

# Pair and quartet correlations in the shell model

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Two-nucleon correlation functions

Alpha-particle correlation functions

Deuteron and alpha clustering

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# Two-nucleon correlators

Two-nucleon correlation operators are defined as

$$\sum_{i<j} \delta(r - \hat{r}_{ij}), \quad \sum_{i<j} \delta(R - \hat{R}_{ij}) \delta(r - \hat{r}_{ij})$$

with

$$\hat{R}_{ij} = \frac{1}{2} |\hat{r}_i + \hat{r}_j|, \quad \hat{r}_{ij} = |\hat{r}_i - \hat{r}_j|$$

Expectation value defines a probability density:

*times  $dr \rightarrow$  probability of two nucleons separated by  $r$ ;*

*times  $drdR \rightarrow$  probability of two nucleons separated by  $r$  and at a distance  $R$  from their centre of mass.*

# Two-nucleon correlation functions

Expectation values for a two-nucleon state

$$\langle abLST | \delta(r - \hat{r}_{12}) | cdLST \rangle = \frac{1}{\sqrt{8}} r^2 \sum_{N\mathcal{L}nn'l} \tilde{a}_{N\mathcal{L}nl,LST}^{n_a l_a n_b l_b} \tilde{a}_{N\mathcal{L}n'l,LST}^{n_c l_c n_d l_d} R_{nl} \left( \frac{r}{\sqrt{2}} \right) R_{n'l} \left( \frac{r}{\sqrt{2}} \right)$$

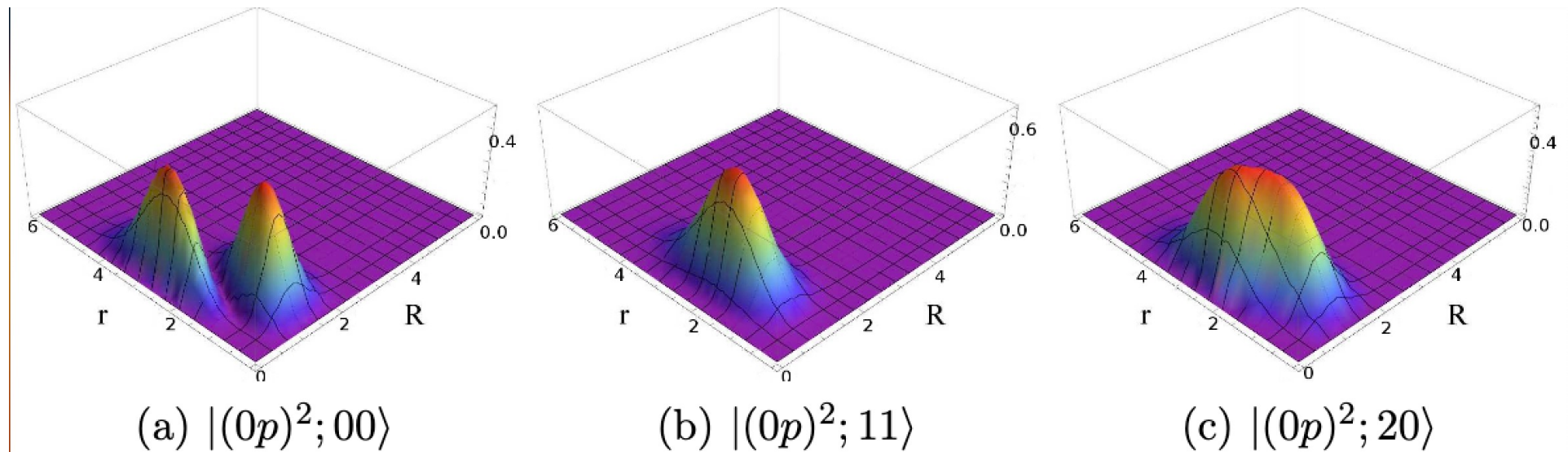
and

$$\begin{aligned} & \langle abLST | \delta(R - \hat{R}_{12}) \delta(r - \hat{r}_{12}) | cdLST \rangle \\ &= R^2 r^2 \sum_{NN'\mathcal{L}} \sum_{nn'l} \tilde{a}_{N\mathcal{L}nl,LST}^{n_a l_a n_b l_b} \tilde{a}_{N'\mathcal{L}n'l,LST}^{n_c l_c n_d l_d} R_{N\mathcal{L}}(\sqrt{2}R) R_{N'\mathcal{L}}(\sqrt{2}R) R_{nl} \left( \frac{r}{\sqrt{2}} \right) R_{n'l} \left( \frac{r}{\sqrt{2}} \right) \end{aligned}$$

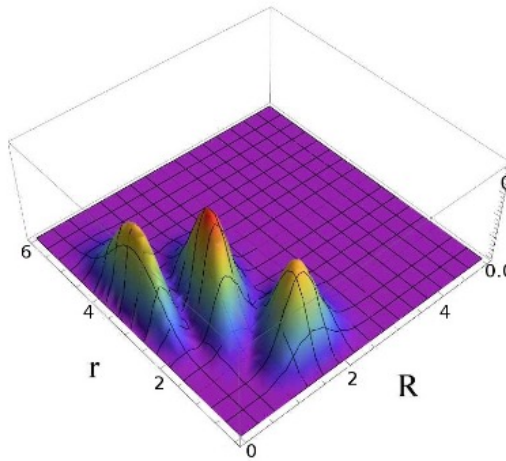
in terms of modified Talmi-Moshinsky brackets

$$\tilde{a}_{N\mathcal{L}nl,LST}^{n_a l_a n_b l_b} \equiv \frac{1 - (-)^{l+S+T}}{\sqrt{2(1 + \delta_{n_a n_b} \delta_{l_a l_b})}} a_{N\mathcal{L}nl,L}^{n_a l_a n_b l_b}$$

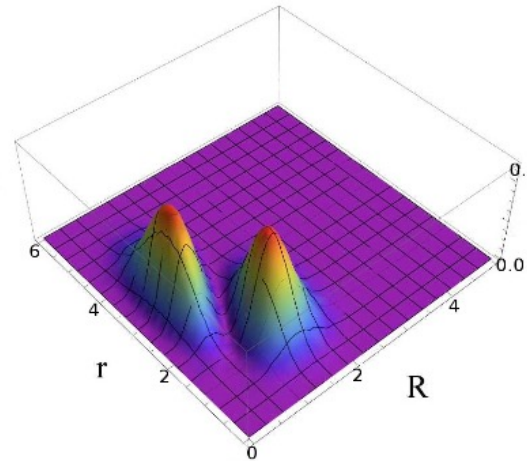
# Two identical nucleons in $0p$ orbital



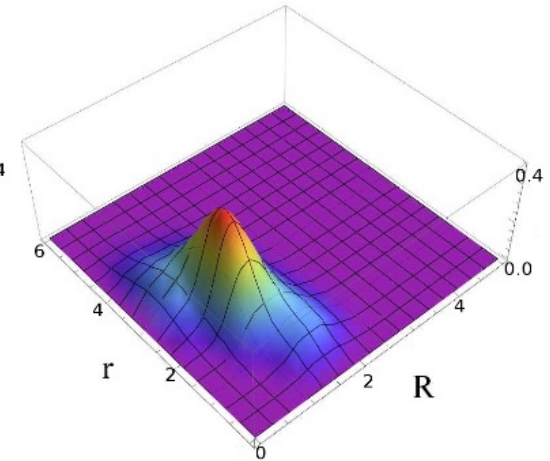
# Two identical nucleons in $0d$ orbital



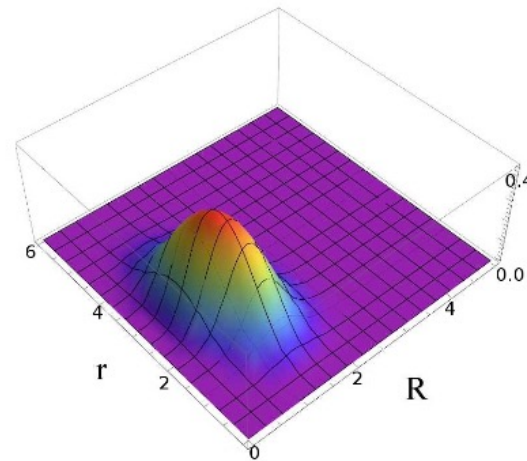
(a)  $|(0d)^2; 00\rangle$



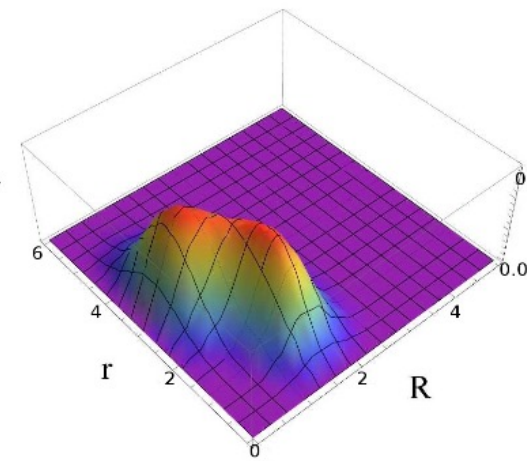
(b)  $|(0d)^2; 11\rangle$



(c)  $|(0d)^2; 20\rangle$



(d)  $|(0d)^2; 31\rangle$



(e)  $|(0d)^2; 40\rangle$

# Four identical nucleons in $0p$

A  $J = 0$  state has  $(LS) = (00)$  or  $(LS) = (11)$  with probability densities given by

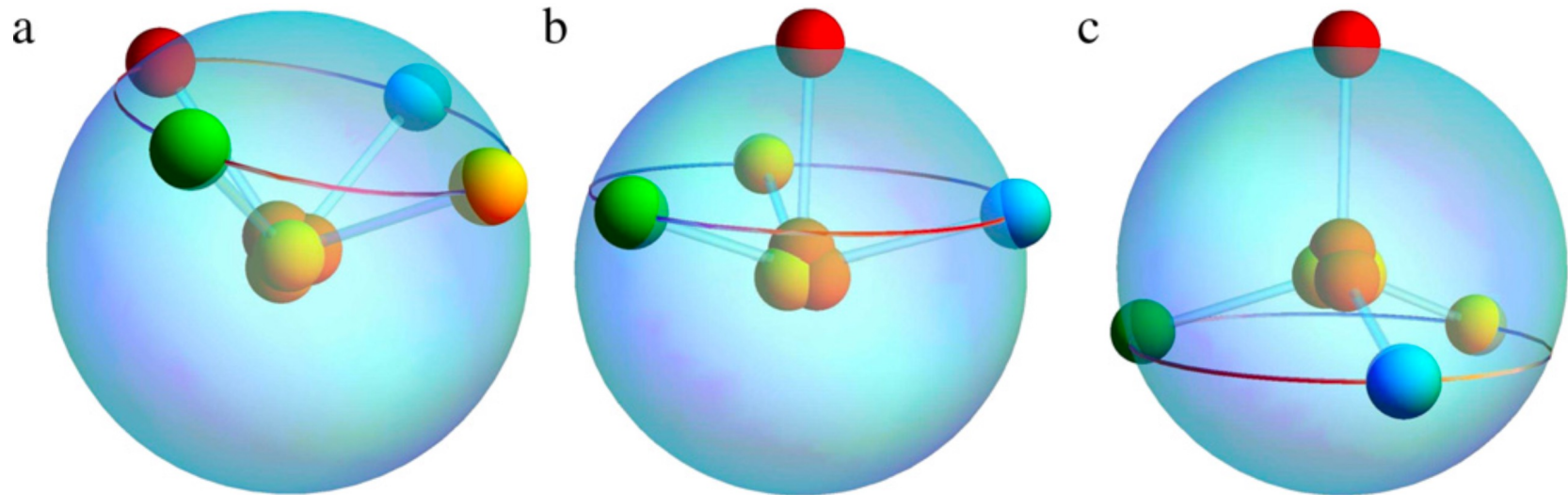
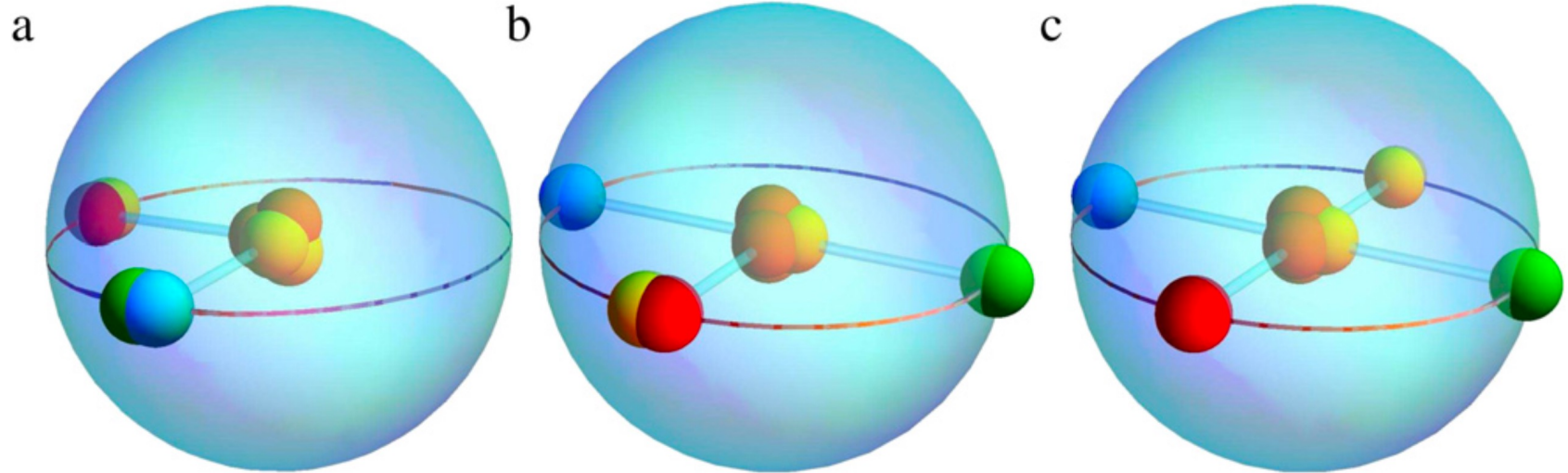
$$\mathcal{P}_{p^4;00} = \frac{9}{1024\pi^4} \sum_{(ij) \neq (kl)} \sin^2 \theta_{ij} \sin^2 \theta_{kl} \cos^2 \theta_{ij,kl}$$

and

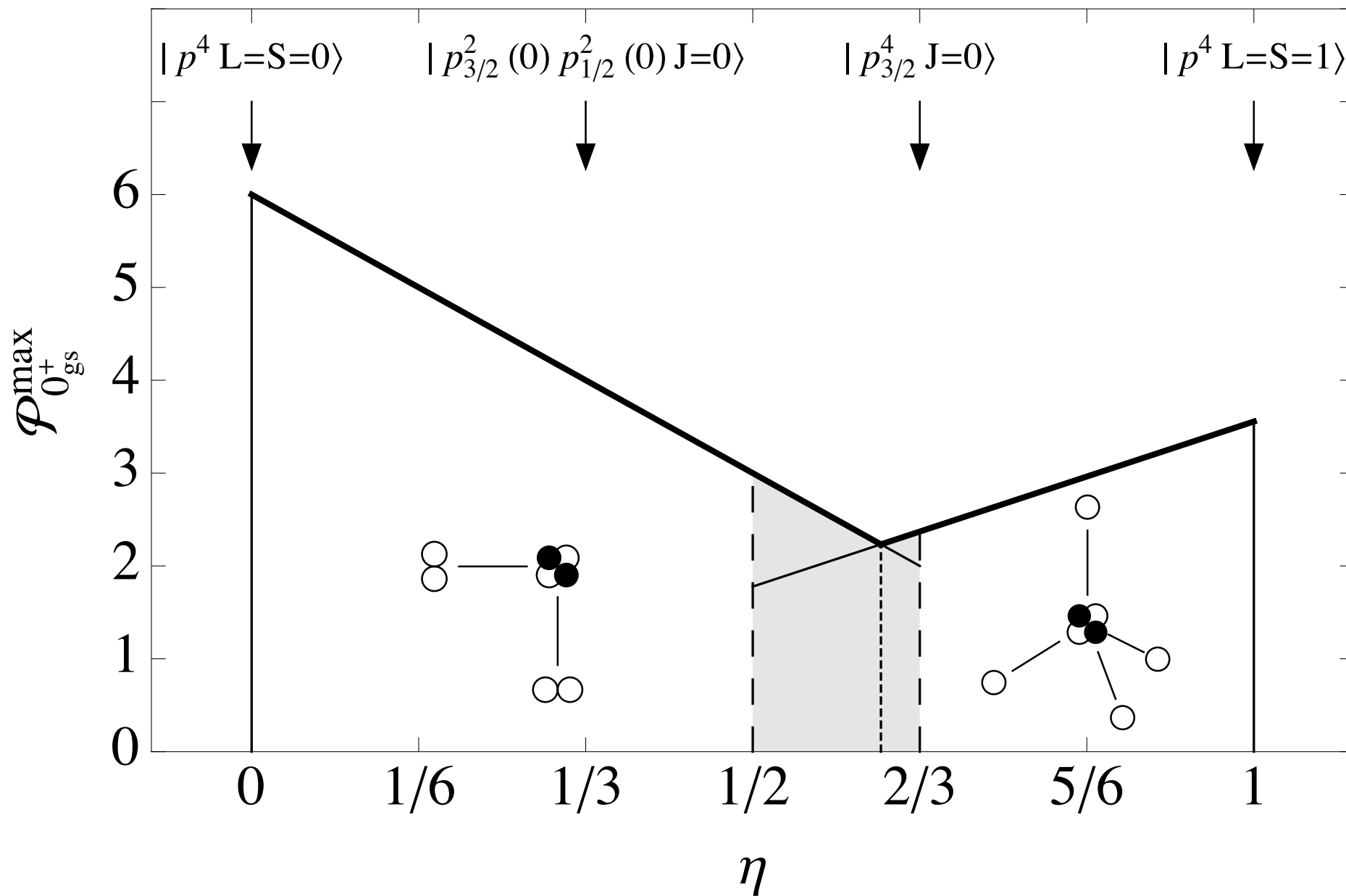
$$\mathcal{P}_{p^4;11} = \frac{9}{2048\pi^4} \sum_{(ij) \neq (kl)} \sin^2 \theta_{ij} \sin^2 \theta_{kl} \sin^2 \theta_{ij,kl}$$

where  $\theta_{ij}$  is the angle between  $\bar{r}_i$  and  $\bar{r}_j$ , and  $\theta_{ij,kl}$  is the angle between  $\bar{r}_{ij}$  and  $\bar{r}_{kl}$ .

# Four identical nucleons in $0p_{3/2}$



# Example: ${}^8\text{He}$





# Four-nucleon $(2\nu, 2\pi)$ correlators

Four-nucleon correlation operators are defined as

$$\hat{\Delta}_{ijkl}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) \equiv \delta(R_{\nu\pi} - \hat{R}_{ij\cdot kl})\delta(r_{\nu\nu} - \hat{r}_{ij})\delta(r_{\pi\pi} - \hat{r}_{kl})$$

with  $(ij \in \nu, kl \in \pi)$  where

$r_{\nu\nu}$  is the distance between the neutrons,

$r_{\pi\pi}$  is the distance between the protons,

$R_{\nu\pi}$  is the distance between the centres of mass of the neutrons and of the protons,

$r_{\nu\pi}$  is the distance between a neutron and a proton.

One has the relation, valid for a tetrahedron,

$$4R_{\nu\pi}^2 = 4r_{\nu\pi}^2 - r_{\nu\nu}^2 - r_{\pi\pi}^2$$

# Four-nucleon correlation function

$$\begin{aligned}
 & \langle ab(L_\nu S_\nu), cd(L_\pi S_\pi); LS | \hat{\Delta}_{1234}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) | a'b'(L'_\nu S'_\nu), c'd'(L'_\pi S'_\pi); LS \rangle \\
 &= \frac{1}{8} R_{\nu\pi}^2 r_{\nu\nu}^2 r_{\pi\pi}^2 \delta_{S_\nu S'_\nu} \delta_{S_\pi S'_\pi} \sum_{\mathcal{L}_\nu \mathcal{L}_\pi} \sum_{\mathcal{L}'_\nu \mathcal{L}'_\pi} \sum_{l_\nu l_\pi} \sum_{\mathcal{L}_\rho l_\rho} \begin{bmatrix} \mathcal{L}_\nu & l_\nu & L_\nu \\ \mathcal{L}_\pi & l_\pi & L_\pi \\ \mathcal{L}_\rho & l_\rho & L \end{bmatrix} \begin{bmatrix} \mathcal{L}'_\nu & l_\nu & L'_\nu \\ \mathcal{L}'_\pi & l_\pi & L'_\pi \\ \mathcal{L}_\rho & l_\rho & L \end{bmatrix} \\
 & \times \sum_{\mathcal{N}_\nu \mathcal{N}_\pi} \left[ \sum_{n_\nu} \tilde{a}_{\mathcal{N}_\nu \mathcal{L}_\nu n_\nu l_\nu, L_\nu S_\nu}^{n_a l_a n_b l_b} R_{n_\nu l_\nu} \left( \frac{r_{\nu\nu}}{\sqrt{2}} \right) \right] \left[ \sum_{n_\pi} \tilde{a}_{\mathcal{N}_\pi \mathcal{L}_\pi n_\pi l_\pi, L_\pi S_\pi}^{n_c l_c n_d l_d} R_{n_\pi l_\pi} \left( \frac{r_{\pi\pi}}{\sqrt{2}} \right) \right] \\
 & \times \sum_{\mathcal{N}'_\nu \mathcal{N}'_\pi} \left[ \sum_{n'_\nu} \tilde{a}_{\mathcal{N}'_\nu \mathcal{L}'_\nu n'_\nu l_\nu, L'_\nu S'_\nu}^{n'_a l'_a n'_b l'_b} R_{n'_\nu l_\nu} \left( \frac{r_{\nu\nu}}{\sqrt{2}} \right) \right] \left[ \sum_{n'_\pi} \tilde{a}_{\mathcal{N}'_\pi \mathcal{L}'_\pi n'_\pi l_\pi, L'_\pi S'_\pi}^{n'_c l'_c n'_d l'_d} R_{n'_\pi l_\pi} \left( \frac{r_{\pi\pi}}{\sqrt{2}} \right) \right] \\
 & \times \sum_{\mathcal{N} \mathcal{L} l} \left[ \sum_n a_{\mathcal{N} \mathcal{L} n l, \mathcal{L}_\rho}^{\mathcal{N}_\nu \mathcal{L}_\nu \mathcal{N}_\pi \mathcal{L}_\pi} R_{n l}(R_{\nu\pi}) \right] \left[ \sum_{n'} a_{\mathcal{N} \mathcal{L} n' l, \mathcal{L}_\rho}^{\mathcal{N}'_\nu \mathcal{L}'_\nu \mathcal{N}'_\pi \mathcal{L}'_\pi} R_{n' l}(R_{\nu\pi}) \right],
 \end{aligned}$$

# The $\alpha$ particle: $0\hbar\omega$

If all nucleons are in the  $0s$  orbital, ( $LST$ ) =  $(000)$ , the probability density is

$$P_{000}^{(0)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{8}{\pi^{3/2}} R_{\nu\pi}^2 r_{\nu\nu}^2 r_{\pi\pi}^2 e^{-R_{\nu\pi}^2 - \frac{1}{2}r_{\nu\nu}^2 - \frac{1}{2}r_{\pi\pi}^2}$$

The probability peaks at  $r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$  and  $R_{\nu\pi} = 1$  or  $r_{\nu\pi} = r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$ .

→ *The most probable geometry of an  $\alpha$  particle is a Platonic tetrahedron.*

# The $\alpha$ particle: $1\hbar\omega$

The isoscalar  $1^-$  state ( $LST$ ) = (100) is spurious and has the probability density

$$P_{100}^{(1)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{8}{\pi^{3/2}} R_{\nu\pi}^2 r_{\nu\nu}^2 r_{\pi\pi}^2 e^{-R_{\nu\pi}^2 - \frac{1}{2}r_{\nu\nu}^2 - \frac{1}{2}r_{\pi\pi}^2}$$

The probability peaks at  $r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$  and  $R_{\nu\pi} = 1$  or  $r_{\nu\pi} = r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$ .

→ *The spurious state has the same density distribution as the  $0\hbar\omega$  ground state since it has the same intrinsic structure.*

# The $\alpha$ particle: $1\hbar\omega$

The isovector  $1^-$  state ( $LST$ ) = (101) has the probability density

$$P_{101}^{(1)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{16}{3\pi^{3/2}} R_{\nu\pi}^4 r_{\nu\nu}^2 r_{\pi\pi}^2 e^{-R_{\nu\pi}^2 - \frac{1}{2}r_{\nu\nu}^2 - \frac{1}{2}r_{\pi\pi}^2}$$

The probability peaks at  $r_{\nu\nu} = r_{\pi\pi} = \sqrt{2}$  and  $R_{\nu\pi} = \sqrt{2}$  or  $r_{\nu\pi} = \sqrt{3}$ .

# The $\alpha$ particle: $1\hbar\omega$

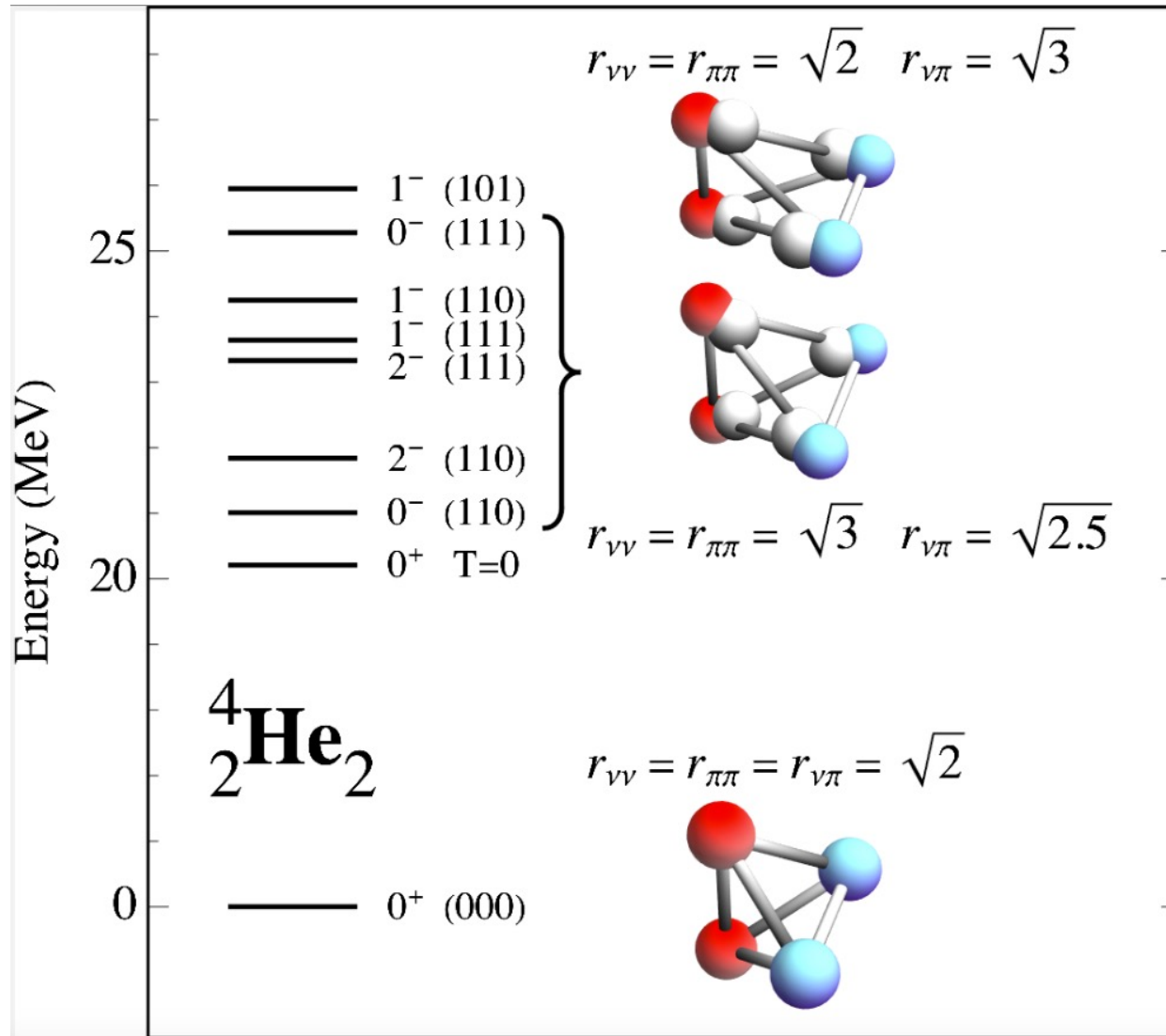
The isoscalar and isovector states with  $(LS) = (11)$  have the probability density

$$P_{110}^{(1)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{4}{3\pi^{3/2}} R_{\nu\pi}^2 r_{\nu\nu}^2 r_{\pi\pi}^2 (r_{\nu\nu}^2 + r_{\pi\pi}^2) e^{-R_{\nu\pi}^2 - \frac{1}{2}r_{\nu\nu}^2 - \frac{1}{2}r_{\pi\pi}^2}$$

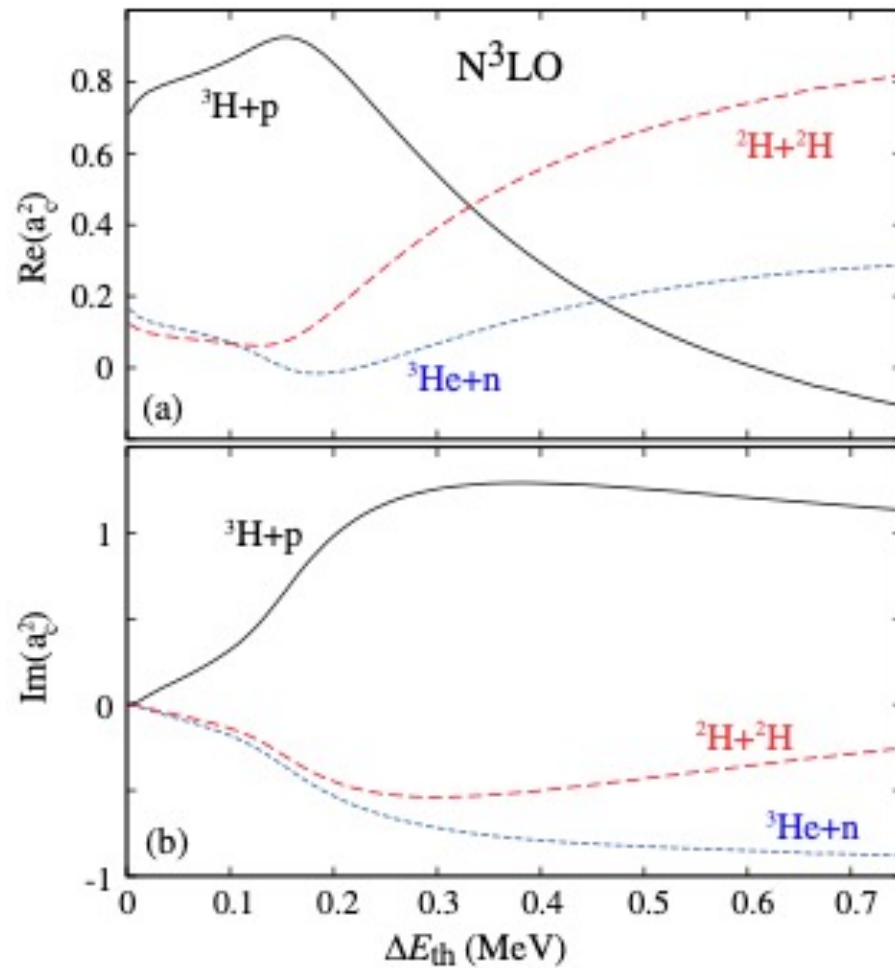
$$P_{111}^{(1)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{4}{3\pi^{3/2}} R_{\nu\pi}^2 r_{\nu\nu}^2 r_{\pi\pi}^2 (r_{\nu\nu}^2 + r_{\pi\pi}^2) e^{-R_{\nu\pi}^2 - \frac{1}{2}r_{\nu\nu}^2 - \frac{1}{2}r_{\pi\pi}^2}$$

The probability peaks at  $r_{\nu\nu} = r_{\pi\pi} = \sqrt{3}$  and  $R_{\nu\pi} = 1$  or  $r_{\nu\pi} = \sqrt{2.5}$ .

# The $\alpha$ particle: $0\hbar\omega + 1\hbar\omega$



# Unbound nuclear states



First-excited  $0^+$  state in  $^4He$  is close to the  $^3H+p$  threshold and adopts its shape. Nuclear chameleonic mimicry.



# The $\alpha$ particle: $0\hbar\omega + 2\hbar\omega$

The tensor force mixes the  $(LS) = (00)$  ground state with  $(LS) = (22)$ , with probability density

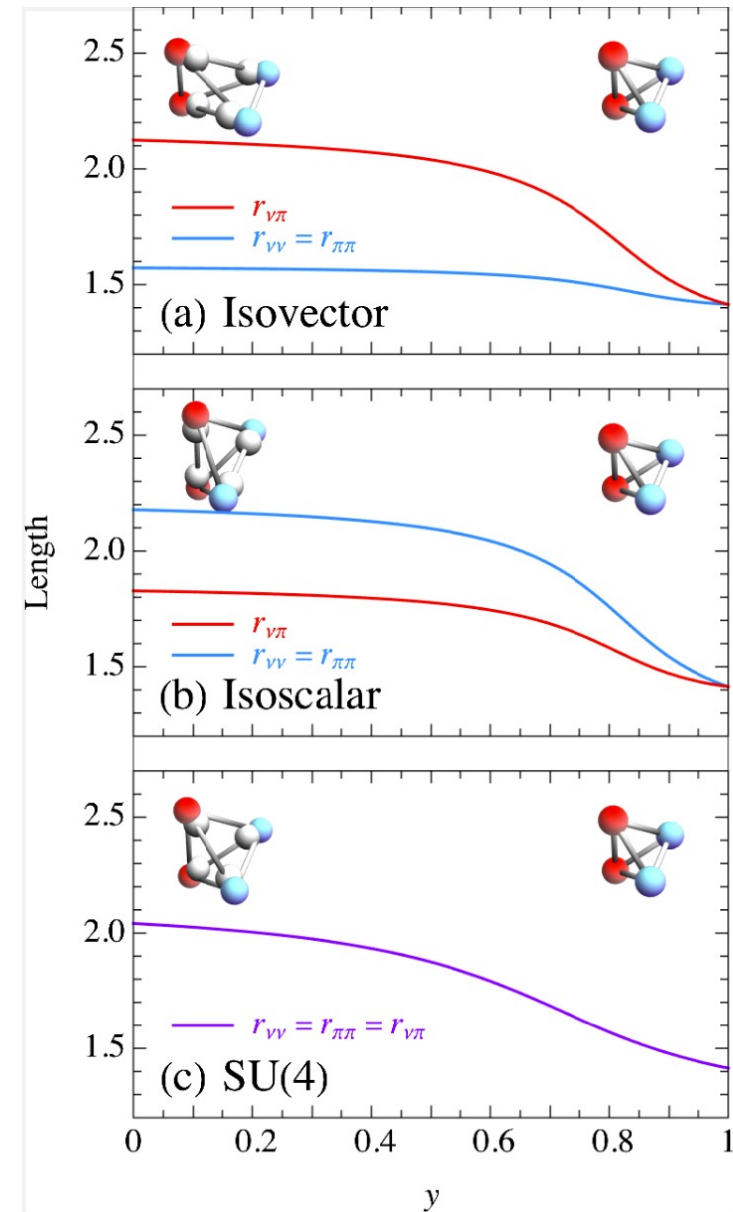
$$P_{220}^{(2)}(R_{\nu\pi}, r_{\nu\nu}, r_{\pi\pi}) = \frac{8}{9\pi^{3/2}} R_{\nu\pi}^2 r_{\nu\nu}^4 r_{\pi\pi}^4 e^{-R_{\nu\pi}^2 - \frac{1}{2}r_{\nu\nu}^2 - \frac{1}{2}r_{\pi\pi}^2}$$

The probability peaks at  $r_{\nu\nu} = r_{\pi\pi} = 2$  and  $R_{\nu\pi} = 1$  or  $r_{\nu\pi} = \sqrt{3}$ .

# The $\alpha$ particle: $0\hbar\omega + 2\hbar\omega$

If  $y = 0$ , orbitals are degenerate,  $\hbar\omega = 0$ .

If  $y = 1$ ,  $\hbar\omega \gg a_T$  and we recover  $s^4$  solution.



# Deuteron and alpha clustering

How to quantify the probability of formation of a deuteron or an  $\alpha$  particle in an arbitrary shell-model state with  $n_\nu$  neutrons and  $n_\pi$  protons?

This probability is obtained by summing over all np pairs or 2n2p quartets with the *intrinsic* structure of a deuteron or an  $\alpha$  particle.

Put all quanta of the pair/quartet in the centre-of-mass coordinate and none in the other degrees of freedom.

# Deuteron cluster probability

Expansion of an eigenstate in an isospin basis

$$|\Phi\Gamma_n\rangle = \sum_{\bar{n}\alpha_n} b_{\bar{n}\alpha_n}^{\Gamma_n} |\Omega^{1\dots n}\bar{n}\alpha_n\Gamma_n\rangle$$

In isospin formalism [ $\hat{P}_d = \sum_{i<j} \hat{P}_d(i,j)$ ]

$$\langle\Phi\Gamma_n|\hat{P}_d|\Phi\Gamma_n\rangle = \frac{n(n-1)}{2} \sum_{\bar{n}\alpha_n} \sum_{\bar{n}'\alpha'_n} b_{\bar{n}\alpha_n}^{\Gamma_n} b_{\bar{n}'\alpha'_n}^{\Gamma_n} \langle\Omega^{1\dots n}\bar{n}\alpha_n\Gamma_n|\hat{P}_d(1,2)|\Omega^{1\dots n}\bar{n}'\alpha'_n\Gamma_n\rangle$$

with

$$\begin{aligned} &\langle\Omega^{1\dots n}\bar{n}\alpha_n\Gamma_n|\hat{P}_d(1,2)|\Omega^{1\dots n}\bar{n}'\alpha'_n\Gamma_n\rangle \\ &= \sum_{\bar{r}\alpha_r\Gamma_r} [\bar{2}\Gamma_d, \bar{r}\alpha_r\Gamma_r|\bar{n}\alpha_n\Gamma_n] [\bar{2}'\Gamma_d, \bar{r}\alpha_r\Gamma_r|\bar{n}'\alpha'_n\Gamma_n] \tilde{a}_{N000,\Gamma_d}^{n_a l n_b l} \tilde{a}_{N'000,\Gamma_d}^{n'_a l' n'_b l'} \end{aligned}$$

# Deuteron cluster probability

Expansion of an eigenstate in an np basis

$$|\Phi\Gamma_n\rangle = \sum_{\Lambda_{nv}\Lambda_{n\pi}} b_{\Lambda_{nv}\Lambda_{n\pi}}^{\Gamma_n} |\Omega_{\nu}^{1_{\nu}\dots n_{\nu}} \Lambda_{nv}, \Omega_{\pi}^{1_{\pi}\dots n_{\pi}} \Lambda_{n\pi}; \Gamma_n\rangle$$

In neutron-proton formalism [ $\hat{P}_d = \sum_{i_{\nu}i_{\pi}} \hat{P}_d(i_{\nu}, i_{\pi})$ ]

$$\begin{aligned} \langle \Phi\Gamma_n | \hat{P}_d | \Phi\Gamma_n \rangle &= n_{\nu} n_{\pi} \sum_{\Lambda_{nv}\Lambda_{n\pi}} \sum_{\Lambda'_{nv}\Lambda'_{n\pi}} b_{\Lambda_{nv}\Lambda_{n\pi}}^{\Gamma_n} b_{\Lambda'_{nv}\Lambda'_{n\pi}}^{\Gamma_n} \\ &\times \langle \Omega_{\nu}^{1_{\nu}\dots n_{\nu}} \Lambda_{nv}, \Omega_{\pi}^{1_{\pi}\dots n_{\pi}} \Lambda_{n\pi}; \Gamma_n | \hat{P}_d(1_{\nu}, 1_{\pi}) | \Omega_{\nu}^{1_{\nu}\dots n_{\nu}} \Lambda'_{nv}, \Omega_{\pi}^{1_{\pi}\dots n_{\pi}} \Lambda'_{n\pi}; \Gamma_n \rangle \end{aligned}$$

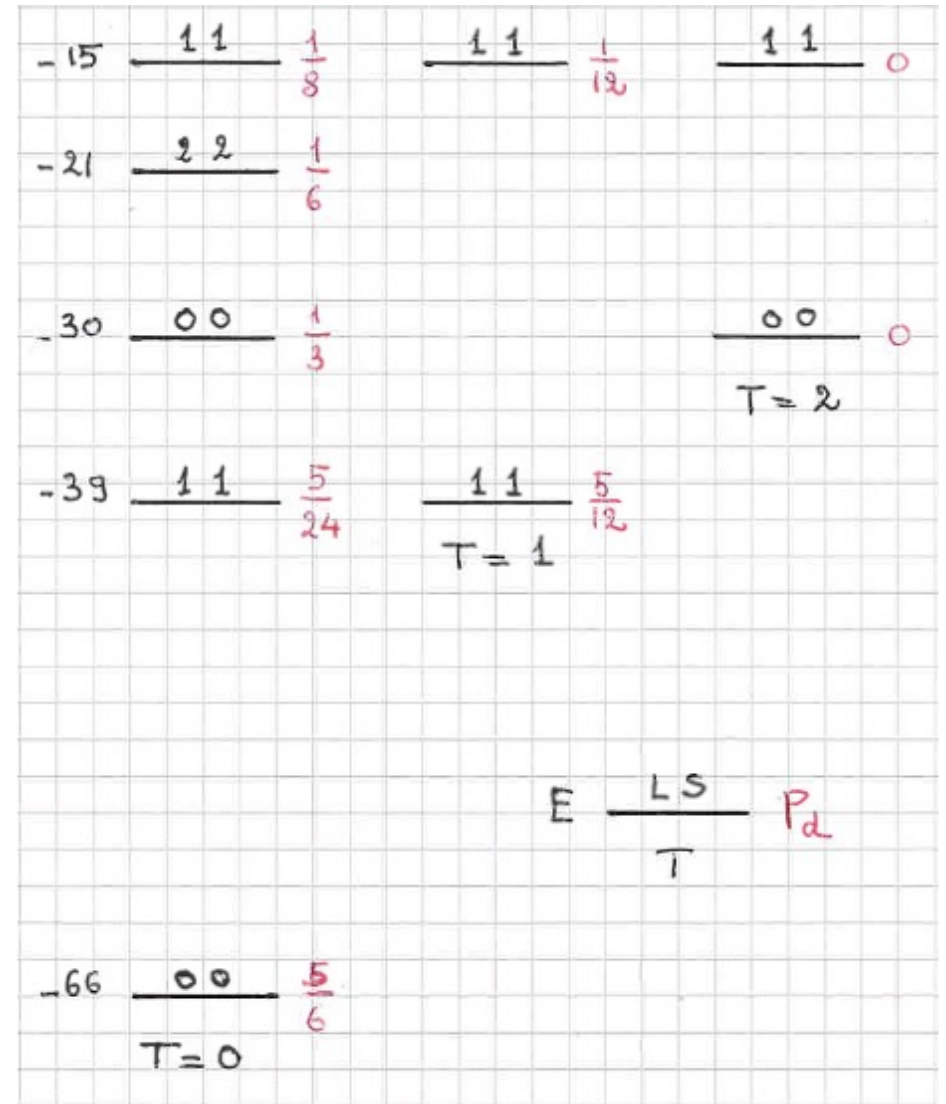
with

$$\begin{aligned} &\langle \Omega_{\nu}^{1_{\nu}\dots n_{\nu}} \Lambda_{nv}, \Omega_{\pi}^{1_{\pi}\dots n_{\pi}} \Lambda_{n\pi}; \Gamma_n | \hat{P}_d(1_{\nu}, 1_{\pi}) | \Omega_{\nu}^{1_{\nu}\dots n_{\nu}} \Lambda'_{nv}, \Omega_{\pi}^{1_{\pi}\dots n_{\pi}} \Lambda'_{n\pi}; \Gamma_n \rangle. \\ &= \sum_{\Lambda_{rv}\Lambda_{r\pi}} [\bar{1}_{\nu}\Gamma_{1\nu}, \Lambda_{rv}] \{\Lambda_{nv}\} [\bar{1}_{\pi}\Gamma_{1\pi}, \Lambda_{r\pi}] \{\Lambda_{n\pi}\} [\bar{1}'_{\nu}\Gamma'_{1\nu}, \Lambda_{rv}] \{\Lambda'_{nv}\} [\bar{1}'_{\pi}\Gamma'_{1\pi}, \Lambda_{r\pi}] \{\Lambda'_{n\pi}\} \\ &\times \sum_{\Gamma_r} \begin{bmatrix} \Gamma_{1\nu} & \Gamma_{rv} & \Gamma_{nv} \\ \Gamma_{1\pi} & \Gamma_{r\pi} & \Gamma_{n\pi} \\ \Gamma_d & \Gamma_r & \Gamma_n \end{bmatrix} \begin{bmatrix} \Gamma'_{1\nu} & \Gamma_{rv} & \Gamma'_{nv} \\ \Gamma'_{1\pi} & \Gamma_{r\pi} & \Gamma'_{n\pi} \\ \Gamma_d & \Gamma_r & \Gamma_n \end{bmatrix} a_{N000,0}^{n_{\nu}l n_{\pi}l} a_{N'000,0}^{n'_{\nu}l' n'_{\pi}l'} \end{aligned}$$

# Example: $(0p)^4 0^+$ states

SDI with isoscalar and isovector strengths  $a_0$  and  $a_1$  and spin-orbit term with strength  $g$ .

For  $g = 0 \rightarrow LS$ -coupled eigenstates.



# $\alpha$ cluster probability

The  $\alpha$ -particle probability of the state

$$|\Phi\Gamma_n\rangle = \sum_{\Lambda_{n\nu}\Lambda_{n\pi}} b_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n} |\Lambda_{n\nu}, \Lambda_{n\pi}; \Gamma_n\rangle$$

is

$$p_\alpha \equiv \langle \Phi\Gamma_n | \hat{P}_\alpha | \Phi\Gamma_n \rangle^2 = \sum_{\Lambda_{r\nu}\Lambda_{r\pi}} \sum_{S=0,1} \left( \sum_{\Lambda_{n\nu}\Lambda_{n\pi}} b_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n} d_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n}^{S\Lambda_{r\nu}\Lambda_{r\pi}} \right)^2$$

$$d_{\Lambda_{n\nu}\Lambda_{n\pi}\Gamma_n}^{S\Lambda_{r\nu}\Lambda_{r\pi}} = \sum_{\Gamma_2} [\Lambda_{2\nu}, \Lambda_{r\nu} | \} \Lambda_{n\nu}] [\Lambda_{2\pi}, \Lambda_{r\pi} | \} \Lambda_{n\pi}] \begin{bmatrix} \Gamma_2 & \Gamma_{r\nu} & \Gamma_{n\nu} \\ \Gamma_2 & \Gamma_{r\pi} & \Gamma_{n\pi} \\ 00 & \Gamma_n & \Gamma_n \end{bmatrix} c_{\Lambda_{2\nu}\Lambda_{2\pi}}^{S00}$$

$$c_{\Lambda_{2\nu}\Lambda_{2\pi}}^{S00} = -\tilde{a}_{\mathcal{N}_\nu L_2 00, L_2 S_2}^{n_{a\nu} l_{a\nu} n_{b\nu} l_{b\nu}} \tilde{a}_{\mathcal{N}_\pi L_2 00, L_2 S_2}^{n_{a\pi} l_{a\pi} n_{b\pi} l_{b\pi}} \begin{bmatrix} 1/2 & 1/2 & S_2 \\ 1/2 & 1/2 & S_2 \\ S & S & 0 \end{bmatrix} a_{\mathcal{N}000,0}^{\mathcal{N}_\nu L_2 \mathcal{N}_\pi L_2}$$

# Example: the *sd* shell

How does the  $\alpha$  cluster probability  $p_\alpha$  change with neutron and proton numbers?

Consider the ground state of even-even nuclei in the *sd* shell.

Three choices of interaction:

- *only isoscalar*
- *only isovector*
- *isoscalar=isovector [SU(4)]*



# Example: the $sd$ shell

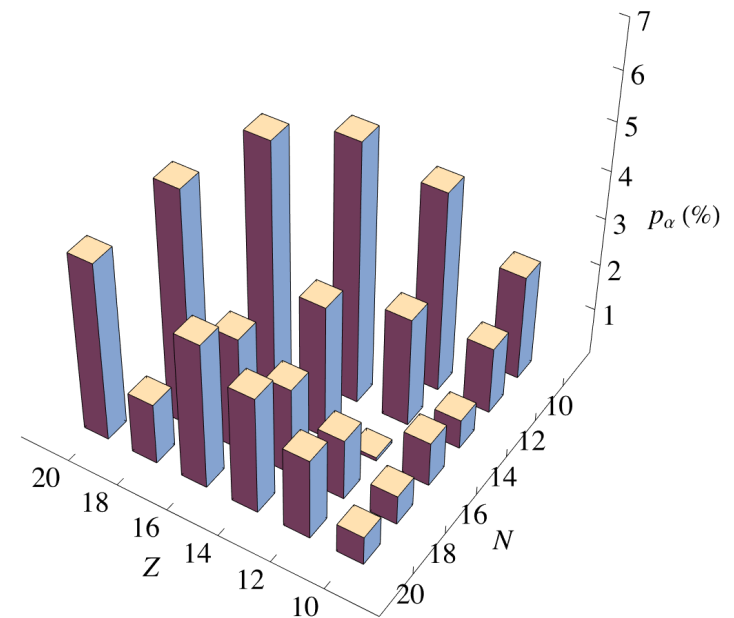
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**sd shell (isoscalar)**



# Example: the $sd$ shell

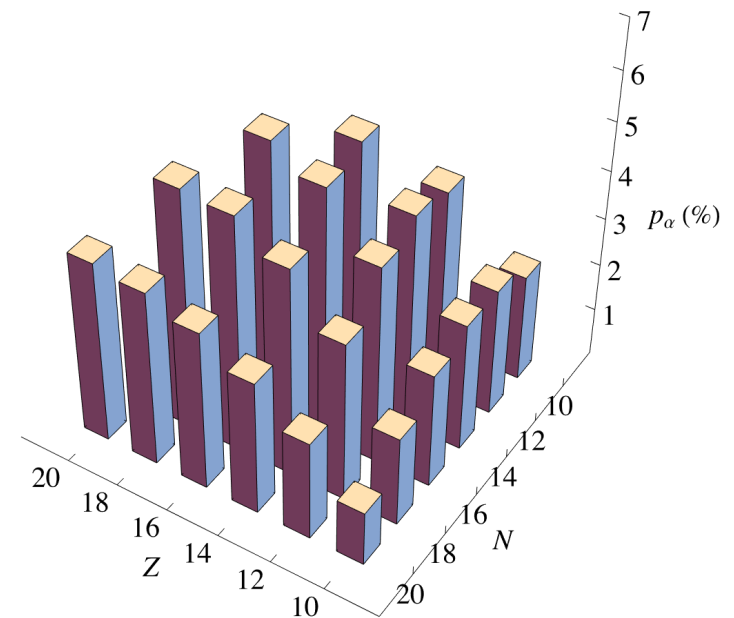
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**sd shell (isovector)**



# Example: the $sd$ shell

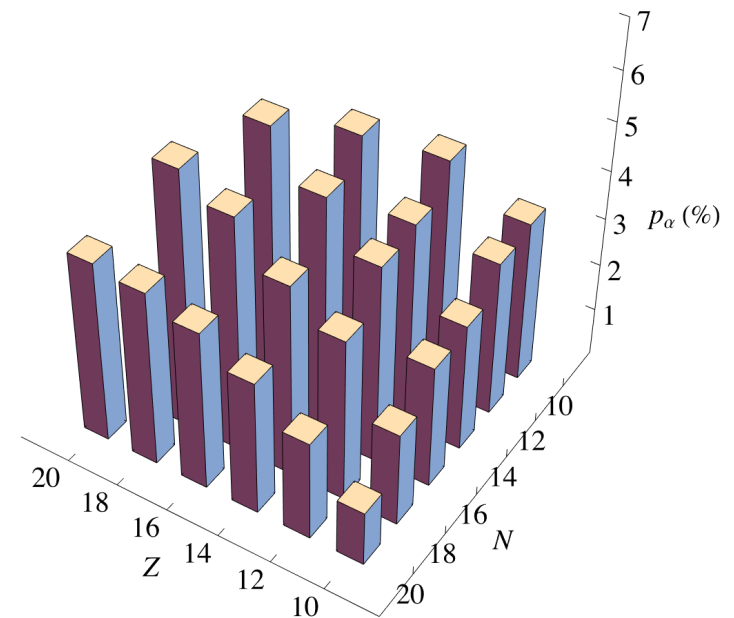
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**sd shell [SU(4)]**



# Closing remarks

Probability of deuteron clustering: Apply to np transfer reactions.

*Experimental program on  $f_{7/2}$  nuclei (M. Assié)*

Probability of  $\alpha$  clustering: What is the effect of cross-shell excitations?

*Example:  $^{20}\text{Ne}$  (Stu Pittel)*

Thank you