The extended Lipkin model: implementation in a quantum platform and machine learning analysis of its phase diagram

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Shapes and Symmetries in Nuclei: from Experiment to Theory (SSNET'24)

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Outline

Outline

Why the Lipkin/Extended Lipkin Model?

- It is a solvable many-body model that allows to deal with nuclear, molecular, solid state or quantum optics systems.
- It can be "mapped" into the Interacting Boson Model of Nuclear Physics.
- Nowadays, it is used to benchmark many-body approximations because of its great flexibility and simplicity to be solved for large systems.
- The model has a rich phase diagram.
- It is a model highly used in Quantum Information Science and, therefore, of great interest.
- It is directly mapped into Pauli matrices without using the Jordan-Wigner mapping.

Why Quantum Computing in Nuclear Physics?

- The U.S. Department of Energy in its document Energy Research Opportunity II (2019) establishes "A broad theory program should be supported, which can, e.g., develop methods to address problems in NP using digital quantum computers and quantum simulators, utilize QIS concepts to better understand nuclear phenomena (such as the nuclear many-body problem and hadronization), and develop new QIS applications of importance to nuclear physics".
- In the future, quantum computers will allow to outperform present computational (classical) capabilities.
- The applications of Quantum Computing in Nuclear Physics are increasing.
- At present (Noisy Intermediate-Scale Quantum era) is a rather active area, but still it is only dealing with small systems and schematic models: Variational Quantum Eigensolver or Quantum Equation of Motion applied to the Lipkin model, Shell-Model applications, implementation/restoration of symmetries.

Why Quantum Computing in Nuclear Physics?

- Variational Quantum Eigensolver or Quantum Equation of Motion applied to the Lipkin model, Shell-Model applications, implementation/restoration of symmetries.
  - "Cloud Quantum Computing of an Atomic Nucleus", PRL **120**, 210501 (2018).
  - "Symmetry-Assisted Preparation of Entangled Many-Body States on a Quantum Computer", PRL **125**, 230502 (2020).
  - "Lipkin model on a quantum computer", PRC **104**, 024305 (2021).
  - "Simulating excited states of the Lipkin model on a quantum computer", PRC **106**, 024319 (2022).
  - "Quantum computing of the ^{6}Li nucleus via ordered unitary coupled cluster", PRC **106** (2022).
  - "Nuclear shell-model simulation in digital quantum computers", Sci. Rep. 13:12291 (2023).

Outline

What a (digital) Quantum Computer is?

A device composed of:

- m **2-level** quantum systems (qubits),
- a set of quantum gates (acting as unitary operators),
- a set of measurement operators (measuring the state of defined subset of qubits),
- a classical control unit which determines which gate should be applied.

I. M. Georgescu, S. Ashhab, and Franco Nori, Rev. Mod. Phys. **86**, 153 (2014).

Implementations

- Trapped ions.
- Superconducting circuits.
- Nuclear spins (NMR).
- Photons.
- Neutral atoms.
- Cavity arrays.

Bits and qubits

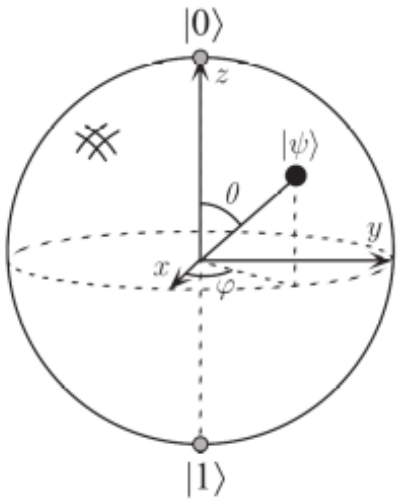
Classical vs quantum

Bits: 0 or 1

Qubits:

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$



Bloch sphere

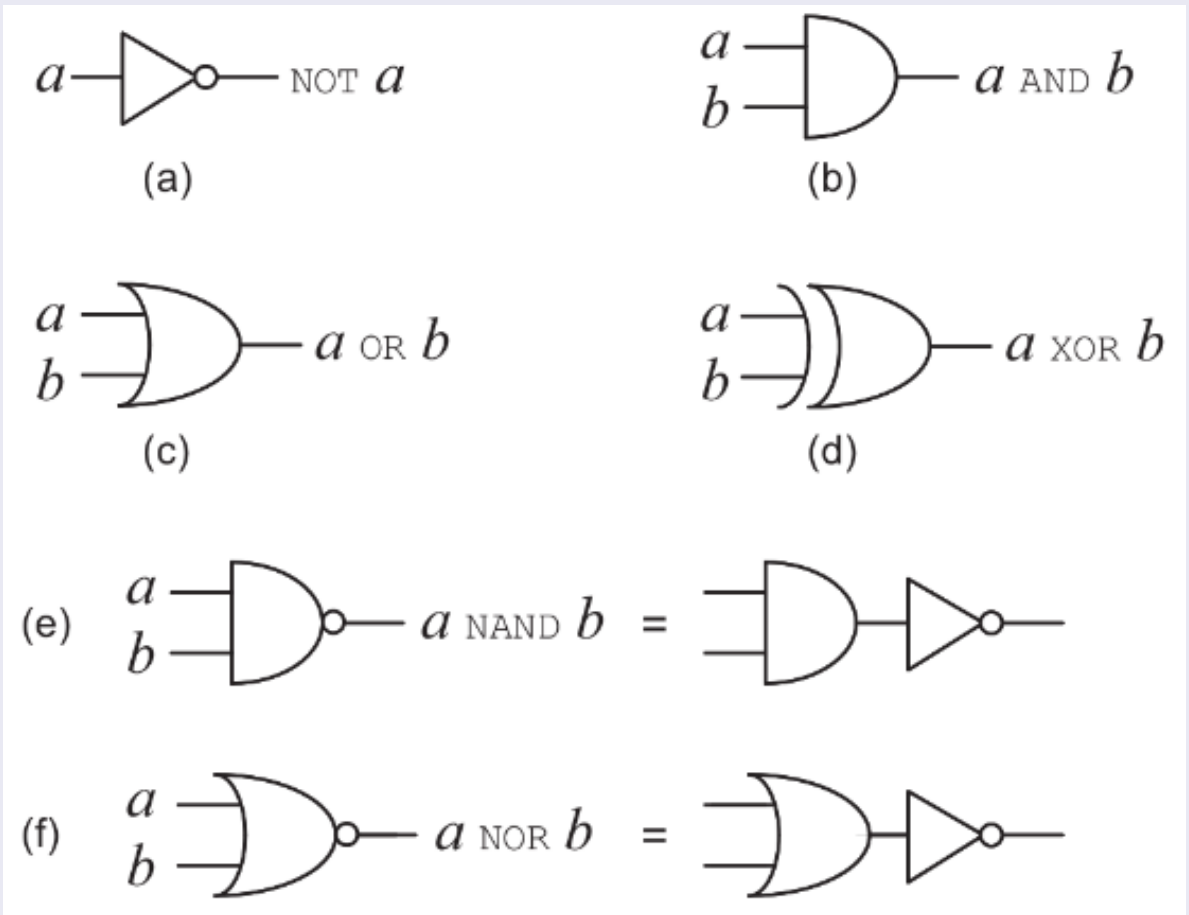
Multiple Qubits:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Bell state: $\dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

Gates

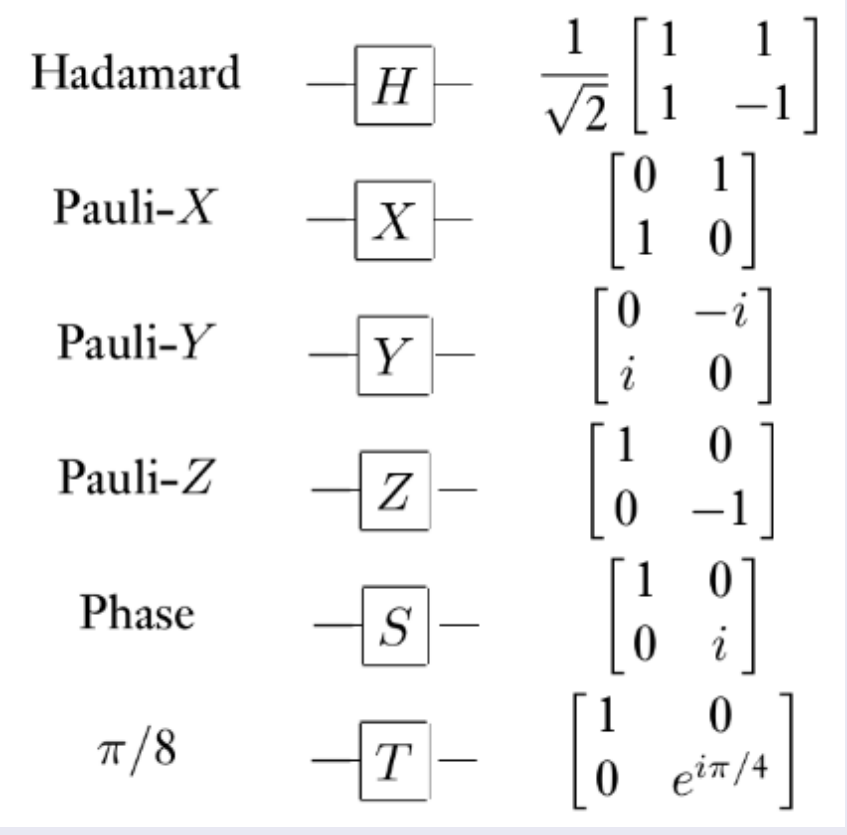
Classical gates



From M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information

Quantum gates

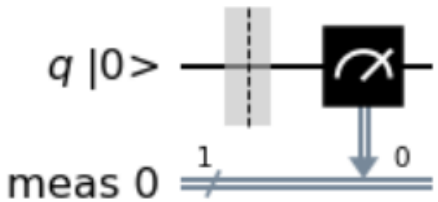
Single qubit gates

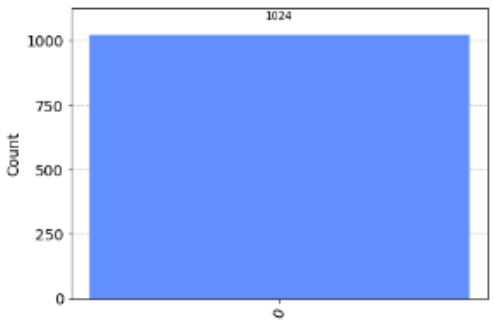


From M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information

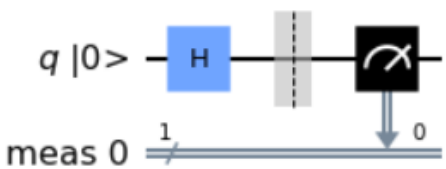
Quantum circuits

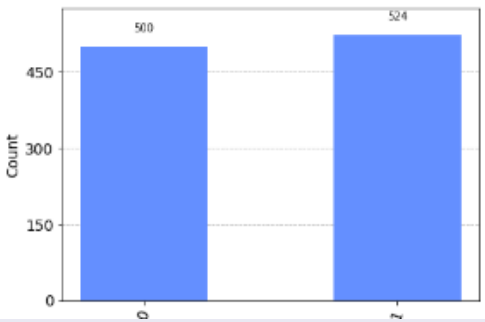
The simplest case





A step further





Results from Qiskit, https://www.ibm.com/quantum/qiskit

Outline

The (Extended) Lipkin Model

The first appearance

H.J. Lipkin, N. Meshkov, and A.J. Glick, "Validity of many-body approximation methods for a solvable model: (I). Exact solutions and perturbation theory", Nucl. Phys. **62**, 188 (1965).

The original Hamiltonian

It describes a set of N particles interacting with a long-range interaction. $\vec{S} = \sum_{i=1}^{N} \vec{s}_i$ and $S = N/2$.

$$H_L = \varepsilon S_z - \frac{V}{2}(S_+^2 + S_-^2) - \frac{W}{2}(S_+ S_- + S_- S_+)$$

A convenient way of rewriting the Hamiltonian

$$H_L = (1-\lambda)(S+S_z) - \frac{\lambda}{N}(S_+ + S_-)^2 = (1-\lambda)(S+S_z) - 4\frac{\lambda}{N}S_x^2$$

The ELM as an approximation of the Interacting Boson Model

The ELM Hamiltonian *à la* CQF

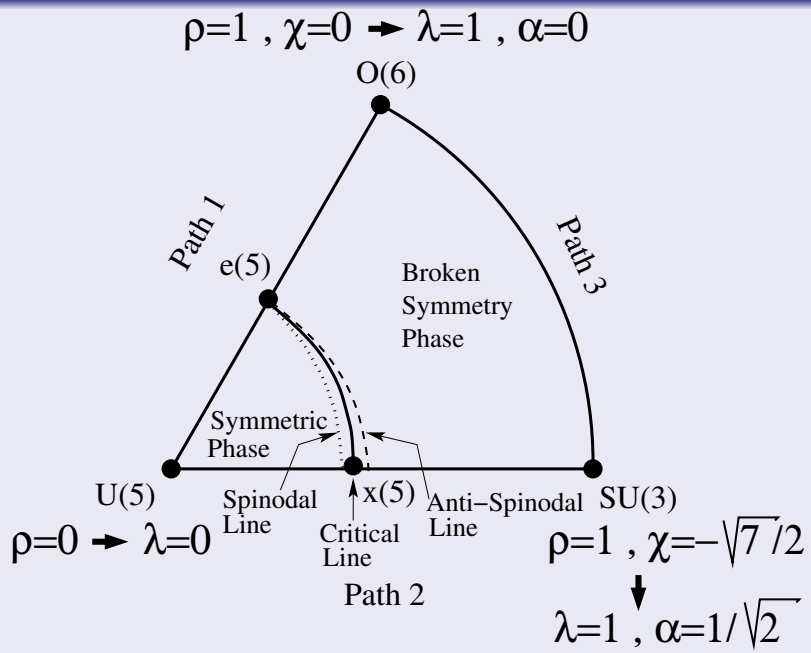
$$H = (1-\lambda)\, n_t - \frac{\lambda}{N}\, Q^{(\alpha)} \cdot Q^{(\alpha)},$$

$n_t = t^\dagger t$ and $Q^{(\alpha)} = (s^\dagger t + t^\dagger s) + \alpha(t^\dagger t)$

Equivalence with the Interacting Boson Model

$$S_+ = t^\dagger s$$
$$S_- = s^\dagger t,$$
$$S_z = \frac{1}{2}(t^\dagger t - s^\dagger s).$$

$$H_{EL} = H_L - \frac{\lambda}{N}\Big[\alpha^2(S+S_z)^2 - 2\alpha\big(S_x(S+S_z) + (S+S_z)S_x\big)\Big],$$

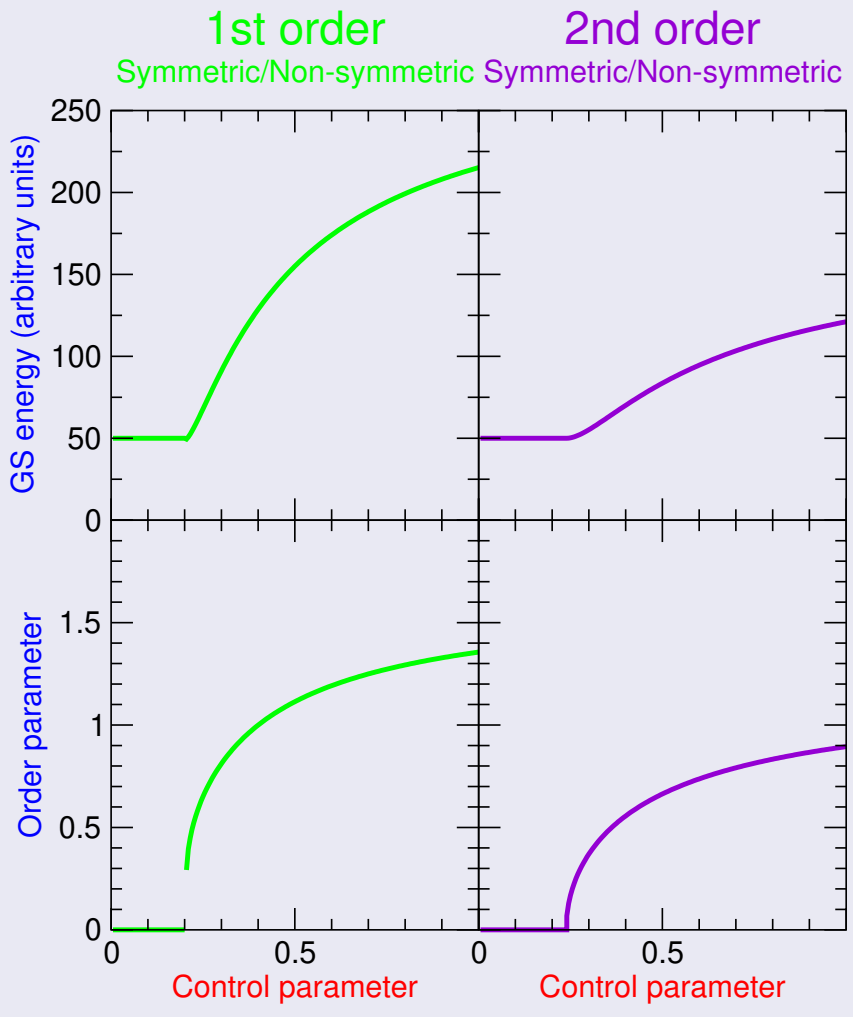


Outline

What a Quantum Phase Transition (QPT) is?

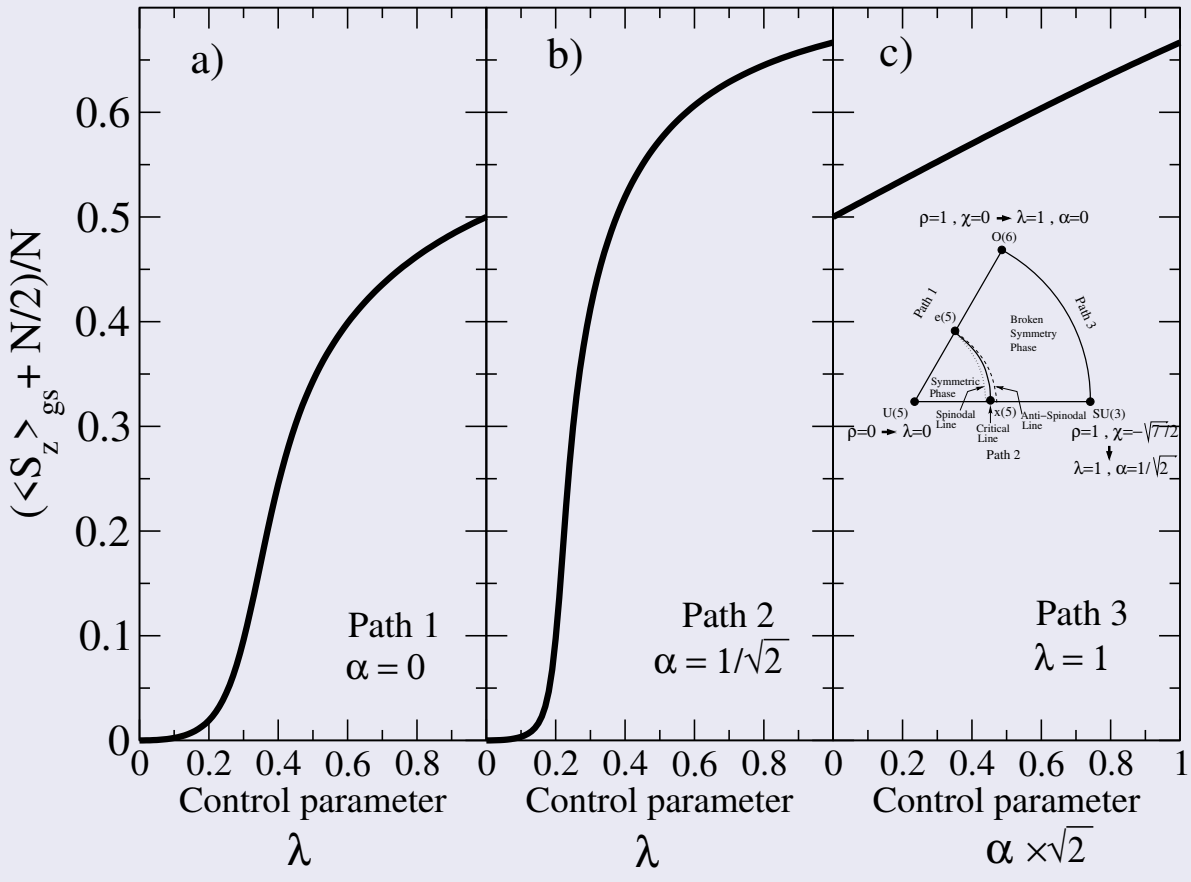
A QPT appears when the ground state a quantum system experiences a sudden change in its structure (order parameter) when a parameter that affects the Hamiltonian (control parameter) slightly changes around its critical value. This transitions are assumed to occurs at zero temperature.

$$\hat{H} = (1-\xi)\hat{H}_1 + \xi\hat{H}_2$$



Precursors of a quantum phase transition

Order parameter



Expectation value of $S_z + N/2$, normalized to N. Calculations are done for $N = 6$.

How to determine the shape/phase of the system?

The obvious things

- Shape is not really an observable.
- The shape of the system is a property of its ground state (it is true that it can be also defined for a excited state).
- It is well defined at the mean-field level.

A different view

- The shape of the system characterizes its spectrum.
- An observable depending on the spectrum could encode the shape of the system. That, in general, will happen for the time evolution of the matrix element of a non-eigenstate.
- Most probably the results will depend of the state and on the used operator. Difficult to determine a priori the best state and operator.
- These types of measurements are the easiest ones in Quantum Computing.

Our *Ansatz*

The shape of the system is encoded in the time evolution of an appropriately chosen operator.

The correlation operator

$$C_\nu(i,j,t) = \langle\Phi_0|U(t)^\dagger \sigma_\nu^i \sigma_\nu^j U(t)|\Phi_0\rangle - \langle\Phi_0|U(t)^\dagger \sigma_\nu^i U(t)|\Phi_0\rangle\langle\Phi_0|U(t)^\dagger \sigma_\nu^j U(t)|\Phi_0\rangle,$$

σ_ν^i is a Pauli matrix, $|\Phi_0\rangle$ is an arbitrary state, not an eigenstate. Also can be used S_z,

$$S_z(t) = \langle\Phi_0|U(t)^\dagger S_z U(t)|\Phi_0\rangle,$$

Time evolution as a tool determine the phase diagram of a system

The evolution operator

$$U(t) = \exp(-i\, H\, t)$$

In practice, it is implemented through the Lie-Trotter-Suzuki decomposition (Trotter in short)

$$U(t) \approx \left(\prod_k e^{-iH_k t/n_T} \right)^{n_T} = U_T(t, n_T),$$

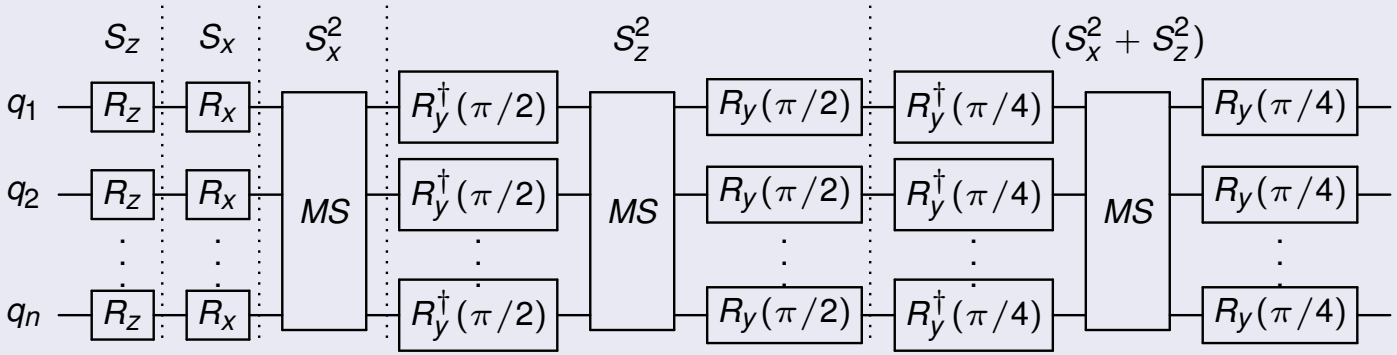
where the error produced will depend on the commutator $[H_i, H_j]$ and scale as $1/n_T$, where n_T denotes the number of Trotter steps.

The Hamiltonian terms

$$H_1 = g_z S_z + g_{zz} S_z^2,$$
$$H_2 = g_x S_x + g_{xx} S_x^2,$$
$$H_3 = g_{xz}(S_x + S_z)^2.$$

The implementation of the time evolution

Quantum circuit



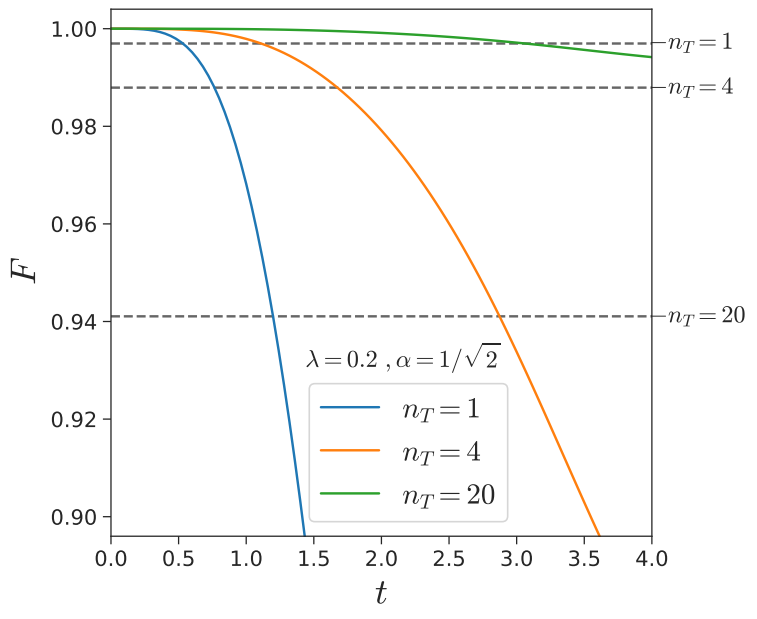
Fidelities of up to 99.9999% for single-qubit gates and up to 99.9% for MS gates.

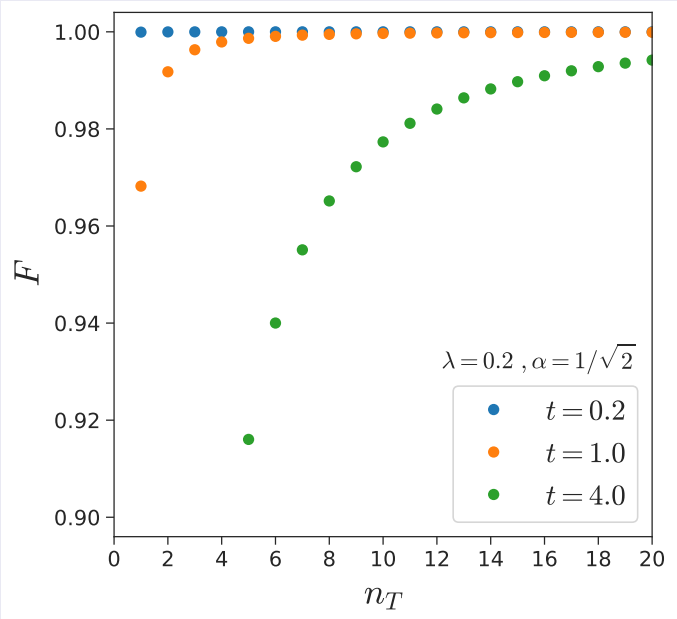
$$f = \left[(f_s)^{6N}(f_{MS})^3\right]^{n_T}.$$

The depth of the circuit depends on n_T.

The fidelity

$$F = |\langle\phi(0)|U_T(t, n_T)U(t)|\phi(0)\rangle|^2.$$

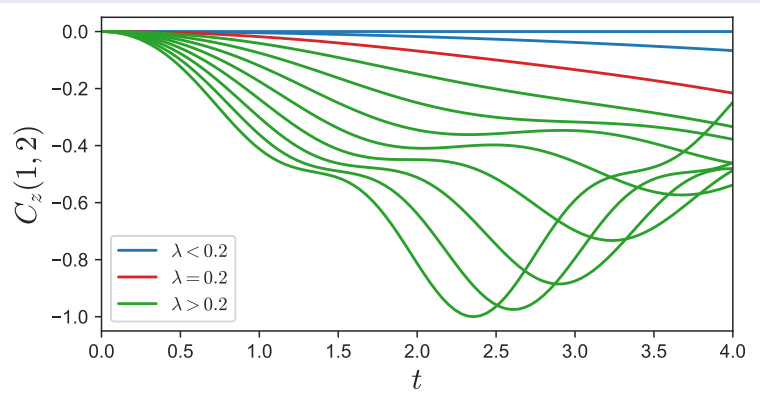


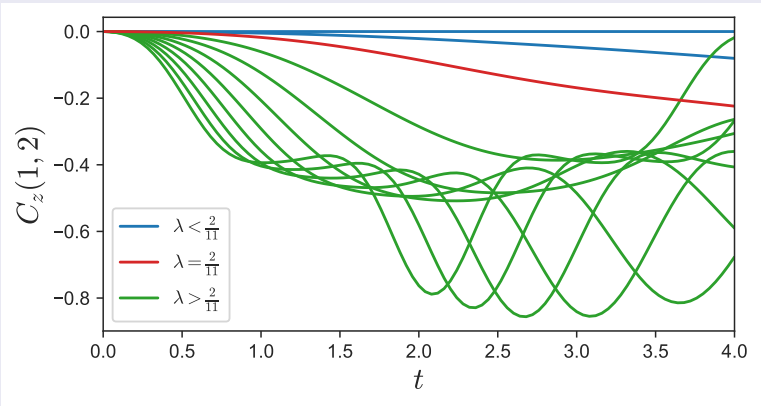


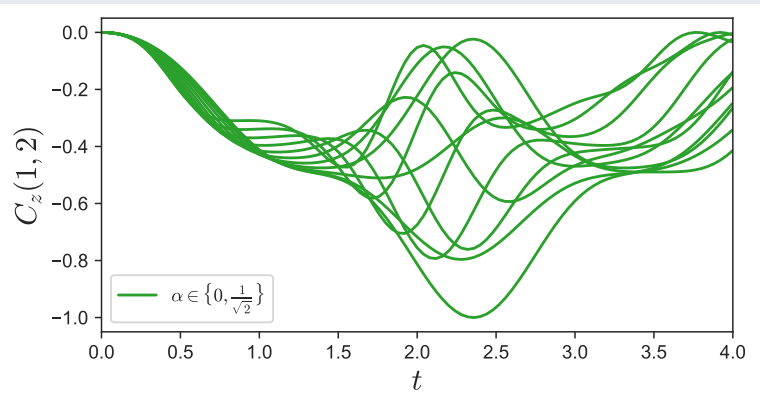
Fidelity of the trotterized evolution. $N = 6$, initial state $|\Phi_0\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$.

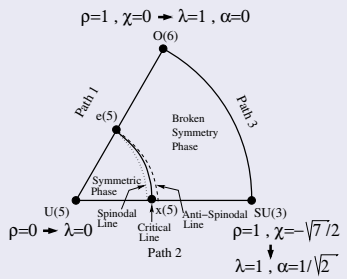
Really useful information from time evolution?

Time evolution of the correlation operator









Time evolution in selected cases for path1, path2, and path3

Machine learning to recognize the shape of the system

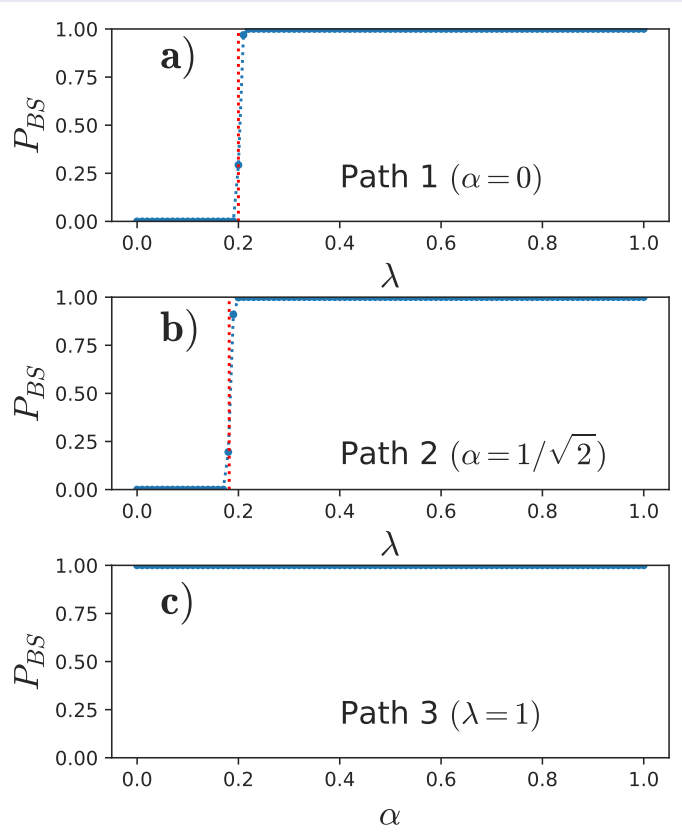
Machine learning in a classical computer

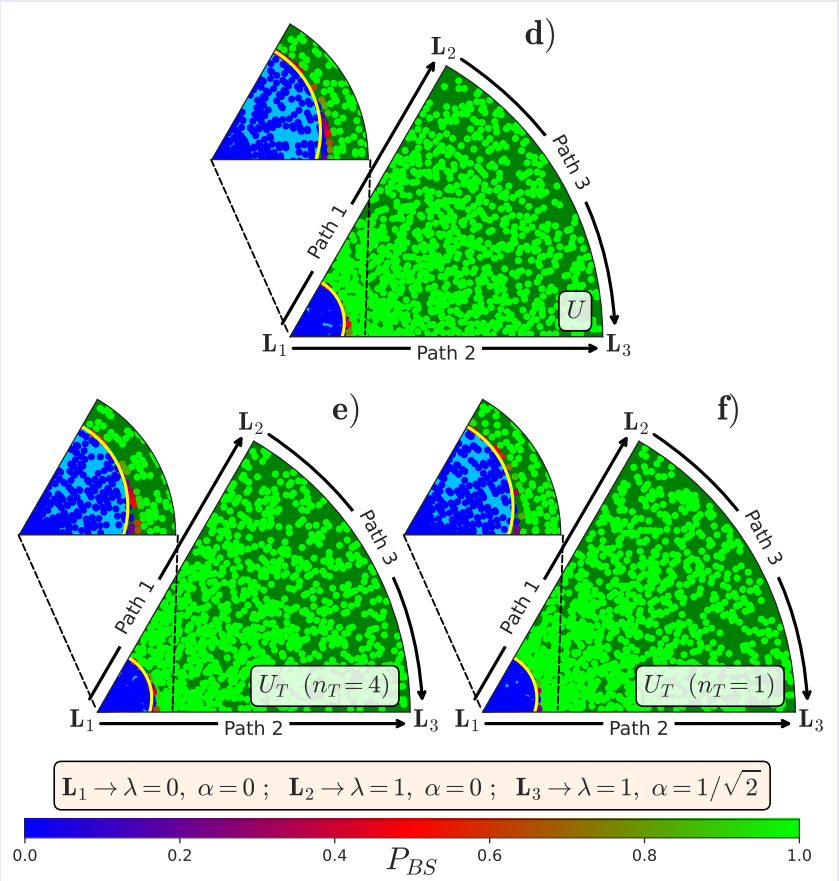
- Regression
- **Clustering**
- Decision Trees
- Reinforced Learning
- Genetic Algorithms
- **Neural Networks**

The options

- To use supervised learning.
- To use supervised learning with partial information.
- To use unsupervised learning.

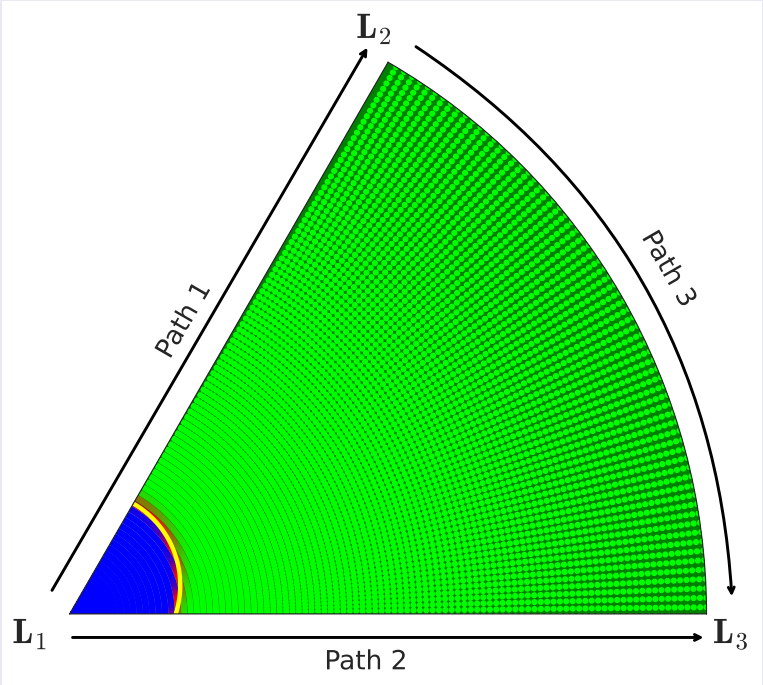
Results from a Convolutional Neural Network





Using $C_z(1, 2, t)$ for a system of size $N = 6$, with initial state $|\Phi_0\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle$.

Results from a CNN with partial information



Training the CNN only with points within path 1 ($\overline{L_1 L_2}$), i.e. $\alpha = 0$.

The unsupervised case

C-means

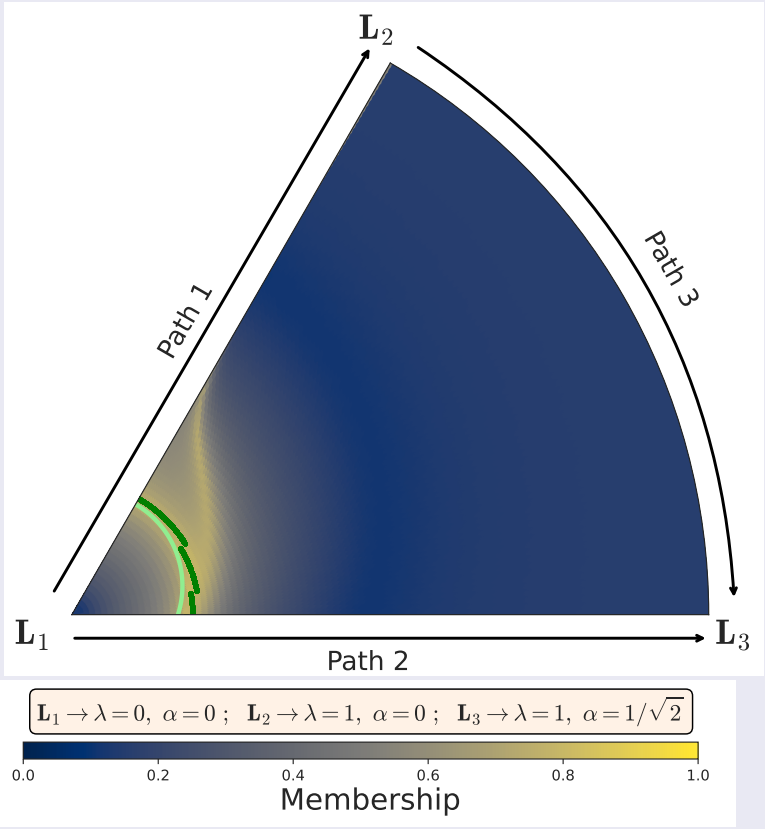
Objective function to be minimized

$$f(\vec{C}) = \sum_{i}^{n}\sum_{j}^{k} w_{i,j}(||\vec{x}_i - \vec{c}_j||)^2$$

where

$$w_{i,j} = \frac{1}{\sum_{j'}^{k}\left(\frac{||\vec{x}_i - \vec{c}_j||}{||\vec{x}_i - \vec{c}_{j'}||}\right)^{\frac{2}{m-1}}},$$

In K-means $w_{i,j} \in \{0, 1\}$.



Cluster 1 membership for the phase diagram of the ELM

Outline

Conclusions and outlook

- It has been presented the implementation of the Extended Likpin model in a quantum simulator.
- The phase diagram of the model has been determined calculating the time evolution and using a Convolutional Neural Network.
- The time evolution method does not need the knowledge of the ground state.
- Supervised and unsupervised flavours have been presented.

Further reading

- P. Pérez-Fernández, J.M. Arias, JEGR, and L. Lamata, “A digital quantum simulation of the Agassi model”, Phys. Lett. B **829**, 137133 (2022).
- A. Sáiz, JEGR, J.M. Arias, L. Lamata, and P. Pérez-Fernández, “Quantum Simulations of an Extended Agassi Model in Trapped Ions using Machine Learning”, Phys. Rev. C **106**, 064322 (2022).
- S. Baid, A. Sáiz, L. Lamata, P. Pérez-Fernández, A. Rios, A.M. Romero, J.M. Arias, and JEGR,“Extended Lipkin model: Proposal for implementation in a quantum platform and machine learning analysis of its phase diagram”, Phys. Rev. C **110**, 044318 (2024).

Thank you for your attention