genus one data

$$A_a(k,l,\tau_a,\epsilon) = \frac{\epsilon^{(k+l)/2}}{\sqrt{kl}}C(k,l,\tau_a) =$$

$$\begin{bmatrix} \epsilon E_2(\tau_a) & 0 & \sqrt{3}\epsilon^2 E_4(\tau_a) & 0 & \cdots \\ 0 & -3\epsilon^2 E_4(\tau_a) & 0 & -5\sqrt{2}\epsilon^3 E_6(\tau_a) & \cdots \\ \sqrt{3}\epsilon^2 E_4(\tau_a) & 0 & 10\epsilon^3 E_6(\tau_a) & 0 & \cdots \\ 0 & -5\sqrt{2}\epsilon^3 E_6(\tau_a) & 0 & -35\epsilon^4 E_8(\tau_a) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A Determinant and the Period Matrix

Consider the infinite matrix $I - A_1 A_2$ where I is the infinite identity matrix and define $\det(I - A_1 A_2)$ by

$$\begin{aligned} \log\det(I - A_1A_2) &= \operatorname{Tr}\log(I - A_1A_2) \\ &= -\sum_{n\geq 1}\frac{1}{n}\operatorname{Tr}((A_1A_2)^n), \end{aligned}$$

as a formal power series in ϵ.

Theorem (Mason-Tuite)

(a) The infinite matrix

$$(I - A_1A_2)^{-1} = \sum_{n\geq 0}(A_1A_2)^n,$$

is convergent for $(\tau_1, \tau_2, \epsilon) \in \mathcal{D}^{\epsilon}$.
(b) $\det(I - A_1A_2)$ *is non-vanishing and holomorphic on* $\mathcal{D}^{\epsilon}$.

Furthermore we may obtain an explicit formula for the genus two period matrix $\Omega = \Omega^{(2)}$ on $\Sigma^{(2)}$

Theorem (Mason-Tuite)

$\Omega = \Omega(\tau_1, \tau_2, \epsilon)$ is holomorphic on $\mathcal{D}^\epsilon$ and is given by

$$\begin{aligned} 2\pi i \Omega_{11} &= 2\pi i \tau_1 + \epsilon (A_2 (I - A_1 A_2)^{-1})(1,1), \\ 2\pi i \Omega_{22} &= 2\pi i \tau_2 + \epsilon (A_1 (I - A_2 A_1)^{-1})(1,1), \\ 2\pi i \Omega_{12} &= -\epsilon (I - A_1 A_2)^{-1}(1,1). \end{aligned}$$

Here $(1,1)$ refers to the $(1,1)$-entry of a matrix.

The Szegö Kernel

The Szegö Kernel is defined by

$$S\begin{bmatrix}\theta\\ \phi\end{bmatrix}(x,y|\Omega)=\frac{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}\left(\int_y^x \nu\right)}{\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0)E(x,y)}\sim\frac{dx^{\frac{1}{2}}dy^{\frac{1}{2}}}{x-y}\quad \text{for } x\sim y,$$

with $\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(0)\neq 0$ for Riemann theta series with real characteristics $\alpha=(\alpha_i)$, $\beta=(\beta_i)$ for $i=1,\ldots,g$

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\Omega)=\sum_{n\in\mathbb{Z}^g}\exp\left(i\pi(n+\alpha).\Omega.(n+\alpha)+(n+\alpha).(z+2\pi i\beta)\right),$$

$$\theta_j=-e^{-2\pi i\beta_j},\quad \phi_j=-e^{2\pi i\alpha_j},\quad j=1,\ldots,g,$$

and $E(x,y)$ is the genus g prime form.

Genus One Szego Kernel

On the torus $\Sigma^{(1)}$ the Szegö kernel for $(\theta, \phi) \neq (1, 1)$ is

$$S^{(1)} \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (x, y|\tau) = P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (x - y, \tau) dx^{\frac{1}{2}} dy^{\frac{1}{2}},$$

where

$$P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau) = \frac{\vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (z, \tau)}{\vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (0, \tau)} \frac{\partial_z \vartheta_1(0, \tau)}{\vartheta_1(z, \tau)},$$

for $\vartheta_1(z, \tau) = \vartheta \left[\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right] (z, \tau)$.

We define 'twisted' modular weight k Eisenstein series [DLM, Mason-Tuite-Z]

$$P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](z,\tau) = \frac{1}{z}-\sum_{k\geq 1}E_k\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau)z^{k-1},$$

$$E_k\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau) = \frac{1}{(2\pi i)^k}\sum_m\left[\sum_n{}' \frac{\theta^m\phi^n}{(m\tau+n)^k}\right].$$

It is also useful to note that

$$P_1\left[\begin{array}{c}\theta\\ \phi\end{array}\right](x-y,\tau)=\frac{1}{x-y}+\sum_{k,l\geq 1}C\left[\begin{array}{c}\theta\\ \phi\end{array}\right](k,l)x^{k-1}y^{l-1},$$

where $C\left[\begin{array}{c}\theta\\ \phi\end{array}\right](k,l,\tau)=(-1)^l\binom{k+l-2}{k-1}E_{k+l-1}\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau)$.

Modular Properties

Define the standard left action of the modular group for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = SL(2,\mathbb{Z})$ on $(z,\tau) \in \mathbb{C} \times \mathbb{H}$ with

$$\gamma.(z,\tau) = (\gamma.z, \gamma.\tau) = \left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right).$$

We also define a *left* action of Γ on (θ, ϕ)

$$\gamma. \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} \theta^a \phi^b \\ \theta^c \phi^d \end{bmatrix}.$$

Then we obtain:

Theorem (Mason-Tuite-Z)

For $(\theta, \phi) \neq (1,1)$ *we have*

$$P_k \left(\gamma. \begin{bmatrix} \theta \\ \phi \end{bmatrix} \right) (\gamma.z, \gamma.\tau) = (c\tau + d)^k P_k \begin{bmatrix} \theta \\ \phi \end{bmatrix} (z,\tau).$$

Modular Properties

Theorem (Mason-Tuite-Z)

For $(\theta, \phi) \neq (1, 1)$, $E_k \left[\begin{array}{c} \theta \\ \phi \end{array} \right]$ *is a modular form of weight* k *where*

$$E_k \left(\gamma . \left[\begin{array}{c} \theta \\ \phi \end{array} \right] \right) (\gamma.\tau) = (c\tau + d)^k E_k \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (\tau).$$

The Szegö Kernel on $\Sigma^{(2)}$ and another Determinant

We may compute $S^{(2)}\begin{bmatrix}\theta\\\phi\end{bmatrix}(x,y)$ for $\theta=(\theta_1,\theta_2)$ in the sewing scheme in terms of the genus one data

$$F_a(k,l)=F_a\begin{bmatrix}\theta_a\\\phi_a\end{bmatrix}(k,l,\tau_a,\epsilon)=\epsilon^{\frac{1}{2}(k+l-1)}C\begin{bmatrix}\theta_a\\\phi_a\end{bmatrix}(k,l,\tau_a).$$

$S^{(2)}$ is described in terms of the infinite matrix $I-Q$ for

$$Q=\begin{bmatrix}0 & \xi\, F_1\begin{bmatrix}\theta_1\\\phi_1\end{bmatrix}\\ -\xi\, F_2\begin{bmatrix}\theta_2\\\phi_2\end{bmatrix} & 0\end{bmatrix},\qquad \xi=\sqrt{-1}.$$

Theorem (Tuite-Z)

(a) The infinite matrix $(I-Q)^{-1}=\sum_{n\geq 0}Q^n$ *is convergent for* $(\tau_1,\tau_2,\epsilon)\in\mathcal{D}^\epsilon$,
(b) $\det(I-Q)$ *is non-vanishing and holomorphic on* $\mathcal{D}^\epsilon$.

A Vertex Operator Superalgebra (VOSA) is a quadruple $(V, Y, \mathbf{1}, \omega)$: $V = V_{\bar{0}} \oplus V_{\bar{1}} = \bigoplus_{n\geq 0} V_n$ is a superspace, Y is a linear map $Y : V \to (\mathrm{End}V)[[z, z^{-1}]]$: so that for any vector (state) $a \in V$,

$$Y(a, z) = \sum_{n\in\mathbb{Z}} a(n) z^{-n-1}, \quad a(k)\mathbf{1} = \delta_{k,-1} a, \quad k \geq -1,$$

$a(n)V_\alpha \subset V_{\alpha+p(a)}$, with locality property for all a, $b \in V$

$$(x - y)^N [Y(a, x), Y(b, y)] = 0;$$

$\mathbf{1} \in V_{\bar{0},0}$ is the vacuum vector, $Y(\mathbf{1}, z) = Id_V$, and $\omega \in V_{\bar{0},2}$ the conformal vector,

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2},$$

where $L(n)$ forms a Virasoro algebra for central charge c

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m,-n}.$$

$L(-1)$ satisfies the translation property

$$Y(L(-1)a, z) = \frac{d}{dz} Y(a, z).$$

$L(0)$ describes the grading with $L(0)a = wt(a)a$, and $V_n = \{a \in V | wt(a) = n\}$.
We quote also the standard commutator property of VOSAs

$$[a(m), Y(b, z)] = \sum_{j \geq 0} \binom{m}{j} Y(a(j)b, z) z^{m-j}.$$

Note also the associativity property for a, $b \in V$,

$$Y(a, x)Y(b, y) = Y(Y(a, x - y)b, y),$$

Heisenberg Continuous Automorphisms and Twisted Modules

The Heisenberg vector zero mode $a(0)$ generates a **continuous** VOSA automorphism

$$g = \exp(2\pi i \alpha a(0)), \quad \alpha \in \mathbb{R}.$$

In particular we define the order two 'fermion number' by $\sigma = \exp(\pi i a(0))$.

We can construct [Li] a g-twisted module as follows. Define

$$\Delta(\alpha, z) = z^{\alpha a(0)} \exp\left(-\alpha \sum_{n\geq 1} a(n) \frac{(-z)^{-n}}{n}\right),$$

and for all $v \in V$

$$Y_g(v, z) = Y(\Delta(-\alpha, z)v, z).$$

Then (V, Y_g) is a g-twisted V-module M_g.

The Genus Two Partition Function for a VOA

For a VOA $V = \oplus_{n \geq 0} V_n$ of central charge c define the genus one partition (trace or characteristic) function by

$$Z_V^{(1)}(q) = Tr_V(q^{L(0)-c/24}) = \sum_{n \geq 0} \dim V_n q^{n-c/24},$$

For the Heisenberg VOA M

$$Z_M^{(1)}(q) = \frac{1}{\eta(\tau)} \quad \text{for } \eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n),$$

For a Heisenberg module $M \otimes e^{\alpha}$ we have

$$Z^{(1)}_{M\otimes e^{\alpha}}(q) = \frac{q^{\alpha^2/2}}{\eta(\tau)}.$$

For a lattice VOA V_L this implies

$$Z^{(1)}_{V_L}(q) = \frac{\theta_L(q)}{\eta(\tau)^c}, \quad \text{for } \theta_L(q) = \sum_{\alpha\in L} \exp(\pi i \tau(\alpha,\alpha)),$$

the lattice theta function $\theta_L(q)$.

1-Point Trace Functions

For $u \in V$ we define

$$Z_V^{(1)}(u, \tau) = Tr(Y(z^{L(0)}u, z)q^{L(0)}),$$

which is independent of z.
Zhu developed recursion relations for these 1-point functions in terms of the square bracket VOA with vertex operators

$$Y[u, z] = Y(q_z^{L(0)}u, q_z - 1) = \sum_n u[n]z^{n-1},$$

(for $q_z = e^z$) and Virasoro vector $\tilde{\omega} = \omega - \frac{c}{24}\mathbf{1}$. $V = \oplus_n V_{[n]}$ with associated $\tilde{\omega}$ grading operator $L[0]$.

The Heisenberg VOA

For the Heisenberg VOA M generated by $a \in V_1$ we may choose a Fock basis

$$u = a[-k_1] \ldots a[-k_n]\mathbf{1},$$

for $k_i \geq 1$ square bracket weight $wt[u] = \sum_i k_i$.
Zhu's recursion relations allows us to compute all 1-point functions in this example:

Theorem (Mason-Tuite)

$$Z_M^{(1)}(u, \tau) = \frac{Q_u(\tau)}{\eta(q)},$$

where $Q_u(\tau)$ is a quasimodular form of weight $wt[u]$ which can be combinatorially expressed in terms of all pairs $C(k_i, k_j, \tau)$.

This can be generalized for 1-point functions $Z^{(1)}_{M\otimes e^\alpha}(u, \tau)$ for any Heisenberg module.

The Genus Two Partition Function

We define the genus two partition function in the earlier sewing scheme in terms of data coming from the two tori, namely the set of 1-point functions $Z_V^{(1)}(u, \tau_a)$ for all $u \in V$.
We assume that V has a nondegenerate invariant bilinear form - the Li-Zamolodchikov metric (which holds if $\dim V_0 = 1$ and V is simple).
Define

$$Z_V^{(2)}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in V_{[n]}} Z_V^{(1)}(u, \tau_1) Z_V^{(1)}(\bar{u}, \tau_2)$$

The inner sum is taken over any basis and $\bar{u}$ is dual to u wrt to the Li-Zamolodchikov metric.

The Heisenberg VOA.

We can compute $Z_M^{(2)}$ using a combinatorial-graphical technique based on the explicit Fock basis and recalling the infinite matrices A_1, A_2:

Theorem (Mason-Tuite)

(a) The genus two partition function for the rank one Heisenberg VOA is

$$Z_M^{(2)}(\tau_1, \tau_2, \epsilon) = \frac{1}{\eta(\tau_1)\eta(\tau_2)} (\det(I - A_1 A_2))^{-1/2},$$

(b) $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ *is holomorphic on the domain* $\mathcal{D}^\epsilon$*,*
(c) $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)^2$ *is automorphic of weight* -1 *wrt the modular group* $G = SL(2, \mathbb{Z}) \wr 2 \subset Sp(4, \mathbb{Z})$ *with a Siegel-form like automorphic factor and multipliers.*
(d) $Z_M^{(2)}(\tau_1, \tau_2, \epsilon)$ *has an infinite product formula.*

We may also consider a pair of Heisenberg modules $M \otimes e^{\alpha_a}$ for $a = 1, 2$. The partition function is then

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon) = \sum_{n \geq 0} \epsilon^n \sum_{u \in M_{[n]}} Z^{(1)}_{M \otimes e^{\alpha_1}}(u, \tau_1) Z^{(1)}_{M \otimes e^{\alpha_2}}(\bar{u}, \tau_2),$$

Let $\alpha.\Omega.\alpha = \sum_{i,j=1,2} \alpha_i \Omega_{ij} \alpha_j$ where Ω_{ij} is the genus two period matrix.

Theorem (Mason-Tuite)

(a)

$$Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon) = e^{i\pi\alpha.\Omega.\alpha} Z^{(2)}_{M}(\tau_1, \tau_2, \epsilon),$$

(b) $Z^{(2)}_{\alpha_1,\alpha_2}(\tau_1, \tau_2, \epsilon)$ *is holomorphic on the domain* $\mathcal{D}^{\epsilon}$.

Lattice VOA

Consider a lattice VOA V_L for a rank l lattice. Viewing $M^l \otimes e^\alpha$ as a simple module for M^l the previous result implies

Theorem (Mason-Tuite)

We have

$$Z^{(2)}_{V_L}(\tau_1, \tau_2, \epsilon) = Z^{(2)}_{M^l}(\tau_1, \tau_2, \epsilon)\theta^{(2)}_L(\Omega),$$

where $\theta^{(2)}_L(\Omega)$ *is the genus two Siegel lattice theta function*

$$\theta^{(2)}_L(\Omega) = \sum_{\alpha,\beta \in L} \exp(\pi i((\alpha,\alpha)\Omega_{11} + 2(\alpha,\beta)\Omega_{12} + (\beta,\beta)\Omega_{22})).$$

4. Rank Two Fermionic Vertex Operator Super Algebra

Consider the Vertex Operator Super Algebra (VOSA) generated by

$$Y(\psi^{\pm}, z) = \sum_{n\in\mathbb{Z}} \psi^{\pm}(n) z^{-n-1},$$

for two vectors $\psi^{\pm}$ with modes satisfying anti-commutation relations

$$[\psi^{+}(m), \psi^{-}(n)]_{+} = \delta_{m,-n-1}, \quad [\psi^{\pm}(m), \psi^{\pm}(n)]_{+} = 0.$$

The VOSA vector space $V = \oplus_{k\geq 0} V_{k/2}$ is a Fock space with basis vectors

$$\Psi(\mathbf{k}, \mathbf{l}) \equiv \psi^{+}(-k_1) \ldots \psi^{+}(-k_s) \psi^{-}(-l_1) \ldots \psi^{-}(-l_t) \mathbf{1},$$

of weight

$$wt[\Psi(\mathbf{k}, \mathbf{l})] = \sum_{i} (k_i + \frac{1}{2}) + \sum_{j} (l_j + \frac{1}{2}),$$

where $1 \leq k_1 < k_2 < \ldots < k_s$ and $1 \leq l_1 < l_2 < \ldots < l_t$ with $\psi^{\pm}(k)\mathbf{1} = 0$ for all $k \geq 0$.

The conformal vector and its

$$\omega = \frac{1}{2}[\psi^+(-2)\psi^-(-1)+\psi^-(-2)\psi^+(-1)]\mathbf{1},$$

whose modes generate a Virasoro algebra of central charge 1. $\psi^\pm$ has $L(0)$-weight $\frac{1}{2}$.
The weight 1 subspace of V is $V_1 = \mathbb{C}a$, for normalized Heisenberg bosonic vector $a = \psi^+(-1)\psi^-(-1)\mathbf{1}$, conformal vector, and Virasoro grading operator are

$$[a(m), a(n)] = m\delta_{m,-n},$$

$$\omega = \frac{1}{2}a(-1)^2\mathbf{1},$$

$$L(0) = \frac{a(0)^2}{2} + \sum_{n>0} a(-n)a(n).$$

We define the genus one partition function for the VOSA by the supertrace

$$Z_V^{(1)}(\tau) = \mathrm{STr}_V(q^{L(0)-\frac{1}{24}}) = \mathrm{Tr}_V(\sigma q^{L(0)-\frac{1}{24}}) = q^{-\frac{1}{24}} \prod_{n\geq 0}(1-q^{n+\frac{1}{2}})^2,$$

where $\sigma u = e^{2\pi i wt(u)}u$.
More generally, we can construct a σg-twisted module $M_{\sigma g}$ for any automorphism $g = e^{2\pi i\beta a(0)}$ generated by the Heisenberg state $a \in V_1$.
We also introduce a second automorphism $h = e^{2\pi i\alpha a(0)}$ and define the orbifold σg-twisted trace by

$$Z_V^{(1)}\left[\begin{array}{c} h \\ g \end{array}\right](q) = \mathrm{STr}_{M_{\sigma g}}(hq^{L(0)-\frac{1}{24}}),$$

to find for $\theta = e^{-2\pi i\alpha}$,

$$Z_V^{(1)}\left[\begin{array}{c} h \\ g \end{array}\right](q) = q^{(\beta+1/2)^2/2-1/24} \prod_{l\geq 1}(1-\theta^{-1}q^{l-\beta-1})(1-\theta q^{l+\beta}).$$

Genus One 1-Point Functions

Each orbifold 1-point function can found from a generalized Zhu reduction formulas as a determinant.

Theorem (Mason-Tuite-Z)

For a Fock vector
$\Psi[\mathbf{k},\mathbf{l}] = \psi^+[-k_1]\ldots\psi^+[-k_n]\psi^-[-l_1]\ldots\psi^-[-l_n]\mathbf{1}$,

$$Z_V^{(1)}\left[\begin{array}{c} h \\ g \end{array}\right](\Psi[\mathbf{k},\mathbf{l}],q) = \det\left(\mathbf{C}\left[\begin{array}{c} \theta \\ \phi \end{array}\right]\right) Z_V^{(1)}\left[\begin{array}{c} h \\ g \end{array}\right](q),$$

where for $i,j=1,2,\ldots,n$

$$\mathbf{C}\left[\begin{array}{c} \theta \\ \phi \end{array}\right](i,j) = C\left[\begin{array}{c} \theta \\ \phi \end{array}\right](k_i,l_j,\tau).$$

Genus One n-Point Functions

In general, we can define the genus one orbifold n-point function for $v_1, \ldots, v_n \in V$ by

$$
\begin{aligned}
&Z_V^{(1)} \left[\begin{array}{c} h \\ g \end{array} \right] ((v_1, z_1), \ldots, (v_n, z_n); q) \\
&\equiv \mathrm{STr}_{M_{\sigma g}} \left(h\, Y(v_1, z_1) \ldots Y(v_n, z_n)\; q^{L(0) - \frac{1}{24}} \right) \\
&= Z_V^{(1)} \left[\begin{array}{c} h \\ g \end{array} \right] (Y[v_1, z_1].Y[v_2, z_2] \ldots Y[v_n, z_n].\mathbf{1}, q).
\end{aligned}
$$

Every orbifold n-point function can be computed using generalized **Zhu reduction** formulas in terms of a determinant with entries arising from the basic 2-point function for ψ^+, ψ^- [Mason-Tuite-Z].

Zhu Reduction Formula

To reduce an n-point function to a sum of $n-1$-point functions we need:

the supertrace property:

$$STr(AB) = p(A,B)STr(BA), \qquad p(A,B) = (-1)^{p(A)p(B)},$$

Borcherds commutation formula:

$$[a(m), Y(b,z)] = \sum\nolimits_{j \geq 0} \binom{m}{j} Y(a(j)b, z) z^{m-j},$$

an expansion for P_1-function:

$$P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (x-y, \tau) = \frac{1}{x-y} + \sum_{k,l \geq 1} C \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (k,l) x^{k-1} y^{l-1}.$$

Generating Function

Generating two-point function (for $(\theta, \phi) \neq (1,1)$) is given by

$$Z_V^{(1)} \left[\begin{array}{c} h \\ g \end{array} \right] ((\psi^+, z_1), (\psi^-, z_2); q) = P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (z_1 - z_2, q) \; Z_V^{(1)} \left[\begin{array}{c} h \\ g \end{array} \right] (q).$$

Theorem (Mason-Tuite-Z)

$$Z_V^{(1)} \left[\begin{array}{c} h \\ g \end{array} \right] ((v_1, z_1), \ldots, (v_n, z_n); q) = Z_V^{(1)} \left[\begin{array}{c} h \\ g \end{array} \right] (q) \; \det M.$$

Theorem (Mason-Tuite-Z)

For n *Fock vectors* $\Psi^{(a)} = \Psi^{(a)}[-\mathbf{k}^{(a)}; -\mathbf{l}^{(a)}]$ *and* $\Psi_h^{(a)} = \Psi^{(a)}[-\mathbf{k}^{(a)}; -\mathbf{l}^{(a)}]_h$ *for* $\mathbf{k}^{(a)} = k_1^{(a)}, \ldots k_{s_a}^{(a)}$ *and* $\mathbf{l}^{(a)} = l_1^{(a)}, \ldots l_{t_a}^{(a)}$ *with* $a = 1 \ldots n$. *Then for* $(\theta, \phi) \neq (1, 1)$ *the corresponding* n*-point functions are non-vanishing provided* $\sum_{a=1}^{n} (s_a - t_a) = 0$, *and*

$$
\begin{aligned}
& Z_V^{(1)} \begin{bmatrix} f \\ g \end{bmatrix} ((\Psi^{(1)}, z_1), \ldots, (\Psi^{(n)}, z_n); \tau) \\
&= Z_{V,h}^{(1)}(f; (\Psi_h^{(1)}, z_1), \ldots, (\Psi_h^{(n)}, z_n); \tau) \\
&= \epsilon \det \mathbf{M}. \ Z_{V,h}^{(1)}(f; \tau).
\end{aligned}
$$

Here $\mathbf{M}$ is the block matrix

$$\mathbf{M} = \left(\begin{array}{ccc} \mathbf{C}^{(11)} & \mathbf{D}^{(12)} \ldots & \mathbf{D}^{(1n)} \\ \mathbf{D}^{(21)} & \mathbf{C}^{(22)} \ldots & \mathbf{D}^{(2n)} \\ \vdots & \ddots & \vdots \\ \mathbf{D}^{(n1)} & \ldots & \mathbf{C}^{(nn)} \end{array} \right),$$

with

$$\mathbf{C}^{(aa)}(i,j) = C \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (k_i^{(a)}, l_j^{(a)}, \tau), \quad (1 \leq i \leq s_a, 1 \leq j \leq t_a),$$

for $s_a, t_a \geq 1$ with $1 \leq a \leq n$ and

$$\mathbf{D}^{(ab)}(i,j) = D \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (k_i^{(a)}, l_j^{(b)}, \tau, z_{ab}), \quad (1 \leq i \leq s_a, 1 \leq j \leq t_b),$$

for $s_a, t_b \geq 1$ with $1 \leq a, b \leq n$ and $a \neq b$. ϵ is the sign of the permutation associated with the reordering of $\psi^{\pm}$ to the alternating ordering.
Furthermore, the n-point function (1) is an analytic function in z_a and converges absolutely and uniformly on compact subsets of the domain $|q| < |q_{z_{ab}}| < 1$.

Following the definition for the bosonic VOA we define for h_a, g_a

$$Z^{(2)} \left[\begin{array}{c} h \\ g \end{array} \right] (q_1, q_2, \epsilon) =$$
$$\sum_{m \in \frac{1}{2}\mathbb{Z}} \epsilon^m \sum_{u \in V_{[m]}} Z^{(1)} \left[\begin{array}{c} h_1 \\ g_1 \end{array} \right] (u, q_1) Z^{(1)} \left[\begin{array}{c} h_2 \\ g_2 \end{array} \right] (\bar{u}, q_2).$$

The inner sum is taken over any $V_{[m]}$ basis and $\bar{u}$ is dual to u with respect to the Li-Zamolodchikov square bracket metric.
$Z_V^{(1)} \left[\begin{array}{c} h_a \\ g_a \end{array} \right] (u, q_a)$ is the genus one orbifold 1-point function.

Recall that the non-zero 1-point functions arise for Fock vectors

$$\Psi[\mathbf{k},\mathbf{l}] = \psi^+[-k_1]\ldots\psi^+[-k_n]\psi^-[-l_1]\ldots\psi^-[-l_n]\mathbf{1},$$

$$m = \mathrm{wt}\ \Psi[\mathbf{k},\mathbf{l}] = \sum_{1\le i\le n}(k_i+l_i+1),$$

$$Z_V^{(1)}\left[\begin{array}{c} h \\ g \end{array}\right](\Psi[\mathbf{k},\mathbf{l}],q) = \det\left(\mathbf{C}\left[\begin{array}{c} \theta \\ \phi \end{array}\right]\right) Z_V^{(1)}\left[\begin{array}{c} h \\ g \end{array}\right](q).$$

The Li-Zamolodchikov metric dual to the Fock vector is

$$\overline{\Psi}[\mathbf{k},\mathbf{l}] = (-1)^n\Psi[\mathbf{l},\mathbf{k}].$$

Recalling the infinite matrix Q we find

Theorem (Tuite-Z)

(a) The genus two orbifold partition function is

$$Z^{(2)}\left[\begin{array}{c} h \\ g \end{array}\right](q_1, q_2, \epsilon) = Z^{(1)}\left[\begin{array}{c} h_1 \\ g_1 \end{array}\right](q_1) Z^{(1)}\left[\begin{array}{c} h_2 \\ g_2 \end{array}\right](q_2) \det(I - Q),$$

(b) $Z^{(2)}\left[\begin{array}{c} h \\ g \end{array}\right](q_1, q_2, \epsilon)$ *is holomorphic on the domain* $\mathcal{D}^{\epsilon}$*,*

(c) $Z^{(2)}\left[\begin{array}{c} h \\ g \end{array}\right](q_1, q_2, \epsilon)$ *has natural modular properties under the action of* G*.*

Bosonization

The genus one orbifold partition function can be alternatively computed by decomposing the VOSA into Heisenberg modules $M \otimes e^m$ indexed by $a(0)$ integer eigenvalues m, i.e., a $\mathbb{Z}$ lattice,

$$\begin{aligned} Z\left[\begin{array}{c} h \\ g \end{array}\right](\tau) &= \sum_{m\in\mathbb{Z}}(-1)^m e^{2\pi i m\alpha}\mathrm{Tr}_{M\otimes e^m}\left(q^{L(0)+\frac{1}{2}(\beta+\frac{1}{2})^2-(\beta+\frac{1}{2})m-\frac{1}{24}}\right) \\ &= \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right](\tau). \end{aligned}$$

Comparing to the fermionic product formula we obtain the standard Jacobi triple product formula:

$$\prod_{n>0}(1-q^n)(1+zq^{n-1})(1+z^{-1}q^n) = \sum_{m\in\mathbb{Z}} z^m q^{m(m-1)/2}.$$

The Genus Two Jacobi Triple Product Formula

The genus two partition function can similarly be computed in the bosonized formalism to obtain a genus two version of the Jacobi triple product formula for the genus two Riemann theta function [Mason-Tuite-Z]

$$Z^{(2)}\left[\begin{array}{c} h \\ g \end{array}\right](q_1, q_2, \epsilon) = \Theta^{(2)}\left[\begin{array}{c} a \\ b \end{array}\right](\Omega) Z_M^{(2)}(q_1, q_2, \epsilon),$$

for an appropriate character valued genus two Riemann theta function

$$\Theta^{(2)}\left[\begin{array}{c} a \\ b \end{array}\right](\Omega) = \sum_{m \in \mathbb{Z}^2} e^{i\pi(m+a).\Omega.(m+a)+2\pi i(m+a).b}.$$

Comparing with the fermionic result we thus find that on $\mathcal{D}^\epsilon$

$$\frac{\Theta^{(2)}\left[\begin{array}{c} a \\ b \end{array}\right](\Omega)}{\vartheta\left[\begin{array}{c} a_1 \\ b_1 \end{array}\right](\tau_1)\vartheta\left[\begin{array}{c} a_2 \\ b_2 \end{array}\right](\tau_2)} = \det(I - A_1 A_2)^{1/2} \det(I - Q).$$

Fay's Trisecant Identity

In a similar fashion we can compute the general $2n$-generating function $G^{(1)}_{2n,h}$ in the bosonic setting to obtain:

Theorem (Tuite-Z)

$$
\begin{aligned}
&G^{(1)}_{2n,h}(f;x_1,\ldots,x_n;y_1,\ldots,y_n;\tau)\\
&\quad=\frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\vartheta\left[\begin{array}{c}-\beta+\frac{1}{2}\\ \alpha+\frac{1}{2}\end{array}\right]\left(\sum_{i=1}^{n}(x_i-y_i),\tau\right)\\
&\quad\cdot\frac{\prod\limits_{1\leq i<j\leq n}K^{(1)}(x_i-x_j,\tau)K^{(1)}(y_i-y_j,\tau)}{\prod\limits_{1\leq i,j\leq n}K^{(1)}(x_i-y_j,\tau)}.
\end{aligned}
$$

Comparing this to fermionic expressions for $(\theta, \phi) \neq (1,1)$ we obtain the classical Frobenius elliptic function version of Fay's Generalized Trisecant Identity [Fay]:

Corollary (Tuite-Z)

For $(\theta, \phi) \neq (1,1)$ *we have*

$$\det(\mathbf{P}) = \frac{\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right]\left(\sum_{i=1}^{n}(x_i - y_i), \tau\right)}{\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right](0,\tau)}$$

$$\cdot \frac{\prod_{1\leq i<j\leq n} K^{(1)}(x_i - x_j, \tau) K^{(1)}(y_i - y_j, \tau)}{\prod_{1\leq i,j\leq n} K^{(1)}(x_i - y_j, \tau)}.$$

Corollary (Tuite-Z)

For $(\theta, \phi) = (1, 1)$,

$$\det(\widetilde{\mathbf{P}}) = -K^{(1)}\left(\sum_{i=1}^{n}(x_i - y_i), \tau\right) \cdot \frac{\prod_{1\leq i<j\leq n} K^{(1)}(x_i - x_j, \tau) K^{(1)}(y_i - y_j, \tau)}{\prod_{1\leq i,j\leq n} K(x_i - y_j, \tau)},$$

with

$$\widetilde{\mathbf{P}} = \begin{pmatrix} P_1(x_1 - y_1, \tau) & \dots & P_1(x_1 - y_n, \tau) & 1 \\ \vdots & \ddots & & \vdots \\ P_1(x_n - y_1, \tau) & & P_1(x_n - y_n, \tau) & 1 \\ 1 & \dots & 1 & 0 \end{pmatrix}.$$

Generalized Fay's Trisecant Identity

We may generalize these identities using [Mason-Tuite]. Consider the general lattice n-point function: [Tuite-Z] For integers $m_i, n_j \geq 0$ satisfying $\sum_{i=1}^{r} m_i = \sum_{j=1}^{s} n_j$, we have

$$Z_V^{(1)}(f; (\mathbf{1}\otimes e^{m_1}, x_1), \ldots (\mathbf{1}\otimes e^{m_r}, x_r), (\mathbf{1}\otimes e^{-n_1}, y_1), \ldots (\mathbf{1}\otimes e^{-n_s}, y_s); \tau)$$
$$= \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)} \vartheta \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] \left(\sum_{i=1}^{r} m_i x_i - \sum_{j=1}^{s} n_j y_j, \tau \right)$$
$$\frac{\prod\limits_{1 \leq i < k \leq r} K(x_i - x_k, \tau)^{m_i m_k} \prod\limits_{1 \leq j < l \leq s} K(y_j - y_l, \tau)^{n_j n_l}}{\prod\limits_{1 \leq i \leq r, 1 \leq j \leq s} K(x_i - y_j, \tau)^{m_i n_j}}.$$

Comparing this to the expression for n-point functions we obtain a new elliptic generalization of Fay's Trisecant Identity:

Corollary (Tuite-Z)

For $(\theta, \phi) \neq (1, 1)$ *we have*

$$\det(\mathbf{M}) = \frac{\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right]\left(\sum_{i=1}^{r} m_i x_i - \sum_{j=1}^{s} n_j y_j, \tau\right)}{\vartheta\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array}\right](0,\tau)} \cdot \frac{\prod_{1\leq i<k\leq r} K(x_i - x_k, \tau)^{m_i m_k} \prod_{1\leq j<l\leq s} K(y_j - y_l, \tau)^{n_j n_l}}{\prod_{1\leq i\leq r, 1\leq j\leq s} K(x_i - y_j, \tau)^{m_i n_j}}.$$

Here $\mathbf{M}$ is the block matrix

$$\mathbf{M} = \begin{pmatrix} \mathbf{D}^{(11)} & \dots & \mathbf{D}^{(1s)} \\ \vdots & \ddots & \vdots \\ \mathbf{D}^{(r1)} & \dots & \mathbf{D}^{(rs)} \end{pmatrix},$$

with $\mathbf{D}^{(ab)}$ the $m_a \times n_b$ matrix

$$\mathbf{D}^{(ab)}(i,j) = D\left[\begin{array}{c}\theta \\ \phi\end{array}\right](i,j,\tau,x_a - y_b), \quad (1 \le i \le m_a, 1 \le j \le n_b),$$

for $1 \le a \le r$ and $1 \le b \le s$, and D-functions are given by the expansion

$$P_1\left[\begin{array}{c}\theta \\ \phi\end{array}\right](z + z_1 - z_2, \tau) = \sum_{k,l \ge 1} D\left[\begin{array}{c}\theta \\ \phi\end{array}\right](k,l,z) z_1^{k-1} z_2^{l-1}.$$

A similar identity for $(\theta, \phi) = (1,1)$ generalizing (1) can also be described.